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Multiplication operators, integration operators and companion operators on weighted Bloch space

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Abstract. Let g be an analytic function on the open unit disk D in the complex plane $\boldsymbol{C}$. We will study the following operator

$$I_g(h)(z) := \int_0^z h'(\zeta)g(\zeta)d\zeta, \quad J_g(h)(z) := \int_0^z h(\zeta)g'(\zeta)d\zeta$$

on the Bloch space. In this paper, we will study the boundedness and compactness of I_g on the α-Bloch space, and the boundedness and compactness of products of I_g and J_g defined on the α-Bloch space. And we will get the relationship of multiplication operator M_g and the operators I_g, J_g defined on the α-Bloch space.

Key words: multiplication operator, integration operator, Bloch space, boundedness, compactness.

1. Introduction

Let $D = \{z \in \boldsymbol{C}\colon\ |z| < 1\}$ denote the open unit disk in the complex plane $\boldsymbol{C}$ and let $\partial D = \{z \in \boldsymbol{C}\colon\ |z| = 1\}$ denote the unit circle. Let $H(D)$ denote the space of analytic functions on D. For $1 \le p < +\infty$, the Lebesgue space $L^p(D, dA)$ is defined to be the Banach space of Lebesgue measurable functions on the open unit disk D with

$$\| f \|_{L^p(dA)} := \left(\int_D |f(z)|^p dA(z) \right)^{1/p} < +\infty,$$

where $dA(z)$ is the normalized area measure on D. The Bergman space $L^p_a(D)$ is defined to be the subspace of $L^p(D, dA)$ consisting of analytic functions. For $0 < p < +\infty$, the Hardy space H^p is defined to be the Banach space of analytic functions f on D with

$$\| f \|_p := \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < +\infty.$$

The space of analytic functions on D of bounded mean oscillation,

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denoted by $BMOA$, consists of functions f in H^2 for which

$$\| f \|_{BMOA} := |f(0)| + \sup_{z\in D} \| f \circ \varphi_z - f(z) \|_2 < +\infty.$$

Let $\alpha \geq 1$. Then the α-Bloch space B^α of D is defined to be the space of analytic functions f on D such that

$$\| f \|_{B^\alpha} := |f(0)| + \sup_{z\in D}(1-|z|^2)^\alpha |f'(z)| < +\infty.$$

And the little α-Bloch space of D, denoted B_0^α, is the closed subspace of B^α consisting of functions f with $(1-|z|^2)^\alpha f'(z) \to 0$ ($|z| \to 1^-$).

Note that B^1, B_0^1 are the Bloch space B, the little Bloch space B_0, respectively.

Let ω be analytic on $\{\,\zeta\colon |1-\zeta| < 1\,\}$. Suppose that $\left|\omega(1-|z|^2)\right| \to 0$ as $z \in D$ and $|z| \to 1^-$. Then the weighted Bloch space B_ω of D is defined to be the space of analytic functions f on D such that

$$\| f \|_{B_\omega} := |f(0)| + \sup_{z\in D}\left|\omega(1-|z|^2)\right| |f'(z)| < +\infty.$$

For g analytic on D, the operator J_g is defined by the following:

$$J_g(h)(z) := \int_0^z h(\zeta)g'(\zeta)d\zeta.$$

If $g(z) = z$, then J_g is the integration operator. If $g(z) = \log 1/(1-z)$, then J_g is the Cesáro operator. And we also define the companion operator I_g, the multiplication operator M_g by the following:

$$I_g(h)(z) := \int_0^z g(\zeta)h'(\zeta)d\zeta, \quad M_g(h)(z) := g(z)h(z).$$

Let X be a Banach space. For an analytic function g on D, g is a multiplier for X if $gX \subset X$, i.e. $fg \in X$ for all $f \in X$. By the closed-graph theorem, $gX \subset X$ if and only if the multiplication operator M_g is bounded on X. Let $S\colon X \to X$ be a linear operator. Then the operator S is said to be compact operator if for every bounded sequence $\{x_n\}$ in X, $\{S(x_n)\}$ has a convergent subsequence. On the other hand, the operator S is said to be weakly compact operator if for every bounded sequence $\{x_n\}$ in X, $\{S(x_n)\}$ has a weakly convergent subsequence. Then the operator S is weakly compact operator if and only if $S^{**}(X^{**}) \subset X$ where S^{**} be the second adjoint of S and X is identified with its image under the natural embedding into its second dual X^{**}.

In [5], Ch. Pommerenke showed that J_g is a bounded operator on Hardy space H^2 if and only if g is in $BMOA$, and this result was extended to the other Hardy space H^p $1 \le p < +\infty$ in [1]. In [2], A. Aleman and A.G. Siskakis studied the operator J_g defined on the weighted (radial weight) Bergman space. Recently, in [7], A.G. Siskakis and R. Zhao studied the operator J_g defined on $BMOA$.

In [9], we showed the following result about the operator J_g defined on the Bloch space B.

Theorem A *For g analytic on D, the operator J_g is bounded on B if and only if*

$$\sup_{z\in D}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)| < +\infty,$$

and the operator J_g is compact on B if and only if

$$\lim_{|z|\to 1^-}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)| = 0.$$

Let $\alpha > 1$. Then the operator J_g is bounded on B^α if and only if $g \in B$, and the operator J_g is compact on B^α if and only if $g \in B_0$.

In this paper, we will study the boundedness and compactness of I_f defined on the α-Bloch space. And we will give the relationship between multiplication operator M_g and the operators I_g, J_g. In some cases, it is advantageous to think of I_g and J_g as distant cousins of Hankel and Toeplitz operators, respectively. In [8], K. Stroethoff and D. Zheng studied products of Hankel and Toeplitz operators. So we will also study the boundedness and compactness of products of I_f and J_g.

Throughout this paper, C will denote positive constant whose value is not necessary the same at each occurrence.

2. The operators I_f defined on the α-Bloch space

In this section, we study the boundedness and compactness of I_f on the α-Bloch space.

Lemma B *Let $\alpha \geq 1$. Then there exist $h_1, h_2 \in B^\alpha$ such that*

$$|h_1'(z)| + |h_2'(z)| \geq \frac{1}{(1-|z|)^\alpha} \quad (z \in D).$$

Proof. See Proposition 5.4 in [6]. □

Theorem 2.1 *Let $\alpha \geq 1$ and f be an analytic function on D. Then the operator I_f is bounded on B^α if and only if $\sup_{z\in D} |f(z)| < +\infty$.*

Proof. Let $\alpha \geq 1$. Let f be an analytic function on D. If $f \in H^\infty$, it is trivial that I_f is bounded on B^α. To prove the converse, suppose that I_f is bounded on B^α. By Lemma B, there exist $h_1, h_2 \in B^\alpha$ such that

$$|h_1'(z)| + |h_2'(z)| \geq \frac{1}{(1-|z|)^\alpha},$$

for all $z \in D$. So for any $z \in D$, we have

$$\begin{aligned} |f(z)| &\leq (1-|z|^2)^\alpha \big(|h_1'(z)| + |h_2'(z)|\big)|f(z)| \\ &\leq \sup_{z\in D}(1-|z|^2)^\alpha |h_1'(z)|\,|f(z)| + \sup_{z\in D}(1-|z|^2)^\alpha |h_2'(z)|\,|f(z)| \\ &= \| I_f h_1 \|_{B^\alpha} + \| I_f h_2 \|_{B^\alpha} \\ &\leq \| I_f \| \, \| h_1 \|_{B^\alpha} + \| I_f \| \, \| h_2 \|_{B^\alpha} < +\infty. \end{aligned}$$

Hence we have $\sup_{z\in D} |f(z)| < +\infty$. □

Corollary 2.2 *For g analytic on D, the following are equivalent:*

(i) $gB \subset B$;
$(i)'$ $gB_0 \subset B_0$;
(ii) *Both I_g and J_g are bounded operators on B.*
$(ii)'$ *Both I_g and J_g are bounded operators on B_0.*
(iii) $g \in H^\infty$, $\sup_{z\in D}(1-|z|^2)\left(\log \frac{1}{1-|z|^2}\right)|g'(z)| < +\infty$.

Let $\alpha > 1$. For g analytic on D, the following are equivalent:

(i) $gB^\alpha \subset B^\alpha$;
$(i)'$ $gB_0^\alpha \subset B_0^\alpha$;
(ii) *I_g is bounded operator on B^α;*
$(ii)'$ *I_g is bounded operator on B_0^α;*
(iii) $g \in H^\infty$.

Proof. The above equivalences of (i), $(i)'$, (iii) were proved by [3] and [12].

The other equivalences are immediate consequences of Theorem 2.1 and Theorem 1 in [3]. □

Theorem 2.3 *Let $\alpha \geq 1$. Let f be an analytic function on D. Then the operator I_f is compact on B^α if and only if* $f \equiv 0 \quad \cdots (*)$.

Proof. Let $\alpha = 1$. Let f be an analytic function on D. Since $|h(z)| \leq C\| h \|_B \log\bigl(1/(1-|z|^2)\bigr)$ for $h \in B$, the unit ball of B is a normal family of analytic functions. By normal family arguments, I_f is compact operator on B if and only if every sequence $\{h_n\}$ in B with $\| h_n \|_B \leq 1$ and $h_n \to 0$ $(n \to +\infty)$ uniformly on compact subsets of D has a subsequence $\{h_{n_k}\}$ in B such that $\| I_f h_{n_k} \|_B \to 0$ $(k \to +\infty)$.

We show that every sequence which goes to zero has a subsequence such that the condition $(*)$ holds when the limit is taken over that sequence. This implies that the condition $(*)$ holds.

Suppose that the operator I_f is compact on B. Let $a_n \to a \in \partial D$ and consider the test functions $h_n(z) := \log\bigl(1/(1-\overline{a_n}z)\bigr)$, $h(z) := \log\bigl(1/(1-\overline{a}z)\bigr)$. Then $h_n \to h$ uniformly on compact subsets of D. Using the fact $|c+d|^2 \leq 2|c|^2 + 2|d|^2$ and the subharmonicity of $|f(z)|$,

$$\sup_{a\in D} \int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2} < \infty,$$

we have

$$\begin{aligned}
&|a_n|^2 |f(a_n)|^2 \\
&\quad \leq C \frac{|a_n|^2}{(1-|a_n|^2)^2} \int_{D(a_n,r)} |f(z)|^2 dA(z) \\
&\quad \leq CK \int_{D(a_n,r)} \left| \left(\log \frac{1}{1-\overline{a_n}z} \right)' \right|^2 |f(z)|^2 dA(z) \\
&\quad = CK \int_{D(a_n,r)} (1-|z|^2)^2 \left| \left(\log \frac{1}{1-\overline{a_n}z} \right)' \right|^2 |f(z)|^2 \frac{dA(z)}{(1-|z|^2)^2} \\
&\quad \leq CK \sup_{z\in D(a_n,r)} (1-|z|^2)^2 \left| \left(\log \frac{1}{1-\overline{a_n}z} \right)' \right|^2 |f(z)|^2 \\
&\qquad \times \int_{D(a_n,r)} \frac{dA(z)}{(1-|z|^2)^2}
\end{aligned}$$

$$
\begin{aligned}
&\le 2CK \int_{D(a_n,r)} \frac{dA(z)}{(1-|z|^2)^2} \\
&\quad \times \sup_{z\in D(a_n,r)} \left| \left(\log \frac{1}{1-\overline{a_n}z}\right)' - \left(\log \frac{1}{1-\overline{a}z}\right)' \right|^2 |f(z)|^2 (1-|z|^2)^2 \\
&\quad + 2CK \int_{D(a_n,r)} \frac{dA(z)}{(1-|z|^2)^2} \\
&\quad \times \sup_{z\in D(a_n,r)} \left| \left(\log \frac{1}{1-\overline{a}z}\right)' \right|^2 |f(z)|^2 (1-|z|^2)^2 \\
&\le 2CK \int_{D(a_n,r)} \frac{dA(z)}{(1-|z|^2)^2} \parallel I_f(h_n - h) \parallel_B^2 \\
&\quad + 2CK \int_{D(a_n,r)} \frac{dA(z)}{(1-|z|^2)^2} \\
&\quad \times \sup_{z\in D(a_n,r)} \left| \log\left(\frac{1}{1-\overline{a}z}\right)' \right|^2 |f(z)|^2 (1-|z|^2)^2 \\
&=: M_1 + M_2.
\end{aligned}
$$

By the compactness of I_f, we have $M_1 \to 0$ $(n \to \infty)$. Since B_0 is a subspace of B and they share the same norm, the compactness of I_f on B implies its compactness on B_0. Hence we see that I_f is weakly compact on B_0. Since $(B_0)^{**} = B$ and $I_f^{**} = I_f$, by using the fact of the introduction, we have $I_f(B) \subset B_0$. Thus we have $I_f(h) \in B_0$. Thus we have

$$
\begin{aligned}
M_2 &= \sup_{z\in D(a_n,r)} \left| \left(\log \frac{1}{1-\overline{a}z}\right)' \right|^2 |f(z)|^2 (1-|z|^2)^2 \\
&= \sup_{z\in D(a_n,r)} \big((1-|z|^2) \big| (I_f(h))'(z) \big| \big)^2 \\
&= \sup_{z\in D} \big(\chi_{D(a_n,r)}(z) (1-|z|^2) \big| (I_f(h))'(z) \big| \big)^2.
\end{aligned}
$$

Hence we have $M_2 \to 0$ $(n \to +\infty)$. So we have $\lim_{|a_n|\to 1^-} |f(a_n)| = 0$. Since $f \in H^\infty$, thus we see $f \equiv 0$. The proof of the converse is trivial.

Let $\alpha > 1$. Let f be an analytic function on D. Since $|h(z)| \le C\parallel h \parallel_{B^\alpha} (1-|z|^2)^{1-\alpha}$ for $h \in B^\alpha$, the unit ball of B^α is a normal family of analytic functions. By normal family arguments, I_f is a compact operator on B^α if and only if every sequence $\{h_n\}$ in B^α with $\parallel h_n \parallel_{B^\alpha} \le 1$ and $h_n \to 0$ $(n \to +\infty)$ uniformly on compact subsets of D has a subsequence $\{h_{n_k}\}$ in B^α such that $\| I_f h_{n_k} \|_{B^\alpha} \to 0$ $(k \to +\infty)$.

We show that every sequence which goes to zero has a subsequence such that the condition $(*)$ holds when the limit is taken over that sequence. This implies that the condition $(*)$ holds.

Suppose that the operator I_f is compact on B^α. Let $a_n \to a \in \partial D$ and consider the test functions $h_{a_n}(z) := (1-\overline{a_n}z)^{1-\alpha}$, $h_a(z) := (1-\overline{a}z)^{1-\alpha}$. Then $h_{a_n}(z) \to h_a(z)$ uniformly on compact subsets of D. Then we have by using $\sup_{a\in D}\int_{D(a,r)} dA(z)/(1-|z|^2)^2 < \infty$

$$\begin{aligned}
&|a_n|^2|f(a_n)|^2 \\
&\quad \le C\int_{D(a_n,r)} \frac{dA(z)}{(1-|z|^2)^2} \sup_{z\in D(a_n,r)} (1-|z|^2)^{2\alpha}|h'_{a_n}(z)|^2|f(z)|^2 \\
&\quad \le 2C\int_{D(a_n,r)} \frac{dA(z)}{(1-|z|^2)^2} \\
&\qquad \times \sup_{z\in D(a_n,r)} (1-|z|^2)^{2\alpha}|h'_{a_n}(z)-h'_a(z)|^2|f(z)|^2 \\
&\qquad + 2C\int_{D(a_n,r)} \frac{dA(z)}{(1-|z|^2)^2} \sup_{z\in D(a_n,r)} (1-|z|^2)^{2\alpha}|h'_a(z)|^2|f(z)|^2 \\
&\quad \le 2C\int_{D(a_n,r)} \frac{dA(z)}{(1-|z|^2)^2} \| I_f(h_{a_n}-h_a) \|^2_{B^\alpha} \\
&\qquad + 2C\int_{D(a_n,r)} \frac{dA(z)}{(1-|z|^2)^2} \sup_{z\in D(a_n,r)} (1-|z|^2)^{2\alpha}|h'_a(z)|^2|f(z)|^2 \\
&\quad =: N_1 + N_2.
\end{aligned}$$

By the compactness of I_f, we have $N_1 \to 0$ $(n\to\infty)$. Since B_0^α is a subspace of B^α and they share the same norm, the compactness of I_f on B^α implies its compactness on B_0^α. Hence we see that I_f is weakly compact on B_0^α. Since $(B_0^\alpha)^{**} = B^\alpha$ (see [12]) and $I_f^{**} = I_f$, by using the fact of the introduction, we have $I_f(B^\alpha) \subset B_0^\alpha$. Thus we have $I_f(h_a) \in B_0^\alpha$. Thus we have

$$\begin{aligned}
N_2 &= \sup_{z\in D(a_n,r)} (1-|z|^2)^{2\alpha}|h'_a(z)|^2|f(z)|^2 \\
&= \sup_{z\in D(a_n,r)} \big((1-|z|^2)^\alpha\big|(I_f(h_a))'(z)\big|\big)^2 \\
&= \sup_{z\in D}\big(\chi_{D(a_n,r)}(z)(1-|z|^2)^\alpha\big|(I_f(h_a))'(z)\big|\big)^2.
\end{aligned}$$

Hence we have $N_2 \to 0$ $(n \to +\infty)$. So we have $\lim_{|a_n|\to 1^-} |f(a_n)| = 0$. Since $f \in H^\infty$, thus we see $f \equiv 0$. The proof of the converse is trivial. $\square$

3. The operators I_f defined on the weighted Bloch space

In this section, we study the boundedness of I_f on the weighted Bloch space B_ω. And we will give the relationship between multiplication operator M_g and the operators I_g, J_g defined on the weighted Bloch space B_ω. The examples of the weighted Bloch space B_ω are the α-Bloch space and $\{\, f \in H(D)\colon \sup_{z\in D}(1-|z|^2)\bigl(\log(1/(1-|z|^2))\bigr)|f'(z)| < \infty \,\}$, and so on.

Theorem 3.1 *Let $0<r<+\infty$. Let ω be analytic on D and non-vanishing on $\{\,\zeta\colon |1-\zeta| < 1\,\}$. Suppose that $\sup_{z,\,a\in D} \left|\omega(1-|z|^2)\right| / |\omega(1-\overline{a}z)| < \infty$, and that for any $a \in D$ there is a constant $C > 0$ (independent of a) such that $|\omega(1-\overline{a}z)| / \left|\omega(1-|z|^2)\right| \le C$ for all $z \in D(a,r)$. Let f be an analytic function on D. Then the operator I_f is bounded on B_ω if and only if*

$$\sup_{z\in D} |f(z)| < +\infty.$$

Proof. Let f be an analytic function on D. Suppose that $\|\, f \,\|_\infty = \sup_{z\in D} |f(z)| < +\infty$. Then

$$\|\, I_f(g) \,\|_{B_\omega} = \sup_{z\in D} \left|\omega(1-|z|^2)\right| |g'(z)f(z)| \le \|\, f \,\|_\infty \|\, g \,\|_{B_\omega}.$$

Hence we see that I_f is bounded on B_ω.

Next, we prove the converse. Suppose that I_f is bounded on B_ω. Put $h_a(z) := \int_0^z 1/\omega(1-\overline{a}\eta)\, d\eta$. Then it is clear that $h_a \in B_\omega$ for all $a \in D$ because of the assumption $\sup_{z,\,a\in D} \left|\omega(1-|z|^2)\right| / |\omega(1-\overline{a}z)| < +\infty$. Since

$$\sup_{a\in D} \int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2} < +\infty \quad \text{and} \quad \frac{1}{\left|\omega(1-|z|^2)\right|} \le \frac{C}{|\omega(1-\overline{a}z)|}$$

for all $z \in D(a,r)$, for any $a \in D$

$$\begin{aligned} &|f(a)|^2 \\ &\quad \le K \int_{D(a,r)} \frac{1}{\left|\omega(1-|z|^2)\right|^2} \left|\omega(1-|z|^2)\right|^2 |f(z)|^2 \frac{dA(z)}{(1-|z|^2)^2} \\ &\quad \le KC^2 \int_{D(a,r)} \frac{1}{\left|\omega(1-\overline{a}z)\right|^2} \left|\omega(1-|z|^2)\right|^2 |f(z)|^2 \frac{dA(z)}{(1-|z|^2)^2} \end{aligned}$$

$$
\begin{aligned}
&= KC^2 \int_{D(a,r)} \left|(h_a(z))'\right|^2 \left|\omega(1-|z|^2)\right|^2 |f(z)|^2 \frac{dA(z)}{(1-|z|^2)^2} \\
&\le KC^2 \int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2} \sup_{z\in D(a,r)} \left|(h_a(z))'\right|^2 \left|\omega(1-|z|^2)\right|^2 |f(z)|^2 \\
&\le KC^2 \int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2} \| I_f h_a \|_{B_\omega}^2 \\
&\le KC^2 \int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2} \| I_f \|^2 \| h_a \|_{B_\omega}^2 \\
&\le KC^2 \int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2} \| I_f \|^2 \sup_{a\in D} \| h_a \|_{B_\omega}^2 < +\infty.
\end{aligned}
$$

Hence we see $\sup_{z\in D} |f(z)| < +\infty$. □

We proved the following proposition in [9].

Proposition 3.2 *Let $0 < r < +\infty$. Let ω be analytic and non-vanishing on $\{\zeta : |1-\zeta| < 1\}$. Suppose that $\sup_{z\in D}\left|\omega(1-|z|^2)\right| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} < +\infty$ and $\sup_{z,a\in D} \frac{|\omega(1-|z|^2)|}{|\omega(1-\bar{a}z)|} < +\infty$, and $\int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} < +\infty$ for any $z \in D$ and $\int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} \to \infty$ $(|z| \to 1^-)$, and that for any $a \in D$ there is a constant $C > 0$ (independent of a) such that $\left|\frac{\omega(1-\bar{a}z)}{\omega(1-|z|^2)}\right| \le C$ for all $z \in D(a,r)$, and that there is a constant $K > 0$ (independent of z) such that $\int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} \le K\left|\int_0^z \frac{1}{\omega(1-\bar{z}\eta)} d\eta\right|$ for all $z \in D$. Then the operator J_g is bounded on B_ω if and only if*

$$
\| g \|_W := \sup_{z\in D}\left|\omega(1-|z|^2)\right| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |g'(z)| < +\infty.
$$

Proof. See [9]. □

Corollary 3.3 *Let $0 < r < +\infty$. Let ω be as* Proposition 3.2. *Then for g analytic on D, the following are equivalent*:

(i) $gB_\omega \subset B_\omega$;

(ii) *Both I_g and J_g are bounded operators on B_ω;*

(iii) $g \in H^\infty$, $\sup_{z\in D} \omega(1-|z|^2) \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |g'(z)| < +\infty.$

Proof. The equivalence of (ii) and (iii) is an immediate consequence of Theorem 3.1 and Proposition 3.2. If (ii) holds, it is trivial that (i) holds.

So it suffices to prove that (i) implies (ii). In fact, suppose that $gB_\omega \subset B_\omega$. For arbitrary $a \in D$, let $h_a(z) := \int_0^z \frac{1}{\omega(1-\overline{a}\eta)} d\eta$. And let $k_a(z) := h_a(z) - h_a(a)$. Then we see that $k_a'(z) = \frac{1}{\omega(1-\overline{a}z)}$, $k_a(a) = 0$ and $k_a \in B_\omega$ because of $\sup_{z,\,a\in D} \frac{|\omega(1-|z|^2)|}{|\omega(1-\overline{a}z)|} < +\infty$. So for g analytic on D, we have $(gk_a)'(a) = g(a)\frac{1}{\omega(1-|a|^2)}$. By using the boundedness of M_g on B_ω and the subharmonic property of $|\omega(1-\overline{a}z)|\,|(gk_a)'(z)|$ and the assumption that for any $a \in D$ there is a constant $C > 0$ (independent of a) such that $\frac{|\omega(1-\overline{a}z)|}{|\omega(1-|z|^2)|} \le C$ for all $z \in D(a, r)$,

$$
\begin{aligned}
|g(a)| &= \left|\omega(1-|a|^2)\right| \left|(gk_a)'(a)\right| \\
&\le K \int_{D(a,r)} |\omega(1-\overline{a}z)| \left|(gk_a)'(z)\right| \frac{dA(z)}{(1-|z|^2)^2} \\
&\le KC \int_{D(a,r)} \left|\omega(1-|z|^2)\right| \left|(gk_a)'(z)\right| \frac{dA(z)}{(1-|z|^2)^2} \\
&\le KC \int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2} \parallel M_g k_a \parallel_{B_\omega} \\
&\le KC \int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2} \parallel M_g \parallel \parallel k_a \parallel_{B_\omega} \\
&\le KC \sup_{a\in D} \int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2} \parallel M_g \parallel \sup_{a\in D} \parallel k_a \parallel_{B_\omega} < +\infty.
\end{aligned}
$$

Hence we have $g \in H^\infty$. By Theorem 3.1, $g \in H^\infty$ if and only if I_g is bounded on B_ω. So by the boundedness of I_g and M_g on B_ω, we see that J_g is bounded on B_ω. □

Remark Carefully examining the proof of the above corollary, we see that the equivalence of (i) and (ii) can be proved by using the assumption of ω in Theorem 3.1 only. □

Based on Corollary 3.3, we furthermore make the following conjecture.

Conjecture Let $0 < r < +\infty$. Let ω be as Proposition 3.2. And suppose that $\frac{|\omega(1-|z|^2)|}{1-|z|^2} \to 0$ ($|z| \to 1^-$). Then for g analytic on D, the following are equivalent:

(i) $gB_\omega \subset B_\omega$;
(ii) I_g is a bounded operator on B_ω;
(iii) $g \in H^\infty$.

Supposing the assumption that there is a constant $C > 0$ (independent of z) such that $\left|\omega(1-|z|^2)\right| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} \le C(1-|z|^2)$ for all $z \in D$ which is stronger than the assumption $\frac{\left|\omega(1-|z|^2)\right|}{1-|z|^2} \to 0$ $(|z| \to 1^-)$, we see that the above conjecture holds.

4. The product of the operators I_f and J_g defined on the α-Bloch space

In this section, we study the boundedness and compactness of products of I_f and J_g defined on the α-Bloch space.

Theorem 4.1 *Let f be an analytic function on D and g be an analytic function on D. Suppose that $\sup_{z\in D}|f(z)| < +\infty$. Then the operator I_fJ_g is bounded on B if and only if*

$$\sup_{z\in D}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|\,|f(z)| < +\infty.$$

Let $\alpha > 1$. Then the operator I_fJ_g is bounded on B^α if and only if

$$\sup_{z\in D}(1-|z|^2)|g'(z)|\,|f(z)| < +\infty.$$

Proof. Let $\alpha = 1$. Let f be a function on D. First, suppose that

$$\sup_{z\in D}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|\,|f(z)| < +\infty.$$

Then we see

$$I_fJ_gh(z) = \int_0^z h(\zeta)g'(\zeta)f(\zeta)d\zeta.$$

Since $|h(z)| \le C\| h \|_B \log\frac{1}{1-|z|^2}$ for $h \in B$, we have

$$\begin{aligned}(1-|z|^2)&|(I_fJ_gh)'(z)|\\ &= (1-|z|^2)|h(z)g'(z)f(z)|\\ &\le \sup_{z\in D}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|\,|f(z)|\,\| h \|_B.\end{aligned}$$

Hence

$$\| I_fJ_gh \|_B \le \sup_{z\in D}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|\,|f(z)|\,\| h \|_B.$$

To prove the converse, we suppose that $I_f J_g$ is bounded on B. For $a \in D$, it is clear that the test function $h(z) := \log(1-\overline{a}z) \in B$. Since $I_f J_g$ is bounded on B, for any $z \in D$

$$\begin{aligned}&(1-|z|^2)\big|\log(1-\overline{a}z)\big|\,|g'(z)|\,|f(z)|\\&\quad=(1-|z|^2)|h(z)|\,|g'(z)|\,|f(z)|\\&\quad\le \|\, I_f J_g h\,\|_B \le \|\, I_f J_g\,\|\,\|\, h\,\|_B \le K < +\infty.\end{aligned}$$

Applying $z = a$, we have

$$(1-|a|^2)\left(\log\frac{1}{1-|a|^2}\right)|g'(a)|\,|f(a)| \le K,$$

for any $a \in D$. Hence we have

$$\sup_{z\in D}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|\,|f(z)| < +\infty.$$

In the case of $\alpha > 1$, we can prove it as well. So we omit it. □

Theorem 4.2 *Let f be function on D and g be an analytic function on D. Suppose that $\sup_{z\in D}|f(z)| < +\infty$. Then the operator $I_f J_g$ is compact on B if and only if*

$$\lim_{|z|\to 1^-}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|\,|f(z)| = 0.$$

Let $\alpha > 1$. Then the operator $I_f J_g$ is compact on B^α if and only if

$$\lim_{|z|\to 1^-}(1-|z|^2)|g'(z)|\,|f(z)| = 0.$$

Proof. Let $\alpha = 1$. Then we have

$$\begin{aligned}\|\, I_f J_g h\,\|_B &\le \sup_{z\in D}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|\,|f(z)|\,\|\, h\,\|_B\\&\le \sup_{z\in D}|f(z)|\,\sup_{z\in D}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|\,\|\, h\,\|_B.\end{aligned}$$

By using the fact that A_1A_2 is compact operator on B for any bounded operator A_1 on B and compact operator A_2 on B and the fact $(I_f J_g)^{**} = I_f J_g$, we can prove this theorem as well as the proof of Theorem 2.3. In the case of $\alpha > 1$, we can prove it as well. □

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