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Operators having commutants endowed with cyclicity-preserving quasiaffinities

(Dedicated to the memory of Katsutoshi Takahashi)

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Abstract. It is shown that there are commutant-cyclic vectors in the ranges of the quasiaffinities belonging to the commutant of any isometry or any quasinormal operator with a dominating unilateral shift part. This property ensures that the commutant-multiplicity is constant in the quasisimilarity orbits of these operators.

Key words: commutant, cyclic vector, multiplicity, quasiaffinity, quasisimilarity, isometry, quasinormal operator, outer function.

1. Introduction

Let $\mathcal{H}$ be a (nonzero, separable, complex) Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the C^*-algebra of all (bounded, linear) operators acting on $\mathcal{H}$. Given a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$, containing the identity operator I, a nonempty vector set $\mathcal{G} \subset \mathcal{H}$ is called *cyclic* for $\mathcal{A}$, if the vectors $\mathcal{A}\mathcal{G} = \{Ag\colon A \in \mathcal{A},\ g \in \mathcal{G}\}$ span the whole space: $\vee\mathcal{A}\mathcal{G} = \mathcal{H}$. The minimum of the cardinalities $|\mathcal{G}|$ of the sets $\mathcal{G}$, cyclic for $\mathcal{A}$, is called the *multiplicity* of $\mathcal{A}$, and is denoted by $\mu(\mathcal{A})$. With an operator $T \in \mathcal{L}(\mathcal{H})$ two algebras can be naturally associated: the algebra $\mathcal{A}_T := \{p(T)\colon p(\lambda) \text{ is a polynomial}\}$ generated by T, and the commutant $\{T\}' := \{C \in \mathcal{L}(\mathcal{H})\colon CT = TC\}$ of T. The multiplicity of $\mathcal{A}_T$ is called the *multiplicity of the operator* T, and is denoted by $\mu(T) := \mu(\mathcal{A}_T)$. The multiplicity of $\{T\}'$ is called the *commutant-multiplicity of* T, and is denoted by $\mu'(T) := \mu(\{T\}')$. A quick inspection in well-known classes of operators convince the reader that while the multiplicity $\mu(T)$ shows great variety, the commutant is usually cyclic, that is $\mu'(T) = 1$. It was W.R. Wogen who showed in [W] that the commutant-multiplicity $\mu'(T)$ can be also arbitrary. Even more, it turned out that, for any cardinal number $1 \leq n \leq \aleph_0$, the set $\mathcal{C}_n(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H})\colon \mu'(T) = n\}$ is norm-dense in the operator space $\mathcal{L}(\mathcal{H})$, provided $\dim\mathcal{H} = \aleph_0$ (see [AFHV, Theorem 11.19]).

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The notion of quasisimilarity was introduced by B. Sz.-Nagy and C. Foias in [NF1], and played important role in giving canonical models for several classes of operators; see, e.g., [Be] or [DH]. We recall that a (bounded) linear transformation $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is a *quasiaffinity*, if X is injective and has dense range. The operator $T_1 \in \mathcal{L}(\mathcal{H}_1)$ is a *quasiaffine transform* of the operator $T_2 \in \mathcal{L}(\mathcal{H}_2)$, in notation: $T_1 \prec T_2$, if the intertwining set $\mathcal{I}(T_1, T_2) := \{C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) : CT_1 = T_2C\}$ contains a quasiaffinity. Finally, the operators T_1 and T_2 are *quasisimilar*, and we use the notation $T_1 \sim T_2$, if $T_1 \prec T_2$ and $T_2 \prec T_1$ hold simultaneously. It is an easy exercise to show that quasisimilarity preserves the multiplicity of the operators. However, the analogous question, asking whether the commutant-multiplicity is also a quasisimilarity invariant, proved to be very hard. This problem was posed by D.A. Herrero in [He], and is still open.

In our previous papers [K1] and [K2] we gave partial answers to that question verifying, among others, that the commutant-multiplicity is really constant in the quasisimilarity orbits of normal operators, C_0-contractions and weak contractions. We showed that all quasinormal operators are commutant-cyclic, and completely settled the case when V is an isometry with a unilateral shift part of finite multiplicity. However, we were able to prove only that $\mu'(T) \leq 2$ if $T \sim V$, when V is an arbitrary isometry. The problem, whether $\mu'(T) = 1$ whenever $T \sim V$, led to a question concerning the range of bounded, analytic, operator-valued functions, which was left open. In the present note we solve this question. Furthermore, focusing on a relevant property of the commutant, we settle Herrero's problem for a large class of quasinormal operators, including all isometries.

The starting idea, how to attack the aformentioned question about operator-valued functions, was communicated to us by Professor Katsutoshi Takahashi during our visit at Hokkaido University in 1996. We were planning a joint research on that field, which was however prevented by Takahashi's tragically sudden death. I got to know Katsutoshi Takahashi as a kind person living in a loving family. He was a brilliant mathematician, who should have proved yet many beautiful theorems. This paper is dedicated to his memory.

2. Commutants with cyclicity-preserving quasiaffinities

We say that the operator $T \in \mathcal{L}(\mathcal{H})$ has a *commutant endowed with strongly cyclicity-preserving quasiaffinities* (shortly, T has CSCQ), if the set $Q\mathcal{G}$ is cyclic for $\{T\}'$, whenever $\mathcal{G} \subset \mathcal{H}$ is cyclic for $\{T\}'$ and $Q \in \{T\}'$ is a quasiaffinity. The following, weaker property will play more important role in the sequel. We say that T has a *commutant endowed with cyclicity-preserving quasiaffinities* (shortly, T has CCQ), if for every quasiaffinity $Q \in \{T\}'$ there exists a set $\mathcal{G} \subset \mathcal{H}$ such that $|\mathcal{G}| = \mu'(T)$ and $Q\mathcal{G}$ is cyclic for $\{T\}'$.

It was shown in [K1] that if $N \in \mathcal{L}(\mathcal{H})$ is a normal operator, then $\mu'(N) = 1$, and a vector $g \in \mathcal{H}$ is cyclic for the commutant $\{N\}'$ if and only if g is separating for the bicommutant $\{N\}''$. It is immediate that any quasiaffinity $Q \in \{N\}'$ transforms a vector g, separating for $\{N\}''$, into a vector Qg, which is also separating for $\{N\}''$. Therefore, all normal operators have CSCQ. The same can be said about the C_0-model operators $\sum_n \oplus S(m_n)$, considered in [K1].

The relevance of the property, having CCQ, to Herrero's problem is shown by the following statement.

Proposition 1 *Let $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ be quasisimilar operators, and let us assume that T_1 has CCQ. Then*
(i) $\mu'(T_2) \leq \mu'(T_1)$, *and*
(ii) $\mu'(T_1) = \mu'(T_2)$ *if and only if T_2 has also CCQ.*

Proof. Let $X \in \mathcal{I}(T_1, T_2)$ and $Y \in \mathcal{I}(T_2, T_1)$ be quasiaffinities. Given any quasiaffinity $Q \in \{T_2\}'$, the product $YQX \in \{T_1\}'$ is also a quasiaffinity. Since T_1 has CCQ, there exists a set $\mathcal{G}_1 \subset \mathcal{H}_1$ such that $|\mathcal{G}_1| = \mu'(T_1)$ and $YQX\mathcal{G}_1$ is cyclic for $\{T_1\}'$. Taking into account that

$$\begin{aligned}\mathcal{H}_2 &= (X\mathcal{H}_1)^- = \big(X \vee \{CYQX\mathcal{G}_1 \colon C \in \{T_1\}'\}\big)^- \\ &= \vee\big\{XCYQX\mathcal{G}_1 \colon C \in \{T_1\}'\big\} \subset \vee\big\{DQX\mathcal{G}_1 \colon D \in \{T_2\}'\big\} \\ &\subset \mathcal{H}_2,\end{aligned}$$

we infer that $Q\mathcal{G}_2$ is cyclic for $\{T_2\}'$, where $\mathcal{G}_2 = X\mathcal{G}_1$. Since $|\mathcal{G}_2| = |\mathcal{G}_1|$, it follows that $\mu'(T_2) \leq \mu'(T_1)$. Furthermore, assuming $\mu'(T_2) = \mu'(T_1)$, we obtain that T_2 has CCQ. In the opposite direction, supposing that T_2 also has CCQ, statement (i) readily yields that $\mu'(T_1) = \mu'(T_2)$. □

As an immediate consequence, we obtain that if the operator T is commutant-cyclic (that is, $\mu'(T) = 1$) and if T has CCQ, then all operators in the quasisimilarity orbit of T are commutant-cyclic and have CCQ. It was shown in [K2] that quasinormal operators are commutant-cyclic. Thus, in order to settle Herrero's problem for the class of quasinormal operators, it is sufficient to show that they have CCQ. In the subsequent sections we verify this property first for all unilateral shifts, and then for a large class of quasinormal operators.

3. Unilateral shifts

Given a Hilbert space $\mathcal{E}$, let us consider the Hardy space $H^2(\mathcal{E})$. We recall that $H^2(\mathcal{E})$ is the Hilbert space of those measurable functions $f\colon \mathbf{T} \to \mathcal{E}$, which are defined on the unit circle $\mathbf{T} = \{z \in \mathbf{C}\colon |z| = 1\}$, are square-integrable with respect to the normalized Lebesgue measure ν on $\mathbf{T}$, and whose Fourier coefficients of negative indices are equal to zero. The inner product on $H^2(\mathcal{E})$ is defined by $\langle f_1, f_2\rangle := \int_{\mathbf{T}}\langle f_1(z), f_2(z)\rangle_{\mathcal{E}}\, d\nu(z)$. The operator $S_{\mathcal{E}} \in \mathcal{L}(H^2(\mathcal{E}))$ of multiplication by the identical function $\chi(z) = z$ is called a *unilateral shift*, its multiplicity is $\mu(S_{\mathcal{E}}) = \dim \mathcal{E}$.

Given two Hilbert spaces $\mathcal{E}$ and $\mathcal{F}$, let $H^\infty(\mathcal{E}, \mathcal{F})$ stand for the Banach space of those measurable functions $\Theta\colon \mathbf{T} \to \mathcal{L}(\mathcal{E}, \mathcal{F})$, whose norm $\|\Theta\|_\infty := \operatorname{ess\,sup}\{\|\Theta(z)\|\colon z \in \mathbf{T}\}$ is finite, and whose Fourier coefficients of negative indices are equal to zero. The elements of $H^2(\mathcal{E})$ and $H^\infty(\mathcal{E}, \mathcal{F})$ can be extended to analytic functions on the open unit disc $\mathbf{D} := \{z \in \mathbf{C}\colon |z| < 1\}$ via the Poisson formula or power series expansion, and they can be recovered from these extensions taking nontangential limits. For further details the reader is referred to [NF1, Chapter V] or [RR, Chapter 3].

Every transformation-valued function $\Theta \in H^\infty(\mathcal{E}, \mathcal{F})$ induces a linear transformation $\Theta_+ \in \mathcal{L}(H^2(\mathcal{E}), H^2(\mathcal{F}))$, defined by $(\Theta_+ f)(z) := \Theta(z)f(z)$. It is known that the commutant of $S_{\mathcal{E}}$ can be identified with the Banach algebra $H^\infty(\mathcal{E}, \mathcal{E})$; more precisely, the mapping $\tau\colon H^\infty(\mathcal{E}, \mathcal{E}) \to \{S_{\mathcal{E}}\}'$, $\Theta \mapsto \Theta_+$ is an isometric algebra-isomorphism (see, e.g., [NF1, Lemma V.3.2]).

The simple unilateral shift $S = S_{\mathbf{C}}$, acting on the scalar-valued Hardy space $H^2 = H^2(\mathbf{C})$, is cyclic, and its cyclic vectors are precisely the outer functions of H^2. We recall that a nonzero *function* $f \in H^2$ is called *outer*, if the values of f on $\mathbf{D}$ are determined by the absolute value of f on $\mathbf{T}$ according to the formula:

$$f(z) = \kappa \exp\left(\int_{\mathbf{T}} \frac{\zeta + z}{\zeta - z} \log|f(\zeta)|\, d\nu(\zeta)\right) \quad (z \in \mathbf{D}),$$

where $\kappa \in \mathbf{T}$. Since the commutant $\{S\}' = \{\theta_+ \colon \theta \in H^\infty\}$ is the closure of $\mathcal{A}_S$ in the weak operator topology, it follows that the commutant-cyclic vectors are also the outer functions. It is also well-known that every nonzero function $f \in H^2$ can be factored into the product $f = f_1 f_2$, where $f_1 \in H^2$ is an outer function and $f_2 \in H^2$ is an *inner function*, which means that $|f_2(z)| = 1$ holds, for almost every $z \in \mathbf{T}$. Furthermore, this canonical factorization is unique up to constant multiples of modulus one. We say that an inner function $u \in H^\infty$ *divides* a function $f \in H^2$, if there exists a function $v \in H^2$ such that $f = uv$. For any nonempty set $\Phi \subset H^2$, let $\wedge\Phi$ stand for the *greatest common inner divisor* of the functions belonging to Φ. This function always exists and is uniquely determined up to a constant multiple of modulus one. In connection with the canonical factorization and the arithmetic in the function space H^∞ we refer to [Ho], [NF1, Chapter III] and [Be].

Let us consider again an arbitrary Hilbert space $\mathcal{E}$, where $1 \leq \dim \mathcal{E} \leq \aleph_0$. With any function $f \in H^2(\mathcal{E})$ and with any vector $e \in \mathcal{E}$, we can associate the scalar-valued function $f_e \in H^2$, defined by $f_e(z) := \langle f(z), e \rangle_{\mathcal{E}}$. The *inner factor of* f is defined by $[f]_i := \wedge\{f_e : e \in \mathcal{E}\}$. Finally, the nonzero *vector-valued function* $f \in H^2(\mathcal{E})$ is called *outer*, if its inner factor $[f]_i$ is constant. It is clear that f is outer, when f_e is an outer function for some vector $e \in \mathcal{E}$. However, f can be outer in such a way too that f_e is not outer for every choice of $e \in \mathcal{E}$; see [K2, Remark 3.7]. Since by Beurling's theorem the nonzero invariant subspaces of S are of the form uH^2, where $u \in H^\infty$ is an inner function, and taking into consideration the structure of the commutant $\{S_{\mathcal{E}}\}'$, it can be easily seen that $\mu'(S_{\mathcal{E}}) = 1$, and that a vector $f \in H^2(\mathcal{E})$ is cyclic for $\{S_{\mathcal{E}}\}'$ if and only if f is outer.

We remind the reader that an *operator-valued function* $\Theta \in H^\infty(\mathcal{E}, \mathcal{F})$ is called *outer*, if the transformation $\Theta_+ \in \mathcal{L}(H^2(\mathcal{E}), H^2(\mathcal{F}))$ has dense range (see [NF1, Chapter V]).

After identifying the commutant-cyclic vectors and the operators with dense ranges in the commutant, we are able to show that the unilateral shift $S_{\mathcal{E}}$ has CSCQ, provided the Hilbert space $\mathcal{E}$ is finite dimensional.

Proposition 2 *Let us assume that* $1 \leq \dim \mathcal{E} < \aleph_0$. *If* $f \in H^2(\mathcal{E})$ *is outer and* $\Theta \in H^\infty(\mathcal{E}, \mathcal{E})$ *is outer, then* $\Theta f \in H^2(\mathcal{E})$ *is also outer.*

Proof. Though this statement has been already proved in [K1] (as Lemma 5.1), we give here a reorganized version of the short proof for the sake of completeness, and in order to fix some notation.

Let $n = \dim \mathcal{E}$, and let us consider the matrix $[\Theta]$ of Θ with respect to a fixed orthonormal basis $\{e_i\}_{i=1}^n$ in $\mathcal{E}$. We recall that $[\Theta] = [\vartheta_{i,j}]_n$ is an n-by-n matrix over H^∞, in notation: $[\Theta] \in H_n^\infty$, which is defined by $\vartheta_{i,j}(z) := \langle \Theta(z)e_j, e_i\rangle$. Let $\Theta^A \in H^\infty(\mathcal{E}, \mathcal{E})$ be the operator-valued function, whose matrix $[\Theta^A] \in H_n^\infty$, with respect to the given basis, is the algebraic adjoint of the matrix $[\Theta]$. It is clear that $\Theta\Theta^A = \Theta^A\Theta = (\det\Theta)I_{\mathcal{E}}$, where $\det\Theta := \det[\Theta]$. (It can be shown that the definitions of Θ^A and $\det\Theta$ are actually independent of the choice of the basis.) Since Θ is outer, it follows that $\det\Theta \in H^\infty$ is outer; see [NF1, Corollary V.6.3]. Taking into account that

$$(\det\Theta)f = \Theta^A(\Theta f) \in [\Theta f]_i H^2(\mathcal{E}),$$

and that $[(\det\Theta)f]_i = [f]_i$ is constant, we infer that $[\Theta f]_i$ must be constant, that is, Θf is outer. □

The previous proof does not work if $\dim\mathcal{E} = \aleph_0$. In fact, it turned out that $S_{\mathcal{E}}$ does not have CSCQ in the infinite-dimensional case; see the example given in [K2, Remark 3.6]. We are going to show, however, that $S_{\mathcal{E}}$ does have CCQ. To achieve this end we shall need some auxiliary results connected with quasiequivalence of matrices over H^∞. This concept was introduced by E.A. Nordgren in [No], and was used also by B. Sz.-Nagy and C. Foias establishing canonical models for certain classes of contractions; see [NF2] and [Na]. We give the definition in terms of operator-valued functions rather than matrices.

The functions Θ_1 and $\Theta_2 \in H^\infty(\mathcal{E}, \mathcal{F})$ are called to be *quasiequivalent*, if for every inner function $\omega \in H^\infty$, there exist functions $\Psi_1, \Psi_1^a \in H^\infty(\mathcal{F}, \mathcal{F})$ and $\Psi_2, \Psi_2^a \in H^\infty(\mathcal{E}, \mathcal{E})$ such that

$$\Psi_1\Theta_1 = \Theta_2\Psi_2,$$
$$\Psi_1\Psi_1^a = \Psi_1^a\Psi_1 = \psi_1 I_{\mathcal{F}},\ \Psi_2\Psi_2^a = \Psi_2^a\Psi_2 = \psi_2 I_{\mathcal{E}},$$

where the nonzero functions $\psi_1, \psi_2 \in H^\infty$ are prime to ω:

$$\psi_1 \wedge \omega = \psi_2 \wedge \omega = 1.$$

It can be easily verified that quasiequivalence is an equivalence relation (i.e., it is reflexive, symmetric and transitive). The following lemma states that it preserves the outer function property.

Lemma 3 *Let $\Theta_1, \Theta_2 \in H^\infty(\mathcal{E}, \mathcal{F})$ be quasiequivalent functions. If Θ_1 is outer, then so is Θ_2.*

Proof. Given any inner function $\omega \in H^\infty$, there exist nonzero functions $\Psi_1, \Psi_1^a \in H^\infty(\mathcal{F}, \mathcal{F})$, $\Psi_2, \Psi_2^a \in H^\infty(\mathcal{E}, \mathcal{E})$ and $\psi_1, \psi_2 \in H^\infty$ such that $\Psi_1\Theta_1 = \Theta_2\Psi_2$, $\Psi_1\Psi_1^a = \Psi_1^a\Psi_1 = \psi_1 I_{\mathcal{F}}$, $\Psi_2\Psi_2^a = \Psi_2^a\Psi_2 = \psi_2 I_{\mathcal{E}}$ and $\psi_1 \wedge \omega = \psi_2 \wedge \omega = 1$. Exploiting these intertwining relations and the assumption that Θ_1 is outer, we obtain that

$$\begin{aligned}(\Theta_2 H^2(\mathcal{E}))^- &\supset (\Theta_2\Psi_2 H^2(\mathcal{E}))^- = (\Psi_1\Theta_1 H^2(\mathcal{E}))^- \\ &= (\Psi_1(\Theta_1 H^2(\mathcal{E}))^-)^- = (\Psi_1 H^2(\mathcal{F}))^- \\ &\supset (\Psi_1\Psi_1^a H^2(\mathcal{F}))^- = (\psi_1 H^2(\mathcal{F}))^- \\ &= [\psi_1]_i H^2(\mathcal{F}).\end{aligned}$$

Therefore, for the given inner function ω there exists a nonzero function $\psi_1 \in H^\infty$ such that

$$(\Theta_2 H^2(\mathcal{E}))^- \supset [\psi_1]_i H^2(\mathcal{F}) \quad \text{and} \quad \psi_1 \wedge \omega = 1.$$

Applying this statement again, replacing ω by $[\psi_1]_i$, we infer that there exists a nonzero function $\psi_1' \in H^\infty$ such that $(\Theta_2 H^2(\mathcal{E}))^- \supset [\psi_1']_i H^2(\mathcal{F})$ and $[\psi_1']_i \wedge [\psi_1]_i = \psi_1' \wedge [\psi_1]_i = 1$. Then

$$\begin{aligned}(\Theta_2 H^2(\mathcal{E}))^- &\supset \big([\psi_1]_i H^2(\mathcal{F})\big) \vee \big([\psi_1']_i H^2(\mathcal{F})\big) \\ &= \big([\psi_1]_i \wedge [\psi_1']_i\big) H^2(\mathcal{F}) = H^2(\mathcal{F}),\end{aligned}$$

and so Θ_2 is an outer function. □

The following lemma will play a crucial role in the proof of our theorem.

Lemma 4 *Let $\Theta \in H^\infty(\mathcal{E}, \mathcal{F})$ be an outer function, and let us assume that* $\dim \mathcal{F} = 2$. *Then, for every inner function $\omega \in H^\infty$, there exist nonzero functions $\Omega \in H^\infty(\mathcal{F}, \mathcal{E})$ and $\delta \in H^\infty$ such that*

$$\Theta\Omega = \delta I_{\mathcal{F}} \quad \text{and} \quad \delta \wedge \omega = 1.$$

Proof. Since $\Theta \in H^\infty(\mathcal{E}, \mathcal{F})$ is outer, it follows that $\Theta(z)$ has dense range for almost every $z \in \mathbf{T}$ (see [NF1, Proposition V.2.4]), and so $2 \leq n := \dim \mathcal{E} \leq \aleph_0$. Let $\{e_i\}_{0 \leq i < n}$ be an orthonormal basis in $\mathcal{E}$, let (f_0, f_1) be an orthonormal basis in $\mathcal{F}$, and let us consider the matrix $[\Theta]$ of Θ with respect to these bases. Applying [Na, Theorem 2] to the transpose of $[\Theta]$, we obtain that there exists a function $\Theta_0 \in H^\infty(\mathcal{E}, \mathcal{F})$, quasiequivalent to Θ and with matrix of the form

$$[\Theta_0] = \begin{bmatrix} \varepsilon_1 & 0 & 0 & \cdots \\ 0 & \varepsilon_2 & 0 & \cdots \end{bmatrix},$$

where $\varepsilon_1, \varepsilon_2 \in H^\infty$ are inner functions. Since Θ_0 is outer by Lemma 3, it follows that $\varepsilon_1 = \varepsilon_2 = 1$. Hence Θ_0 is a constant coisometry from the space $\mathcal{E}$ onto the space $\mathcal{F}$. The quasiequivalence of Θ and Θ_0 yields the existence of nonzero functions $\Psi_1, \Psi_1^a \in H^\infty(\mathcal{F}, \mathcal{F})$, $\Psi_2, \Psi_2^a \in H^\infty(\mathcal{E}, \mathcal{E})$ and $\psi_1, \psi_2 \in H^\infty$ such that $\Theta\Psi_2 = \Psi_1\Theta_0$, $\Psi_1\Psi_1^a = \Psi_1^a\Psi_1 = \psi_1 I_{\mathcal{F}}$, $\Psi_2\Psi_2^a = \Psi_2^a\Psi_2 = \psi_2 I_{\mathcal{E}}$ and $\psi_1 \wedge \omega = \psi_2 \wedge \omega = 1$. Considering the functions $\Omega := \Psi_2\Theta_0^*\Psi_1^a \in H^\infty(\mathcal{F}, \mathcal{E})$ and $\delta := \psi_1 \in H^\infty$, we conclude that

$$\Theta\Omega = \Theta\Psi_2\Theta_0^*\Psi_1^a = \Psi_1\Theta_0\Theta_0^*\Psi_1^a = \Psi_1\Psi_1^a = \delta I_{\mathcal{F}},$$

and so the proof is complete. □

Now, we are ready to prove our theorem, claiming the existence of an outer function in each 2-dimensional section of the range of an operator-valued outer function. To give the exact formulation we need some notation. For any $n \in \mathbf{N}$, let $\operatorname{Lat}_n \mathcal{E}$ denote the set of all n-dimensional subspaces of the Hilbert space $\mathcal{E}$. Furthermore, for any subspace $\mathcal{M}$ of $\mathcal{E}$, let $P_{\mathcal{M}} \in \mathcal{L}(\mathcal{E})$ stand for the orthogonal projection onto $\mathcal{M}$. We note that $P_{\mathcal{M}}$ can be considered also as a constant element of $H^\infty(\mathcal{E}, \mathcal{E})$, and that $(P_{\mathcal{M}})_+ \in \mathcal{L}(H^2(\mathcal{E}))$ is the orthogonal projection onto the subspace $H^2(\mathcal{M})$.

Theorem 5 *If the function $\Theta \in H^\infty(\mathcal{E}, \mathcal{E})$ is outer and $2 \le \dim \mathcal{E} \le \aleph_0$, then, for every subspace $\mathcal{M} \in \operatorname{Lat}_2 \mathcal{E}$, there exists a vector $g \in H^2(\mathcal{E})$ such that $P_{\mathcal{M}}\Theta g$ is an outer function.*

Proof. Let us give an arbitrary subspace $\mathcal{M} \in \operatorname{Lat}_2 \mathcal{E}$, and let us consider an orthonormal basis $\{e_i\}_{0 \le i < n}$ in $\mathcal{E}$, where $n := \dim \mathcal{E}$. Since

$$\begin{aligned} H^2(\mathcal{M}) &= P_{\mathcal{M}} H^2(\mathcal{E}) = P_{\mathcal{M}}(\Theta H^2(\mathcal{E}))^- = (P_{\mathcal{M}}\Theta H^2(\mathcal{E}))^- \\ &= \vee\{\xi_i P_{\mathcal{M}}\Theta e_i \colon \xi_i \in H^\infty,\ 0 \le i < n\}, \end{aligned}$$

we can find different indices k and l such that the vectors $(P_{\mathcal{M}}\Theta(z)e_k, P_{\mathcal{M}}\Theta(z)e_l)$ are linearly independent for all z in a set $\alpha \subset \mathbf{T}$ of positive Lebesgue measure. Let us consider the subspace $\mathcal{N} := \vee\{e_k, e_l\} \in \operatorname{Lat}_2 \mathcal{E}$ and the function $\Theta_1 \in H^\infty(\mathcal{N}, \mathcal{M})$, defined by $\Theta_1(z) := P_{\mathcal{M}}\Theta(z)|\mathcal{N}$. Given an orthonormal basis (e_1', e_2') in $\mathcal{M}$, as well, we can form the matrix $[\Theta_1]$ with respect to the pair of bases (e_k, e_l) and (e_1', e_2'). Since the function $\delta_1 := \det[\Theta_1] \in H^\infty$ does not vanish on α, it must be nonzero almost everywhere on $\mathbf{T}$. Let $\Omega_1 \in H^\infty(\mathcal{M}, \mathcal{N})$ be the function whose matrix, with respect to the bases (e_1', e_2') and (e_k, e_l), is the algebraic adjoint of the matrix $[\Theta_1]$. It follows that $\Theta_1\Omega_1 = \delta_1 I_{\mathcal{M}}$ and $\Omega_1\Theta_1 = \delta_1 I_{\mathcal{N}}$. Applying Lemma 4 for the outer function $\Theta_2 \in H^\infty(\mathcal{E}, \mathcal{M})$, defined by $\Theta_2(z) = P_{\mathcal{M}}\Theta(z)$, with the inner function $\omega = [\delta_1]_i$, we obtain that there exist nonzero functions $\Omega_2 \in H^\infty(\mathcal{M}, \mathcal{E})$ and $\delta_2 \in H^\infty$ such that $\Theta_2\Omega_2 = \delta_2 I_{\mathcal{M}}$ and $\delta_1 \wedge \delta_2 = 1$. Let us consider the vectors $g_1 := \Omega_1 e_1' \in H^2(\mathcal{N}) \subset H^2(\mathcal{E})$, $g_2 := \Omega_2 e_2' \in H^2(\mathcal{E})$, and let $g := g_1 + g_2$. Since

$$\begin{aligned} P_{\mathcal{M}}\Theta g &= P_{\mathcal{M}}\Theta g_1 + P_{\mathcal{M}}\Theta g_2 = \Theta_1\Omega_1 e_1' + \Theta_2\Omega_2 e_2' \\ &= \delta_1 e_1' + \delta_2 e_2', \end{aligned}$$

and since $\delta_1 \wedge \delta_2 = 1$, we infer that $[P_{\mathcal{M}}\Theta g]_i = 1$, that is, $P_{\mathcal{M}}\Theta g$ is an outer function. □

It is evident that $[\Theta g]_i$ divides $[P_{\mathcal{M}}\Theta g]_i$, and so Θg is outer, when $P_{\mathcal{M}}\Theta g$ is outer. Therefore, *the unilateral shift $S_{\mathcal{E}}$ has CCQ, but does not have CSCQ, if* $\dim \mathcal{E} = \aleph_0$. We guess that Theorem 5 is not true when the 2-dimensional subspace $\mathcal{M}$ is replaced by a 1-dimensional subspace.

4. Quasinormal operators

Quasinormal operators were first studied by A. Brown in [Br], where a simple canonical model was given for them. We recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called *quasinormal*, if $T(T^*T) = (T^*T)T$ holds, or equivalently, if the factors in the polar decomposition of T commute. It is clear that all normal operators are quasinormal. The *quasinormal operator T* is called *pure*, if there is no nonzero reducing subspace $\mathcal{H}'$ of T such that the restriction $T|\mathcal{H}'$ is a normal operator. Pure quasinormal operators can be constructed in the following way. Given any nonzero Hilbert space $\mathcal{E}$, let $\mathcal{L}_+(\mathcal{E})$ denote the set of all strictly positive operators acting on $\mathcal{E}$, that is, the set of those operators A, satisfying the condition $\langle Ax, x\rangle > 0$, for every $0 \neq x \in \mathcal{E}$. Every operator $A \in \mathcal{L}_+(\mathcal{E})$ can be considered also as a constant element of the function space $H^\infty(\mathcal{E}, \mathcal{E})$. For any $A \in \mathcal{L}_+(\mathcal{E})$, let us consider the operator $W(A) \in \mathcal{L}(H^2(\mathcal{E}))$, defined by $W(A) := A_+ S_{\mathcal{E}} = S_{\mathcal{E}} A_+$. It is easy to check that the unilateral weighted shift $W(A)$, with the constant operator-weight A, is a pure quasinormal operator. More importantly, it was shown in [Br] that every pure quasinormal operator is unitarily equivalent to an operator of this form. Furthermore, the operators $W(A_1)$ and $W(A_2)$, with $A_1 \in \mathcal{L}_+(\mathcal{E}_1)$ and $A_2 \in \mathcal{L}_+(\mathcal{E}_2)$, are unitarily equivalent, if and only if A_1 and A_2 are unitarily equivalent.

If the quasinormal operator T is not normal, then it is unitarily equivalent to an operator of the form $N \oplus W(A) \in \mathcal{L}(\mathcal{K} \oplus H^2(\mathcal{E}))$, where $N \in \mathcal{L}(\mathcal{K})$ is normal, $A \in \mathcal{L}_+(\mathcal{E})$, $0 \leq \dim \mathcal{K} \leq \aleph_0$ and $1 \leq \dim \mathcal{E} \leq \aleph_0$. The operators N and $W(A)$, appearing here, are uniquely determined up to unitary equivalence, and are called the *normal* and *pure parts of* T, respectively. We refer to [C, Section II.3] for details and further references.

The description of the similarity and quasisimilarity relations between quasinormal operators proved to be a hard problem. The complete characterization was given by K.-Y. Chen, D.A. Herrero and P.Y. Wu in [CHW]. We shall heavily rely on their deep result.

First, we consider pure quasinormal operators. The next lemma deals with the particular case $W(\alpha I_{\mathcal{E}})$, where $\alpha \in (0, \infty)$. Let us observe that $W(\alpha I_{\mathcal{E}}) = \alpha S_{\mathcal{E}}$.

We recall that an operator $T_1 \in \mathcal{L}(\mathcal{H}_1)$ is *densely intertwined to* an operator $T_2 \in \mathcal{L}(\mathcal{H}_2)$, in notation: $T_1 \stackrel{\mathrm{d}}{\prec} T_2$, if the intertwining set $\mathcal{I}(T_1, T_2)$ contains a transformation having dense range. Let us say that T_1 is *collectively densely intertwined to* T_2, and we use the notation: $T_1 \stackrel{\mathrm{cd}}{\prec} T_2$, if $\vee\{\operatorname{ran} X : X \in \mathcal{I}(T_1, T_2)\} = \mathcal{H}_2$.

Lemma 6 *Let $\mathcal{E}$ and $\mathcal{F}$ be nonzero, separable Hilbert spaces, and let us assume that $0 < \alpha < \beta$. Then*

$$\mathcal{I}(\alpha S_{\mathcal{E}}, \beta S_{\mathcal{F}}) = \{0\} \quad \textit{and} \quad \beta S_{\mathcal{F}} \stackrel{\mathrm{cd}}{\prec} \alpha S_{\mathcal{E}}.$$

Proof. Let $X \in \mathcal{I}(\alpha S_{\mathcal{E}}, \beta S_{\mathcal{F}})$ be arbitrary. Since $X(\alpha/\beta)^n S_{\mathcal{E}}^n = S_{\mathcal{F}}^n X$ holds for every $n \in \mathbf{N}$, it follows that

$$\|Xh\| = \|S_{\mathcal{F}}^n X h\| = \left(\frac{\alpha}{\beta}\right)^n \|X S_{\mathcal{E}}^n h\| \leq \left(\frac{\alpha}{\beta}\right)^n \|X\| \|h\|$$

is valid for any $h \in H^2(\mathcal{E})$ and $n \in \mathbf{N}$. Taking into account that

$$\lim_{n\to\infty} \left(\frac{\alpha}{\beta}\right)^n = 0,$$

we conclude that $X = 0$.

Since $\|(\alpha/\beta)S_{\mathcal{E}}\| = \alpha/\beta < 1$ and $\dim H^2(\mathcal{E}) = \aleph_0$, it follows that the minimal isometric dilation V of $(\alpha/\beta)S_{\mathcal{E}}$ is unitarily equivalent to $S_{\mathcal{K}}$, where $\dim \mathcal{K} = \aleph_0$ (see [NF1, Theorems II.1.2 and II.2.1]). Taking into account that $V \overset{\text{d}}{\prec} (\alpha/\beta)S_{\mathcal{E}}$ (see [NF1, Theorem I.4.1]), we infer that $S_{\mathcal{K}} \overset{\text{d}}{\prec} (\alpha/\beta)S_{\mathcal{E}}$. On the other hand, it can be readily seen that $S_{\mathcal{F}} \overset{\text{cd}}{\prec} S_{\mathcal{K}}$. Thus, $S_{\mathcal{F}} \overset{\text{cd}}{\prec} (\alpha/\beta)S_{\mathcal{E}}$ is also true, that is $\beta S_{\mathcal{F}} \overset{\text{cd}}{\prec} \alpha S_{\mathcal{E}}$. □

We note here that conditions, under which a unilateral shift can be injectively intertwined to an operator, were extensively studied by K. Takahashi in [T].

The following statement claims a slightly stronger property than having CCQ for the pure quasinormal operator $W(A)$, when the largest spectrum-point $\|A\|$ of $A \in \mathcal{L}_+(\mathcal{E})$ is an eigenvalue. For the sake of easy reference, let us say in that case that $W(A)$ *has a dominating shift*. Then, considering the spectral measure E_A of A and the spectral subspaces $\mathcal{E}_0 := E_A((0, \|A\|))\mathcal{E}$, $\mathcal{E}_1 := E_A(\{\|A\|\})\mathcal{E} \neq \{0\}$, it is clear that $W(A) = \|A\|S_{\mathcal{E}}$ if $\mathcal{E}_0 = \{0\}$, and $W(A) = W(A_0) \oplus (\|A\|S_{\mathcal{E}_1})$ if $\mathcal{E}_0 \neq \{0\}$, where $A_0 = A|\mathcal{E}_0$.

Proposition 7 *Let us assume that the pure quasinormal operator $W(A) \in \mathcal{L}(H^2(\mathcal{E}))$ has a dominating shift. If the operator $Q \in \{W(A)\}'$ has dense range, then there exists a vector $g \in H^2(\mathcal{E})$ such that Qg is cyclic for the commutant $\{W(A)\}'$.*

Proof. By the main result, Theorem 3.1 of [CHW], we can find a diagonal operator $A_0 \in \mathcal{L}_+(\mathcal{E})$ such that $W(A)$ is quasisimilar to $W(A_0)$, and $\|A_0\| = \|A\|$ is an eigenvalue of A_0. Since $\mu'(W(A)) = \mu'(W(A_0)) = 1$ (see [K2, Theorem 2.1]), we infer by an analogue of Proposition 1 (with Q having dense range replacing Q quasiaffinity) that $W(A)$ and $W(A_0)$ have the required property at the same time. Thus, we may assume in the sequel that A is a diagonal operator.

Let $\alpha_0 := \|A\|$. If $A = \alpha_0 I_{\mathcal{E}}$, then the commutant of $W(A) = \alpha_0 S_{\mathcal{E}}$ coincides with $\{S_{\mathcal{E}}\}'$, and so Proposition 2 and Theorem 5 imply the statement. We can suppose, therefore, that the point spectrum $\sigma_p(A)$ of A contains more than one point. For any eigenvalue $\alpha \in \sigma_p(A)$, let us consider the eigenspace $\mathcal{E}_\alpha := \ker(A - \alpha I_{\mathcal{E}})$. We know that $\mathcal{E} = \sum_\alpha \oplus \mathcal{E}_\alpha$, and that the orthogonal decomposition $H^2(\mathcal{E}) = \sum_\alpha \oplus H^2(\mathcal{E}_\alpha)$ is reducing for $W(A)$, and $W(A)|H^2(\mathcal{E}_\alpha) = \alpha S_\alpha$, where $S_\alpha := S_{\mathcal{E}_\alpha}$ $(\alpha \in \sigma_p(A))$. The operator $P_\alpha := (P_{\mathcal{E}_\alpha})_+$ is the orthogonal projection onto $H^2(\mathcal{E}_\alpha)$ in $H^2(\mathcal{E})$, and belongs to the commutant of $W(A)$.

Let $Q \in \{W(A)\}'$ be an arbitrary operator with dense range. Since $P_{\alpha_0}Q|H^2(\mathcal{E}_\alpha) \in \mathcal{I}(\alpha S_\alpha,\, \alpha_0 S_{\alpha_0})$, and since $\mathcal{I}(\alpha S_\alpha,\, \alpha_0 S_{\alpha_0}) = \{0\}$ by Lemma 6 if $\alpha < \alpha_0$, we infer that $P_{\alpha_0}QH^2(\mathcal{E}) = P_{\alpha_0}QH^2(\mathcal{E}_{\alpha_0})$. Thus, the operator $Q_0 := P_{\alpha_0}Q|H^2(\mathcal{E}_{\alpha_0}) \in \{\alpha_0 S_{\alpha_0}\}' = \{S_{\alpha_0}\}'$ has dense range, and so — as we have already seen — there exists a vector $g \in H^2(\mathcal{E}_{\alpha_0})$ such that $Q_0 g$ is cyclic for $\{\alpha_0 S_{\alpha_0}\}'$.

Let us introduce the subspace $\mathcal{H} = \vee\{DQg\colon D \in \{W(A)\}'\}$. For any $C \in \{S_{\alpha_0}\}'$, the operator $\widetilde{C} := \sum_\alpha \oplus C_\alpha$ belongs to $\{W(A)\}'$, where $C_{\alpha_0} = C$ and $C_\alpha = 0$ for $\alpha \neq \alpha_0$. Hence $\mathcal{H}$ contains the subspace $\vee\{\widetilde{C}Qg = CQ_0 g\colon C \in \{S_{\alpha_0}\}'\} = H^2(\mathcal{E}_{\alpha_0})$. On the other hand, given any $\alpha \in \sigma_p(A) \setminus \{\alpha_0\}$, for every intertwining mapping $X \in \mathcal{I}(\alpha_0 S_{\alpha_0},\, \alpha S_\alpha)$, the transformation $\widetilde{X} \in \mathcal{L}(H^2(\mathcal{E}))$, defined by $\widetilde{X}|H^2(\mathcal{E}_{\alpha_0}) := X$ and $\widetilde{X}|H^2(\mathcal{E}_\alpha) := 0$ for $\alpha \neq \alpha_0$, also belongs to the commutant $\{W(A)\}'$. Taking into account that $\alpha_0 S_{\alpha_0} \overset{cd}{\prec} \alpha S_\alpha$ is true by Lemma 6, we conclude that

$$
\begin{aligned}
H^2(\mathcal{E}_\alpha) &= \vee\{XH^2(\mathcal{E}_{\alpha_0})\colon X \in \mathcal{I}(\alpha_0 S_{\alpha_0},\, \alpha S_\alpha)\} \\
&= \vee\{\widetilde{X}\widetilde{C}Qg\colon X \in \mathcal{I}(\alpha_0 S_{\alpha_0},\, \alpha S_\alpha),\ C \in \{\alpha_0 S_{\alpha_0}\}'\} \subset \mathcal{H}.
\end{aligned}
$$

Consequently, $\mathcal{H}$ must coincide with the whole space $H^2(\mathcal{E})$, and so the vector Qg is cyclic for $\{W(A)\}'$. □

Now, we prove our main result.

Theorem 8 *Let $T \in \mathcal{L}(\mathcal{H})$ be a quasinormal operator, and let us assume that T is either normal or its pure part has a dominating shift. Then T has CCQ.*

Proof. If T is normal, then it has even CSCQ, as we have seen in Section 2. On the other hand, if T is pure, then Proposition 7 can be applied. Since unitary equivalence obviously preserves the property having CCQ, we may assume that T is of the form $T = N \oplus W(A) \in \mathcal{L}(\mathcal{H} = \mathcal{K} \oplus H^2(\mathcal{E}))$, with nonzero $\mathcal{K}$ and $\mathcal{E}$, where $N \in \mathcal{L}(\mathcal{K})$ is normal and $W(A) \in \mathcal{L}(H^2(\mathcal{E}))$ has a dominating shift. As in Proposition 7, the proof can be reduced again to the case, when $A \in \mathcal{L}_+(\mathcal{E})$ is a diagonal operator. We may suppose, furthermore, that $\|A\| = 1$, since $\{cT\}' = \{T\}'$ holds, for any nonzero c.

Let E_N denote the spectral measure of N, and let us consider the spectral subspaces $\mathcal{K}_0 := E_N(\mathbf{D})\mathcal{K}$, $\mathcal{K}_u := E_N(\mathbf{T})\mathcal{K}$, $\mathcal{K}_1 := E_N(\mathbf{C} \setminus \mathbf{D}^-)\mathcal{K}$, and the restrictions $N_0 := N|\mathcal{K}_0$, $N_u := N|\mathcal{K}_u$, $N_1 := N|\mathcal{K}_1$. It is known that $\mathcal{K}_u$ can be decomposed into the orthogonal sum $\mathcal{K}_u = \mathcal{K}_a \oplus \mathcal{K}_s$, reducing for N_u, such that $N_a := N|\mathcal{K}_a$ is an absolutely continuous unitary operator and $N_s := N|\mathcal{K}_s$ is a singular unitary operator. The subspaces $\mathcal{K}_0$, $\mathcal{K}_a$, $\mathcal{K}_s$ and $\mathcal{K}_1$ are all invariant for the commutant $\{N\}'$.

We shall assume that $\sigma_p(A)$ contains more than one point; the simpler case $\sigma_p(A) = \{1\}$ can be treated with obvious modifications. For any $\alpha \in \sigma_p(A)$, let $\mathcal{E}_\alpha := \ker(A - \alpha I_\mathcal{E})$ and $S_\alpha := S_{\mathcal{E}_\alpha}$. Considering the decomposition $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, where $\mathcal{E}_0 := \sum_{\alpha<1} \oplus \mathcal{E}_\alpha$, we obtain that $W(A) = W(A_0) \oplus S_1$, where $A_0 = A|\mathcal{E}_0$ and $W(A_0) = \sum_{\alpha<1} \oplus \alpha S_\alpha$.

Now, we examine which intertwining sets reduce to the trivial one in connection with the restrictions of T to the reducing subspaces introduced before. First, we know by [CHW, Lemma 4.2] that $\mathcal{I}(N, W(A)) = \{0\}$.

For any $\varepsilon > 0$, let $\mathcal{K}_{1,\varepsilon} := E_N(\mathbf{C} \setminus (1+\varepsilon)\mathbf{D})\mathcal{K}$ and $N_{1,\varepsilon} := N|\mathcal{K}_{1,\varepsilon}$. Taking into account that

$$\inf\{\|N_{1,\varepsilon}x\| : x \in \mathcal{K}_{1,\varepsilon}, \|x\| = 1\} \geq 1 + \varepsilon > 1 = \|W(A)\|,$$

we infer by [CHW, Lemma 3.2] that $\mathcal{I}(W(A), N_{1,\varepsilon}) = \{0\}$. (An argumentation, similar to the proof of Lemma 6 could be also applied here.) Since $\varepsilon > 0$ was arbitrary, it follows that $\mathcal{I}(W(A), N_1) = \{0\}$. Considering the decomposition $W(A_0) = \sum_{\alpha<1} \oplus \alpha S_\alpha$, the relations $\mathcal{I}(W(A_0), N_u) = \{0\}$ and $\mathcal{I}(W(A_0), S_1) = \{0\}$ can be proved in a similar way (the latter one can be also derived directly from Lemma 6).

Since $W(A)$ is a completely nonunitary contraction and N_s is a singular unitary operator, the Lifting Theorem results in that $\mathcal{I}(W(A), N_s) = \{0\}$; see [NF1, Theorems II.2.3 and II.6.4] and [D, Section 4].

On the other hand, the unilateral shift S_1 can be abundantly intertwined to some restrictions. Namely, $S_1 \overset{\text{cd}}{\prec} W(A_0)$ holds by Lemma 6. The relation $S_1 \overset{\text{cd}}{\prec} N_0$ is true, since the minimal isometric dilation of N_0 is a unilateral shift. Finally, $S_1 \overset{\text{cd}}{\prec} N_a$ can be verified taking the functional model of N_a, see, e.g., [K3].

Let us consider now an arbitrary quasiaffinity $Q \in \{T\}'$. In view of the prohibited nontrivial intertwiners listed before, the matrix of Q, with respect to the decomposition

$$\mathcal{H} = \mathcal{K}_0 \oplus \mathcal{K}_a \oplus \mathcal{K}_s \oplus \mathcal{K}_1 \oplus H^2(\mathcal{E}_0) \oplus H^2(\mathcal{E}_1),$$

is of the form:

$$Q = \begin{bmatrix} Q_1 & 0 & 0 & 0 & Q_{1,5} & Q_{1,6} \\ 0 & Q_2 & 0 & 0 & 0 & Q_{2,6} \\ 0 & 0 & Q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_5 & Q_{5,6} \\ 0 & 0 & 0 & 0 & 0 & Q_6 \end{bmatrix}.$$

The structure of the operator Q shows that the entries $Q_3 \in \{N_s\}'$, $Q_4 \in \{N_1\}'$ are quasiaffinities, and $Q_6 \in \{S_1\}'$ has dense range. Since the normal operators N_s and N_1 have CSCQ, there exist vectors $g_3 \in \mathcal{K}_s$ and $g_4 \in \mathcal{K}_1$ such that $Q_3 g_3$ is cyclic for $\{N_s\}'$, and $Q_4 g_4$ is cyclic for $\{N_1\}'$. On the other hand, Proposition 2 and Theorem 5 ensure the existence of a vector $g_6 \in H^2(\mathcal{E}_1)$ such that $Q_6 g_6$ is cyclic for $\{S_1\}'$. Let us consider the vector

$$g := 0 \oplus 0 \oplus g_3 \oplus g_4 \oplus 0 \oplus g_6 \in \mathcal{H},$$

and the subspace $\mathcal{H}' := \vee\{CQg\colon C \in \{T\}'\}$. Taking into account that, for any $C_3 \in \{N_s\}'$, $C_4 \in \{N_1\}'$ and $C_6 \in \{S_1\}'$, the operator $0 \oplus 0 \oplus C_3 \oplus C_4 \oplus 0 \oplus C_6$ belongs to the commutant $\{T\}'$, and that

$$Qg = Q_{1,6}g_6 \oplus Q_{2,6}g_6 \oplus Q_3 g_3 \oplus Q_4 g_4 \oplus Q_{5,6}g_6 \oplus Q_6 g_6,$$

we infer that

$$\mathcal{H}' \supset \mathcal{K}_s \oplus \mathcal{K}_1 \oplus H^2(\mathcal{E}_1).$$

Furthermore, since $\mathcal{H}' \supset H^2(\mathcal{E}_1)$, the relation $S_1 \overset{\text{cd}}{\prec} N_0 \oplus N_a \oplus W(A_0)$ implies that

$$\mathcal{H}' \supset \mathcal{K}_0 \oplus \mathcal{K}_a \oplus H^2(\mathcal{E}_0).$$

Hence $\mathcal{H}' = \mathcal{H}$, and so the vector Qg is cyclic for $\{T\}'$. □

The following statement is an immediate consequence of Proposition 1 and Theorem 8.

Corollary 9 *Let us assume that the quasinormal operator T is either normal or its pure part has a dominating shift. Then every operator T', which is quasisimilar to T, is commutant-cyclic.*

Remark 10 We note that the pattern of the proof of Theorem 8 actually yields a more general statement than Corollary 9. Namely, let us be given an operator $T \in \mathcal{L}(\mathcal{H})$, which is the orthogonal sum of a normal operator

$N \in \mathcal{L}(\mathcal{K})$, an absolutely continuous contraction $R \in \mathcal{L}(\mathcal{R})$, and a unilateral shift $S_{\mathcal{E}} \in \mathcal{L}(H^2(\mathcal{E}))$. (We recall that the absolute continuity of R means that no nonzero subspace reduces R to a singular unitary operator.)

If the intertwining relations

$$S \overset{\text{cd}}{\prec} R \quad \text{and} \quad \mathcal{I}(R, S) = \{0\}$$

hold between the simple unilateral shift S and R, then T has CCQ, $\mu'(T) = 1$, and so $\mu'(T') = 1$ is true whenever $T' \sim T$.

The proof needs only the following adjustment in the reasoning. Taking into account that $\mathcal{I}(N_1, R) = \{0\}$ does not necessarily hold, we can infer only that the operator $Q_4 \in \{N_1\}'$ has dense range. However, it follows by [D, Lemma 4.1] that the subspace $(\ker Q_4)^\perp$ reduces N_1 to an operator, which is unitarily equivalent to N_1. Let $Z \in \{N_1\}'$ be an isometry such that $\operatorname{ran} Z = (\ker Q_4)^\perp$. Then $Q_4' = Q_4 Z \in \{N_1\}'$ is a quasiaffinity and $\operatorname{ran} Q_4' = \operatorname{ran} Q_4$. Hence, there exists a vector $g_4 \in \mathcal{K}_1$ such that $Q_4 g_4 = Q_4' Z^* g_4$ is cyclic for $\{N_1\}'$.

We note also that applying the previous method for the component Q_3 too, we may obtain that, for every operator $Q \in \{T\}'$ with dense range, there exists a vector $g \in \mathcal{H}$ such that Qg is cyclic for the commutant $\{T\}'$.

We close this paper by posing the following questions.

Questions 11

(a) *Do all quasinormal operators have CCQ?*

(b) *Is the property having CCQ a quasisimilarity invariant, in general?*

(c) *Do all operators have CCQ?*

Affirmative answer for the last question would settle Herrero's problem in the positive (see Proposition 1).

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