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Backward shift invariant subspaces in the bidisc

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Abstract. Suppose that T_ϕ is a Toeplitz operator with a symbol ϕ on the Hardy space H^2 on the bidisc. Let N be a backward shift invariant subspace of H^2, that is, N is an invariant subspace under T_z^* and T_w^*. Let P be the orthogonal projection from H^2 onto N. For ϕ in H^∞, put $S_\phi = PT_\phi|N$. In this paper, we give a characterization of a backward shift invariant subspace which satisfies $S_z S_w^* = S_w^* S_z$.

Key words: bidisc, Hardy space, backward shift, invariant subspace, double commuting.

1. Introduction

Let T^2 be the torus that is the Cartesian product of two unit circles T in $\mathbf{C}$. Let $p=2$ or $p=\infty$. The usual Lebesgue spaces, with respect to the Haar measure m on T^2, are denoted by $L^p = L^p(T^2)$, and $H^p = H^p(T^2)$ is the space of all f in L^p whose Fourier coefficients

$$\hat{f}(j,\ell) = \int_{T^2} f(z,w)\bar{z}^j \bar{w}^\ell dm(z,w)$$

are 0 as soon as at least one component of (j,ℓ) is negative. Then H^p is called the Hardy space. As $T^2 = (z,T)\times(w,T)$, $H^p(z,T)$ and $H^p(w,T)$ denote the one variable Hardy spaces.

Let P_{H^2} be the orthogonal projection from L^2 onto H^2. For ϕ in L^∞, the Toeplitz operator T_ϕ is defined by

$$T_\phi f = P_{H^2}(\phi f) \quad (f \in H^2).$$

A closed subspace N of H^2 is said to be backward shift invariant if

$$T_z^* N \subset N \quad \text{and} \quad T_w^* N \subset N.$$

A closed subspace M of H^2 is said to be shift invariant if $T_z M \subset M$ and $T_w M \subset M$. The orthogonal complement of N is shift invariant. Let P_N

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and P_M be the orthogonal projections from H^2 onto N and M, respectively. For ϕ in H^∞, put

$$S_\phi = P_N T_\phi P_N | N \quad \text{and} \quad V_\phi = P_M T_\phi P_M | M.$$

It is known in [2] that $V_z V_w^* = V_w^* V_z$ if and only if $M = qH^2$ for some inner function q in H^∞. In this paper, we are interested in backward shift invariant subspaces N which satisfy $S_z S_w^* = S_w^* S_z$. Let $M = H^2 \ominus N$. We will write $P = P_N$ and $Q = I - P_N$, where I is the identity operator on H^2. In this paper, we also study two operators

$$A = QT_z P \quad \text{and} \quad B = PT_w^* Q.$$

In §2, we show that $AB|M = V_w^* V_z - V_z V_w^*$ and $BA|N = S_z S_w^* - S_w^* S_z$. Then $AB = 0$ is equivalent to $V_z V_w^* = V_w^* V_z$, and $BA = 0$ is equivalent to $S_z S_w^* = S_w^* S_z$. Moreover we determine backward shift invariant subspaces satisfying $A = 0$ or $B = 0$. In §3, we give a characterization of backward shift invariant subspaces satisfying $BA = 0$, equivalently $S_z S_w^* = S_w^* S_z$. And we give simple sufficient conditions to be $S_z S_w^* = S_w^* S_z$. In §4, we give a conjecture, that is, the sufficient condition is also necessary one.

Throughout this paper, for a subset H of H^2, $[H]_2$ denotes the closed linear span of H and $[H]$ the linear span of H.

2. Invariant subspace with $A = 0$ or $B = 0$

Let N be a backward shift invariant subspace and M be the orthogonal complement of N in H^2. Put $P = P_N$ and $Q = I - P_N$, then Q is the orthogonal projection from H^2 onto M.

Lemma 2.1

(1) $AB = QT_w^* QT_z Q - QT_z QT_w^* Q$ *and so* $AB|M = V_w^* V_z - V_z V_w^*$.

(2) $BA = PT_z PT_w^* P - PT_w^* PT_z P$ *and so* $BA|N = S_z S_w^* - S_w^* S_z$.

(3) $\ker A = \{f \in N : T_z f \in N\} \oplus M$.

(4) $\ker B = \{f \in M : T_w^* f \in M\} \oplus N$.

Proof. (1) Since $T_z Q = QT_z Q$ and $T_z T_w^* = T_w^* T_z$,

$$\begin{aligned} AB &= QT_z PT_w^* Q \\ &= QT_z T_w^* Q - QT_z QT_w^* Q \\ &= QT_w^* QT_z Q - QT_z QT_w^* Q. \end{aligned}$$

(2) Since $T_w^* P = PT_w^* P$ and $T_w^* T_z = T_z T_w^*$,

$$\begin{aligned} BA &= PT_w^* Q T_z P \\ &= PT_w^* T_z P - PT_w^* P T_z P \\ &= PT_z PT_w^* P - PT_w^* PT_z P. \end{aligned}$$

The properties (3) and (4) are clear. □

Theorem 2.2

(1) $A = 0$ *if and only if* $N = H^2$ *or* $N = H^2 \ominus qH^2$ *where* q *is a one variable inner function with* $q = q(w)$.

(2) $B = 0$ *if and only if* $M = [0]$ *or* $M = qH^2$ *where* q *is a one variable inner function with* $q = q(z)$.

(3) $A = B = 0$ *if and only if* $N = [0]$ *or* $N = H^2$.

Proof. (2) follows from (1). We will show (1). We have $H^2 = N \oplus M$ and $T_z M \subset M$. Suppose $A = 0$. By Lemma 2.1 (3), $T_z N \subset N$. Put $N_0 = N \ominus T_z N$ and $M_0 = M \ominus T_z M$. Then

$$H^2 = \sum_{n=0}^{\infty} \oplus (N_0 \oplus M_0) z^n = \sum_{n=0}^{\infty} \oplus H^2(w, T) z^n$$

because $zH^2 = zN \oplus zM$ and so $N_0 \oplus M_0 = H^2(w, T)$. By Lemma 2.1 (1), $V_w^* V_z = V_z V_w^*$ and so $V_z^* V_w = V_w V_z^*$ because $AB = 0$. Hence $V_w(\ker V_z^*) \subseteq \ker V_z^*$ and $\ker V_z^* = M_0$. Therefore by a theorem of Beurling [1], if $M_0 \neq [0]$, $M_0 = qH^2(w, T)$ and q is a one variable inner function with $q = q(w)$. Hence $M = qH^2$ and so $N = H^2 \ominus qH^2$. If $M_0 = [0]$, then $M = [0]$, and so $N = H^2$. □

3. Invariant subspace with $AB = 0$ or $BA = 0$

Suppose that N is a backward shift invariant subspace and $M = H^2 \ominus N$. By Lemma 2.1, $AB = 0$ if and only if $V_w^* V_z = V_z V_w^*$, and $BA = 0$ if and only if $S_z S_w^* = S_w^* S_z$. Hence we know (see [2], [3], [4]) that $AB = 0$ if and only if $M = qH^2$ for some inner function q. In this section, we study N when $BA = 0$, that is, $S_z S_w^* = S_w^* S_z$.

Lemma 3.1

$$[\operatorname{ran} A]_2 = \{M \ominus zM\} \ominus \{H^2(w, T) \cap M\}$$

and

$$\ker B = \{H^2(z,T) \cap M\} \oplus wM \oplus N.$$

Proof. Since $(T_w^* f, g) = (f, wg)$ if $f, g \in H^2$,

$$\{f \in M;\ T_w^* f \in M\} = M \cap \{H^2 \ominus wN\} = \{H^2(z,T) \cap M\} \oplus wM,$$

because $H^2 \ominus wN = (H^2 \ominus wH^2) \oplus w(H^2 \ominus N)$ and $N = H^2 \ominus M$. Hence by Lemma 2.1 (4), $\ker B = \{f \in M;\ T_w^* f \in M\} \oplus N = \{H^2(z,T) \cap M\} \oplus wM \oplus N$. By the same argument, $\ker A^* = \{H^2(w,T) \cap M\} \oplus zM \oplus N$ and so

$$\begin{aligned}[\operatorname{ran} A]_2 &= H^2 \ominus \ker A^* \\ &= \{M \ominus zM\} \ominus \{H^2(w,T) \cap M\}.\end{aligned}$$

□

Lemma 3.2
(1) $A = 0$ *if and only if* $M = \{H^2(w,T) \cap M\} \oplus zM$.
(2) $B = 0$ *if and only if* $M = \{H^2(z,T) \cap M\} \oplus wM$.
(3) $BA = 0$ *if and only if* $\{H^2(z,T) \cap M\} \oplus wM \supseteq \{M \ominus zM\} \ominus \{H^2(w,T) \cap M\}$.

Proof. These follow from Lemma 3.1. □

For a subset H of H^2, let $H_k = \sum_{i+j=k} z^i w^j H$ for $k \geq 0$.

Theorem 3.3 *Let N be a backward shift invariant subspace of H^2 and M its orthogonal complement. Suppose $N \neq H^2$.*

(1) *$S_z S_w^* = S_w^* S_z$ if and only if $M = H + M_1$ and if and only if $M = \sum_{j=0}^{k-1} H_j + M_k$ for any $k \geq 1$, where $H = H_0 = H^2(z,T) \cap M + H^2(w,T) \cap M$. If $S_z S_w^* = S_w^* S_z$, then $H \neq [0]$.*

(2) *When $M \cap H^2(z,T) = [0]$ or $M \cap H^2(w,T) = [0]$, $S_z S_w^* = S_w^* S_z$ if and only if $M = qH^2 + M_k$ for any $k \geq 1$ where q is a one variable inner function such that $M \cap H^2(z,T) = qH^2(z,T)$ or $M \cap H^2(w,T) = qH^2(w,T)$.*

(3) *When $M \cap H^2(z,T) \neq [0]$ and $M \cap H^2(w,T) \neq [0]$, $S_z S_w^* = S_w^* S_z$ if and only if $M = q_1 H^2 + q_2 H^2 + M_k$ for any $k \geq 1$ where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable inner functions such that $M \cap H^2(z,T) = q_1 H^2(z,T)$ and $M \cap H^2(w,T) = q_2 H^2(w,T)$.*

Proof. (1) Since $S_zS_w^* = S_w^*S_z$ is equivalent to $BA = 0$, $S_zS_w^* = S_w^*S_z$ if and only if $M = H + M_1$ by Lemma 3.2 (3). It is easy to see that $M = H + M_1 = \sum_{j=0}^{k-1} H_j + M_k$ for any $k \geq 1$. If $H = [0]$, then $M = M_k$ and hence $M = [0]$. This contradicts $N \neq H^2$.

(2) We may assume that $M \cap H^2(z,T) = [0]$ and $M \cap H^2(w,T) \neq [0]$. By a theorem of Beurling [1], $M \cap H^2(w,T) = qH^2(w,T)$ for some one variable inner function $q = q(w)$. By (1), $S_zS_w^* = S_w^*S_z$ if and only if $M = qH^2(w,T) + M_1$ if and only if

$$M = q\sum_{j=0}^{k-1} \oplus H^2(w,T)z^j + M_k$$

for any $k \geq 1$. This is equivalent to $M = qH^2 + M_k$ for any $k \geq 1$. For, $M_k \supset qz^kH^2$.

(3) By a theorem of Beurling, $M \cap H^2(z,T) = q_1H^2(z,T)$ and $M \cap H^2(w,T) = q_2H^2(w,T)$ where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable inner functions. By (1), $S_zS_w^* = S_w^*S_z$ if and only if $M = q_1H^2(z,T) + q_2H^2(w,T) + M_1$ if and only if

$$M = q_1\sum_{j=0}^{k-1} \oplus H^2(z,T)w^j + q_2\sum_{j=0}^{k-1} \oplus H^2(w,T)z^j + M_k$$

for any $k \geq 1$. This is equivalent to $M = q_1H^2 + q_2H^2 + M_k$ for any $k \geq 1$. For, $M_k \supseteq q_1w^kH^2 + q_2z^kH^2$. □

Corollary 3.4

(1) *$AB = BA = 0$ if and only if $A = 0$ or $B = 0$.*

(2) *If $N = H^2 \ominus qH^2$ and q is an inner function and $S_zS_w^* = S_w^*S_z$, then q is a one variable.*

Proof. (1) If $AB = BA = 0$, then by Lemma 2.1 (1) $V_w^*V_z = V_zV_w^*$ and so $M = qH^2$ for some inner function q (see [2], [4]). On the other hand, by Theorem 3.3 (1), $M \cap H^2(z,T) \neq [0]$ or $M \cap H^2(w,T) \neq [0]$ because $S_zS_w^* = S_w^*S_z$. Hence q is one variable. By Theorem 2.2, $A = 0$ or $B = 0$.

(2) is clear by (1). □

Corollary 3.5 *Let N be a backward shift invariant subspace and $N \neq H^2$.*

(1) *If $S_zS_w^* = S_w^*S_z$, then $N \subseteq H^2 \ominus qH^2$ for some one variable inner function q.*

(2) *If* $N = H^2 \ominus qH^2$ *for some one variable inner function* q*, then* $S_z S_w^* = S_w^* S_z$.

Proof. (1) By Theorem 3.3, if $S_z S_w^* = S_w^* S_z$ then $M \supseteq qH^2$ for some one variable inner function q. Hence $N \subseteq H^2 \ominus qH^2$.

(2) is clear by Theorem 3.3 (3). □

Corollary 3.6 *Suppose that* $A \neq 0$ *and* $B \neq 0$.

(1) *If* $S_z S_w^* = S_w^* S_z$*, then* $N \subseteq (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$ *where* $q_1 = q_1(z)$ *and* $q_2 = q_2(w)$ *are one variable inner functions.*

(2) *If* $N = (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$ *where* $q_1 = q_1(z)$ *and* $q_2 = q_2(w)$ *are one variable inner functions, then* $S_z S_w^* = S_w^* S_z$.

Proof. By Theorem 2.2, we can prove (1) as in the proof of Corollary 3.5 (1).

(2) Since $q_1 H^2 + q_2 H^2 = [q_1, q_2] + (q_1 H_1^2 + q_2 H_1^2)$, $M = [q_1, q_2] + (zM + wM) = q_1 H^2 + q_2 H^2 + M_k$ for any $k \geq 1$. It is easy to see that $M \cap H^2(z, T) = q_1 H^2(z, T)$ and $M \cap H^2(w, T) = q_2 H^2(w, T)$. Hence by Theorem 3.3 (3) $S_z S_w^* = S_w^* S_z$. □

4. Conjecture

By Corollary 3.5 (2) and Corollary 3.6, if $N = H^2$, $N = H^2 \ominus qH^2$ for some one variable inner function q or $N = (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$ for some one variable inner functions $q_1 = q_1(z)$ and $q_2 = q_2(w)$, then $S_z S_w^* = S_w^* S_z$. Because of Theorem 3.3, we have the following conjecture. In this section, we study this conjecture.

Conjecture If $S_z S_w^* = S_w^* S_z$, then $N = H^2$, $N = H^2 \ominus qH^2$ for some one variable inner function q or $N = (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$, where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable inner functions.

Proposition 4.1 *If* $M = q_1 H^2 + q_2 H^2 + M_k$ *for any* $k \geq 1$*, where* $M \cap H^2(z, T) = q_1 H^2(z, T)$ *and* $M \cap H^2(w, T) = q_2 H^2(w, T)$*, then* $M = q_1(H^2 \ominus w^k H^2) + q_2(H^2 \ominus w^k H^2) + w^k M$ *for any* $k \geq 1$*. The converse is also true.*

Proof. Since $wM \supseteq q_2 w H^2(w, T)$, by Lemma 3.2 (3) and Theorem 3.3 (3),

$$q_1 H^2(z, T) \oplus wM \supseteq K_2 \oplus q_2 w H^2(w, T),$$

where $M \ominus zM = K_2 \oplus q_2 H^2(w, T)$. Thus

$$q_1 H^2(z,T) \oplus wM \supseteq \sum_{j=0}^{\infty} \oplus \{K_2 \oplus q_2 w H^2(w,T)\} z^j.$$

Since

$$M = \sum_{j=0}^{\infty} \oplus (M \ominus zM) z^j = \sum_{j=0}^{\infty} \oplus \{K_2 \oplus q_2 H^2(w,T)\} z^j,$$

we have

$$q_1 H^2(z,T) + q_2 H^2(z,T) + wM \supseteq M.$$

Hence $M = q_1 H^2(z,T) + q_2 H^2(z,T) + wM$. This leads our assertion. □

Corollary 4.2 *If $M = qH^2 + M_k$ for any $k \geq 1$ where $M \cap H^2(z,T) = qH^2(z,T)$ and $M \cap H^2(w,T) = [0]$, then $M = qH^2$.*

Proof. By Proposition 4.1 and its proof, $M = q(H^2 \ominus w^k H^2) + w^k M$ for any $k \geq 1$. This implies that $M = qH^2$, because $q(H^2 \ominus w^k H^2)$ is orthogonal to $w^k M$. □

It is not difficult to prove that $q_1 H^2 + q_2 H^2$ is closed when $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable. Hence our conjecture is equivalent to the following one. If $S_z S_w^* = S_w^* S_z$, then $M = [0]$, $M = qH^2$ or $M = q_1 H^2 + q_2 H^2$. Even if N is of finite dimension, $S_z S_w^* \neq S_w^* S_z$ may happen. In fact, $N = \{1, z, w\}$ is such an example.

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