Hokkaido Mathematical Journal Vol. 33 (2004) p. 429–441

The nullity of a compact minimal hypersurface in a compact symmetric space of rank one

Tohru GOTOH

(Received September 5, 2002)

Abstract. We determine a compact minimal hypersuface with the least nullity in the Cayley projective plane. Combining this with the preceding results, we conclude the following: Let X be a compact symmetric space of rank one and M a compact minimal hypersuface in X. Then the nullity of M is bounded from below by the dimension of X. When the nullity of M is equal to the dimension of X, M must be a minimal geodesic hypersphere in X. Conversely,the nullity of a minimal geodesic hypersphere in X is equal to the dimension of X.

Key words: minimal submanifolds, nullity, Cayley projective space, compact symmetric spaces of rank one..

1. Introduction

In this article, we will prove the following:

Theorem 1.1 *Let M be a compact minimal hypersurface in the Cayley projective plane. Then its nullity satisfies* $\mathrm{nul}(M) \geqq 16$. *When the nullity of M is equal to* 16, *then M must be a minimal geodesic hypersphere.*

Similar results on the nullity of minimal hypersurfaces in spheres were obtained by Simons([S]), in real projective space by Ohnita([O]) and in complex or quaternion projective spaces by the present author([G1], [G2]). Those results are summarized as follows:

Theorem 1.2 *Let X be a compact symmetric space of rank one and M a compact minimal hypersuface in X. Then the nullity of M is bounded from below by the dimension of X. When the nullity of M is equal to the dimension of X, M must be a minimal geodesic hypersphere in X. Conversely, the nullity of a minimal geodesic hypersphere in X is equal to the dimension of X.*

In Section 2, we give a brief review on the Jordan algebra and the group

---

1991 Mathematics Subject Classification : 53C42.

F_4 so as to describe the Cayley projective plane. In Section 3, we give a proof of Theorem 1.1. The key ingredient of our proof is the computation of the first eigenvalue of the Laplacian on S^{15} with a canonically varied metric. The first eigenvalue of the Laplacian with respect to such metric was studied by Berard Bergery and Bourguignon([BB]). In Section 4, following their method, we compute the first eigenvalue of a geodesic hypersphere and the nullity of a minimal geodesic hypersphere in the Cayley projective plane.

2. Preliminaries

2.1. The Jordan algebra and the group F_4

For detailed account on this subsection, see [Y, Chapter 5]. We denote the field of Cayley numbers by $\mathbb{C}ay$ and its standard basis by $e_0 = 1, e_1, \ldots, e_7$. We also denote by $\operatorname{Re}\mathbb{C}ay$ (resp. $\operatorname{Im}\mathbb{C}ay$) the linear subspace of $\mathbb{C}ay$ spaned by e_0 (resp. $e_1, \ldots, e_7$) over $\mathbb{R}$. The Jordan algebra $\mathfrak{J}$ is a 27-dimensional real algebra consisting of all Hermitian 3-matrices with entries in $\mathbb{C}ay$. The Jordan product $\circ$ is defined on $\mathfrak{J}$ by $X \circ Y = (XY + YX)/2$, $X, Y \in \mathfrak{J}$. For triples $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$ and $\boldsymbol{x} = (x_1, x_2, x_3)$ in $\mathbb{C}ay^3$, we put

$$X(\boldsymbol{\xi}, \boldsymbol{x}) = \begin{pmatrix} \xi_1 & x_3 & \bar{x_2} \\ \bar{x_3} & \xi_2 & x_1 \\ x_2 & \bar{x_1} & \xi_3 \end{pmatrix}$$

and, for $x \in \mathbb{C}ay$,

$$\begin{array}{ll} E_1 = X((1, 0, 0), \mathbf{0}), & E_2 = X((0, 1, 0), \mathbf{0}), \\ E_3 = X((0, 0, 1), \mathbf{0}) & F_1(x) = X(\mathbf{0}, (x, 0, 0)), \\ F_2(x) = X(\mathbf{0}, (0, x, 0)), & F_3(x) = X(\mathbf{0}, (0, 0, x)). \end{array}$$

Then the set $\{E_1, E_2, E_3, F_1(e_i), F_2(e_i), F_3(e_i), i = 0, 1, \ldots, 7\}$ forms a real basis of $\mathfrak{J}$.

Let F_4 be the group of autmorphisms of the Jordan algebra. We denote by $(\ ,\ )_{\mathbb{C}ay}$ the inner product on $\mathbb{C}ay$ with respect to which the standard basis $e_0, \ldots, e_7$ becomes an orthonormal basis. An inner product on $\mathfrak{J}$ is defined by $(X, Y)_{\mathfrak{J}} = \operatorname{Tr}(X \circ Y)$ for $X, Y \in \mathfrak{J}$. Precisely

$$(X(\boldsymbol{\xi}, \boldsymbol{x}), X(\boldsymbol{\eta}, \boldsymbol{y}))_{\mathfrak{J}} = \sum_{i=1}^{3} \left(\xi_i \eta_i + 2(x_i, y_i)_{\mathbb{C}ay}\right).$$

It is known that every element in F_4 preserves the trace and F_4 is contained in the orthogonal group $O(\mathfrak{J})$.

The Lie algebra of F_4 is

$$\mathfrak{f}_4 = \{D \in \mathrm{End}_{\mathbb{R}}(\mathfrak{J}) \mid D(X \circ Y) = D(X) \circ Y + X \circ D(Y),\ X, Y \in \mathfrak{J}\},$$

which has following structures. Let $\mathfrak{M}^-$ be the set of all skew-Hermitian 3-matrices with entries in $\mathbb{C}ay$. Then we have $[\mathfrak{M}^-, \mathfrak{J}] \subset \mathfrak{J}$ and $[\mathfrak{J}, \mathfrak{J}] \subset \mathfrak{M}^-$. For $A \in \mathfrak{M}^-$, define $\tilde{A} : \mathfrak{J} \to \mathfrak{J}$ by $\tilde{A}(X) = [A, X]$. Then $\tilde{A}$ belongs to $\mathfrak{f}_4$ if $\mathrm{Tr}\, A = 0$. We set

$$\tilde{\mathcal{A}} = \{\tilde{A} \in \mathfrak{f}_4 \mid A \in \mathfrak{M}^- \quad \text{and} \quad \mathrm{diag}(A) = 0\}.$$

We put for $a \in \mathbb{C}ay$,

$$A_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\bar{a} & 0 \end{pmatrix}, A_2(a) = \begin{pmatrix} 0 & 0 & -\bar{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, A_3(a) = \begin{pmatrix} 0 & a & 0 \\ -\bar{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\tilde{\mathcal{A}}_i = \{\tilde{A}_i(a) \in \mathfrak{f}_4 \mid a \in \mathbb{C}ay\}$. Then $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2 \oplus \tilde{\mathcal{A}}_3$.

Next we set

$$\mathfrak{d}_4 = \{\delta \in \mathfrak{f}_4 \mid \delta(E_i) = 0,\ i = 1, 2, 3\},$$

which is described as follows. Let $so(\mathbb{C}ay)$ be the set of skew-symmetric endomorphisms of the real vector space $\mathbb{C}ay$ with respect to $(\ ,\ )_{\mathbb{C}ay}$.

The Principle of Triality in $so(\mathbb{C}ay)$

(1) For each $D_1 \in so(\mathbb{C}ay)$, there exist D_2 and $D_3 \in so(\mathbb{C}ay)$ satisfying

$$(D_1 a)b + a(D_2 b) = D_3(ab), \quad a, b \in \mathbb{C}ay.$$

Those D_2 and D_3 are determined uniquely by D_1.

(2) For $D_1, D_2, D_3 \in so(\mathbb{C}ay)$, suppose the equality

$$(D_1 a)b + a(D_2 b) = \overline{D_3(\overline{ab})}, \quad a, b \in \mathbb{C}ay$$

holds. Then we also have

$$(D_2 a)b + a(D_3 b) = \overline{D_1(\overline{ab})} \quad \text{and} \quad (D_3 a)b + a(D_1 b) = \overline{D_2(\overline{ab})}, \quad a, b \in \mathbb{C}ay.$$

Now for $D_1 \in so(\mathbb{C}ay)$, choose D_2 and $D_3 \in so(\mathbb{C}ay)$ satisfying the triality equality $(D_1a)b + a(D_2b) = \overline{D_3(\overline{ab})}$, $a, b \in \mathbb{C}ay$, and then define a map $\delta(D_1) : \mathfrak{J} \to \mathfrak{J}$ by

$$\delta(D_1)(X(\boldsymbol{\xi}, \boldsymbol{x})) = X(\mathbf{0}, (D_1x_1, D_2x_2, D_3x_3)).$$

The map $\delta(D_1)$ belongs to $\mathfrak{d}_4$ and the correspondence $D_1 \mapsto \delta(D_1)$ gives an isomorphism between the real Lie algebras $so(\mathbb{C}ay)$ and $\mathfrak{d}_4$. Especially, $\dim \mathfrak{d}_4 = 28$.

Finally $\mathfrak{f}_4$ is decomposed as

$$\mathfrak{f}_4 = \mathfrak{d}_4 \oplus \tilde{\mathcal{A}} = \mathfrak{d}_4 \oplus \tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2 \oplus \tilde{\mathcal{A}}_3$$

as vector spaces. For $a, s \in \mathbb{C}ay$, define three endomorphisms $D_i^{(a,s)}$ $(i = 1, 2, 3)$ of $\mathbb{C}ay$ by

$$\begin{aligned} D_1^{(a,s)}(x) &= (xa)\bar{s} - (xs)\bar{a}, \\ D_2^{(a,s)}(x) &= 4((a, x)_{\mathbb{C}ay}s - (s, x)_{\mathbb{C}ay}a), \\ D_3^{(a,s)}(x) &= \bar{s}(ax) - \bar{a}(sx). \end{aligned}$$

Then those $D_i^{(a,s)}$'s $(i = 1, 2, 3)$ belong to $so(\mathbb{C}ay)$ and satisfy the triality equality

$$(D_1^{(a,s)}x)y + x(D_2^{(a,s)}y) = \overline{D_3^{(a,s)}(\overline{xy})}.$$

By means of those endomorphisms, Lie bracket operations in $\mathfrak{f}_4$ are computed as follows:

$$\begin{aligned} &[\delta(D_1), \tilde{A}_i(a)] = \tilde{A}_i(D_i(a)), \quad i = 1, 2, 3, \\ &[\tilde{A}_1(a), \tilde{A}_1(s)] = \delta(D_2^{(a,s)}), \\ &[\tilde{A}_2(a), \tilde{A}_2(s)] = \delta(D_1^{(a,s)}), \\ &[\tilde{A}_3(a), \tilde{A}_3(s)] = \delta(D_3^{(a,s)}), \\ &[\tilde{A}_1(a), \tilde{A}_2(s)] = -\tilde{A}_3(\overline{as}), \\ &[\tilde{A}_1(a), \tilde{A}_3(s)] = \tilde{A}_2(\overline{sa}), \\ &[\tilde{A}_2(a), \tilde{A}_3(s)] = -\tilde{A}_1(\overline{as}). \end{aligned} \tag{2.1.1}$$

2.2. Spinor groups and the Cayley projective plane

Let $SO(\mathbb{C}ay)$ and $SO(\operatorname{Im}\mathbb{C}ay)$ be the special orthogonal groups for $\mathbb{C}ay$ and $\operatorname{Im}\mathbb{C}ay$ with respect to the inner product $(\ ,\ )_{\mathbb{C}ay}$. We regard

$SO(\mathrm{Im}\,\mathbb{C}ay)$ as a subgroup of $SO(\mathbb{C}ay)$ consisting of elements each of which fixes the unit e_0 of $\mathbb{C}ay$.

For $\alpha_1, \alpha_2, \alpha_3 \in SO(\mathbb{C}ay)$, define the map $f(\alpha_1, \alpha_2, \alpha_3) : \mathfrak{J} \to \mathfrak{J}$ by

$$f(\alpha_1, \alpha_2, \alpha_3)(X(\boldsymbol{\xi}, \boldsymbol{x})) = X(\boldsymbol{\xi}, (\alpha_1(x_1), \alpha_2(x_2), \alpha_3(x_3))).$$

If the three α_i's satisfy the triality equality $\alpha_1(x)\alpha_2(y) = \overline{\alpha_3(\overline{xy})}$ for all $x, y \in \mathbb{C}ay$, then $f(\alpha_1, \alpha_2, \alpha_3)$ belongs to F_4. Thus we set

$$\mathrm{Spin}(8) = \left\{ f(\alpha_1, \alpha_2, \alpha_3) \in F_4 \;\middle|\; \begin{array}{l} \alpha_1, \alpha_2, \alpha_3 \in SO(\mathbb{C}ay) \quad \text{satisfying} \\ \alpha_1(x)\alpha_2(y) = \overline{\alpha_3(\overline{xy})} \\ \qquad\qquad \text{for all} \quad x, y \in \mathbb{C}ay \end{array} \right\}.$$

Then it is known that Spin(8) becomes a simply connected closed subgroup of F_4 and the map $f(\alpha_1, \alpha_2, \alpha_3) \mapsto \alpha_1$ defines a double covering map from Spin(8) onto $SO(\mathbb{C}ay)$.

We also set

$$\mathrm{Spin}(7) = \left\{ f(\alpha_1, \alpha_2, \alpha_3) \in \mathrm{Spin}(8) \mid \alpha_3 \in SO(\mathrm{Im}\,\mathbb{C}ay) \right\}.$$

Then it is known that Spin(7) becomes a simply connected closed subgroup of F_4 and the map $f(\alpha_1, \alpha_2, \alpha_3) \mapsto \alpha_3$ defines a double covering map from *Spin*(7) onto $SO(\mathrm{Im}\,\mathbb{C}ay)$.

Now the Cayley projective plane is defined by

$$\mathbb{C}ay\mathbb{P} = \left\{ X \in \mathfrak{J} \mid X \circ X = X \quad \text{and} \quad \mathrm{Tr}(X) = 1 \right\}.$$

The group F_4 acts transitively on $\mathbb{C}ay\mathbb{P}$. The isotropy subgroup at E_1 has a structure of a simply connected double covering space over $SO(9)$. For this reason, we denote it by

$$\mathrm{Spin}(9) = \left\{ f \in F_4 \mid f(E_1) = E_1 \right\}.$$

The Lie algebras of those spinor groups are

$$\begin{aligned} &\mathrm{spin}(9) = \delta_4 \oplus \tilde{\mathcal{A}}_1, \\ &\mathrm{spin}(8) = \delta_4, \\ &\mathrm{spin}(7) = \left\{ \delta(D_1) \in \delta_4 \mid D_3 \in so(\mathrm{Im}\,\mathbb{C}ay) \right\}. \end{aligned}$$

$(F_4, \mathrm{Spin}(9))$ is a compact symmetric pair with the canonical decomposition

$$\mathfrak{f}_4 = (\delta_4 \oplus \tilde{\mathcal{A}}_1) \oplus (\tilde{\mathcal{A}}_2 \oplus \tilde{\mathcal{A}}_3).$$

We identify the tangent space $T_{E_1}\mathbb{C}ay\mathbb{P}$ to $\mathbb{C}ay\mathbb{P}$ at E_1 with $\mathfrak{p} = \tilde{\mathcal{A}}_2 \oplus \tilde{\mathcal{A}}_3$. The inner product $(\ ,\ )_{\mathfrak{p}}$ on $\mathfrak{p}$ is given by

$$(\tilde{A}_2(a) + \tilde{A}_3(b),\ \tilde{A}_2(s) + \tilde{A}_3(t))_{\mathfrak{p}} = (a,\ s)_{\mathbb{C}ay} + (b,\ t)_{\mathbb{C}ay}, \qquad a,\ b,\ s,\ t \in \mathbb{C}ay.$$

The linear isotropy action of Spin(9) on $T_{E_1}\mathbb{C}ay\mathbb{P}$ is equal to the adjoint action of Spin(9) on $\mathfrak{p}$. For later use, we write down here the ajoint action of Spin(7) on $\mathfrak{p} = \tilde{\mathcal{A}}_2 \oplus \tilde{\mathcal{A}}_3$.

$$\begin{aligned} &\mathrm{Ad}(f(\alpha_1,\ \alpha_2,\ \alpha_3))(\tilde{A}_2(a)) = \tilde{A}_2(\alpha_2(a)), \\ &\mathrm{Ad}(f(\alpha_1,\ \alpha_2,\ \alpha_3))(\tilde{A}_3(a)) = \tilde{A}_3(\alpha_3(a)), \quad a \in \mathbb{C}ay. \end{aligned} \tag{2.2.1}$$

Let R be the curvature tensor of the symmetric space $\mathbb{C}ay\mathbb{P} = F_4/\mathrm{Spin}(9)$. Then, at the base point E_1, $R_{E_1}(X,\ Y)$, $X,\ Y \in \mathfrak{p}$, is given by $R_{E_1}(X,\ Y) = -\,\mathrm{ad}[X,\ Y]$. Hence we obtain from (2.1.1)

$$\begin{aligned} &R_{E_1}(\tilde{A}_2(s),\ \tilde{A}_2(t))(\tilde{A}_2(u)) = -\tilde{A}_2(D_2^{(s,t)}(u)), \\ &R_{E_1}(\tilde{A}_2(s),\ \tilde{A}_2(t))(\tilde{A}_3(u)) = -\tilde{A}_3(D_3^{(s,t)}(u)), \\ &R_{E_1}(\tilde{A}_2(s),\ \tilde{A}_3(t))(\tilde{A}_2(u)) = -\tilde{A}_3(\bar{u}(st)), \\ &R_{E_1}(\tilde{A}_2(s),\ \tilde{A}_3(t))(\tilde{A}_3(u)) = \tilde{A}_2((st)\bar{u}), \\ &R_{E_1}(\tilde{A}_3(s),\ \tilde{A}_3(t))(\tilde{A}_3(u)) = -\tilde{A}_3(D_2^{(s,t)}(u)), \end{aligned} \tag{2.2.2}$$

for $s,\ t,\ u \in \mathbb{C}ay$. As a consequence, we have

Proposition 2.1 *There exists no totally umbilical hypersuface (even locally) in* $\mathbb{C}ay\mathbb{P}$.

Proof. Suppose there would be a totally umbilical hypersuface M in $\mathbb{C}ay\mathbb{P}$. We may assume that M contains E_1, and the tangent space $T_{E_1}M$ is identified with $\tilde{\mathcal{A}}_2 \oplus \mathrm{Im}\,\tilde{\mathcal{A}}_3$ under the identification $T_{E_1}\mathbb{C}ay\mathbb{P} = \mathfrak{p} = \tilde{\mathcal{A}}_2 \oplus \tilde{\mathcal{A}}_3$, where we put $\mathrm{Im}\,\tilde{\mathcal{A}}_3 = \{\tilde{A}_3(a) \mid a \in \mathrm{Im}\,\mathbb{C}ay\}$. By means of Codazzi equation, for $X,\ Y,\ Z \in \mathfrak{p}$, if X and Y are perpendicular to Z, then $R_{E_1}(X,\ Y)Z$ must be tangent to M. However if we take $X = \tilde{A}_2(e_1)$, $Y = \tilde{A}_3(e_1)$ and $Z = \tilde{A}_2(e_0)$, then (2.2.2) implies $R(X,\ Y)Z = \tilde{A}_3(e_0)$, which is perpendicular to M. This contradiction shows the assertion. □

We note that this proposition is already known. In [C], B.Y. Chen proved that there is no totally umbilical submanifolds of dimension greater than 8 in the Cayley projective plane. Moreover, if a submanifold is totally umbilical, then it has parallel second fundamental form, hence the classification of such submanifolds follows from the classification of parallel submanifolds in the Cayley projective plane. Such a classification was completed by K. Tsukada in [T].

Let M be a compact minimal hypersurface in $\mathbb{C}ay\mathbb{P}$. Then, because of (2.2.2), the Jacobi operator $\mathcal{J}_M : \Gamma(NM) \to \Gamma(NM)$ of M (e.g. see [G2] for definition) is given by

$$\mathcal{J}_M(V) = -\Delta^{NM} V - (36 + \|B\|^2)V. \tag{2.2.3}$$

Here NM is the normal bundle of M, $\Gamma(NM)$ is the space of sections of NM, Δ^{NM} is the negative rough Laplacian with respect to the normal connection on NM, and B is the second fundamental form of M. Since $\mathcal{J}_M$ is a strongly elliptic operator, its kernel is of finite dimensional. The nullity of M is defined to be the dimension of the kernel of $\mathcal{J}_M$.

3. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. For this purpose, we recall the results obtained in our preceding paper [G2].

Let (G, H) be a compact symmetric pair and $\mathfrak{g} = \mathfrak{h} + \tilde{\mathfrak{m}}$ be its canonical decomposition. Let M be a compact minimal submanifold of G/H which contains the origin o of G/H. We denote by $\mathfrak{m}$ the subspace of $\tilde{\mathfrak{m}}$ that corresponds to the tangent space T_oM under the canonical identification between T_oG/H and $\tilde{\mathfrak{m}}$. Define a linear map $\Psi : \mathfrak{h} \to \mathrm{Hom}(\mathfrak{m}, \mathfrak{m}^\perp)$ by $\Psi(Z)(X) = (\mathrm{ad}(Z)(X))^\perp$.

Theorem 3.1 ([G2]) *The nullity of M satisfies the inequality*

$$\mathrm{nul}(M) \geqq \mathrm{codim}(M) + \dim \mathrm{Im}\,\Psi.$$

If $\mathrm{nul}(M) = \mathrm{codim}(M) + \dim \mathrm{Im}\,\Psi$, *then we have*

(1) *M is an orbit of a closed subgroup of G.*

(2) *Let H_0 be the subgroup of H generated by* $\mathrm{Ker}\,\Psi$. *Then H_0 leaves M invariant.*

We are now in a position to prove Theorem 1.1. Let M be a compact minimal hypersuface in $\mathbb{C}ay\mathbb{P}$. We may assume that M contains E_1, and $T_{E_1}M = \tilde{\mathcal{A}}_2 \oplus \mathrm{Im}\,\tilde{\mathcal{A}}_3$. We also put $\mathrm{Re}\,\tilde{\mathcal{A}}_3 = \{\tilde{A}_3(a) \mid a \in \mathrm{Re}\,\mathbb{C}ay\}$. In this case, the linear map $\Psi : \mathrm{spin}(9) \to \mathrm{Hom}(\tilde{\mathcal{A}}_2 \oplus \mathrm{Im}\,\tilde{\mathcal{A}}_3,\ \mathrm{Re}\,\tilde{\mathcal{A}}_3)$ is given by

$$\Psi(Z)(B) = ([Z,\ B],\ \tilde{A}_3(e_0))_{\mathfrak{p}}\tilde{A}_3(e_0).$$

For $Z = \delta(D_1) + \tilde{A}_1(a)$ in $\mathrm{spin}(9)$ and $B = \tilde{A}_2(s) + \tilde{A}_3(t)$ in $\tilde{\mathcal{A}}_2 \oplus \mathrm{Im}\,\tilde{\mathcal{A}}_3$, where $a,\ s \in \mathbb{C}ay$, $t \in \mathrm{Im}\,\mathbb{C}ay$, we have

$$\begin{aligned}([Z,\ B],\ \tilde{A}_3(e_0))_{\mathfrak{p}} &= (\tilde{A}_2(D_2(s) + \overline{ta}) + \tilde{A}_3(D_3(t) - \overline{as}),\ \tilde{A}_3(e_0))_{\mathfrak{p}} \\ &= (D_3(t) - \overline{as},\ e_0)_{\mathbb{C}ay} \\ &= \mathrm{Re}(D_3(t) - \overline{as}).\end{aligned}$$

This implies that $\Psi(Z) = 0$ if and only if $a = 0$ and $D_3 \in so(\mathrm{Im}\,\mathbb{C}ay)$. Hence we have $\mathrm{Ker}\,\Psi = \mathrm{spin}(7)$ and $\dim \mathrm{Im}\,\Psi = \dim \mathrm{Spin}(9) - \dim \mathrm{Ker}\,\Psi = 36 - 21 = 15$. By virture of Theorem 3.1, we obtain

$$\mathrm{nul}(M) \geqq \mathrm{codim}\, M + \dim \mathrm{Im}\,\Psi = 16.$$

We now assume $\mathrm{nul}(M) = 16$ in what follows. Let ν be a unit normal field of M with $\nu_{E_1} = \tilde{A}_3(e_0) \in \mathrm{Re}\,\tilde{\mathcal{A}}_3$, and S^{ν} the shape operator of M associated to ν. By the assumption on the nullity of M, Spin(7) acts on M as stated in Theorem 3.1. Hence we have

$$\mathrm{Ad}(f) \circ S = S \circ \mathrm{Ad}(f) \quad \text{for all} \quad f \in \mathrm{Spin}(7), \tag{3.1}$$

where we put $S = S^{\nu_{E_1}}$. We shall show that the shape operator S leaves each of $\tilde{\mathcal{A}}_2$ and $\mathrm{Im}\,\tilde{\mathcal{A}}_3$ invariant. For this purpose, denote the unit elements in $SO(\mathbb{C}ay)$ and $SO(\mathrm{Im}\,\mathbb{C}ay)$ by $\mathbf{1}$ and $\mathbf{1}'$ respectively. Then $f(-\mathbf{1},\ -\mathbf{1},\ \mathbf{1}')$ belongs to Spin(7), and because of (2.2.1), it acts on $\tilde{\mathcal{A}}_2$ by $(-1)\times$ identity and on $\mathrm{Im}\,\tilde{\mathcal{A}}_3$ by identity. This, combined with (3.1), shows that the space $S(\tilde{\mathcal{A}}_2)$ is perpendicular to $\mathrm{Im}\,\tilde{\mathcal{A}}_3$. Therefore both spaces $\tilde{\mathcal{A}}_2$ and $\mathrm{Im}\,\tilde{\mathcal{A}}_3$ are leaved invariant by S respectively.

The adjoint actions of Spin(7) on $\tilde{\mathcal{A}}_2$ and $\mathrm{Im}\,\tilde{\mathcal{A}}_3$ are given in (2.2.1). It is then known that those actions are transitive on the unit spheres in both $\tilde{\mathcal{A}}_2$ and $\mathrm{Im}\,\tilde{\mathcal{A}}_3$. Hence the number of the distinct eigenvalues of S is at most two. Because M cannot be umbilic (Proposition 2.1), we conclude that M has two distinct constant principal curvatures.

According to Iwata [I] or Kollross [K], the codimension one orbits of some closed groups acting on the Cayley projective plane are essentially the following two cases.

1. One of those is a geodesic hypershere, which has two distinct constant principal curvatures.

2. Another one is given by a cirtain action of $Sp(3) \times Sp(1)$ on $\mathbb{C}ay\mathbb{P}$ with the orbit $(Sp(3) \times Sp(1))/(Sp(1) \times Sp(1) \times Sp(1))$. An elementary calculation shows that this orbit has four distinct constant principal curvatures.

We finally conclude that M must be a minimal geodesic hypersphere. □

In the next section, we will show that the nullity of a minimal geodesic hypersphere in $\mathbb{C}ay\mathbb{P}$ is actually equal to 16.

4. A minimal geodesic hypersphere in $\mathbb{C}ay\mathbb{P}$ and its nullity

All the notation which appeared in the preceding sections are used in the present section. We also denote by Δ_M the Laplacian of a Riemannian manifold M acting on the space of smooth functions. The λ-eigenspace of Δ_M is denoted by $E_\lambda(\Delta_M)$.

4.1. Geodesic hyperspheres in $\mathbb{C}ay\mathbb{P}$

From (2.2.2), the Ricci transformation $\mathrm{Ric}_{\tilde{A}_3(e_0)}$ in the direction $\tilde{A}_3(e_0)$ is given by

$$\mathrm{Ric}_{\tilde{A}_3(e_0)} = \begin{cases} \mathrm{id} & \text{on} \quad \tilde{\mathcal{A}}_2, \\ 4\,\mathrm{id} & \text{on} \quad \mathrm{Im}\,\tilde{\mathcal{A}}_3, \\ 0 & \text{on} \quad \mathrm{Re}\,\tilde{\mathcal{A}}_3. \end{cases} \tag{4.1.1}$$

We denote by $\gamma_{\tilde{A}_3(e_0)}$ the geodesic in $\mathbb{C}ay\mathbb{P}$ with $\gamma_{\tilde{A}_3(e_0)}(0) = E_1$ and $\gamma'_{\tilde{A}_3(e_0)}(0) = \tilde{A}_3(e_0)$. Let $B = B_2 + B_3 \in \tilde{\mathcal{A}}_2 \oplus \mathrm{Im}\,\tilde{\mathcal{A}}_3$ and $B_i(t)$ the parallel vector field along $\gamma_{\tilde{A}_3(e_0)}$ with $B_i(0) = B_i$ $(i = 2, 3)$. Then, from (4.1.1), the vector field

$$Z(t) = (\sin t)B_2(t) + \frac{1}{2}(\sin 2t)B_3(t) \tag{4.1.2}$$

is the Jacobi field along $\gamma_{\tilde{A}_3(e_0)}$, which satisfies the initial conditions $Z(0) = 0$ and $\nabla_t Z(0) = B$. This implies that the injectivity radius of $\mathbb{C}ay\mathbb{P}$ is $\pi/2$.

Consider the initial value problem

$$\begin{aligned} &\nabla_t^2 \mathfrak{S}_t + \mathrm{Ric}_{\gamma'_{\tilde{A}_3(e_0)}(t)} \circ \mathfrak{S}_t = 0, \\ &\mathfrak{S}_0 = 0, \\ &\nabla_t \mathfrak{S}_0 = \mathrm{id}_{\tilde{\mathcal{A}}_2 \oplus \mathrm{Im}\,\tilde{\mathcal{A}}_3}, \end{aligned} \tag{4.1.3}$$

for a differentiable function $t \mapsto \mathfrak{S}_t \in \mathrm{End}(\tilde{\mathcal{A}}_2(t) \oplus \mathrm{Im}\,\tilde{\mathcal{A}}_3(t))$. Here $\tilde{\mathcal{A}}_2(t)$

and $\operatorname{Im}\tilde{\mathcal{A}}_3(t)$ denote the spaces obtained by parallel translation of $\tilde{\mathcal{A}}_2$ and $\operatorname{Im}\tilde{\mathcal{A}}_3$ along $\gamma_{\tilde{A}_3(e_0)}$. Because the Jacobi fields along $\gamma_{\tilde{A}_3(e_0)}$ are given by (4.1.2), the solution of (4.1.3) is given by

$$\mathfrak{S}_t = \begin{cases} (\sin t)\,\mathrm{id} & \text{on} \quad \tilde{\mathcal{A}}_2(t) \\ \frac{1}{2}(\sin 2t)\,\mathrm{id} & \text{on} \quad \operatorname{Im}\tilde{\mathcal{A}}_3(t), \end{cases} \tag{4.1.4}$$

and its covariant derivative is given by

$$\nabla_t \mathfrak{S}_t = \begin{cases} (\cos t)\,\mathrm{id} & \text{on} \quad \tilde{\mathcal{A}}_2(t) \\ (\cos 2t)\,\mathrm{id} & \text{on} \quad \operatorname{Im}\tilde{\mathcal{A}}_3(t). \end{cases} \tag{4.1.5}$$

Now let $\mathbb{S}_t$ be the geodesic hypersphere in $\mathbb{C}ay\mathbb{P}$ centered at E_1 and with radius t $(0 < t < \pi/2)$. It follows from (4.1.4) and (4.1.5) that the shape operator of $\mathbb{S}_t$ at $\gamma_{\tilde{A}_3(e_0)}(t)$ associated to the unit normal vector $\gamma'_{\tilde{A}_3(e_0)}(t)$ is given by

$$S^{\gamma'_{\tilde{A}_3(e_0)}(t)} = (\nabla_t \mathfrak{S}_t) \circ \mathfrak{S}_t^{-1} = \begin{cases} (\cot t)\,\mathrm{id} & \text{on} \quad \tilde{\mathcal{A}}_2(t) \\ 2(\cot 2t)\,\mathrm{id} & \text{on} \quad \operatorname{Im}\tilde{\mathcal{A}}_3(t). \end{cases}$$

This implies

(4.1.6) $\mathbb{S}_t$ has two constant principal curvatures $\cot t$ and $2\cot 2t$.

(4.1.7) $\mathbb{S}_t$ is a minimal hypersurface if and only if $\cot^2 t = 7/15$.

(4.1.8) If $\mathbb{S}_t$ is a minimal hypersuface, then we have $\|S^{\gamma_{\tilde{A}_3(e_0)}(t)}\|^2 = 8$ and the Jacobi operator $\mathcal{J}_{\mathbb{S}_t} : \Gamma(N\mathbb{S}_t) \to \Gamma(N\mathbb{S}_t)$ is given by

$$\mathcal{J}_{\mathbb{S}_t}(V) = -\Delta^{N\mathbb{S}_t} V - 44V,$$

because of (2.2.3) and (4.1.7). Especially its nullity is equal to $\dim E_{44}(\Delta_{\mathbb{S}_t})$, since $\mathbb{S}_t$ is orientable.

4.2. The first eigenvalue of the Laplacian of S^{15} with a canonically varied metric

We denote by $S^n(r)$ the Euclidean n-sphere with radius r, and by $g_{S^n(r)}$ its canonical metric. It is then known that there is a Riemannian submersion with totally geodesic fibers:

$$\begin{array}{ccc} S^7(r) & \longrightarrow & S^{15}(r) \\ & & \big\downarrow \pi \\ & & S^8(r/2). \end{array}$$

The tangent space $T_pS^{15}(r)$ to $S^{15}(r)$ at p is decomposed as $T_pS^{15}(r) = H_p \oplus V_p$, where H_p and V_p are the horizontal and the vertical space with respect to the Riemannian submersion. Define a metric $g_s(s>0)$ on S^{15} by

$$\begin{aligned} &g_s \mid V_p \times H_p = 0, \\ &g_s \mid V_p = s^2 g_{S^{15}(r)} \mid V_p, \\ &g_s \mid H_p = g_{S^{15}(r)} \mid H_p. \end{aligned}$$

In [BB], such a metric g_s is called a canonical variation of $S^{15}(r)$ associated with the Riemannian submersion.

We investigate the first eigenvalue of the Laplacian $\Delta^s_{S^{15}(r)}$ of $(S^{15}(r), g_s)$. The vertical and the horizontal Laplacians Δ^s_v and Δ^s_h for g_s are defined by

$$\begin{aligned} (\Delta^s_v f)(p) &= (\Delta^s_{F_p} f)(p) \quad \text{for} \quad f \in C^\infty(S^{15}(r)), \\ \Delta^s_h &= \Delta^s_{S^{15}(r)} - \Delta^s_v. \end{aligned}$$

Here F_p denote the fiber through p and $\Delta^s_{F_p}$ its Laplacian. After Berard Bergery and Bourguignon [BB], we set, for non-negative real numbers b and ϕ,

$$\boldsymbol{H}^s(b, \phi) = \{ f \in C^\infty(S^{15}(r)) \mid \Delta^s_h f = bf,\ \Delta^s_v f = \phi f \},$$

which is contained in $E_{b+\phi}(\Delta^s_{S^{15}(r)})$. One of the important observations in [BB] is that the three Laplacians $\Delta^s_{S^{15}(r)}$, Δ^s_v and Δ^s_h are commutative with each other and hence the Hilbert space $L^2(S^{15}(r))$ admits a Hilbert basis consisting of elements in those spaces $\boldsymbol{H}^s(b, \phi)$. Furthermore they have shown

$$\boldsymbol{H}^1(b, \phi) = \boldsymbol{H}^s(b,\ s^{-2}\phi) \quad \text{for all} \quad s > 0. \tag{4.2.1}$$

Now the spectra of the Laplacians of $S^{15}(r)$, $S^7(r)$ and $S^8(r/2)$ are given by

$$\begin{aligned} \operatorname{Spec}(S^{15}(r)) &= \left\{ \mu_k = \frac{k(k+14)}{r^2} \;\middle|\; k = 0, 1, \dots \right\}, \\ \operatorname{Spec}(S^7(r)) &= \left\{ \phi_k = \frac{k(k+6)}{r^2} \;\middle|\; k = 0, 1, \dots \right\}, \\ \operatorname{Spec}(S^8(r/2)) &= \left\{ \frac{4k(k+7)}{r^2} \;\middle|\; k = 0, 1, \dots \right\}. \end{aligned} \tag{4.2.2}$$

For each $k \geqq 1$ and $l = 0, 1, \dots, L_k = \max\{ l \mid \phi_l \leqq \mu_k \}$, we put $b_{k,l} = \mu_k -$

ϕ_l. Precisely L_k is given by the table below:

k	1	$2 \leqq k \leqq 3$	$4 \leqq k \leqq 13$	$14 \leqq k$
L_k	1	$k+1$	$k+2$	$k+3$

Then, each μ_k may be decomposed into the form

$$\mu_k = b_{k,\,l} + \phi_l \quad \text{and} \quad \boldsymbol{H}^1(b_{k,\,l}, \phi_l) \neq 0. \tag{4.2.3}$$

Note that the above range of l is only a possibility. Accoding to the decomposition (4.2.3), we set

$$\mu_{k,\,l}(s) = b_{k,\,l} + \frac{1}{s^2}\phi_l, \quad l = 0,\, 1,\, \ldots,\, L_k.$$

Then $\mu_{k,\,l}(s)$ is an eigenvalue of $\Delta^s_{S^{15}(r)}$ provided such a decomposition (4.2.3) is possible. Using the notation as above, we have the following.

Proposition 4.1 *When $\sqrt{42}/12 < s < 1$, the first eigenvalue of $\Delta^s_{S^{15}(r)}$ is equal to $\mu_{1,\,1}(s) = r^{-2}(8+7s^{-2})$. Its eigenspace coincides with $E_{\mu_1}(\Delta_{S^{15}(r)})$, and hence its multiplicity is* 16.

Proof. As was mentioned in [BB], $\boldsymbol{H}^1(b,\,0) = \{\bar{f} \circ \pi \mid \bar{f} \in E_b(\Delta_{S^8(r/2)})\}$. Thus the decomposition $\mu_1 = b_{1,0} + \phi_0$ is impossible bacause of (4.2.2). On the contrary, the decomposition $\mu_1 = b_{1,\,1} + \phi_1$ is possible, because eigenfunctions for μ_1 and ϕ_1 are the restrictions of linear 1-forms on $\mathbb{R}^{16}$ and $\mathbb{R}^8$ to $S^{15}(r)$ and $S^7(r)$. It is now easy to see that $\mu_{1,\,1}(s) < \mu_{k,\,l}(s)$ for $s,\, k,\, l$ with $\sqrt{42}/12 < s < 1$, $k = 2,\, 3,\, \ldots,\, l = 0,\, 1,\, \ldots,\, L_k$. Hence $\mu_{1,\,1}(s)$ is the first eigenvalue of $\Delta^s_{S^{15}(r)}$. Moreover we obtain from (4.2.1) that $E_{\mu_1}(\Delta_{S^{15}(r)}) = \boldsymbol{H}^1(8/r^2,\, 7/r^2) = \boldsymbol{H}^s(8/r^2,\, 7/(sr)^2) = E_{\mu_{1,\,1}(s)}(\Delta^s_{S^{15}(r)})$. □

4.3. The nullity of a minimal geodesic hypershpere in $\mathbb{C}ay\mathbb{P}$

Let $U_{E_1}\mathbb{C}ay\mathbb{P}$ be the fiber of the unit sphere bundle of $\mathbb{C}ay\mathbb{P}$ at E_1. Then a map $\mathbb{S}_t \to \mathbb{S}_{\pi/2}$ $(0 < t < \pi/2)$ defined by $\gamma_v(t) \mapsto \gamma_v(\pi/2)$ for $v \in U_{E_1}\mathbb{C}ay\mathbb{P}$ gives a Riemannian submersion with totally geodesic fiberes. Here $\mathbb{S}_{\pi/2}$ is the cut locus of E_1. This fibration is equivalent to

$$\begin{array}{ccc} S^7(\cos t \sin t) & \longrightarrow & (S^{15}(\sin t), g_{\cos t}) \\ & & \downarrow \\ & & S^8(\sin t/2), \end{array}$$

where $g_{\cos t}$ is a canonical variation of the metric on $S^{15}(\sin t)$ with respect to the fibration $S^7(\sin t) \to S^{15}(\sin t) \to S^8(\sin t/2)$.

Now minimality condition for $\mathbb{S}_t$ is $\cot^2 t = 7/15$ (see (4.1.7)), when the value of variation parameter is $\cos t = \sqrt{154}/22$. Hence Proposition 4.1 is applicable, and together with (4.1.8), we have

When $\cos t = \sqrt{154}/22$, the first eigenvalue of $\Delta_{\mathbb{S}_t}$ is equal to that of $\Delta^{\cos t}_{S^{15}(\sin t)}$, and hence it is equal to $\mu_{1,1}(\sqrt{154}/22) = 44$ and its multiplicity is 16. We conclude the nullity of a minimal geodesic hypersphere is equal to 16.

References

[BB] Berard B.L. and Bourguignon J-P., *Laplacians and Riemannian submersions with totally geodesic fibres.* Illinois J. Math. **26** (1982), 181–200.

[C] Chen B.Y., *Totally umbilical submanifolds of Cayley plane.* Soochow J. Math. Natur. Sci. **3** (1977), 1–7.

[G1] Gotoh T., *The nullity of compact minimal real hypersufaces in a complex projective space.* Tokyo J. Math. **17** (1994), 201–209.

[G2] Gotoh T., *The nullity of a compact minimal real hypersufaces in a quaternion projective space.* Geometriae Dedicata **76** (1999), 53–64.

[I] Iwata K., *Compact transformation groups on rational cohomology Cayley projective planes.* Tohoku Math. J. **33** (1981), 429–442.

[K] Kollross A., *A classification of hyperporlar and cohomogeneity one action.* Trans. Amer. Math. Soc. **354** (2001), 571–612.

[O] Ohnita Y., *On stability of minimal submanifolds in compact symmetric spaces.* Compositio Math. **64** (1987), 157–189.

[S] Simons J., *Minimal varieties in Riemannian manifolds.* Ann. Math. **88** (1968), 62–105.

[T] Tsukada K., *Parallel submanifolds of Cayley plane.* Sci. Rep. Niigata Univ. Ser. A **21** (1985), 19–32.

[Y] Yokota I., *Gun to hyogen.* (in Japanese), Shokabo, 1973.

Department of Mathematics
The National Defense Academy
1-10-20 Hashirimisu
Yokosuka City, Kanagawa
239-8686, Japan
E-mail: tgotoh@nda.ac.jp