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On the new sequence spaces which include the spaces c_0 and c

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Abstract. In the present paper, the sequence spaces a_0^r and a_c^r of non-absolute type which are the BK-spaces including the spaces c_0 and c have been introduced and proved that the spaces a_0^r and a_c^r are linearly isomorphic to the spaces c_0 and c, respectively. Additionally, the α-, β- and γ-duals of the spaces a_0^r and a_c^r have been computed and their basis have been constructed. Finally, the necessary and sufficient conditions on an infinite matrix belonging to the classes $(a_c^r : \ell_p)$ and $(a_c^r : c)$ have been determined and the characterizations of some other classes have also been derived by means of a given basic lemma, where $1 \le p \le \infty$.

Key words: Sequence spaces of non-absolute type, duals and basis of a sequence space, matrix transformations.

1. Preliminaries, background and notation

By w, we shall denote the space of all real valued sequences. Any vector subspace of w is called as a *sequence space*. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p; we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively; where $1 < p < \infty$.

A sequence space λ with a linear topology is called a *K-space* provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A K-space λ is called an *FK-space* provided λ is a complete linear metric space. An FK-space whose topology is normable is called a *BK-space* (see Choudhary and Nanda [5, pp. 272–273]).

Let λ, μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform

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of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}). \tag{1.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

For a sequence space λ, the *matrix domain* λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in w \colon Ax \in \lambda\}. \tag{1.2}$$

We shall denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has been recently employed by Wang [11], Ng and Lee [9], Malkowsky [8] and Altay and Başar [2]. They introduced the sequence spaces $(\ell_p)_{N_q}$ in [11], $(\ell_p)_{C_1} = X_p$ in [9], $(\ell_\infty)_{R^t} = r_\infty^t$, $c_{R^t} = r_c^t$ and $(c_0)_{R^t} = r_0^t$ in [8] and $(\ell_p)_{E^r} = e_p^r$ in [2]; where N_q, C_1, R^t and E^r denote the Nörlund, arithmetic, Riesz and Euler means, respectively and $1 \le p \le \infty$. In the present paper, following [11], [9], [8] and [2], we introduce the sequence spaces a_0^r and a_c^r of non-absolute type and derive some results related to those sequence spaces. Furthermore, we have constructed the basis and computed the α-, β- and γ-duals of the spaces a_0^r and a_c^r. Finally, we have essentially characterized the matrix classes $(a_c^r : \ell_p)$, $(a_c^r : c)$ and also derived the characterizations of some other classes by means of a given basic lemma, where $1 \le p \le \infty$.

2. The sequence spaces a_0^r and a_c^r of non-absolute type

We introduce the sequence spaces a_0^r and a_c^r, as the set of all sequences such that A^r-transforms of them are in the spaces c_0 and c, respectively, that is

$$a_0^r = \left\{x = (x_k) \in w \colon \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} (1 + r^k) x_k = 0\right\}$$

and

$$a_c^r = \left\{ x = (x_k) \in w\colon \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} (1 + r^k) x_k \quad \text{exists} \right\};$$

where A^r denotes the matrix $A^r = (a_{nk}^r)$ defined by

$$a_{nk}^r = \begin{cases} \dfrac{1+r^k}{n+1}, & (0 \leq k \leq n) \\ 0, & (k > n) \end{cases}.$$

It is known by Başar [3] that the method A^r is regular for $0 < r < 1$ and is stronger than the Cesàro method C_1. We assume unless stated otherwise that $0 < r < 1$. With the notation of (1.2), we may redefine the spaces a_0^r and a_c^r as follows:

$$a_0^r = (c_0)_{A^r} \quad \text{and} \quad a_c^r = c_{A^r}. \tag{2.1}$$

It is trivial that $a_0^r \subset a_c^r$. Define the sequence $y = \{y_k(r)\}$, which will be frequently used, as the A^r-transform of a sequence $x = (x_k)$, i.e.,

$$y_k(r) = \sum_{j=0}^{k} \frac{1+r^j}{k+1} x_j\,; \quad (k \in \mathbb{N}). \tag{2.2}$$

Now, we may begin with the following theorem which is essential in the text:

Theorem 2.1 *The sets a_0^r and a_c^r are the linear spaces with the co-ordinatewise addition and scalar multiplication which are the BK-spaces with the norm $\|x\|_{a_0^r} = \|x\|_{a_c^r} = \|A^r x\|_{\ell_\infty}$.*

Proof. The first part of the theorem is a routine verification and so we omit it. Furthermore, since (2.1) holds and c_0, c are the BK-spaces with respect to their natural norms (see Maddox [7, pp. 217–218]), and the matrix $A^r = (a_{nk}^r)$ is normal, Theorem 4.3.2 of Wilansky [12, p. 61] gives the fact that the spaces a_0^r, a_c^r are the BK-spaces. □

Therefore, one can easily check that the absolute property does not hold on the spaces a_0^r and a_c^r, since $\|x\|_{a_0^r} \neq \big\| |x| \big\|_{a_0^r}$ and $\|x\|_{a_c^r} \neq \big\| |x| \big\|_{a_c^r}$ for at least one sequence in the spaces a_0^r and a_c^r; where $|x| = (|x_k|)$. This says that a_0^r and a_c^r are the sequence spaces of non-absolute type.

Theorem 2.2 *The sequence spaces a_0^r and a_c^r of non-absolute type are linearly isomorphic to the spaces c_0 and c, respectively, i.e., $a_0^r \cong c_0$ and $a_c^r \cong c$.*

Proof. To prove this, we should show the existence of a linear bijection between the spaces a_0^r and c_0. Consider the transformation T defined, with the notation of (2.2), from a_0^r to c_0 by $x \mapsto y = Tx$. The linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.

Let $y \in c_0$ and define the sequence $x = \{x_k(r)\}$ by

$$x_k(r) = \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1+j}{1+r^k} y_j\,; \quad (k \in \mathbb{N}).$$

Then, we have

$$\lim_{n\to\infty} \left\{ \frac{1}{n+1} \sum_{i=0}^{n} (1+r^i) \sum_{j=i-1}^{i} (-1)^{i-j} \frac{1+j}{1+r^i} y_j \right\} = \lim_{n\to\infty} y_n = 0$$

which says us that $x \in a_0^r$. Additionally, we observe that

$$\begin{aligned}\|x\|_{a_0^r} =& \sup_{n\in\mathbb{N}} \left| \frac{1}{n+1} \sum_{i=0}^{n} (1+r^i) \sum_{j=i-1}^{i} (-1)^{i-j} \frac{1+j}{1+r^i} y_j \right| \\ =& \sup_{n\in\mathbb{N}} |y_n| = \|y\|_{c_0} < \infty.\end{aligned}$$

Consequently, we see from here that T is surjective and is norm preserving. Hence, T is a linear bijection which therefore shows us that the spaces a_0^r and c_0 are linearly isomorphic, as was desired.

It is clear here that if the spaces a_0^r and c_0 are respectively replaced by the spaces a_c^r and c, then we obtain the fact that $a_c^r \cong c$. This completes the proof. □

Now, we may give the theorem on the inclusion relations concerning with the spaces a_0^r and a_c^r.

Theorem 2.3 *Although the inclusions $c_0 \subset a_0^r$ and $c \subset a_c^r$ strictly hold, neither of the spaces a_0^r and ℓ_∞ includes the other one.*

Proof. To prove the validity of the inclusion $c_0 \subset a_0^r$, let us take any $y \in c_0$. Then, bearing in mind the regularity of the method A^r we immediately observe that $A^r y \in c_0$ which means that $y \in a_0^r$. Hence, the inclusion $c_0 \subset a_0^r$ holds. Furthermore, let us consider the gap sequence $u = \{u_k(r)\}$ defined by

$$u_k(r) = \begin{cases} \dfrac{\sqrt[3]{k}}{1+r^k}, & (k = m^3,\ m = 0,\ 1,\ 2, \ldots) \\ 0, & (k \neq m^3,\ m = 0,\ 1,\ 2, \ldots) \end{cases} \tag{2.3}$$

for all $k \in \mathbb{N}$. Then, since

$$\begin{aligned}(A^r u)_n &= \frac{1}{n+1}\sum_{k=0}^{n}(1+r^k)u_k(r) \\ &= \frac{1}{n+1}\sum_{k=0}^{m} k = \frac{m(m+1)}{2(n+1)}; \quad (m^3 \leq n < (m+1)^3,\ m \in \mathbb{N}),\end{aligned}$$

$A^r u \in c_0$ which implies that u is in a_0^r but not in c_0. This shows that the inclusion $c_0 \subset a_0^r$ is strict. By the similar discussion, one can see that the strict inclusion $c \subset a_c^r$ also holds.

To establish the second part of Theorem, let us consider the sequences $u = \{u_k(r)\}$ defined by (2.3) and $x = e = (1, 1, 1, \ldots)$. Then, since $(A^r x)_n = 1 + (1 - r^{n+1})/[(1-r)(n+1)]$, one can easily see that x is in ℓ_∞ but not in a_0^r and u is in a_0^r but not in ℓ_∞. Hence, the sequence spaces a_0^r and ℓ_∞ overlap but neither contains the other. This completes the proof. □

3. The basis for the spaces a_0^r and a_c^r

In the present section, we give two sequences of the points of the spaces a_0^r and a_c^r which form the basis for the spaces a_0^r and a_c^r.

Firstly, we define the Schauder basis of a normed space. If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty}\left\| x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n)\right\| = 0$$

then (b_n) is called a *Schauder basis* (or briefly *basis*) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$.

Theorem 3.1 *Define the sequence* $b^{(k)}(r) = \{b_n^{(k)}(r)\}_{n\in\mathbb{N}}$ *of the elements of the space* a_0^r *by*

$$b_n^{(k)}(r) = \begin{cases} (-1)^{n-k}\dfrac{1+k}{1+r^n}, & (k \le n \le k+1) \\ 0, & (n < k \text{ or } n > k+1) \end{cases} \tag{3.1}$$

for every fixed $k \in \mathbb{N}$. *Then,*

(a) *The sequence* $\{b^{(k)}(r)\}_{k\in\mathbb{N}}$ *is a basis for the space* a_0^r *and any* $x \in a_0^r$ *has a unique representation of the form*

$$x = \sum_k \lambda_k(r) b^{(k)}(r). \tag{3.2}$$

(b) *The set* $\{b, b^{(k)}(r)\}$ *is a basis for the space* a_c^r *and any* $x \in a_c^r$ *has a unique representation of the form*

$$x = lb + \sum_k [\lambda_k(r) - l] b^{(k)}(r); \tag{3.3}$$

where $b = (1/(1+r^k))$ *and* $\lambda_k(r) = (A^r x)_k$ *for all* $k \in \mathbb{N}$, *and*

$$l = \lim_{k\to\infty} (A^r x)_k. \tag{3.4}$$

Proof. (a) It is clear that $\{b^{(k)}(r)\} \subset a_0^r$, since

$$A^r b^{(k)}(r) = e^{(k)} \in c_0, \quad (k = 0, 1, 2, \ldots); \tag{3.5}$$

where $e^{(k)}$ is the sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$.

Let $x \in a_0^r$ be given. For every non-negative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(r) b^{(k)}(r).$$

Then, we obtain by applying A^r to $x^{[m]}$ that

$$A^r x^{[m]} = \sum_{k=0}^{m} \lambda_k(r) A^r b^{(k)}(r) = \sum_{k=0}^{m} (A^r x)_k e^{(k)}$$

and

$$\{A^r(x - x^{[m]})\}_i = \begin{cases} 0, & (0 \le i \le m) \\ (A^r x)_i, & (i > m) \end{cases} \quad ; \quad (i, m \in \mathbb{N}).$$

Given $\varepsilon > 0$, then there is an integer m_0 such that

$$\left|(A^r x)_m\right| < \frac{\varepsilon}{2}$$

for all $m \geq m_0$. Hence,

$$\begin{aligned}\left\|x - x^{[m]}\right\|_{a_0^r} &= \sup_{n \geq m} \left|(A^r x)_n\right| \\ &\leq \sup_{n \geq m_0} \left|(A^r x)_n\right| \leq \frac{\varepsilon}{2} < \varepsilon\end{aligned}$$

for all $m \geq m_0$ which proves that $x \in a_0^r$ is represented as in (3.2).

Let us show that the uniqueness of the representation for $x \in a_0^r$ given by (3.2). Suppose, on the contrary, that there exists a representation $x = \sum_k \mu_k(r) b^{(k)}(r)$. Since the linear transformation T, from a_0^r to c_0, used in the proof of Theorem 2.2 is continuous we have at this stage that

$$\left(A^r x\right)_n = \sum_k \mu_k(r) \left(A^r b^{(k)}(r)\right)_n = \sum_k \mu_k(r) e_n^{(k)} = \mu_n(r); \quad (n \in \mathbb{N})$$

which contradicts the fact that $(A^r x)_n = \lambda_n(r)$ for all $n \in \mathbb{N}$. Hence, the representation (3.2) of $x \in a_0^r$ is unique. Thus, the proof of the first part of theorem is completed.

(b) Since $\{b^{(k)}(r)\} \subset a_0^r$ and $b \in c$, the inclusion $\left\{b,\, b^{(k)}(r)\right\} \subset a_c^r$ trivially holds. Let us take $x \in a_c^r$. Then, there uniquely exists an l satisfying (3.4). We thus have the fact that $u \in a_0^r$ whenever we set $u = x - lb$. Therefore, we deduce by the part (a) of the present theorem that the representation of u is unique. This leads us to the fact that the representation of x given by (3.3) is unique and this step concludes the proof. □

4. The α-, β- and γ-duals of the spaces a_0^r and a_c^r

In this section, we state and prove the theorems determining the α-, β- and γ-duals of the sequence spaces a_0^r and a_c^r of non-absolute type.

For the sequence spaces λ and μ, define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \left\{z = (z_k) \in w : xz = (x_k z_k) \in \mu \quad \text{for all} \quad x \in \lambda\right\}. \tag{4.1}$$

With the notation of (4.1), the α-, β- and γ-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ, are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \quad \lambda^\beta = S(\lambda, cs) \quad \text{and} \quad \lambda^\gamma = S(\lambda, bs).$$

We shall begin with to quote the lemmas, due to Stieglitz and Tietz [10], which are needed in proving Theorems 4.3-4.5, below.

Lemma 4.1 *$A \in (c_0 : \ell_1) = (c : \ell_1)$ if and only if*

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty.$$

Lemma 4.2 *$A \in (c : c)$ if and only if*

$$\lim_{n \to \infty} a_{nk} = a_k; \quad (k \in \mathbb{N}), \tag{4.2}$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}| < \infty, \tag{4.3}$$

$$\lim_{n \to \infty} \sum_k a_{nk} \quad \textit{exists}. \tag{4.4}$$

Theorem 4.3 *The α-dual of the spaces a_0^r and a_c^r is the set*

$$d_1^r = \left\{ a = (a_k) \in w : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} (-1)^{n-k} \frac{1+k}{1+r^n} a_n \right| < \infty \right\}.$$

Proof. Let $a = (a_n) \in w$ and define the matrix $B = (b_{nk}^r)$ via the sequence $a = (a_n)$ by

$$b_{nk}^r = \begin{cases} (-1)^{n-k} \dfrac{1+k}{1+r^n} a_n, & (n-1 \leq k \leq n) \\ 0, & (0 \leq k < n-1 \text{ or } k > n) \end{cases} ; \quad (n, k \in \mathbb{N}).$$

Bearing in mind the relation (2.2) we immediately derive that

$$a_n x_n = \sum_{k=n-1}^{n} (-1)^{n-k} \frac{1+k}{1+r^n} a_n y_k = (By)_n, \quad (n \in \mathbb{N}). \tag{4.5}$$

We therefore observe by (4.5) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in a_0^r$ or a_c^r if and only if $By \in \ell_1$ whenever $y \in c_0$ or c. Then, we derive by Lemma 4.1 that

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} (-1)^{n-k} \frac{1+k}{1+r^n} a_n \right| < \infty$$

which yields the consequence that $\{a_0^r\}^\alpha = \{a_c^r\}^\alpha = d_1^r$. □

Theorem 4.4 *Define the sets d_2^r, d_3^r and d_4^r by*

$$d_2^r = \left\{a = (a_k) \in w : \sum_k \left| \triangle\left(\frac{a_k}{1+r^k}\right)(k+1)\right| < \infty\right\},$$

$$d_3^r = \left\{a = (a_k) \in w : \left(\frac{a_k}{1+r^k}\right) \in cs\right\}$$

and

$$d_4^r = \left\{a = (a_k) \in w : \left(\frac{k+1}{1+r^k} a_k\right) \in \ell_\infty\right\},$$

where

$$\triangle\left(\frac{a_k}{1+r^k}\right) = \frac{a_k}{1+r^k} - \frac{a_{k+1}}{1+r^{k+1}} \quad \textit{for all} \quad k \in \mathbb{N}.$$

Then, $\{a_0^r\}^\beta = d_2^r \cap d_4^r$ and $\{a_c^r\}^\beta = d_2^r \cap d_3^r$.

Proof. Because of the proof may also be obtained for the space a_0^r in the similar way, we omit it and give the proof only for the space a_c^r. Consider the equation

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\left[\sum_{j=k-1}^{k} (-1)^{k-j}\frac{j+1}{1+r^k} y_j\right] a_k \\ &= \sum_{k=0}^{n-1} \triangle\left(\frac{a_k}{1+r^k}\right)(k+1) y_k + \frac{n+1}{1+r^n} a_n y_n = (Ty)_n; \\ &\qquad (n \in \mathbb{N}), \end{aligned} \tag{4.6}$$

where $T = (t_{nk}^r)$ is defined by

$$t_{nk}^r = \begin{cases} \triangle\left(\frac{a_k}{1+r^k}\right)(k+1), & (0 \le k \le n-1) \\ \frac{n+1}{1+r^n} a_n, & (k = n) \\ 0, & (k > n) \end{cases} ; \quad (n,\, k \in \mathbb{N}). \tag{4.7}$$

Thus, we deduce from Lemma 4.2 with (4.6) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in a_c^r$ if and only if $Ty \in c$ whenever $y = (y_k) \in c$. It is obvious that the columns of that matrix T are in the space c. Therefore, we derive the consequences from (4.3) and (4.4) that

$$\sum_k \left| \triangle\left(\frac{a_k}{1+r^k}\right)(k+1) \right| < \infty, \tag{4.8}$$

$$\left(\frac{k+1}{1+r^k}a_k\right) \in \ell_\infty \tag{4.9}$$

and

$$\left(\frac{a_k}{1+r^k}\right) \in cs, \tag{4.10}$$

respectively. But the condition (4.9) is redundant, since it may be obtained by combining the conditions (4.8) and (4.10). This shows that $\{a_c^r\}^\beta = d_2^r \cap d_3^r$. □

Theorem 4.5 *The γ-dual of the spaces a_0^r and a_c^r is the set $d_2^r \cap d_4^r$.*

Proof. This is obtained in the similar way used in the proof of Theorem 4.4 and so we leave the detail to the reader. □

5. Some matrix mappings related to the space a_c^r

In this section, we characterize the matrix mappings from a_c^r into some of the known sequence spaces and into the Euler, difference, Riesz, Cesàro sequence spaces. We directly prove the theorems characterizing the classes $(a_c^r : \ell_p)$, $(a_c^r : c)$ and derive the other characterizations from them by means of a given basic lemma, where $1 \le p \le \infty$.

We shall write throughout for brevity that

$$\tilde{a}_{nk} = \triangle\left(\frac{a_{nk}}{1+r^k}\right)(k+1) = \left(\frac{a_{nk}}{1+r^k} - \frac{a_{n,k+1}}{1+r^{k+1}}\right)(k+1)$$

for all $n, k \in \mathbb{N}$. We will also use the similar notation with other letters and use the convention that any term with negative subscript is equal to naught. We shall begin with two lemmas due to Wilansky [12, p. 57 and p. 128] which are needed in the proof of our theorems.

Lemma 5.1 *The matrix mappings between the BK-spaces are continuous.*

Lemma 5.2 *$A \in (c : \ell_p)$ if and only if*

$$\sup_{F\in\mathcal{F}} \sum_n \left| \sum_{k\in F} a_{nk} \right|^p < \infty, \quad (1 \le p < \infty). \tag{5.1}$$

Theorem 5.3 *$A \in (a_c^r : \ell_p)$ if and only if*

(i) *For $1 \leq p < \infty$,*

$$\sup_{F \in \mathcal{F}} \sum_n \left| \sum_{k \in F} \tilde{a}_{nk} \right|^p < \infty, \tag{5.2}$$

$$\sum_k |\tilde{a}_{nk}| < \infty \quad \textit{for all} \quad n \in \mathbb{N}, \tag{5.3}$$

$$\left\{ \frac{a_{nk}}{1+r^k} \right\}_{k \in \mathbb{N}} \in cs \quad \textit{for all} \quad n \in \mathbb{N}. \tag{5.4}$$

(ii) *For $p = \infty$, (5.4) holds, and*

$$\sup_{n \in \mathbb{N}} \sum_k |\tilde{a}_{nk}| < \infty \ . \tag{5.5}$$

Proof. Suppose the conditions (5.2)-(5.4) hold and take any $x \in a_c^r$. Then, $\{a_{nk}\}_{k \in \mathbb{N}} \in \{a_c^r\}^\beta$ for all $n \in \mathbb{N}$ and this implies that Ax exists. Let us define the matrix $B = (b_{nk})$ with $b_{nk} = \tilde{a}_{nk}$ for all $n, k \in \mathbb{N}$. Then, since (5.1) is satisfied for that matrix B we have $B \in (c : \ell_p)$. Let us now consider the following equality obtained from the m^{th} partial sum of the series $\sum_k a_{nk}x_k$:

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m-1} \tilde{a}_{nk}y_k + \frac{1+m}{1+r^m} a_{nm}y_m; \quad (n, m \in \mathbb{N}). \tag{5.6}$$

Following the way that used in the proof of Theorem 4.4, one can derive by combining the conditions (5.3) and (5.4) that $\{(1+m)a_{nm}/(1+r^m)\}_{m \in \mathbb{N}} \in c_0$ for each $n \in \mathbb{N}$. Thus, bearing in mind this fact if we pass to limit in (5.6) as $m \to \infty$ then the second term on the right hand tends to zero and we derive that

$$\sum_k a_{nk}x_k = \sum_k \tilde{a}_{nk}y_k, \quad (n \in \mathbb{N}) \tag{5.7}$$

which yields by taking ℓ_p-norm that

$$\|Ax\|_{\ell_p} = \|By\|_{\ell_p} < \infty.$$

This means that $A \in (a_c^r : \ell_p)$.

Conversely, suppose that $A \in (a_c^r : \ell_p)$. Then, since a_c^r and ℓ_p are the BK-spaces we have from Lemma 5.1 that there exists some real constant $K > 0$ such that

$$\|Ax\|_{\ell_p} \leq K \cdot \|x\|_{a_c^r} \tag{5.8}$$

for all $x \in a_c^r$. Since the inequality (5.8) is also satisfied for the sequence $x = (x_k) = \sum_{k \in F} b^{(k)}(r)$ belonging to the space a_c^r, where $b^{(k)}(r) = \left\{b_n^{(k)}(r)\right\}$ is defined by (3.1), we thus have for any $F \in \mathcal{F}$ that

$$\|Ax\|_{\ell_p} = \left(\sum_n \left|\sum_{k \in F} \tilde{a}_{nk}\right|^p\right)^{1/p} \leq K \cdot \|x\|_{a_c^r}$$

which shows the necessity of (5.2).

Since A is applicable to the space a_c^r by the hypothesis, the necessities of (5.3) and (5.4) are trivial. This completes the proof of the part (i) of Theorem.

Since the part (ii) may also be proved in the similar way that of the part (i), we leave the detailed proof to the reader. □

Theorem 5.4 *$A \in (a_c^r : c)$ if and only if* (5.4) *and* (5.5) *hold, and*

$$\lim_{n \to \infty} \tilde{a}_{nk} = \alpha_k \quad \textit{for each} \quad k \in \mathbb{N}, \tag{5.9}$$

$$\lim_{n \to \infty} \sum_k \tilde{a}_{nk} = \alpha. \tag{5.10}$$

Proof. Suppose that A satisfies the conditions (5.4), (5.5), (5.9) and (5.10). Let us take any $x = (x_k)$ in a_c^r. Then, Ax exists and it is trivial that the sequence $y = (y_k)$ connected with the sequence $x = (x_k)$ by the relation (2.2) is in c such that $y_k \to l$ as $k \to \infty$. At this stage, we observe from (5.9) and (5.5) that

$$\sum_{j=0}^{k} |\alpha_j| \leq \sup_{n \in \mathbb{N}} \sum_j |\tilde{a}_{nj}| < \infty$$

holds for every $k \in \mathbb{N}$. This leads us to the consequence that $(\alpha_k) \in \ell_1$. Considering (5.7), let us write

$$\sum_k a_{nk} x_k = \sum_k \tilde{a}_{nk}(y_k - l) + l \sum_k \tilde{a}_{nk}. \tag{5.11}$$

In this situation, by letting $n \to \infty$ in (5.11) we see that the first term on the right tends to $\sum_k \alpha_k (y_k - l)$ by (5.5) and (5.9), and the second term tends to $l\alpha$ by (5.10) and we thus have that

$$(Ax)_n \to \sum_k \alpha_k(y_k - l) + l\alpha \tag{5.12}$$

which shows that $A \in (a_c^r : c)$.

Conversely, suppose that $A \in (a_c^r : c)$. Then, since the inclusion $c \subset \ell_\infty$ holds, the necessities of (5.4) and (5.5) are immediately obtained from Theorem 5.3. To prove the necessity of (5.9), consider the sequence $x = x^{(k)} = \{x_n^{(k)}(r)\}_{n\in\mathbb{N}} \in a_c^r$ defined by

$$x_n^{(k)}(r) = \begin{cases} (-1)^{n-k}\dfrac{1+k}{1+r^k}, & (k \le n \le k+1) \\ 0, & (0 \le n \le k-1 \text{ or } n > k+1) \end{cases} \tag{5.13}$$

for each $k \in \mathbb{N}$. Since Ax exists and is in c for every $x \in a_c^r$, one can easily see that $Ax^{(k)} = \{\tilde{a}_{nk}\}_{n\in\mathbb{N}} \in c$ for each $k \in \mathbb{N}$ which shows the necessity of (5.9).

Similarly by putting $x=e$ in (5.7), we also obtain that $Ax=\{\sum_k \tilde{a}_{nk}\}_{n\in\mathbb{N}}$ which belongs to the space c and this shows the necessity of (5.10). This step concludes the proof. □

Let us define the concept of s-multiplicativity of a limitation matrix. When there is some notion of limit or sum in λ and μ, we shall say that the method $A \in (\lambda : \mu)$ is multiplicative s if every $x \in \lambda$ is A-summable to s times of lim x, for any fixed real number s and denote the class of all s-multiplicative matrices by $(\lambda : \mu)_s$. It is of course that the class $(a_c^r : c)_s$ of s-multiplicative matrices reduces to the classes $(a_c^r : c_0)$ and $(a_c^r : c)_{reg}$ in the cases $s = 0$ and $s = 1$, respectively; where $(a_c^r : c)_{reg}$ denotes the class of all matrix mappings from a_c^r to c such that $A - \lim\ x = \lim\ x$ for all $x \in a_c^r$. Now, we may give the corollary to Theorem 5.4, without proof.

Corollary 5.5 *$A \in (a_c^r : c)_s$ if and only if* (5.4), (5.5) *hold,* (5.9) *and* (5.10) *also hold with $\alpha_k = 0$ for each k and $\alpha = s$, respectively.*

Now, we may present our basic lemma given by Altay and Başar ([2, Lemma 6.6]) which is useful for obtaining the characterization of some new matrix classes from Theorems 5.3, 5.4 and Corollary 5.5.

Lemma 5.6 *Let λ, μ be any two sequence spaces, A be an infinite matrix and B a triangle matrix. Then, $A \in (\lambda : \mu_B)$ if and only if $BA \in (\lambda : \mu)$.*

It is trivial that Lemma 5.6 has several consequences. Indeed, combining the Lemma 5.6 with Theorems 5.3, 5.4 and Corollary 5.5, one can easily derive the following results:

Corollary 5.7 *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = \sum_{j=0}^{n} \binom{n}{j} (1-r)^{n-j} r^j a_{jk}; \quad (n,\, k \in \mathbb{N}).$$

Then, the necessary and sufficient conditions in order for A belongs to anyone of the classes $(a_c^r : e_\infty^r)$, $(a_c^r : e_p^r)$, $(a_c^r : e_c^r)$ and $(a_c^r : e_c^r)_s$ are obtained from the respective ones in Theorems 5.3, 5.4 and Corollary 5.5 by replacing the entries of the matrix A by those of the matrix C; where e_c^r denotes the Euler space of all sequences whose E^r-transforms are in the space c and is studied by B. Altay and F. Başar in a separate paper.

Corollary 5.8 *Let $A = (a_{nk})$ be an infinite matrix and $t = (t_k)$ be a sequence of positive numbers and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = \frac{1}{T_n} \sum_{j=0}^{n} t_j a_{jk}; \quad (n,\, k \in \mathbb{N}),$$

where $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$. Then, the necessary and sufficient conditions in order for A belongs to anyone of the classes $(a_c^r : r_\infty^t)$, $(a_c^r : r_p^t)$, $(a_c^r : r_c^t)$ and $(a_c^r : r_c^t)_s$ are obtained from the respective ones in Theorems 5.3, 5.4 and Corollary 5.5 by replacing the entries of the matrix A by those of the matrix C; where r_p^t is defined in [1] as the space of all sequences whose R^t-transforms are in the space ℓ_p and is derived from the paranormed space $r^t(p)$ in the case $p_k = p$ for all $k \in \mathbb{N}$.

Since the spaces r_∞^t and r_p^t reduce in the case $t = e$ to the Cesàro sequence spaces X_∞ and X_p of non-absolute type, respectively, Corollary 5.8 also includes the characterizations of the classes $(a_c^r : X_\infty)$ and $(a_c^r : X_p)$.

Corollary 5.9 *Let $A = (a_{nk})$ be an infinite matrix and define the matrices $C = (c_{nk})$ and $D = (d_{nk})$ by $c_{nk} = a_{nk} - a_{n+1,k}$ and $d_{nk} = a_{nk} - a_{n-1,k}$ for all $n,\, k \in \mathbb{N}$. Then, the necessary and sufficient conditions in order for A belongs to anyone of the classes $(a_c^r : \ell_\infty(\triangle))$, $(a_c^r : c(\triangle))$, $(a_c^r : c(\triangle))_s$ and $(a_c^r : bv_p)$ are obtained from the respective ones in Theorems 5.3, 5.4* *and Corollary* 5.5 *by replacing the entries of the matrix A by those of the matrices C and D; where $\ell_\infty(\triangle)$, $c(\triangle)$ denote the difference spaces of all bounded, convergent sequences and introduced by Kızmaz* [6], *and bv_p also denotes the space of all sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$ and is studied by Başar and Altay in* [4].

Corollary 5.10 *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by $c_{nk} = \sum_{j=0}^{n} a_{jk}$ for all $n, k \in \mathbb{N}$. Then, the necessary and sufficient conditions in order for A belongs to anyone of the classes $(a_c^r : bs)$, $(a_c^r : cs)$ and $(a_c^r : cs)_s$ are obtained from the respective ones in Theorems* 5.3, 5.4 *and Corollary* 5.5 *by replacing the entries of the matrix A by those of the matrix C.*

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