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Randers spaces of constant flag curvature induced by almost contact metric structures

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Abstract. We investigate the Randers space induced by an almost contact metric structure. We show that a connected almost contact Riemannian manifold of odd dimension $n \geq 3$ whose automorphism group has maximum dimension induces a natural structure of Randers space of constant flag curvature.

Key words: Finsler metric, Randers space, locally Minkowski space, flag curvature, almost contact metric structure, Sasakian manifold, cosymplectic manifold, Kenmotsu manifold.

Introduction

The study of metric spaces of constant sectional curvature is a fundamental problem in differential geometry. In Riemannian geometry this problem leads to the Hopf classification theorem for Riemannian space forms ([BCS]).

If one tries to study Finsler spaces of constant flag curvature (the Finslerian analog of the Riemannian space forms), then the problem is very different from the Riemannian case. In the general Finslerian case, it is very difficult to make a classification of Finsler spaces of constant flag curvature ([Br], [BCS]).

However, in the case of a class of special Finsler spaces, namely Randers spaces ([AIM], [Ma1], [SS]), a classification theorem was given in 1977 by H. Yasuda and H. Shimada ([YS]). The proof contains tremendous calculations, but the result is a very interesting one.

Yasuda-Shimada's theorem has a long and complicated history. It was initially proved by H. Yasuda and H. Shimada in 1977 ([YS]), and by M. Matsumoto in 1989 ([Ma1]). However, D. Bao and C. Robles had shown last year that both proofs contain some errors and they gave new necessary and sufficient conditions for a Randers space to be of constant flag curvature [BR].

The first concrete examples of Randers space of constant flag curvature

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are the Finslerian Poincaré disk ([BCS]), and the 3-dimensional sphere S^3 ([BS]). See also ([BR]) for other examples generated using a method of Z. Shen.

Bejancu and Farran have shown ([BF]) that a Sasakian space form induces the structure of a Randers space of positive constant flag curvature (an RCT space). The result of [BF] is true even though there is a missing argument in their proof. Inspired by this result, we have investigated if there are other almost contact structures that induce Randers spaces of constant flag curvature. We have found out that a cosymplectic space form $M(c)$ of constant φ-sectional curvature $c = 0$ induces a natural structure of Randers space of constant flag curvature $K = 0$ (actually, this is a locally Minkowski space), and that a Kenmotsu space induces a structure of Randers space of negative constant flag curvature, but this Randers space is not a positive definite one.

In 1968, S. Tanno ([T]) gave a classification theorem (§2, Theorem 2.1) of almost contact Riemannian space forms. Based on Tanno's result we were able to prove that some connected almost contact Riemannian manifold of odd dimension $n \geq 3$ whose automorphism group has maximum dimension induces a natural structure of Randers space of constant flag curvature with $curl_{bi} = 0$ (see (1.7) for definition).

To be more precise here are our main results:

Theorem A *Let M be a connected Riemannian manifold of dimension $n = 2m + 1$ endowed with an almost contact metric structure (φ, ξ, η, a).*

(i) *If $M(c)$ be a cosymplectic space form of constant φ-sectional curvature $c = 0$, then there exists a Randers metric on $M(c)$ of constant flag curvature $K = 0$ (actually this is a locally Minkowski structure).*

(ii) *If $M(c)$ be a Kenmotsu space form of constant φ-sectional curvature c, then there exists a Randers metric on $M(c)$ of negative constant flag curvature $K = -\frac{1}{4}$.*

These Randers metrics are projectively flat.

From here it follows that on some special Riemannian manifolds there exists always a natural structure of Randers space of constant flag curvature.

In [BF] it is shown that on any odd dimensional sphere S^{2m+1}, $m \geq 1$, there is a natural structure of Randers space of positive constant flag curvature.

Besides this, we prove:

Theorem B (i) *Let M be one of the global Riemannian products*

$$T \times CE^m \qquad\qquad L \times CE^m,$$

where CE^m is a complex Euclidean space, T and L denote a circle and a line, respectively. Then there exists on M a natural structure of Randers space of zero constant flag curvature. In fact this structure is a locally Minkowski one.

(ii) *Let M be the warped product space $L \times_f CE^m$, where $f(t) = ce^t$. Then on M there is a natural structure of Randers space of negative constant flag curvature $-\frac{1}{4}$.*

These theorems are interesting because they show the correspondence between connected almost contact Riemannian manifolds of constant sectional curvature and Randers space of constant flag curvature with $curl_{bi} = 0$.

Our paper is organized as follows. In §1 we recall the basics about Randers spaces and the classification theorem for Randers space of constant flag curvature by Yasuda and Shimada. In §2, we recall some results on almost contact metric structures and in §3 we show how an almost contact metric structure induces a Randers structure. This is the section where we prove our main results.

1. Randers spaces of constant flag curvature

Let M be an n-dimensional, real, differentiable manifold, and $\pi : TM \to M$ the tangent bundle of M. We denote by $u = (x^i, y^i)$ the local coordinates of a point $u \in TM$ induced from a covering $\{U, (x^i)\}$ by a system of coordinate neighbourhoods on M.

A function $F : TM \to R$ is called a *Finsler metric* if it satisfies the following conditions:

(1) $F(x, y) \geq 0$ and $F(x, y) = 0$ if and only if $y = 0$.

(2) $F(x, y)$ is smooth on $\widetilde{TM} = TM \setminus \{0\}$.

(3) $F(x, ky) = kF(x, y)$ for $\forall k > 0$.

(4) The fundamental tensor field

$$g_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{1.1}$$

is positive definite.

In his Ph.D. Thesis, Paul Finsler used these conditions, but further development of physics shows that for the most important physical applications these assumptions are too restrictive ([I]). Conditions (1) and (4) are inconsistent with the pseudo-Euclidean character of physical space-time (existence of light cone) ([AIM]). The notion of Finsler space can be formulated only by the conditions (2) and (3).

Let $F^n = (M, F)$ be an n-dimensional Finsler space. The fundamental function $F(x, y)$ is called an (α, β)-*metric* if $F = F(\alpha, \beta)$ is a homogeneous function of α and β of degree one, where $\alpha^2 = a(y, y) = a_{ij}(x)y^i y^j$, $y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M$ is a Riemannian metric, and $\beta = b_i(x)y^i$ is a linear 1-form on $\widetilde{TM}$.

In the present paper we will consider the case of a *Randers space*, which is a Finsler space whose fundamental function is given by

$$\bar{F}(x, y) = \alpha(x, y) + \beta(x, y). \tag{1.2}$$

This metric was introduced by G. Randers in 1941 to discuss the asymmetrical metric in the four dimensional space of general relativity. It has been studied latter by many physicists and mathematicians as the simplest Finslerian deformation of a Riemannian space (M, a) by a linear one form β. This metric was called *Randers metric* by R. Ingarden in 1957 and used to study the optical representation in the electron microscope ([AIM], [I]).

The 1-homogeneity of the fundamental function $\bar{F}(x, y) = F(\alpha, \beta)$ implies

$$\begin{aligned} &\alpha L_\alpha + \beta L_\beta = 2L, \quad \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = L_\alpha, \\ &\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta, \quad \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L, \end{aligned} \tag{1.3}$$

where we have put $L := F^2(\alpha, \beta)$, $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha \partial \alpha}$, etc.

The following result is known ([AIM], [BCS]):

Proposition 1.1 *The fundamental function F of a Randers space is positive and its fundamental tensor g_{ij} is positive definite if and only if*

$$||b||^2 = a^{ij}(x)b_i b_j < 1. \tag{1.4}$$

However, we remark that the classical study of Randers spaces does not need this condition. Indeed, the applications in Physics ([AIM]) as well as the proofs of constant curvature properties of Randers spaces do not need the restriction (1.4).

Let us consider the 0-homogeneous frame (b_i, l_i), where

$$l_i = \frac{1}{\alpha} y_i = \frac{1}{\alpha} a_{ij} y^j, \tag{1.5}$$

and $b_i = \frac{\partial \beta}{\partial y^i}$.

If we denote by $r_{ij}(x, y)$ the fundamental tensor of the Randers space $(M, \alpha + \beta)$, then we obtain ([BCS], [SS]):

Proposition 1.2 *For a positive definite Randers space we have*

(i) $\det ||r_{ij}|| = \left(\frac{\alpha+\beta}{\alpha}\right)^{n+1} \cdot \det ||a_{ij}||$,

(ii) $r_{ij} = \frac{\alpha+\beta}{\alpha} \overset{\circ}{h}_{ij} + d_i d_j$,

(iii) $r^{ij} = \frac{\alpha}{\alpha+\beta} a^{ij} - \frac{\alpha^2}{(\alpha+\beta)^2}(b^i l^j + b^j l^i) + \frac{\alpha^2(\alpha b^2 + \beta)}{(\alpha+\beta)^3} l^i l^j$,

where we have denoted by $\overset{\circ}{h}_{ij} := a_{ij} - l_i l_j$ *the angular metric of the Riemannian space* (M, a), *and* $d_i := b_i + l_i$.

Let us denote by $\bar{R}_i{}^h{}_{js}(x, y)$ the coefficients of the hh-curvature ([AIM]) of the Randers space $(M, \bar{F})$. If we put

$$\bar{R}_{ij} = \bar{R}_i{}^s{}_{js}, \quad K = \frac{1}{(n-1)} \frac{1}{F^2} \bar{R}_0{}^s{}_{0s}$$

where the index 0 means the contraction by y^i, then the space $(M, \bar{F})$ is called to be of *scalar curvature* if the relation

$$F^2 K h^i_j = \bar{R}_0{}^i{}_{0j} \tag{1.6}$$

holds in any point of TM, and to be of *constant curvature* if, furthermore, the scalar K is constant. Here $h^i_j = g^{ik} h_{kj}$, and h_{kj} is the angular tensor of the Randers space $(M, \bar{F})$.

The following result is known as Yasuda-Shimada's theorem in the revised form ([BR]):

Theorem 1.3 *Let* $F^n = (M, \alpha+\beta)$ *be a non-Riemannian Randers metric on a smooth manifold* M *of dimension* $n \geq 2$. *Let* a_{ij} *be the underlying Riemannian metric and* $\beta = b_i y^i$ *the drift* 1*-form, both globally defined on*

M. Let us introduce the quantity

$$curl_{bi} := b^s(b_{s|i} - b_{i|s}) \tag{1.7}$$

where | is the covariant derivative with respect to the Levi Civita connection of the Riemannian metric a_{ij}. Then

A) *F^n has constant negative flag curvature K and $curl_{bi} = 0$ if and only if*

(i) *The Riemannian space (M, a) is of negative constant sectional curvature $c := -R^2$, where R is a non-zero constant,*

(ii) $b_{i|j} = R(a_{ij} - b_i b_j)$,

where | represents the covariant derivative with respect to Levi Civita connection of the Riemannian space form (M, a).

In this case we have $K = -\frac{R^2}{4}$ and the Randers space is called an RCG-space.

B) *F^n is flat (i.e., $K \equiv 0$) and $curl_{bi} = 0$ if and only if it is locally Minkowski.*

C) *F^n has constant positive flag curvature K and $curl_{bi} = 0$ if and only if*

(i) *The Riemannian curvature tensor $R = R(x)$ of the Riemannian space (M, a) satisfies the relation*

$$\begin{aligned} R_{hikj} &= K(b_h b_j a_{ik} - b_k b_h a_{ij}) + K(b_i b_k a_{hj} - b_i b_j a_{kh}) \\ &\quad + K(||b||^2 - 1)(a_{kh} a_{ij} - a_{hj} a_{ik}) \\ &\quad + 2b_{i|h} b_{j|k} - b_{i|k} b_{h|j} - b_{i|j} b_{k|h}, \end{aligned} \tag{1.8}$$

(ii) *$||b||$ is a constant, and b is not parallel,*

(iii) $b_{i|j} + b_{j|i} = 0$.

Then the flag curvature of the Randers space is equal to K and the Randers space is called an RCT-space.

Remark The revised version of Yasuda-Shimada theorem, clarifies the fact that $curl_{bi} = 0$ is not a tautology but a restriction. In [BR] it is explained why the condition $curl_{bi} = 0$ cannot be removed. Let us point out that the Riemannian curvature tensor R_{hikj} from (1.8) is different from the one used by Yasuda and Shimada only by sign. We used here the form from [BR].

The name "RCG" stands for Randers space of constant curvature with gradient. Indeed, in the case A), the vector field b_i is gradient because

(ii′) $b_{i|j} - b_{j|i} = 0.$

On the other hand, "RCT" stands for Randers space of constant curvature with translation, because b_i satisfies

$$b_{i|j} + b_{j|i} = 0.$$

and has constant length.

A Randers space $F^n = (M, F = \alpha + \beta)$ is called *projectively related* to the Riemannian space (M, a) if its geodesics coincide with the geodesics of the associated Riemannian space (M, a). It is known ([BM]) that a Randers space $F^n = (M, F = \alpha + \beta)$ is projectively related to (M, a) if and only if the linear one-form β is closed, i.e. $d\beta = 0$.

2. Almost contact metric structures

Let M be an $n = (2m+1)$-dimensional manifold, and let φ, ξ, η be a tensor field of type (1,1), a vector field, a 1-form on M, respectively. If (φ, ξ, η) satisfies the relations

$$\eta(\xi) = 1, \qquad \varphi^2 X = -X + \eta(X)\xi \tag{2.1}$$

for any vector field $X \in \mathfrak{X}(M)$, then M is said to have an *almost contact structure* (φ, ξ, η) and it is called an *almost contact manifold.*

There exists a Riemannian metric a on M compatible with an almost contact structure, i.e.:

$$a(\varphi X, \varphi Y) = a(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any $X, Y \in \mathfrak{X}(M)$. In this case M is said to have an *almost contact metric structure* (φ, ξ, η, a).

The following relations hold ([Bl], [YK]):

Proposition 2.1

$$\varphi\xi = 0, \qquad \eta(\varphi X) = 0, \tag{2.3}$$

$$a(\varphi X, Y) + a(X, \varphi Y) = 0, \tag{2.4}$$

$$a(X, \xi) = \eta(X), \tag{2.5}$$

$$a(\xi, \xi) = 1. \tag{2.6}$$

Let us denote by ∇ the Levi Civita connection of the Riemannian metric a on M.

If ∇ satisfies the relation:

$$(\nabla_X \varphi)Y = \eta(Y)X - a(X,Y)\xi, \tag{2.7}$$

then M is called a *Sasakian manifold.*

Remark In the paper [BF], the Sasakian manifold is defined by the relation $(\nabla_X \varphi)Y := a(X,Y)\xi - \eta(Y)X$, which is different from ours only by the sign.

We have ([Bl], [YK], [H]):

Proposition 2.2 *On Sasakian manifolds the following relations are known*

$$\nabla_X \xi = \varphi X, \tag{2.8}$$

$$(\nabla_X \eta)Y = a(\varphi X, Y), \tag{2.9}$$

$$(\nabla_X \eta)Y + (\nabla_Y \eta)X = 0, \tag{2.10}$$

$$(\nabla_Z \nabla_X \eta)Y = a(Y,Z)\eta(X) - a(X,Z)\eta(Y). \tag{2.11}$$

Lemma 2.3 *In a Sasakian manifold with the almost contact metric structure* (φ, ξ, η, a)*, the relation*

$$curl_{bi} := \eta^s(\eta_{s|i} - \eta_{i|s}) = 0, \tag{2.12}$$

always holds good, where $|$ *is the covariant derivative with respect to the connection* ∇ *given in* (2.7).

Proof. The relation (2.10) reads locally

$$\eta_{i|j} + \eta_{j|i} = 0, \tag{2.13}$$

On the other hand, taking the covariant derivative $|$ of (2.6) it follows

$$\eta^s \eta_{s|i} = 0. \tag{2.14}$$

This relation is true for any almost contact metric structure.

Now, from (2.13) and (2.14) the statement follows immediately. □

A plane section in T_xM is called a *φ-section* if there exists a unit vector X in T_xM orthogonal to ξ such that $\{X, \varphi X\}$ is an orthonormal basis of the plane section. Then, the sectional curvature

$$H(X) := a(R(X, \varphi X)\varphi X, X) \tag{2.15}$$

is called *φ-sectional curvature*. It is known that on a Sasakian manifold the φ-sectional curvatures determine the curvature completely. A Sasakian manifold of constant *φ-sectional curvature* c is called a *Sasakian space form* and it is denoted by $M(c)$.

One has ([K]):

Proposition 2.4 *The Riemannian curvature tensor R of Sasakian space form $M(c)$ is given by*

$$\begin{aligned} R(X,Y)Z = {} & \frac{c+3}{4}[a(Y,Z)X - a(X,Z)Y] \\ & + \frac{c-1}{4}\big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + a(X,Z)\eta(Y)\xi \\ & - a(Y,Z)\eta(X)\xi + a(X,\varphi Z)\varphi Y - a(Y,\varphi Z)\varphi X \\ & + 2a(X,\varphi Y)\varphi Z\big], \end{aligned} \tag{2.16}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

It is known that if the Levi-Civita connection of the Riemannian metric a of an almost contact metric structure (φ, ξ, η, a) on M satisfies the relations

$$\nabla_X \varphi = 0, \qquad \nabla_X \eta = 0 \tag{2.17}$$

then M is called *cosymplectic manifold* ([L]).

Proposition 2.5 *In a cosymplectic manifold the following relations hold good:*

$$\nabla_X \xi = 0, \qquad (\nabla_Z \nabla_X \eta)Y = 0. \tag{2.18}$$

We have:

Lemma 2.6 *In a cosymplectic manifold with the almost contact metric structure (φ, ξ, η, a) the relation (2.12) always holds good, where $|$ is the covariant derivative with respect to the Levi-Civita connection defined in (2.17).*

Proof. It is straightforward from (2.17). □

A cosymplectic manifold of constant φ-sectional curvature c is called a *cosymplectic space form* and it is denoted by $M(c)$.

Then we have ([L]):

Proposition 2.7 *The Riemannian curvature tensor R of a cosymplectic space form $M(c)$ is given by*

$$\begin{aligned} R(X,Y)Z = &-\frac{c}{4}[a(X,Z)Y - a(Y,Z)X] \\ &-\frac{c}{4}[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + a(Y,Z)\eta(X)\xi \\ &- a(X,Z)\eta(Y)\xi + a(Y,\varphi Z)\varphi X \\ &- a(X,\varphi Z)\varphi Y - 2a(X,\varphi Y)\varphi Z] \end{aligned} \tag{2.19}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

On the other hand, if the Levi Civita connection ∇ of the Riemannian metric a on M satisfies the relation

$$(\nabla_X \varphi)Y = \eta(Y)\varphi X - a(\varphi X, Y)\xi \tag{2.20}$$

then M is called a *Kenmotsu manifold* ([K, P]).

Remark In the original paper [K],

$$(\nabla_X \varphi)Y = -\eta(Y)\varphi X + a(\varphi X, Y)\xi$$

which is different from our definition only by sign.

We have ([K, P]):

Proposition 2.8 *In a Kenmotsu manifold the following relations hold good*

$$\nabla_X \xi = -X + \eta(X)\xi \tag{2.21}$$

$$(\nabla_X \eta)Y = -[a(X,Y) - \eta(X)\eta(Y)] \tag{2.22}$$

$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0 \tag{2.23}$$

$$(\nabla_Z \nabla_X \eta)Y = -a(X,Z)\eta(Y) - a(Y,Z)\eta(X) + 2\eta(X)\eta(Y)\eta(Z). \tag{2.24}$$

Lemma 2.9 *In a Kenmotsu manifold with the almost contact metric structure (φ, ξ, η, a) the relation (2.12) always holds good, where this time* |

is the covariant derivative with respect to the connection ∇ *given by* (2.20).

Proof. The statement is obvious because of (2.23). □

As in the case of Sasakian manifolds, a Kenmotsu manifold of constant φ-sectional curvature c is called a *Kenmotsu space form* $M(c)$.

Then we have ([K]):

Proposition 2.10 *The Riemannian curvature tensor* R *of a Kenmotsu space form* $M(c)$ *is given by*

$$\begin{aligned} R(X,Y)Z = \frac{c-3}{4}&[a(Y,Z)X - a(X,Z)Y] \\ + \frac{c+1}{4}&[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + a(X,Z)\eta(Y)\xi \\ &- a(Y,Z)\eta(X)\xi + a(X,\varphi Z)\varphi Y - a(Y,\varphi Z)\varphi X \\ &+ 2a(X,\varphi Y)\varphi Z]. \end{aligned} \tag{2.25}$$

An interesting property follows ([K]):

Theorem 2.11 *If* M *is a Kenmotsu space form* $M(c)$ *of dimension* $n \geq 3$, *then* M *is a Riemannian space form of constant sectional curvature* $c = -1$.

Remark One can remark that because of the Theorem 2.11 there is a big difference between Sasakian and Kenmotsu space forms. Namely, in the first case the φ-sectional curvature determines the Riemannian curvature tensor of this space, but in the second case, the only possibility is $c = -1$.

In 1968, S. Tanno has classified connected almost contact Riemannian manifolds whose automorphism group has maximum dimension ([T]). Namely, he proved:

Theorem 2.12 (S. Tanno) *Let* M *be a connected almost contact Riemannian manifold of dimension* $n = 2m + 1$. *Then the maximum dimension of the automorphism group is* $(m+1)^2$. *This maximum is attained if and only if the sectional curvature for 2-planes which contain* ξ *is a constant* c *and* M *is one of the following spaces:*

(i) $c > 0$: M *is a homogeneous Sasakian manifold of constant* φ-*sectional curvature,*

(ii) $c = 0$: M *is a global Riemannian product of a line or a circle and a Kählerian space form,*

(iii) $c < 0$: *M is the warped product* $L \times_f \mathbb{C}E^m$, *where* $f(t) = c \cdot e^t$, $\mathbb{C}E^m$ *is an m-dimensional complex Euclidean space, and L is a line.*

We recall that the warped product $M = L \times_f \mathbb{C}E^m$ is the manifold $L \times \mathbb{C}E^m$ endowed with Riemannian structure a such that

$$a(X, Y) = dt(p_*X)dt(p_*Y) + f^2(px)a_0(\pi_*X, \pi_*Y), \tag{2.26}$$

for every $X, Y \in \mathfrak{X}(M)$, where a_0 is the canonical metric on $\mathbb{C}E^m$, $p : L \times \mathbb{C}E^m \to L$, and $\pi : L \times \mathbb{C}E^m \to \mathbb{C}E^m$ are the canonical projections of the product manifold $L \times \mathbb{C}E^m$.

Remark The classes of connected almost contact Riemannian manifolds whose automorphism group has maximum dimension (i), (ii), and (iii) given in Theorem 2.12 are exactly the Sasakian, cosymplectic and Kenmotsu space forms discussed above in the present paragraph, respectively.

3. The Randers space induced by an almost contact metric structure

Let us restrict ourselves to a differential manifold M of odd dimension $n = 2m + 1$, $m \geq 1$, endowed with an almost contact metric structure (φ, ξ, η, a). In this way we have a Riemannian metric a and a 1-form $\eta = \eta_i(x)dx^i$ on M, so we can construct in a natural way a Randers metric $F = \alpha + \beta$, where we put $\beta := \eta_i(x)y^i$. This metric will be called in the following the *Randers metric induced* by the almost contact metric structure (φ, ξ, η, a).

More general, one can construct an (α, β)-metric from an almost contact metric structure on M. However, in the present paper we restrict our considerations only to Randers metrics.

One can remark that from (2.6) it follows $||\eta|| = 1$, so the positive definiteness condition is not satisfied in general by a Randers metric $F = \alpha + \beta$, where β is constructed as above. However in some cases this problem can be solved by scaling properly the linear 1-form β, for example one can consider $\beta := \varepsilon\eta_i(x)y^i$, $0 < \varepsilon < 1$ instead of $\eta_i(x)y^i$. Unfortunately this is not always possible, as it will be seen.

Recently, Bejancu and Farran have shown the following ([BF]):

Theorem 3.1 *Let $M(c)$ be a Sasakian space form of constant φ-sectional curvature $c \in (-3, 1)$. Then for any constant $K > 0$ there exists a Randers* *metric F on $M(c)$ such that $(M(c), F)$ has constant flag curvature K and is not projectively flat.*

Namely, starting with a Sasakian space form structure (φ, ξ, η, a) on M, and taking $\beta := \frac{\sqrt{1-c}}{2}\eta_i y^i$ on M, the function $F = \alpha + \beta$ satisfies the conditions in Theorem 1.3, C). Indeed, by writing (2.16) in local coordinates it results (i), the Riemannian length of β is a constant, and (2.10) written in local coordinates gives (iii) of C). Taking into account Lemma 2.3 it follows that the Randers space with the fundamental metric $F = \alpha + \beta$ is an RCT space of constant flag curvature $K = 1$. Now, if one puts

$$F^*(x, y) = \frac{1}{\sqrt{K}} F(x, y) \tag{3.1}$$

then the Randers space (M, F^*) satisfies the conditions of Theorem 1.3, C). Moreover, this Randers space is positive definite.

We mention that the authors of [BF] were not aware, that time, of the fact that Yasuda-Shimada theorem holds good only in the form given in Theorem 1.3, that means only if the supplementary condition $curl_{bi} = 0$ is added, so they did not check this fact.

Fortunately, our Lemma 2.3 shows that the condition $curl_{bi} = 0$ is satisfied for any Sasakian manifold, so their result is completely true.

They give also the following:

Corollary 3.2 ([BF]) *On any odd dimensional sphere S^{2m+1}, $m \geq 1$, there is a natural structure of Randers space of positive constant flag curvature.*

The RCT space induced by the Sasakian space form structure is remarkable because it shows the existence of an RCT structure on any odd dimensional sphere generalizing in this way the result in [BS] about S^3.

Inspired by this result, we study the Randers space induced by a cosymplectic space form and a Kenmotsu space form $M(c)$.

Proof of Theorem A. (i) Let $(M, F = \alpha + \beta)$ be the Randers space induced by the cosymplectic space form $M(c)$, where $\alpha^2 := a(y, y)$ and $\beta := \varepsilon\eta_i(x)y^i$, $0 < \varepsilon < 1$. In this case the Riemannian length $||\beta||$ is less than one, i.e. the Randers space $(M, F = \alpha + \beta)$ is a positive defined one.

From (2.17) it follows that the Randers space induced as above by a cosymplectic space is a Berwald space, and therefore its Chern connection coincides with the Levi Civita connection ∇ of the Riemannian manifold (M, a), defined by the relations (2.17). Since our cosymplectic space form is of φ-sectional curvature $c = 0$, using this fact and (2.19) it follows the induced Randers space is a locally Minkowski space.

The projectively flatness of this Randers space (M, F) can be seen as follows. A locally Minkowski space means that there is a coordinate system in which $F = F(y)$. Since the fundamental tensor g_{ij} of the space has no x dependence, it results that the formal Christoffel symbols Γ^k_{ij} of g_{ij} must all vanish. Hence the geodesics of constant Finslerian speed of the Randers space (M, F) satisfy the equation $\frac{d^2x^i}{dt^2} = 0$. In other words, in this particular coordinate system, the geodesics are straight lines.

Then, from Theorem 1.3, B) it results that $(M, F = \alpha + \beta)$ is a Randers space of constant flag curvature $K = 0$, and $curl_{bi} = 0$. In this case, the relation $curl_{bi} = 0$ also can be shown directly.

(ii) Let us consider the Randers space $(M, F = \alpha + \beta)$ induced by the given Kenmotsu structure. We will show that F satisfies all the conditions of the Theorem 1.3, A).

Since $M(c)$ is a Kenmotsu space form, from Theorem 2.11. It follows that (M, a) is a Riemannian space form of constant sectional curvature $c = -R^2 = -1$. It follows $R = 1$ or $R = -1$.

On the other hand, from (2.22) it follows that (ii) of A) in Theorem 1.3 holds good for $R = -1$. Now, taking into account Lemma 2.9, all the necessary conditions being satisfied, from Theorem 1.3, A) it follows that the induced Randers metric is a RCG space of constant flag curvature $K = -\frac{1}{4}$.

From (2.22) it follows $d\eta(X, Y) = \frac{1}{2}((\nabla_X \eta)Y - (\nabla_Y \eta)X) = 0$, so this Randers space is projectively related to the Kenmotsu manifold (M, a). On the other hand, since the Riemannian space (M, a) is of constant sectional curvature $K = -\frac{1}{4}$ it follows from Beltrami's theorem for Riemannian space forms that (M, a) is projectively flat. Finally, it results that the Randers space is projectively flat being projectively related to a flat Riemannian space. □

Proof of Theorem B. (i) From Theorem 2.12 (ii) it follows that the base manifold it has to be one of the following manifolds

$$T \times CP^m, \qquad T \times CE^m, \qquad T \times CD^m,$$
$$L \times CP^m, \qquad L \times CE^m, \qquad L \times CD^m,$$

where T and L are a circle and a line, respectively, and CP^m, CE^m, and CD^m are a complex projective space, an unitary Euclidean space and an open ball, respectively. Among these, the only ones with zero constant φ-sectional curvature are the manifolds $T \times CE^m$ and $L \times CE^m$, so the statement follows from Theorem A, (i).

(ii) It follows immediately from [K] and (ii) of Theorem A. □

Remark Let M be an almost contact Riemannian manifold with the structure (φ, ξ, η, a). Let us consider the deformed equation

$$(\nabla_X \varphi)Y = k\{\eta(Y)\varphi X - a(\varphi X, Y)\eta\} \tag{3.2}$$

instead of (2.20), where k is a positive constant. Then we have

$$\nabla_X \xi = -k\{X - \eta(X)\xi\}, \tag{3.3}$$

$$(\nabla_X \eta)Y = -k\{a(X,Y) - \eta(X)\eta(Y)\}. \tag{3.4}$$

If M is of constant φ-sectional curvature, then M is of constant sectional curvature $c = -k^2$. Therefore, because of Theorem 1.3, A) we can reformulate (ii) in our Theorem A as follows.

Proposition 3.3 *Let $M(c)$ be a Kenmotsu space form of constant φ-sectional curvature c. Then for any real constant $k \in \mathbb{R}$ there exists a Randers metric on M of constant flag curvature $K = -\frac{k^2}{4}$.*

We recall that the 1-form η has unit Riemannian length. In the case of RCG spaces, this length cannot be made less than 1 by any means because of the following reason.

Lemma 3.4 *In an RCG space $(M, F = \alpha + \beta)$, if $||b||$ is a constant, then this constant has to be equal to 1. In other words if $||b|| =$ constant then the RCG space $(M, F = \alpha + \beta)$ cannot be positive definite.*

Proof. Let us start from the relation

$$||b||^2 = a^{ij}(x)b_i b_j = p, \quad p \in \mathbb{R}.$$

By covariant differentiation with respect to the Levi Civita connection of (M, a) we obtain

$$(a^{ij}(x)b_j)b_{i|k} = 0$$

and using (ii) from Theorem 1.3, A) it follows

$$a^{ij}b_j(a_{ik} - b_ib_k) = 0$$

since $R \neq 0$. In other words $(\delta^j_k - a^{ij}b_ib_k)b_j = 0$, or $b_k(||b||^2 - 1) = 0$.

If $b_k \neq 0$ it follows $||b|| = 1$ and the proposition is proved. The case $b_k = 0$ is not possible because in this case (ii) implies that for this k we have

$$b_{k|i} = R(a_{ki} - b_kb_i), \qquad \forall i \in \{1, 2, \ldots, n\},$$

what is equivalent with $a_{ki} = 0$, $\forall i \in \{1, 2, \ldots, n\}$. This is impossible because the Riemannian metric a cannot be degenerate. □

Summarizing some results from [K] we obtain:

Corollary 3.5 *Let M be a Kenmotsu manifold of dimension $n = 2m+1$. If M is one of the following:*

(a) *a locally symmetric Riemannian space,*

(b) *a conformally flat space of dimension $n > 3$,*

then on M there exists an RCG structure.

Proof. Corollary 6 in [K] states that if M is a Kenmotsu manifold that is locally symmetric, then M is of constant negative curvature -1. It follows that it is a Kenmotsu manifold of constant φ-sectional curvature and therefore from Theorem A, (ii) we obtain (a).

In the same way, Proposition 11 in [K] states that if M is a Kenmotsu manifold of dimension $n > 3$ that it is conformally flat, then M is a space of constant negative curvature. By the same argument as above, we obtain (b). □

4. Conclusion

We have studied the way an almost contact metric structure (φ, ξ, η, a) induces a Randers metric on the manifold M of dimension $n = 2m+1$. Precisely, a Sasakian space form structure induces an RCT-structure ([BF]), a cosymplectic space form induces a locally Minkowski structure and a Kenmotsu space form induces an RCG space, in a natural way, i.e. $F := \alpha + \beta$, where $\alpha^2 = a(y, y)$ and $\beta = \eta_i(x)y^i$. However, the Riemannian length of η is equal to one, so these Randers metrics are not positive definite. This is no trouble in the first two cases because one can scale the length of β by defining $\beta = \varepsilon\eta_i(x)y^i$, $0 < \varepsilon < 1$, but in the third case the induced Randers space cannot be made a positive definite one.

S. Tanno's classification theorem shows that an almost contact metric structure of constant φ-sectional curvature induces a Randers space of constant flag curvature with $curl_{bi} = 0$. Moreover, on the manifolds: S^{2m+1}, $T \times CE^m$, $L \times CE^m$, $L \times_f CE^m$ we can construct a natural structure of Randers space of constant flag curvature with $curl_{bi} = 0$.

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