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A lower bound for the curvature invariant $p(G/K)$ associated with a Riemannian symmetric space G/K

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Abstract. We investigate the curvature invariant $p(G/K)$ associated with a Riemannian symmetric space G/K, which was introduced in [3] in order to estimate the least dimension of the Euclidean space $\boldsymbol{R}^N$ into which G/K can be locally isometrically imbedded. We calculate, in a systematic method, a lower bound of $p(G/K)$ for any compact irreducible Riemannian symmetric space G/K. Further, we calculate $p(G/K)$ for compact rank one Riemannian symmetric spaces and establish a non-existence theorem of isometric imbeddings. It is conjectured that the lower bound obtained by our method coincides with $p(G/K)$ for almost every compact irreducible Riemannian symmetric space G/K.

Key words: curvature invariant, isometric imbedding, Riemannian symmetric space.

1. Introduction

Let M be a Riemannian manifold. In our paper [3], we defined a $\boldsymbol{Z}$-valued function p_M on M, which is a curvature invariant of M. As we have shown, p_M is effective to estimate the least dimension of the Euclidean space into which M can be locally isometrically imbedded (see Proposition 1.1 of [3]).

In the special case where M is a Riemannian symmetric space, it is shown that the function p_M can be reformulated in terms of Lie algebras as follows: Let $M = G/K$ be a Riemannian symmetric space and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of the Lie algebra $\mathfrak{g}$ of G associated with the Riemannian symmetric pair (G, K). Take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$ and denote by $\mathfrak{k}_0$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, i.e., $\mathfrak{k}_0 = \{X \in \mathfrak{k} \mid [X, \mathfrak{a}] = 0\}$. We call a subspace V of $\mathfrak{m}$ *pseudo-abelian* if $[V, V] \subset \mathfrak{k}_0$. By $p(G/K)$ we denote the maximum of the dimensions of pseudo-abelian subspaces in $\mathfrak{m}$, which we call the *pseudo-nullity* of G/K. Then it is shown that the function p_M coincides with $p(G/K)$ everywhere on $M = G/K$ (see Proposition 2.1 of [3]). Applying Proposition 1.1 of [3] to $M = G/K$, we have

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Theorem 1.1 ([3]) *Let G/K be a Riemannian symmetric space. Then, any open set of G/K cannot be isometrically imbedded into the Euclidean space $\boldsymbol{R}^N$ with $N \leq 2\dim G/K - p(G/K) - 1$.*

In this paper, we consider the problem to determine the pseudo-nullity $p(G/K)$. For this purpose, it is essential to calculate $p(G/K)$ for compact irreducible Riemannian symmetric spaces G/K. In fact, we have shown that: (i) If G/K is (locally) isomorphic to a Riemannian product of two Riemannian symmetric spaces $G_i/K_i\,(i=1,2)$, i.e., $G/K \cong G_1/K_1 \times G_2/K_2$, then $p(G/K) = p(G_1/K_1) + p(G_2/K_2)$; (ii) If G/K is of Euclidean type, then $p(G/K) = \dim G/K$; (iii) If G/K is of non-compact type, then $p(G/K) = p((G/K)^*)$, where $(G/K)^*$ is the compact dual of G/K (see [3]).

In [3] and [4] we have calculated the pseudo-nullities $p(G/K)$ for the following compact irreducible Riemannian symmetric spaces:

(1) The spheres S^n $(n \geq 2)$.

(2) Those spaces G/K satisfying $\mathrm{rank}(G/K) = \mathrm{rank}(G)$, i.e.,

$$AI,\ CI,\ EI,\ EV,\ EVIII,\ FI,\ G,$$
$$BI : SO(2n+1)/SO(n+1) \times SO(n)\ (n \geq 2),$$
$$DI : SO(2n)/SO(n) \times SO(n)\ (n \geq 3).$$

(3) Compact Lie groups:

$$Sp(n)\ (n \geq 1),\quad SU(n)\ (2 \leq n \leq 5),\quad SO(n)\ (3 \leq n \leq 9,\ n \neq 4),\quad G_2.$$

As we have stated, for each symmetric space G/K listed above, we obtain an estimate on the least dimension of the Euclidean space into which G/K can be (locally) isometrically imbedded. Especially, by our results we know that in the case where G/K is $CI : Sp(n)/U(n)$ $(n \geq 1)$ or $Sp(n)$ $(n \geq 1)$ the canonical isometric imbedding defined in Kobayashi [13] gives the least dimensional isometric imbedding of G/K.

Unfortunately, we cannot so easily get the estimate stated above for the other compact irreducible Riemannian symmetric spaces G/K, because it is, in general, a hard algebraic problem to calculate the pseudo-nullities $p(G/K)$. In this paper, in order to approach the pseudo-nullity $p(G/K)$ we propose a systematic method to obtain a lower bound for $p(G/K)$.

Our method is divided into two steps. The first step is to localize the problem. Let Σ be the set of all non-zero restricted roots associated with the Riemannian symmetric pair (G, K). We denote by $\mathfrak{m}(\mu)$ the root subspace of $\mathfrak{m}$ corresponding to a non-zero restricted root $\mu \in \Sigma$. By $n(\mu)$ we denote the maximum of the dimensions of pseudo-abelian subspaces contained in $\mathfrak{m}(\mu)$ and call it the *local pseudo-nullity* of μ. Our first task is to describe $n(\mu)$ by using the multiplicity $m(\mu)$ of μ (see Theorem 3.2). Our results of this step are summarized in Table 3.

The second step is explained as follows: Let Γ be a strongly orthogonal subset of Σ (for the definition of strongly orthogonal subsets, see §4). Let $V(\mu)$ be a pseudo-abelian subspace contained in $\mathfrak{m}(\mu)$ with $\dim V(\mu) = n(\mu)$. Then, the sum $\sum_{\mu \in \Gamma} V(\mu)$ plus a suitable subspace of $\mathfrak{a}$ forms a pseudo-abelian subspace of $\mathfrak{m}$ (see Proposition 4.1). By $p_{cat}(G/K)$ we denote the maximum of the dimensions of all pseudo-abelian subspaces constructed in the manner stated above. We call $p_{cat}(G/K)$ the *categorical pseudo-nullity* of G/K, which gives a lower bound for the pseudo-nullity $p(G/K)$. Our second task is to calculate the categorical pseudo-nullity $p_{cat}(G/K)$ by viewing the result of the classification of strongly orthogonal subsets in Σ (see [5]). In Table 4 and Table 5 we exhibit the results of this task.

Although the categorical pseudo-nullity $p_{cat}(G/K)$ does not directly serve to determine the least dimensional (local) isometric imbeddings of G/K, it gives a fairly good estimate on $p(G/K)$. It will be shown that the equality $p(G/K) = p_{cat}(G/K)$ holds for Riemannian symmetric spaces G/K listed above (see Table 4 and Table 5). In §5, we will determine the pseudo-nullities $p(G/K)$ for compact rank one Riemannian symmetric spaces G/K. As a result, we know that the equality $p(G/K) = p_{cat}(G/K)$ holds for any compact rank one Riemannian symmetric spaces except the 2-dimensional complex projective space $P^2(\boldsymbol{C})$ (see Theorem 5.1). On the basis of this result we obtain an estimate on the least dimension of the Euclidean space into which compact rank one Riemannian symmetric spaces G/K can be locally isometrically imbedded (see Theorem 5.6). In the case where $G/K = P^n(\boldsymbol{H})$ $(n \geq 2)$ or $P^2(\boldsymbol{Cay})$, Theorem 5.6 improves the former estimate obtained in [2].

It is expected that the equality $p(G/K) = p_{cat}(G/K)$ holds for a wider class of Riemannian symmetric spaces G/K, whose proof will be investigated as a main subject in our future work.

2. Restricted roots and multiplicities

In this and the subsequent sections we follow the notations in the introduction. Let G/K be a compact irreducible Riemannian symmetric space with G simple. In this section we recall the multiplicities of the restricted roots associated with the Riemannian symmetric pair (G, K).

Let B be the Killing form of $\mathfrak{g}$. We introduce an inner product $(\ ,\ )$ of $\mathfrak{g}$ by

$$(X, Y) = -B(X, Y), \quad X, Y \in \mathfrak{g}.$$

Let $\mathfrak{t}$ be a Cartan subalgebra satisfying $\mathfrak{t} \supset \mathfrak{a}$ and set $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$. Then we have

$$\mathfrak{t} = \mathfrak{a} + \mathfrak{b} \quad (\textit{orthogonal direct sum}).$$

Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$. For each $\alpha \in \mathfrak{t}$ we define a subspace $\mathfrak{g}_\alpha$ of $\mathfrak{g}^c$ by

$$\mathfrak{g}_\alpha = \{Z \in \mathfrak{g}^c \mid [H, Z] = \sqrt{-1}(\alpha, H)Z, \ \forall H \in \mathfrak{t}\}.$$

An element $\alpha \in \mathfrak{t}$ is called a *root* of $\mathfrak{g}^c$ if $\mathfrak{g}_\alpha \neq 0$. By Δ we denote the set of non-zero roots of $\mathfrak{g}^c$.

Let $\lambda \in \mathfrak{t}$. By $\lambda_\mathfrak{a}$ we mean the $\mathfrak{a}$-component of λ with respect to the orthogonal decomposition $\mathfrak{t} = \mathfrak{a} + \mathfrak{b}$. An element $\mu \in \mathfrak{a}$ is called a *restricted root* if there is a root α such that $\alpha_\mathfrak{a} = \mu$. Let us denote by Σ the set of all non-zero restricted roots. As is well-known, Σ forms an irreducible (possibly non-reduced) root system.

Let $\mu \in \Sigma$. We denote by $\Delta(\mu)$ the set of all roots $\alpha \in \Delta$ such that $\alpha_\mathfrak{a} = \mu$. The cardinality $\#\Delta(\mu)$ of $\Delta(\mu)$ is called the *multiplicity* of $\mu \in \Sigma$ and is denoted by $m(\mu)$.

Let $\mu \in \Sigma$. We define two subspaces $\mathfrak{k}(\mu) \subset \mathfrak{k}$ and $\mathfrak{m}(\mu) \subset \mathfrak{m}$ by setting

$$\mathfrak{k}(\mu) = \{X \in \mathfrak{k} \mid \mathrm{ad}(H)^2(X) = -(\mu, H)^2 X, \ \forall H \in \mathfrak{a}\},$$
$$\mathfrak{m}(\mu) = \{Y \in \mathfrak{m} \mid \mathrm{ad}(H)^2(Y) = -(\mu, H)^2 Y, \ \forall H \in \mathfrak{a}\}.$$

As is easily seen, we have $\mathfrak{k}(-\mu) = \mathfrak{k}(\mu)$ and $\mathfrak{m}(-\mu) = \mathfrak{m}(\mu)$. For convenience we set $\mathfrak{k}(0) = \mathfrak{k}_0$, $\mathfrak{m}(0) = \mathfrak{a}$ and $\mathfrak{k}(\mu) = \mathfrak{m}(\mu) = 0$ if $\mu \notin \Sigma \cup \{0\}$.

Let θ be the involution of $\mathfrak{g}$ induced from the geodesic symmetry at the origin of G/K. Let "$<$" be a linear order of $\mathfrak{a}$. We extend "$<$" to a linear order " < "of $\mathfrak{t}$ in such a way

$$H > 0, \quad H \notin \mathfrak{b} \Longrightarrow \theta H < 0.$$

Let Δ^+ (resp. Σ^+) be the set of positive roots of Δ (resp. Σ) with respect to "<". Then the following assertion is well-known (see [14]):

Proposition 2.1 (1) $\dim \mathfrak{k}(\mu) = \dim \mathfrak{m}(\mu) = m(\mu)$.
(2) *The following decompositions hold*:

$$\mathfrak{m} = \mathfrak{a} + \sum_{\mu \in \Sigma^+} \mathfrak{m}(\mu) \quad (\textit{orthogonal direct sum}),$$

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\mu \in \Sigma^+} \mathfrak{k}(\mu) \quad (\textit{orthogonal direct sum}).$$

(3) *Let* μ_1, $\mu_2 \in \Sigma \cup \{0\}$. *Then*:

$$\begin{aligned} \left[\mathfrak{k}(\mu_1), \mathfrak{k}(\mu_2)\right] &\subset \mathfrak{k}(\mu_1 + \mu_2) + \mathfrak{k}(\mu_1 - \mu_2), \\ \left[\mathfrak{m}(\mu_1), \mathfrak{m}(\mu_2)\right] &\subset \mathfrak{k}(\mu_1 + \mu_2) + \mathfrak{k}(\mu_1 - \mu_2), \\ \left[\mathfrak{k}(\mu_1), \mathfrak{m}(\mu_2)\right] &\subset \mathfrak{m}(\mu_1 + \mu_2) + \mathfrak{m}(\mu_1 - \mu_2). \end{aligned}$$

Since the restricted root system Σ is an irreducible root system, Σ contains at most three sorts of roots with different lengths. Let us divide Σ into three subsets Σ_i $(i = 1, 2, 3)$ according as the lengths of restricted roots. In the case where Σ is not reduced, i.e., Σ is of type BC_n $(n \geq 1)$, we denote by Σ_2 (resp. Σ_3) the set of multipliable (resp. divisible) restricted roots and set $\Sigma_1 = \Sigma \setminus (\Sigma_2 \cup \Sigma_3)$. Recall that a restricted root $\mu \in \Sigma$ is called *multipliable* (resp. *divisible*) if $2\mu \in \Sigma$ (resp. $(1/2)\mu \in \Sigma$) (see Helgason [12]). In the case where Σ is reduced, we denote by Σ_2 the set of short restricted roots and set $\Sigma_1 = \Sigma \setminus \Sigma_2$ and $\Sigma_3 = \emptyset$. (By definition, a restricted root $\mu \in \Sigma$ is called *short* if Σ contains a restricted root longer than μ.) In any case, each Σ_i $(i = 1, 2, 3)$ is composed of restricted roots of the same length if $\Sigma_i \neq \emptyset$ and $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ (disjoint union). Since two restricted roots of the same length have the same multiplicity (see Appendix of [2]), it follows that the multiplicity $m(\mu)$ $(\mu \in \Sigma_i)$ takes a constant value on each subset Σ_i.

Let m_i $(i = 1, 2, 3)$ be the multiplicity of the restricted roots in Σ_i. (m_i is assumed to be 0, if $\Sigma_i = \emptyset$.) As we have stated above, by the triplet $\mathcal{M}(G/K) = \{m_1, m_2, m_3\}$ we can recover the multiplicities of all restricted roots $\mu \in \Sigma$. In Table 3 we list the triplets $\mathcal{M}(G/K)$ for all compact, irreducible Riemannian symmetric spaces G/K with G simple, which can be easily read from the classification table in Araki [9].

3. Local pseudo-nullities $n(\mu)$

As in the previous section we assume that G/K is a compact, irreducible Riemannian symmetric space with G simple. Let $\mathcal{P}$ denote the family of pseudo-abelian subspaces of $\mathfrak{m}$. Let $U \in \mathcal{P}$. U is called a *local* pseudo-abelian subspace if U is contained in some root subspace $\mathfrak{m}(\mu)$ $(\mu \in \Sigma)$. Let $\mu \in \Sigma$. We denote by $n(\mu)$ the maximum dimension of local pseudo-abelian subspaces contained in $\mathfrak{m}(\mu)$, i.e.,

$$n(\mu) = \max\{\dim U \mid U \subset \mathfrak{m}(\mu),\ U \in \mathcal{P}\}.$$

The integer $n(\mu)$ is called the *local pseudo-nullity* of $\mu \in \Sigma$.

Considering the action of the Weyl group $W(\Sigma)$, we have the following basic property of $n(\mu)$.

Proposition 3.1 *The local pseudo-nullity is invariant under the action of the Weyl group $W(\Sigma)$, i.e., $n(w\mu) = n(\mu)$ holds for $w \in W(\Sigma)$ and $\mu \in \Sigma$. Consequently, if two restricted roots μ and μ' are of the same length, i.e., $|\mu| = |\mu'|$, then $n(\mu) = n(\mu')$.*

Proof. Let $w \in W(\Sigma)$. As is well-known, there is an element $k \in K$ such that $\mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}$ and $\mathrm{Ad}(k)\mu = w\mu$. Then it can be easily observed that $\mathrm{Ad}(k)\mathfrak{m}(\mu) = \mathfrak{m}(w\mu)$ and $\mathrm{Ad}(k)\mathfrak{k}_0 = \mathfrak{k}_0$. Therefore, a subspace U of $\mathfrak{m}(\mu)$ is pseudo-abelian if and only if $\mathrm{Ad}(k)U$ is a pseudo-abelian subspace of $\mathfrak{m}(w\mu)$. This implies $n(w\mu) = n(\mu)$. If two restricted roots μ and μ' are of the same length, we can find an element of $w \in W(\Sigma)$ such that $\mu' = w\mu$. Therefore, we have $n(\mu') = n(\mu)$. □

Let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ be the decomposition defined in the previous section. In view of Proposition 3.1, we know that the pseudo-nullity $n(\mu)$ $(\mu \in \Sigma_i)$ takes a constant value on each subset Σ_i $(i = 1, 2, 3)$. Let n_i $(i = 1, 2, 3)$ be the local pseudo-nullity of the restricted roots in Σ_i. (As in the case of the multiplicity, n_i is assumed to be 0 if $\Sigma_i = \emptyset$.) It is clear that by the triplet $\mathcal{N}(G/K) = \{n_1, n_2, n_3\}$ we can recover the local pseudo-nullities of all restricted roots $\mu \in \Sigma$.

The following theorem shows that the local pseudo-nullities $\mathcal{N}(G/K)$ are completely determined by the multiplicities $\mathcal{M}(G/K)$.

Theorem 3.2 *Let G/K be a compact irreducible Riemannian symmetric space with G simple. Let $\mathcal{M}(G/K) = \{m_1, m_2, m_3\}$ (resp. $\mathcal{N}(G/K) = \{n_1, n_2, n_3\}$) be the multiplicities (resp. local pseudo-nullities) of G/K. Then, the following equalities hold:*

$$n_1 = m_1, \quad n_2 = m_2/(1+m_3), \quad n_3 = m_3.$$

Before proceeding to the proof of Theorem 3.2, we note

Lemma 3.3 (1) $[\mathfrak{m}(\mu), \mathfrak{m}(\mu)] \subset \mathfrak{k}(2\mu) + \mathfrak{k}_0$.
(2) $[\mathfrak{k}(2\mu), \mathfrak{m}(\mu)] \subset \mathfrak{m}(\mu)$.

Proof. From (3) of Proposition 2.1 the assertion (1) follows directly. We also have the assertion (2), because $3\mu \notin \Sigma$. □

We now start the proof of Theorem 3.2. First assume that μ is not multipliable, i.e., $2\mu \notin \Sigma$. Then we have $[\mathfrak{m}(\mu), \mathfrak{m}(\mu)] \subset \mathfrak{k}_0$ and hence $\mathfrak{m}(\mu) \in \mathcal{P}$. Consequently, if $\mu \in \Sigma_1 \cup \Sigma_3$ then it follows $n(\mu) = m(\mu)$. This proves that $n_1 = m_1$ and $n_3 = m_3$. Similarly, if $\mu \in \Sigma_2$ and $\Sigma_3 = \emptyset$ (equivalently $m_3 = 0$), then we have $n_2 = m_2$.

Next we assume that μ is multipliable, i.e., $\mu \in \Sigma_2$ and $\Sigma_3 \neq \emptyset$. This case occurs only in the case where Σ is of type BC_n. In view of Table 3, we know that such restricted roots are exhausted by the following G/K:

$$AIII_{p,q}\ (p > q \geq 1),\ CII_{p,q}\ (p > q \geq 1),$$
$$DIII_n\ (n = 2m+1),\ EIII,\ FII.$$

We also know that the multiplicity m_3 is equal to 1, 3 or 7 and that restricted roots μ satisfying $m(2\mu) > 1$ can be found only in $CII_{p,q}$ $(p > q \geq 1)$: $G^{p,q}(\boldsymbol{H})$ and $FII : P^2(\boldsymbol{Cay})$.

Now let $X \in \mathfrak{k}(2\mu)$. Then, by (2) of Lemma 3.3, we know that $\operatorname{ad} X$ induces a linear endomorphism of $\mathfrak{m}(\mu)$, which we denote by $X^\dagger$, i.e.,

$$X^\dagger(Y) = [X, Y], \quad Y \in \mathfrak{m}(\mu).$$

It is easy to see that $X^\dagger$ is a skew-symmetric endomorphism of $\mathfrak{m}(\mu)$ with respect to the inner product (,).

Lemma 3.4 *Let V be a subspace of $\mathfrak{m}(\mu)$. Then V is pseudo-abelian if and only if $(X^\dagger(V), V) = 0$ holds for any $X \in \mathfrak{k}(2\mu)$.*

Proof. Let $X \in \mathfrak{k}(2\mu)$. Since

$$(X^\dagger(Y), Z) = ([X, Y], Z) = (X, [Y, Z]), \quad \forall Y, \forall Z \in \mathfrak{m}(\mu),$$

we have $(X^\dagger(V), V) = (X, [V, V])$. Hence, if $[V, V] \subset \mathfrak{k}_0$, then we have $(X^\dagger(V), V) = 0$. Conversely, if $(X^\dagger(V), V) = 0$ holds for any $X \in \mathfrak{k}(2\mu)$, then by the above equality we have $(\mathfrak{k}(2\mu), [V, V]) = 0$. This implies that $[V, V] \subset \mathfrak{k}_0$ (see (1) of Lemma 3.3). This completes the proof. □

The set $\mathfrak{k}(2\mu)^\dagger$ composed of all $X^\dagger$ $(X \in \mathfrak{k}(2\mu))$, which is a subspace of the space of endomorphisms of $\mathfrak{m}(\mu)$, has the following special feature.

Theorem 3.5 *Let $\mu \in \Sigma$. Assume that μ is multipliable, i.e., $2\mu \in \Sigma$. Then:*
(1) *If $m(2\mu) = 1$, then $\dim \mathfrak{m}(\mu)$ is even and there is an element I of $\mathfrak{k}(2\mu)$ such that $I^\dagger$ determines a complex structure of $\mathfrak{m}(\mu)$, i.e., ${I^\dagger}^2 = -\mathbf{1}_{\mathfrak{m}(\mu)}$.*
(2) *If $m(2\mu) = 3$, i.e., if G/K is of type $CII_{p,q}$ $(p > q \geq 1)$, then $\dim \mathfrak{m}(\mu)$ is a multiple of 4 and there are elements I, J, K of $\mathfrak{k}(2\mu)$ such that the triplet $\{I^\dagger, J^\dagger, K^\dagger\}$ determines a quaternion structure of $\mathfrak{m}(\mu)$, i.e.,*

$$\begin{aligned} &{I^\dagger}^2 = {J^\dagger}^2 = {K^\dagger}^2 = -\mathbf{1}_{\mathfrak{m}(\mu)}, \quad I^\dagger J^\dagger = -J^\dagger I^\dagger = K^\dagger, \\ &J^\dagger K^\dagger = -K^\dagger J^\dagger = I^\dagger, \quad K^\dagger I^\dagger = -I^\dagger K^\dagger = J^\dagger. \end{aligned}$$

In the above (1) *and* (2), $\mathbf{1}_{\mathfrak{m}(\mu)}$ *implies the identity mapping of* $\mathfrak{m}(\mu)$.

The proof of Theorem 3.5 will be given in §6.

By virtue of Theorem 3.5, the determination of $n(\mu)$ can be reduced to an easy problem.

First consider the case where $m(2\mu) = 3$, i.e., G/K is of type $CII_{p,q}$ $(p > q \geq 1)$. Let U be an arbitrary local pseudo-abelian subspace of $\mathfrak{m}(\mu)$. Denote by V the sum of four subspaces U, $I^\dagger(U)$, $J^\dagger(U)$ and $K^\dagger(U)$, i.e.,

$$V = U + I^\dagger(U) + J^\dagger(U) + K^\dagger(U).$$

We now prove that the above summation is orthogonal and that $\dim V = 4 \dim U$. In fact, by Lemma 3.4 we have

$$(U, I^\dagger(U)) = (U, J^\dagger(U)) = (U, K^\dagger(U)) = 0.$$

Moreover, since $I^\dagger$ is an orthogonal endomorphism of $\mathfrak{m}(\mu)$, we can prove

$$(I^\dagger(U), J^\dagger(U)) = (U, I^\dagger J^\dagger(U)) = (U, K^\dagger(U)) = 0.$$

Similarly, we have $(J^\dagger(U), K^\dagger(U)) = (K^\dagger(U), I^\dagger(U)) = 0$. Consequently, V is an orthogonal direct sum of four subspaces U, $I^\dagger(U)$, $J^\dagger(U)$ and $K^\dagger(U)$. Moreover, since ${I^\dagger}^2 = {J^\dagger}^2 = {K^\dagger}^2 = -\mathbf{1}_{\mathfrak{m}(\mu)}$, we have $\dim I^\dagger(U) = \dim J^\dagger(U) = \dim K^\dagger(U) = \dim U$. Therefore, we have $\dim V = 4\dim U \leq \dim \mathfrak{m}(\mu) = m(\mu)$, which proves that $\dim U \leq m(\mu)/4$. Since U is an arbitrary local pseudo-abelian subspace contained in $\mathfrak{m}(\mu)$, we obtain $n(\mu) \leq m(\mu)/4$.

We now show the converse. Utilizing the quaternion structure $\{I^\dagger, J^\dagger, K^\dagger\}$, we can get a subspace U_0 of $\mathfrak{m}(\mu)$ such that $\dim U_0 = m(\mu)/4$ and

$$\mathfrak{m}(\mu) = U_0 + I^\dagger(U_0) + J^\dagger(U_0) + K^\dagger(U_0) \quad (\textit{orthogonal direct sum}).$$

Since $(U_0, I^\dagger(U_0)) = (U_0, J^\dagger(U_0)) = (U_0, K^\dagger(U_0)) = 0$ and since $\{I, J, K\}$ forms a basis of $\mathfrak{k}(2\mu)$, we have $(U_0, X^\dagger(U_0)) = 0$ for any $X \in \mathfrak{k}(2\mu)$. This proves that U_0 is a local pseudo-abelian subspace contained in $\mathfrak{m}(\mu)$ and hence $n(\mu) \geq \dim U_0 = m(\mu)/4$. Therefore, we get the equality $n(\mu) = m(\mu)/4$ if $m(2\mu) = 3$.

In a similar manner, we can also prove that $n(\mu) = m(2\mu)/2$ for those symmetric spaces G/K satisfying $m(2\mu) = 1$.

Finally, we consider the case where G/K is of type *FII*, i.e., $G/K = P^2(\boldsymbol{Cay})$. We first prove

Proposition 3.6 *Assume that $G/K = P^2(\boldsymbol{Cay})$ and that μ, $2\mu \in \Sigma$. Let Y be a non-zero element of $\mathfrak{m}(\mu)$. Then, $\dim\big[\mathfrak{k}(2\mu), Y\big] = 7$.*

Proof. Since $\dim \mathfrak{k}(2\mu) = m(2\mu) = 7$, it suffices to prove that $\big[X, Y\big] \neq 0$ for each $X \in \mathfrak{k}(2\mu)$ with $X \neq 0$. Now suppose $\big[X, Y\big] = 0$ holds for some $X \in \mathfrak{k}(2\mu)$. Since $\operatorname{ad}\mu$ gives an isomorphism between $\mathfrak{m}(2\mu)$ and $\mathfrak{k}(2\mu)$, we can find $Y' \in \mathfrak{m}(2\mu)$ such that $X = \big[\mu, Y'\big]$. Consequently, we have

$$\big[Y, \big[\mu, Y'\big]\big] = 0. \tag{3.1}$$

Applying $\operatorname{ad}\mu$ to (3.1), we have

$$\big[\big[\mu, Y\big], \big[\mu, Y'\big]\big] = 4(\mu,\mu)^2\big[Y, Y'\big]. \tag{3.2}$$

(Note that $[\mu, [\mu, Y']] = -4(\mu,\mu)^2 Y'$.) In §5, we will prove that $Y \in \mathfrak{m}(\mu)$

and $Y' \in \mathfrak{m}(2\mu)$ must satisfy the following equality (see (1) of Lemma 5.3):

$$\big[[\mu, Y], [\mu, Y']\big] = 2(\mu, \mu)^2 [Y, Y']. \tag{3.3}$$

Comparing the equalities (3.2) and (3.3), we can easily conclude $[Y, Y'] = 0$. This implies that two vectors Y and Y' span an abelian subspace of $\mathfrak{m}$. However, since $\mathrm{rank}(P^2(\boldsymbol{Cay})) = 1$, Y' must be a scalar multiple of Y. Hence we get $Y' = 0$, because $Y \in \mathfrak{m}(\mu)$ and $Y' \in \mathfrak{m}(2\mu)$. Thus, $X = [\mu, Y'] = 0$, proving the proposition. □

We now proceed to the determination of the local pseudo-nullity $n(\mu)$ for a multipliable restricted root in $G/K = P^2(\boldsymbol{Cay})$. Let V be an arbitrary local pseudo-abelian subspace in $\mathfrak{m}(\mu)$. Let Y be a non-zero element of V. By Lemma 3.4 we know that V is necessarily orthogonal to $[\mathfrak{k}(2\mu), Y]$. Since $\dim[\mathfrak{k}(2\mu), Y] = 7$ and $m(\mu) = 8$, we have $\dim V \leq 1$. This proves $n(\mu) = 1 = 8/(1+7)$.

By the above discussions, we complete the proof of Theorem 3.2. □

We will give in Table 3 the local pseudo-nullity $\mathcal{N}(G/K) = \{n_1, n_2, n_3\}$ for each compact irreducible Riemannian symmetric space G/K with G simple.

4. Categorical pseudo-nullities $p_{cat}(G/K)$

In this section, as the second step to estimate the pseudo-nullities $p(G/K)$, we construct pseudo-abelian subspaces of $\mathfrak{m}$ by summing up suitable local pseudo-abelian subspaces.

Let Γ be a subset of the restricted root system Σ. Γ is called a *strongly orthogonal subset* (= SOS) in Σ if it satisfies the following:

$$\alpha,\ \beta \in \Gamma,\ \alpha \neq \beta \Longrightarrow \alpha \pm \beta \notin \Sigma \cup \{0\}.$$

The notion of the strongly orthogonal subsets was first introduced by Harish-Chandra (see [11]) and has been used in many places concerning geometric or representation theoretic problems. For each irreducible root system Σ we have determined the equivalence classes of maximal strongly orthogonal subsets in Σ under the action of the Weyl group $W(\Sigma)$ (see [5]).

Now let us define the notion of categorical subspace of $\mathfrak{m}$. Let V be a subspace of $\mathfrak{m}$. V is called *categorical* if the following two conditions are satisfied:

(1) V is represented by

$$V = V(0) + \sum_{\mu \in \Sigma^+} V(\mu) \quad (\textit{direct sum}),$$

where $V(\mu) = V \cap \mathfrak{m}(\mu)$ $(\mu \in \Sigma^+ \cup \{0\})$.

(2) The *support* Γ of V, which is defined by $\Gamma = \{\mu \in \Sigma^+ \mid V(\mu) \neq 0\}$, is a SOS in Σ.

The following proposition assures that there are many categorical pseudo-abelian subspaces of $\mathfrak{m}$.

Proposition 4.1 *Let $V = V(0) + \sum_\mu V(\mu)$ be a categorical subspace of $\mathfrak{m}$ and let Γ the support of V. Then, V is pseudo-abelian if and only if it satisfies:*

(1) *$V(0) \perp \Gamma$, i.e., $(V(0), \Gamma) = 0$.*

(2) *For each $\mu \in \Gamma$, $V(\mu)$ is a local pseudo-abelian subspace.*

Proof. Let μ_1, μ_2 be two distinct restricted roots in Γ. Then, by Proposition 2.1, we have

$$[V(\mu_1), V(\mu_2)] \subset \mathfrak{k}(\mu_1 + \mu_2) + \mathfrak{k}(\mu_1 - \mu_2).$$

Since Γ is a SOS, we have $\mu_1 \pm \mu_2 \notin \Sigma \cup \{0\}$. Hence, $[V(\mu_1), V(\mu_2)] = 0$. Therefore, it is easy to see that V is pseudo-abelian if and only if

$$[V(0), V(\mu)] \subset \mathfrak{k}(0), \tag{4.1}$$

$$[V(\mu), V(\mu)] \subset \mathfrak{k}(0) \tag{4.2}$$

hold for each $\mu \in \Gamma$. Obviously, (4.2) implies that $V(\mu)$ is a local pseudo-abelian subspace. On the other hand, (4.1) is equivalent to $[V(0), V(\mu)] = 0$, because $[V(0), V(\mu)] \subset [\mathfrak{a}, \mathfrak{m}(\mu)] \subset \mathfrak{k}(\mu)$. It is easy to verify that $[V(0), V(\mu)] = 0$ holds if and only if $(V(0), \mu) = 0$. □

Let $\mathcal{P}_{cat}$ denote the family of all categorical pseudo-abelian subspaces of $\mathfrak{m}$. We denote by $p_{cat}(G/K)$ the maximum dimension of categorical pseudo-abelian subspaces of $\mathfrak{m}$, i.e.,

$$p_{cat}(G/K) = \max\{\dim V \mid V \in \mathcal{P}_{cat}\}.$$

The integer $p_{cat}(G/K)$ is called the *categorical pseudo-nullity* of G/K. In the following we will determine the pseudo-nullities $p_{cat}(G/K)$ for all compact, irreducible Riemannian symmetric spaces G/K with G simple.

Let $\mathcal{SOS}$ be the family of all SOS's in Σ. For each $\Gamma \in \mathcal{SOS}$ we define a quantity $b(\Gamma) \in \boldsymbol{Z}$ by

$$b(\Gamma) = \operatorname{rank}(G/K) - \#\Gamma + \sum_{\mu \in \Gamma} n(\mu).$$

Then we can prove

Proposition 4.2 $p_{cat}(G/K) = \max\{b(\Gamma) \mid \Gamma \in \mathcal{SOS}\}$.

Proof. Let V be an element of $\mathcal{P}_{cat}$ and Γ be the support of V. We note that since any distinct elements of Γ are mutually orthogonal, we have $\dim \boldsymbol{R}\Gamma = \#\Gamma$ (see [5]). Consequently, we have $\dim V(0) \leq \dim \mathfrak{a} - \#\Gamma$ and $\dim V(\mu) \leq n(\mu)$ for each $\mu \in \Gamma$ (see Proposition 4.1). Therefore, $\dim V \leq b(\Gamma)$ and hence $p_{cat}(G/K) \leq \max_\Gamma\{b(\Gamma)\}$.

Now we show the converse. Let $\Gamma \in \mathcal{SOS}$. Then, there is a categorical pseudo-abelian subspace V whose support coincides with Γ. In fact, define V by $V = \boldsymbol{R}\Gamma^\perp + \sum_{\mu\in\Gamma} U(\mu)$, where $\boldsymbol{R}\Gamma^\perp$ denotes the orthogonal complement of $\boldsymbol{R}\Gamma$ in $\mathfrak{a}$ and $U(\mu)$ a pseudo-abelian subspace of $\mathfrak{m}(\mu)$ such that $\dim U(\mu) = n(\mu)$. Then, it is easy to see that $V \in \mathcal{P}_{cat}$ and $\dim V = b(\Gamma)$. This proves $p_{cat}(G/K) \geq \max_\Gamma\{b(\Gamma)\}$. □

The following assertion is fundamental to calculate $p_{cat}(G/K)$.

Proposition 4.3 (1) *b is invariant under the action of the Weyl group of Σ, i.e., $b(w\Gamma) = b(\Gamma)$ holds for any $w \in W(\Sigma)$ and $\Gamma \in \mathcal{SOS}$.*
(2) *Let Γ, $\Gamma' \in \mathcal{SOS}$. Suppose that $\Gamma \subset \Gamma'$. Then $b(\Gamma) \leq b(\Gamma')$.*

Proof. It is obvious that $w\Gamma \in \mathcal{SOS}$, $\#(w\Gamma) = \#\Gamma$. Since $n(w\mu) = n(\mu)$ (see Proposition 3.1), we have $b(w\Gamma) = b(\Gamma)$. This proves (1).

By the definition, we easily have

$$b(\Gamma') - b(\Gamma) = -\#(\Gamma' \setminus \Gamma) + \sum_{\mu \in \Gamma'\setminus\Gamma} n(\mu) = \sum_{\mu \in \Gamma'\setminus\Gamma} (n(\mu) - 1).$$

Since every one-dimensional subspace of $\mathfrak{m}(\mu)$ is pseudo-abelian, we have $n(\mu) \geq 1$, and hence we get $b(\Gamma') \geq b(\Gamma)$. □

In view of (2) of Proposition 4.3 we know that in order to determine $p_{cat}(G/K)$ we have only to calculate $b(\Gamma)$ for maximal SOS's in Σ. In [5], for each irreducible root system Σ, we determined the equivalence classes of maximal SOS's in Σ under the action of the Weyl group $W(\Sigma)$ and obtained the representative maximal SOS for each equivalence class (see §§3–5 of [5]). By (1) of Proposition 4.3 we can also restrict our calculations to these representatives Γ.

We now recall the conclusions of [5] more closely. Let Γ be a maximal SOS in Σ. First assume that Σ is isomorphic to a reduced, irreducible root system X_n ($X = A \sim G$), where n implies the rank of G/K. Then, in the terminology in [5], Γ is equivalent to $\Gamma(X_n)^s$, where the superscript s indicates the cardinality of the set of short roots contained in Γ, i.e., $s = \#(\Gamma \cap \Sigma_2)$ (see §3 and §5 of [5]). According to the type of Σ, s takes a value in Table 1.

Table 1. Range of s.

Type of Σ	Range
$A_n\,(n \geq 1)$, $D_n\,(n \geq 4)$, $E_i\,(i = 6, 7, 8)$	$s = 0$
$B_n\,(n = 2m+1,\, m \geq 1)$, G_2	$s = 1$
$B_n\,(n = 2m,\, m \geq 1)$, F_4	$0 \leq s \leq 1$
$C_n\,(n \geq 3)$	$0 \leq s \leq [n/2]$

Next assume that Σ is not reduced, i.e., Σ is isomorphic to BC_n ($n = \mathrm{rank}(G/K)$). Then Γ is equivalent to $\Gamma(BC_n)^{r,s}$ ($0 \leq r \leq 1$, $0 \leq s \leq [(n-r)/2]$), where the superscript r implies the number of multipliable roots in Γ and s implies the number of roots in Γ which are not multipliable nor divisible (see §4 of [5]). In our terminology, we have $r = \#(\Gamma \cap \Sigma_2)$ and $s = \#(\Gamma \cap \Sigma_1)$.

These being prepared, we prove the main result of this paper:

Theorem 4.4 *Let Σ be the restricted root system of a compact irreducible Riemannian symmetric space G/K with G simple. Let $\mathcal{N}(G/K) = \{n_1, n_2, n_3\}$ be the local pseudo-nullities of G/K. Then:*
(1) *Assume that Σ is isomorphic to a reduced, irreducible root system X_n ($X = A \sim G$), where $n = \mathrm{rank}(G/K)$. Then:*

$$p_{cat}(G/K) = \mathrm{rank}(G/K) + \max_{s}\{(n_1 - 1)\#\Gamma(X_n)^s + (n_2 - n_1)s\},$$

where s runs through the range listed in Table 1.

(2) *Assume that Σ is isomorphic to BC_n ($n = \mathrm{rank}(G/K)$). Then:*

$$p_{cat}(G/K) = n_3 \,\mathrm{rank}(G/K) + \max_{(r,s)}\{(n_2 - n_3)r + (n_1 - 2n_3 + 1)s\},$$

where the pair (r, s) satisfies $0 \leq r \leq 1$ and $0 \leq s \leq [(n-r)/2]$.

Proof. Let Γ be a maximal SOS in Σ. We set $\Gamma_i = \Gamma \cap \Sigma_i$ ($i = 1, 2, 3$). Then by the definition of $b(\Gamma)$ we easily have

$$b(\Gamma) = \mathrm{rank}(G/K) - \#\Gamma + n_1\#\Gamma_1 + n_2\#\Gamma_2 + n_3\#\Gamma_3.$$

If Σ is isomorphic to X_n ($X = A \sim G$) and if Γ is equivalent to $\Gamma(X_n)^s$, then we have $\#\Gamma = \#\Gamma(X_n)^s$, $\#\Gamma_3 = 0$, $\#\Gamma_2 = s$ and $\#\Gamma_1 = \#\Gamma - s$. Putting these equalities into the above formula of $b(\Gamma)$, we obtain the assertion (1).

On the other hand, if Σ is isomorphic to BC_n and if Γ is equivalent to $\Gamma(BC_n)^{r,s}$, then we have $\#\Gamma = \#\Gamma(BC_n)^{r,s} = n - s$ (see Theorem 4.1 in [5]), $\#\Gamma_1 = s$, $\#\Gamma_2 = r$ and $\#\Gamma_3 = \#\Gamma - r - s$. Putting these equalities into the formula of $b(\Gamma)$, we have the assertion (2). □

The result of the calculations of $p_{cat}(G/K)$ is summarized in Table 4 and Table 5. Details are left to the reader.

The categorical pseudo-nullity $p_{cat}(G/K)$ gives a fairly good estimate of the pseudo-nullity $p(G/K)$. It is expected that the equality $p(G/K) = p_{cat}(G/K)$ holds for many compact irreducible Riemannian symmetric spaces G/K. Here we show the examples satisfying the above equality.

Example 1 (Case of the spheres S^p ($p \geq 2$)) In view of Table 4, we have $p_{cat}(S^p) = p - 1$ ($p \geq 2$) (see the types BII_p and DII_p). This proves that $p(S^p) = p_{cat}(S^p)$ ($p \geq 2$).

Example 2 (Case of G/K with $\mathrm{rank}(G/K) = \mathrm{rank}(G)$) For these spaces G/K we have proved $p(G/K) = \mathrm{rank}(G/K)$ (see Proposition 2.3 of [3]). Since, in our terminology, $\mathfrak{a}$ is a categorical pseudo-abelian subspace of $\mathfrak{m}$, we have $p_{cat}(G/K) \geq \dim \mathfrak{a} = \mathrm{rank}(G/K)$, proving $p_{cat}(G/K) = p(G/K)$.

In the next section we will prove that the equality $p(G/K) = p_{cat}(G/K)$ holds for all compact rank one Riemannian symmetric spaces G/K except $P^2(\boldsymbol{C})$.

In the rest of this section we consider the case where G/K is a compact simple Lie group. Let G be a compact connected simple Lie group and $\mathfrak{g}$ be the Lie algebra of G. As is well-known, G endowed with a bi-invariant metric can be represented by a compact, irreducible Riemannian symmetric space $G = \widetilde{G}/\widetilde{K}$, where $\widetilde{G} = G \times G$ and $\widetilde{K}$ denotes the diagonal subgroup of $G \times G$. In this and the previous sections we have developed our discussions for compact irreducible Riemannian symmetric spaces G/K with G simple. We note that these discussions are also valid for $\widetilde{G}/\widetilde{K}$. For example, the notions of restricted roots, multiplicities, (local) pseudo-nullities and strongly orthogonal subsets, etc. can also be well defined. In addition, the notion of categorical pseudo-nullities $p_{cat}(\widetilde{G}/\widetilde{K})$ can also be defined and Proposition 4.2 is true under this situation.

In [3] we have proved an inequality concerning $p(\widetilde{G}/\widetilde{K})$, which can be expressed as $p(\widetilde{G}/\widetilde{K}) \geq \operatorname{rank}(G) + s_0(G)$ in the terminology of this paper (see Proposition 6.3 of [3]), where $s_0(G)$ denotes the integer given by

$$s_0(G) = \begin{cases} [(n+1)/2], & \textit{if } \mathfrak{g} \cong \mathfrak{su}(n+1), \\ 2[n/2], & \textit{if } \mathfrak{g} \cong \mathfrak{o}(2n), \\ 4, & \textit{if } \mathfrak{g} \cong \mathfrak{e}_6, \\ \operatorname{rank}(G), & \textit{otherwise}. \end{cases}$$

Here, let us reconsider the above estimate $p(\widetilde{G}/\widetilde{K}) \geq \operatorname{rank}(G) + s_0(G)$ in the line of this paper. Then we can prove:

Proposition 4.5 *Let $G = \widetilde{G}/\widetilde{K}$ be a compact simple Lie group. Then*

$$p_{cat}(\widetilde{G}/\widetilde{K}) = \operatorname{rank}(G) + s_0(G).$$

Proof. It is well-known that the Lie algebra $\widetilde{\mathfrak{g}}$ (resp. $\widetilde{\mathfrak{k}}$) of $\widetilde{G}$ (resp. $\widetilde{K}$) is given by $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$ (resp. $\widetilde{\mathfrak{k}} = \{(X, X) \in \mathfrak{g} \oplus \mathfrak{g} \mid X \in \mathfrak{g}\}$). Putting $\widetilde{\mathfrak{m}} = \{(X, -X) \in \mathfrak{g} \oplus \mathfrak{g} \mid X \in \mathfrak{g}\}$, we get the canonical decomposition $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} \oplus \widetilde{\mathfrak{m}}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $\widetilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathfrak{t}$ defines a Cartan subalgebra of $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{a}} = \{(H, -H) \in \widetilde{\mathfrak{m}} \mid H \in \mathfrak{t}\}$ defines a maximal abelian subspace of $\widetilde{\mathfrak{m}}$.

Let Δ (resp. $\widetilde{\Delta}$) be the set of non-zero roots of $\mathfrak{g}^c$ (resp. $\widetilde{\mathfrak{g}}^c$) with respect to $\mathfrak{t}$ (resp. $\widetilde{\mathfrak{t}}$). As is known, $\widetilde{\Delta}$ is composed of roots written in the form $\alpha^+ = (\alpha, 0)$ or $\alpha^- = (0, -\alpha)$, where $\alpha \in \Delta$. Since $\alpha^+{}_{\widetilde{\mathfrak{a}}} = \alpha^-{}_{\widetilde{\mathfrak{a}}} = 1/2 \cdot (\alpha, -\alpha)$, the set of non-zero restricted roots associated with the Riemannian symmetric pair $(\widetilde{G}, \widetilde{K})$ can be written by $\widetilde{\Sigma} = \{1/2 \cdot (\alpha, -\alpha) \mid \alpha \in \Delta\}$.

By these facts we can verify the following:

(1) $m(\mu) = 2$ holds for each restricted root $\mu \in \widetilde{\Sigma}$.

(2) $\widetilde{\Sigma}$ does not contain any multipliable root. Accordingly, $n(\mu) = m(\mu) = 2$ holds for each $\mu \in \widetilde{\Sigma}$.
(3) For each SOS $\widetilde{\Gamma}$ in $\widetilde{\Sigma}$, there is a SOS Γ in Δ such that $\widetilde{\Gamma} = \{1/2 \cdot (\gamma, -\gamma) \mid \gamma \in \Gamma\}$.

By Proposition 4.2 and the above (1), (2) and (3) we have $p_{cat}(\widetilde{G}/\widetilde{K}) = \mathrm{rank}(\mathfrak{g}) + \max_{\Gamma}\{\#\Gamma\}$, where Γ runs over all SOS's in Δ. In [5], we have determined maximal SOS's in Δ for all irreducible root systems Δ. In view of Theorems 3.1 and 5.1 of [5], we easily get the equality $\max_{\Gamma}\{\#\Gamma\} = s_0(G)$, which proves our proposition. □

In [3] and [4], we have shown that $p(\widetilde{G}/\widetilde{K}) = \mathrm{rank}(G) + s_0(G)$ holds for the following compact simple Lie groups G:

$$SU(n)\,(2 \leq n \leq 5),\quad SO(n)\,(3 \leq n \leq 9,\ n \neq 4),\quad Sp(n)\,(n \geq 1),\quad G_2.$$

Proposition 4.5 indicates that $p(\widetilde{G}/\widetilde{K}) = p_{cat}(\widetilde{G}/\widetilde{K})$ holds for these compact simple Lie groups. We conjecture that the equality $p(\widetilde{G}/\widetilde{K}) = p_{cat}(\widetilde{G}/\widetilde{K})$ holds for all compact simple Lie groups $G = \widetilde{G}/\widetilde{K}$.

5. Compact rank one symmetric spaces

Let G/K be a compact rank one Riemannian symmetric space not isomorphic to any sphere S^n, i.e., G/K is one of the following Riemannian symmetric spaces:

(1) The complex projective spaces $P^n(\boldsymbol{C})$ $(n \geq 2)$.
(2) The quaternion projective spaces $P^n(\boldsymbol{H})$ $(n \geq 2)$.
(3) The Cayley projective plane $P^2(\boldsymbol{Cay})$.

The purpose of this section is to calculate the pseudo-nullities $p(G/K)$ for G/K listed above. We prove

Theorem 5.1 *Let G/K be a compact rank one Riemannian symmetric space not isomorphic to any sphere S^n. Then:*

$$p(G/K) = \begin{cases} p_{cat}(G/K) & \textit{if } G/K = P^n(\boldsymbol{C})\ (n \geq 3), \\ & \qquad P^n(\boldsymbol{H})\ (n \geq 2) \ \textit{or}\ P^2(\boldsymbol{Cay}), \\ 2 & \textit{if } G/K = P^2(\boldsymbol{C}). \end{cases}$$

Before proceeding to the proof, we exhibit several basic data on G/K. In view of Table 3, we know that the restricted root system Σ of G/K is isomorphic to BC_1. Now let us take and fix a multipliable root $\mu \in \Sigma$. Then we have the following decomposition:

$$\mathfrak{m} = \mathfrak{a} + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu) \ \ (\textit{orthogonal direct sum}), \quad \mathfrak{a} = \boldsymbol{R}\mu.$$

Further, from Tables 3, 4 and 5, we get

Table 2. Basic data for rank one symmetric spaces.

Type	G/K	$n(\mu)$	$n(2\mu)\,(= m(2\mu))$	$p_{cat}(G/K)$
$AIII_{n,1}$	$P^n(\boldsymbol{C})$ $(n \geq 2)$	$n-1$	1	$n-1$
$CII_{n,1}$	$P^n(\boldsymbol{H})$ $(n \geq 2)$	$n-1$	3	$\max\{3, n-1\}$
FII	$P^2(\boldsymbol{Cay})$	1	7	7

We now proceed to the proof of Theorem 5.1. To prove the theorem we have to estimate the dimensions of non-categorical pseudo-abelian subspaces. It can be shown that the dimension of any non-categorical pseudo-abelian subspace is fairly small. In fact, we have

Proposition 5.2 (1) *Let V be a non-categorical pseudo-abelian subspace of $\mathfrak{m}$. Then the inequality* $\dim V \leq 2$ *holds.*
(2) *If $G/K = P^n(\boldsymbol{C})\,(n \geq 2)$, there is a non-categorical pseudo-abelian subspace V satisfying* $\dim V = 2$.

As is easily seen, Theorem 5.1 immediately follows from this proposition and Table 2.

For the proof of Proposition 5.2 we prepare several lemmas.

Lemma 5.3 *Let $Y_1 \in \mathfrak{m}(\mu)$ and $Y_2 \in \mathfrak{m}(2\mu)$. Then:*
(1) $\big[[\mu, Y_1], [\mu, Y_2]\big] = 2(\mu,\mu)^2[Y_1, Y_2]$.
(2) $\big[[\mu, Y_2], Y_1\big] = 2\big[[\mu, Y_1], Y_2\big]$.

Proof. We first note that $[Y_1, Y_2] \in \mathfrak{k}(\mu)$ (see Proposition 2.1). Consequently, we have

$$(\operatorname{ad}\mu)^2[Y_1, Y_2] = -(\mu,\mu)^2[Y_1, Y_2].$$

Since $(\operatorname{ad}\mu)^2 Y_1 = -(\mu,\mu)^2 Y_1$, $(\operatorname{ad}\mu)^2 Y_2 = -4(\mu,\mu)^2 Y_2$, we have

$$\begin{aligned}(\operatorname{ad}\mu)^2[Y_1,Y_2] &= [(\operatorname{ad}\mu)^2Y_1,Y_2] + 2[[\mu,Y_1],[\mu,Y_2]] + [Y_1,(\operatorname{ad}\mu)^2Y_2] \\ &= 2[[\mu,Y_1],[\mu,Y_2]] - 5\,(\mu,\mu)^2[Y_1,Y_2].\end{aligned}$$

Hence we immediately get the equality (1).

The equality (2) can be easily obtained by applying $\operatorname{ad}\mu$ to the equality (1). □

Lemma 5.4 *Let V be an arbitrary pseudo-abelian subspace of* $\mathfrak{m}$.

(1) *If* $V \not\subset \mathfrak{m}(\mu)$, *then* $\dim V \leq 1 + m(2\mu)$.

(2) *If* $V \not\subset \mathfrak{m}(2\mu)$, *then* $\dim V \leq 1 + n(\mu)$.

Proof. First we show the assertion (1). Since $V \not\subset \mathfrak{m}(\mu)$, V contains an element $Y = H + Y_1 + Y_2\,(H \in \mathfrak{a},\, Y_1 \in \mathfrak{m}(\mu),\, Y_2 \in \mathfrak{m}(2\mu))$ such that $H + Y_2 \neq 0$. Let Y' be an arbitrary element of $V \cap \mathfrak{m}(\mu)$. Then we have

$$[Y,Y'] = [H+Y_2,Y'] + [Y_1,Y'] \in \mathfrak{k}_0.$$

Since $[H+Y_2,Y'] \in \mathfrak{k}(\mu)$ and $[Y_1,Y'] \in \mathfrak{k}_0 + \mathfrak{k}(2\mu)$ (see Proposition 2.1), we have $[H+Y_2,Y'] = 0$. This implies that the subspace spanned by $H+Y_2$ and Y' is an abelian subspace of $\mathfrak{m}$. Since $\operatorname{rank}(G/K) = 1$, it follows that Y' must be a scalar multiple of $H+Y_2$. This proves $Y' = 0$, because $H + Y_2 \in \mathfrak{a} + \mathfrak{m}(2\mu)$, $Y' \in \mathfrak{m}(\mu)$. Hence we have $V \cap \mathfrak{m}(\mu) = 0$. Consequently, we have $\dim V \leq \dim(\mathfrak{a} + \mathfrak{m}(2\mu)) = 1 + m(2\mu)$.

Next we show the assertion (2). As in the proof of (1), we can prove that $V \cap \mathfrak{m}(2\mu) = 0$. Set $r = \dim V$. Since $\dim(V \cap (\mathfrak{m}(\mu) + \mathfrak{m}(2\mu))) \geq r-1$, we get elements $Y^i = Y_1^i + Y_2^i \in V$ $(1 \leq i \leq r-1)$ such that $Y_1^i \in \mathfrak{m}(\mu)$ and $Y_2^i \in \mathfrak{m}(2\mu)$. Moreover, since $V \cap \mathfrak{m}(2\mu) = 0$, we may assume that the vectors $\{Y_1^i\,(1 \leq i \leq r-1)\}$ are linearly independent. Now, since $[Y^i, Y^j] \in \mathfrak{k}_0$, we have

$$[Y_1^i + Y_2^i, Y_1^j + Y_2^j] \in \mathfrak{k}_0.$$

On the other hand, since $[Y_2^i, Y_2^j] \in \mathfrak{k}_0$, $[Y_1^i, Y_1^j] \in \mathfrak{k}_0 + \mathfrak{k}(2\mu)$ and $[Y_1^i, Y_2^j] + [Y_2^i, Y_1^j] \in \mathfrak{k}(\mu)$, we have

$$[Y_1^i, Y_1^j] \in \mathfrak{k}_0, \quad 1 \leq i, j \leq r-1.$$

This implies that the subspace spanned by $\{Y_1^i\,(1 \leq i \leq r-1)\}$ is a local pseudo-abelian subspace in $\mathfrak{m}(\mu)$. This proves that $r - 1 \leq n(\mu)$, completing the proof of the assertion (2). □

Lemma 5.5 *Let V be a non-categorical pseudo-abelian subspace of $\mathfrak{m}$. Then*:

$$\dim V \leq \min\{1+m(2\mu), 1+n(\mu)\}.$$

Proof. Since V is not categorical, it follows that $V \not\subset \mathfrak{m}(\mu)$ and $V \not\subset \mathfrak{m}(2\mu)$. Therefore, by the above lemma we have $\dim V \leq 1+m(2\mu)$ and $\dim V \leq 1+n(\mu)$. □

Proof of Proposition 5.2. By Lemma 5.5, we can prove (1) of Proposition 5.2 for the spaces $P^n(\boldsymbol{C})$ $(n \geq 2)$, $P^2(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$. In fact, we have $m(2\mu) = 1$ if $G/K = P^n(\boldsymbol{C})$ $(n \geq 2)$ and $n(\mu) = 1$ if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$.

Next, we directly show (1) for the remaining spaces $G/K = P^n(\boldsymbol{H})$ $(n \geq 3)$. Suppose that there is a non-categorical pseudo-abelian subspace V with $\dim V \geq 3$. As in the proof of Lemma 5.4, we may assume that there are two elements $Y = Y_1+Y_2$, $Y' = Y_1'+Y_2' \in V$ $(Y_1, Y_1' \in \mathfrak{m}(\mu), Y_2, Y_2' \in \mathfrak{m}(2\mu))$ such that Y_1 and Y_1' are linearly independent and the subspace $\{Y_1, Y_1'\}$ is pseudo-abelian. Further, since $[Y_2, Y_2'] \in \mathfrak{k}_0+\mathfrak{k}(2\mu)$ and $[Y_2, Y_1'] + [Y_1, Y_2'] \in \mathfrak{k}(\mu)$, we have

$$[Y_2, Y_1'] = [Y_2', Y_1]. \tag{5.1}$$

By Lemma 3.4 we know that the condition $[Y_1, Y_1'] \in \mathfrak{k}_0$ is equivalent to

$$(Y_1', I^\dagger Y_1) = (Y_1', J^\dagger Y_1) = (Y_1', K^\dagger Y_1) = 0. \tag{5.2}$$

(Note that $\mathfrak{k}(2\mu)$ is spanned by I, J and K.) Applying $\operatorname{ad}\mu$ to the equality (5.1), we have

$$[[\mu, Y_2], Y_1'] + [Y_2, [\mu, Y_1']] = [[\mu, Y_2'], Y_1] + [Y_2', [\mu, Y_1]].$$

Using (2) of Lemma 5.3, we have

$$X_2^\dagger(Y_1') = X_2'^\dagger(Y_1), \tag{5.3}$$

where we set $X_2 = [\mu, Y_2]$, $X_2' = [\mu, Y_2']$. Applying $X_2^\dagger$ to the both sides of (5.3), we have

$$(X_2^\dagger)^2(Y_1') = X_2^\dagger X_2'^\dagger(Y_1). \tag{5.4}$$

Since $X_2^\dagger$ and $X_2'^\dagger$ are linear combinations of $I^\dagger$, $J^\dagger$ and $K^\dagger$, it follows that $(X_2^\dagger)^2 = c\ \mathbf{1}_{\mathfrak{m}(\mu)}$ $(c \in \boldsymbol{R},\ c \neq 0)$ and $X_2^\dagger X_2'^\dagger$ is written as a linear combination of $\mathbf{1}_{\mathfrak{m}(\mu)}$, $I^\dagger$, $J^\dagger$ and $K^\dagger$. Consequently, by (5.4) we know that Y_1' can be written as a linear combination of Y_1, $I^\dagger(Y_1)$, $J^\dagger(Y_1)$ and $K^\dagger(Y_1)$. This together with (5.2), we can conclude that Y_1' is written by a scalar multiple of Y_1. This contradicts the assumption that Y_1 and Y_1' are linearly independent. Therefore, we have $\dim V \leq 2$.

Finally, we prove (2) of Proposition 5.2. Assume that $G/K = P^n(\boldsymbol{C})$ $(n \geq 2)$. Take a non-zero element $Y \in \mathfrak{m}(\mu)$ satisfying $(Y, Y) = 2(\mu, \mu)^2(I, I)$ and consider a subspace $V \subset \mathfrak{m}$ spanned by $\mu + Y$ and $[I, \mu - 2Y]$. Then it is easily shown that $\dim V = 2$, because $(0 \neq)[I, \mu - 2Y] \in \mathfrak{m}(\mu) + \mathfrak{m}(2\mu)$ but $\mu + Y \notin \mathfrak{m}(\mu) + \mathfrak{m}(2\mu)$. Let us show V is pseudo-abelian. To show this we have to prove

$$[\mu + Y, [I, \mu - 2Y]] \in \mathfrak{k}_0.$$

By (1) of Lemma 5.3 and $(\operatorname{ad}\mu)^2 I = -4(\mu, \mu)^2 I$ we have

$$\begin{aligned}
[I, [\mu, Y]] &= \left[-\frac{1}{4(\mu, \mu)^2}[\mu, [\mu, I]], [\mu, Y]\right] \\
&= -\frac{2(\mu, \mu)^2}{4(\mu, \mu)^2}[[\mu, I], Y] \\
&= -\frac{1}{2}[[\mu, I], Y].
\end{aligned}$$

Consequently, we have

$$[\mu, [I, Y]] = [[\mu, I], Y] + [I, [\mu, Y]] = \frac{1}{2}[[\mu, I], Y].$$

Therefore, by a simple calculation we have

$$[\mu + Y, [I, \mu - 2Y]] = 2\left\{2(\mu, \mu)^2 I - [Y, [I, Y]]\right\}.$$

We note that the right hand side of the above equality is contained in $\mathfrak{k}_0 + \mathfrak{k}(2\mu)$. Since $\mathfrak{k}(2\mu) = \boldsymbol{R}I$ and since

$$\begin{aligned}
(I, 2(\mu, \mu)^2 I - [Y, [I, Y]]) &= 2(\mu, \mu)^2(I, I) - ([I, Y], [I, Y]) \\
&= 2(\mu, \mu)^2(I, I) - (Y, Y) = 0,
\end{aligned}$$

we have $[\mu + Y, [I, \mu - 2Y]] \in \mathfrak{k}_0$. Therefore, we get (2) of Proposition 5.2. □

By Theorem 5.1 we obtain the non-existence theorem:

Theorem 5.6 *Let G/K be a compact rank one Riemannian symmetric space not isomorphic to any sphere S^n. Define an integer $q(G/K)$ by*

$$q(G/K) = \begin{cases} \min\{4n-2, 3n+1\}, & \text{if } G/K = P^n(\boldsymbol{C}) \ (n \geq 2), \\ \min\{8n-3, 7n+1\}, & \text{if } G/K = P^n(\boldsymbol{H}) \ (n \geq 2), \\ 25, & \text{if } G/K = P^2(\boldsymbol{Cay}). \end{cases}$$

Then, any open set of G/K cannot be isometrically imbedded into the Euclidean space $\boldsymbol{R}^N$ with $N \leq q(G/K) - 1$.

Finally, we refer to the result of Agaoka [1] concerning the non-existence of isometric imbeddings of $P^n(\boldsymbol{C})$. He investigated directly the solvability of the Gauss equation associated with isometric imbeddings of $P^n(\boldsymbol{C})$, and obtained the following

Proposition 5.7 ([1]) *Any open set of the complex projective space $P^n(\boldsymbol{C})$ cannot be isometrically imbedded into the Euclidean space $\boldsymbol{R}^N$ with $N \leq [16n/5] - 1$.*

As is easily seen, Agaoka's result is stronger than ours in case n is large enough ($n \geq 10$). It is noted that in such a case the least dimension of Euclidean spaces into which $P^n(\boldsymbol{C})$ is (locally) isometrically imbedded cannot be determined only by $p(P^n(\boldsymbol{C}))$. This is an interesting phenomenon compared with the spaces $Sp(n)/U(n)$ and $Sp(n)$, where the least dimensions are just determined by $p(G/K)$ (see [3] and [4]).

For the spaces $P^2(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$, we can get stronger results than Theorem 5.6, which will be shown in the forthcoming papers [7] and [8].

6. Proof of Theorem 3.5

In this section we prove Theorem 3.5. Before starting the proof, we prepare some lemmas. We follow the notations used in Introduction and §2.

Let G/K be a compact irreducible Riemannian symmetric space with G simple. Let τ be the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$. As is known (see [2]), there is a set of vectors $\{Z_\alpha \in \mathfrak{g}_\alpha \mid \alpha \in \Delta\}$ of $\mathfrak{g}^c$ satisfying

(1) $\theta Z_\alpha = Z_{\theta\alpha}, \quad \tau Z_\alpha = Z_{-\alpha}$,

(2) $[Z_\alpha, Z_{-\alpha}] = 2\sqrt{-1}\,\alpha/(\alpha, \alpha)$.

Let α, $\beta \in \Delta$. We define an integer $A_{\alpha,\beta}$ by $A_{\alpha,\beta} = 2(\alpha, \beta)/(\beta, \beta)$. The following formula, which is a well-known fact in the theory of Lie algebras (see [12]):

Lemma 6.1 *Assume that* $\alpha + \beta \notin \Delta \cup \{0\}$. *Then*:

$$\mathrm{ad}\, Z_\beta (\mathrm{ad}\, Z_{-\beta})^k (Z_\alpha) = k\,(-A_{\alpha,\beta} + k - 1)\,(\mathrm{ad}\, Z_{-\beta})^{k-1}(Z_\alpha), \qquad k \in \boldsymbol{Z},\ k \geq 0.$$

Let us set $\Delta_0 = \Delta \cap \mathfrak{b}$. For a root $\alpha \in \Delta \backslash \Delta_0$, we define a subspace $\mathfrak{g}(\alpha)$ of $\mathfrak{g}^c$ by

$$\mathfrak{g}(\alpha) = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + \mathfrak{g}_{\theta\alpha} + \mathfrak{g}_{-\theta\alpha}.$$

As is easily seen, $\mathfrak{g}(\alpha)$ satisfies the following properties:

Lemma 6.2 *Let* $\alpha \in \Delta \backslash \Delta_0$. *Then*:
(1) $\mathfrak{g}(\alpha) = \mathfrak{g}(-\alpha) = \mathfrak{g}(\theta\alpha) = \mathfrak{g}(-\theta\alpha)$.
(2) $\dim \mathfrak{g}(\alpha) = 4$ *if* $\theta\alpha \neq -\alpha$; $\dim \mathfrak{g}(\alpha) = 2$ *if* $\theta\alpha = -\alpha$.
(3) *Let* $\beta \in \Delta \backslash \Delta_0$ *satisfy* $\beta \neq \pm\alpha$, $\pm\theta\alpha$. *Then* $\mathfrak{g}(\beta)$ *is orthogonal to* $\mathfrak{g}(\alpha)$ *with respect to the inner product* $(\ ,\ )$, *i.e.*, $(\mathfrak{g}(\alpha), \mathfrak{g}(\beta)) = 0$.

We also have the following lemma whose proof is left to the reader.

Lemma 6.3 *Let* Σ *be the restricted root system of* G/K *and* $\mu \in \Sigma$. *Then*:
(1) *Let* $\alpha \in \Delta(\mu)$. *Then* $-\theta\alpha \in \Delta(\mu)$.
(2) *The following decomposition holds*:

$$\mathfrak{k}(\mu)^c + \mathfrak{m}(\mu)^c = \sum_{\alpha \in \Delta(\mu),\ -\theta\alpha \leq \alpha} \mathfrak{g}(\alpha) \quad (\textit{orthogonal direct sum}).$$

(3) *Let* $\alpha \in \Delta(\mu)$. *Define vectors* $X(\alpha)^\pm$ *and* $Y(\alpha)^\pm$ *of* $\mathfrak{g}(\alpha)$ *by*

$$\begin{aligned} X(\alpha)^+ &= Z_\alpha + Z_{-\alpha} + Z_{\theta\alpha} + Z_{-\theta\alpha}, \\ X(\alpha)^- &= \sqrt{-1}\,(Z_\alpha - Z_{-\alpha} + Z_{\theta\alpha} - Z_{-\theta\alpha}), \\ Y(\alpha)^+ &= \sqrt{-1}\,(Z_\alpha - Z_{-\alpha} - Z_{\theta\alpha} + Z_{-\theta\alpha}), \\ Y(\alpha)^- &= Z_\alpha + Z_{-\alpha} - Z_{\theta\alpha} - Z_{-\theta\alpha}. \end{aligned}$$

Then, it holds that $X(\alpha)^\pm \in \mathfrak{k}(\mu)$, $Y(\alpha)^\pm \in \mathfrak{m}(\mu)$ *and* $X(-\theta\alpha)^\pm = \pm X(\alpha)^\pm$, $Y(-\theta\alpha)^\pm = \pm Y(\alpha)^\pm$.
(4) *The set of vectors* $\{X(\alpha)^\pm \mid \alpha \in \Delta(\mu)\}$ (*resp.* $\{Y(\alpha)^\pm \mid \alpha \in \Delta(\mu)\}$) *spans* $\mathfrak{k}(\mu)$ (*resp.* $\mathfrak{m}(\mu)$).

These being prepared, we start the proof of Theorem 3.5. In the following we assume that Σ is of type BC_n and $\mu \in \Sigma$ is a multipliable root, i.e., $2\mu \in \Sigma$. Under this assumption, we have $m(\mu) = \mathit{even}$, $m(2\mu) = \mathit{odd}$ and $\mu \notin \Delta$, $2\mu \in \Delta$ (see [9] or Table 3).

We first prove

Proposition 6.4 (1) *Let* $\alpha \in \Delta(\mu)$. *Then* $A_{\alpha,2\mu} = 1$ *and* $\alpha - 2\mu \in \Delta$, *but* $\alpha + 2\mu \notin \Delta \cup \{0\}$.
(2) *Set* $I = Z_{2\mu} + Z_{-2\mu}$. *Then* $I \in \mathfrak{k}(2\mu)$ *and* $I^\dagger$ *determines a complex structure of* $\mathfrak{m}(\mu)$, *i.e.*, ${I^\dagger}^2 = -\mathbf{1}_{\mathfrak{m}(\mu)}$.

Proof. It is clear that $I \in \mathfrak{k}(2\mu)$. Now let $\alpha \in \Delta(\mu)$. We consider the 2μ-series of roots containing α. Since $A_{\alpha,2\mu} = 2(\alpha, 2\mu)/(2\mu, 2\mu) = 1$, it follows that $\alpha - 2\mu \in \Delta$. On the contrary, since the $\mathfrak{a}$-component of $\alpha + 2\mu$ is equal to 3μ, it follows that $\alpha + 2\mu \notin \Delta \cup \{0\}$. Therefore, by Lemma 6.1 we have

$$\begin{aligned}(\operatorname{ad} I)^2(Z_\alpha) &= \big[Z_{2\mu}, \big[Z_{-2\mu}, Z_\alpha\big]\big] \\ &= -Z_\alpha.\end{aligned}$$

Moreover, since $\operatorname{ad} I \cdot \theta = \theta \cdot \operatorname{ad} I$ and $\operatorname{ad} I \cdot \tau = \tau \cdot \operatorname{ad} I$, we have $(\operatorname{ad} I)^2(Z_{\alpha'}) = -Z_{\alpha'}$, where $\alpha' = \pm\alpha$ or $\pm\theta\alpha$. Since the vectors $Y(\alpha)^\pm$ $(\alpha \in \Delta(\mu))$ generate $\mathfrak{m}(\mu)$, we have ${I^\dagger}^2 = -\mathbf{1}_{\mathfrak{m}(\mu)}$. □

The above lemma shows the assertion (1) of Theorem 3.5. In what follows, we may assume that $m(2\mu) = 3$. We first consider the sets $\Delta(\mu)$ and $\Delta(2\mu)$.

Lemma 6.5 (1) *There is a root* $\nu \in \Delta_0$ *such that* $\Delta(2\mu) = \{2\mu, 2\mu \pm \nu\}$ *and* $(\nu, \nu) = 4(\mu, \mu)$.
(2) *Let* $\alpha \in \Delta(\mu)$. *Then* $(\alpha, \alpha) = 4(\mu, \mu)$.
(3) *Let* α, $\alpha' \in \Delta(\mu)$. *Assume that* $\alpha' \neq \alpha$, $-\theta\alpha$. *Then, one of the following* (a) *and* (b) *holds.*

(a) $A_{\alpha',\alpha} = 1$, $A_{\alpha',-\theta\alpha} = 0$.
(b) $A_{\alpha',\alpha} = 0$, $A_{\alpha',-\theta\alpha} = 1$.

Proof. In Appendix of [2], we have proved that for a restricted root $\psi \in \Sigma$ satisfying $m(\psi) = \mathit{odd}$ and $m(\psi) > 1$, there is a root $\nu \in \Delta_0$ such that $\psi \pm \nu \in \Delta$. Applying this to the case $\psi = 2\mu$, we have the first part of the assertion (1). Since $(2\mu + \nu) \pm (2\mu - \nu) \notin \Delta \cup \{0\}$, we have $(2\mu + \nu, 2\mu - \nu) = 0$. This shows that $(\nu, \nu) = 4(\mu, \mu)$.

We now prove the assertion (2). Consider the α-series of roots containing 2μ. By (1) of Proposition 6.4, we have $2\mu+\alpha \notin \Delta \cup \{0\}$ and $2\mu-\alpha \in \Delta$. On the other hand, by the fundamental property of symmetric spaces we have $2\mu - 2\alpha = -(\alpha + \theta\alpha) \notin \Delta \cup \{0\}$. Therefore we have $A_{2\mu,\alpha} = 1$. Since $A_{\alpha,2\mu} = 1$ (see (1) of Proposition 6.4) we have $(\alpha, \alpha) = (2\mu, 2\mu)$.

Finally, we prove (3). Since $\alpha - \theta\alpha = 2\mu$ and since $(\alpha, \alpha) = (\theta\alpha, \theta\alpha) = (2\mu, 2\mu)$, we have

$$\begin{aligned} A_{\alpha',\alpha} + A_{\alpha',-\theta\alpha} &= \frac{2(\alpha', \alpha)}{(\alpha, \alpha)} + \frac{2(\alpha', -\theta\alpha)}{(\theta\alpha, \theta\alpha)} = \frac{2(\alpha', \alpha - \theta\alpha)}{(2\mu, 2\mu)} \\ &= A_{\alpha',2\mu} = 1. \end{aligned}$$

We also have $|A_{\alpha',\alpha}| \leq 1$ and $|A_{\alpha',-\theta\alpha}| \leq 1$, because $\alpha' \neq \pm\alpha$, $\pm\theta\alpha$, $(\alpha', \alpha') = (\alpha, \alpha)$. Then the assertion (3) immediately follows from these facts. □

In the following discussion we fix an element $\nu \in \Delta_0$ stated in (1) of Lemma 6.5.

Lemma 6.6 *Let* $\alpha \in \Delta(\mu)$. *Then*:

(1) $A_{\alpha,\nu} = \pm 1$. *Moreover,*

  (a) $A_{\alpha,\nu} = 1 \Longleftrightarrow \alpha - \nu \in \Delta$.

  (b) $A_{\alpha,\nu} = -1 \Longleftrightarrow \alpha + \nu \in \Delta$.

(2) $\alpha \pm 2\nu \notin \Delta \cup \{0\}$.

Proof. Since $\alpha - \theta\alpha = 2\mu \in \Delta$ and $2\mu + \nu \in \Delta$, it follows that $[Z_\nu, [Z_\alpha, Z_{-\theta\alpha}]] \neq 0$. Hence, we have either $[Z_\nu, Z_\alpha] \neq 0$ or $[Z_\nu, Z_{-\theta\alpha}] \neq 0$. Therefore, we have either $\alpha + \nu \in \Delta$ or $-\theta\alpha + \nu \in \Delta$.

Now assume that $\alpha + \nu \in \Delta$. Then we have $\alpha + \nu \in \Delta(\mu)$ and hence by Lemma 6.5 we have $(\alpha + \nu, \alpha + \nu) = 4(\mu, \mu)$. Since $(\alpha, \alpha) = 4(\mu, \mu)$ and $(\nu, \nu) = 4(\mu, \mu)$, we have $(\alpha, \nu) = -2(\mu, \mu)$. This implies $A_{\alpha,\nu} = -1$. Conversely, if $A_{\alpha,\nu} = -1$, then we have $\alpha+\nu \in \Delta$. This proves the assertion (b).

Next assume that $-\theta\alpha + \nu \in \Delta$. Then, since $-\theta\alpha + \nu = -\theta(\alpha - \nu)$, we have $\alpha - \nu \in \Delta$. In this case, by the same method stated above, we have $(\alpha, \nu) = 2(\mu, \mu)$ and hence $A_{\alpha,\nu} = 1$. Conversely, if $A_{\alpha,\nu} = 1$, then we have $\alpha - \nu \in \Delta$, which proves the assertion (a).

Finally, we show (2). In view of (1), we know that the length of ν-series containing α is just equal to 2. Hence we have $\alpha \pm 2\nu \notin \Delta \cup \{0\}$. □

Now we define an action of $\mathfrak{k}_0$ on $\mathfrak{m}(\mu)$. Since $[\mathfrak{k}_0, \mathfrak{m}(\mu)] \subset \mathfrak{m}(\mu)$ (see Proposition 2.1), each element $\mathrm{ad}\, X\, (X \in \mathfrak{k}_0)$ induces a skew-symmetric endomorphism of $\mathfrak{m}(\mu)$, which is also denoted by $X^\dagger$. This together with the action of $\mathfrak{k}(2\mu)$ defined in §3, we get the action of $\mathfrak{k}_0 + \mathfrak{k}(2\mu)$ on $\mathfrak{m}(\mu)$. By the definition we directly have

$$[X, X']^\dagger = X^\dagger X'^\dagger - X'^\dagger X^\dagger = [X^\dagger, X'^\dagger], \quad X, X' \in \mathfrak{k}_0 + \mathfrak{k}(2\mu).$$

Set $\widehat{\nu} = 2\nu/(\nu,\nu)$, $P = Z_\nu + Z_{-\nu}$ and $Q = \sqrt{-1}(Z_\nu - Z_{-\nu})$. Then we easily have $\widehat{\nu}$, P, $Q \in \mathfrak{k}_0$ and

$$[\widehat{\nu}, P] = 2Q, \quad [\widehat{\nu}, Q] = -2P, \quad [P, Q] = 2\widehat{\nu}.$$

We now prove

Proposition 6.7 *The triplet $\{\widehat{\nu}^\dagger, P^\dagger, Q^\dagger\}$ determines a quaternion structure of $\mathfrak{m}(\mu)$, i.e.,*

$$(\widehat{\nu}^\dagger)^2 = {P^\dagger}^2 = {Q^\dagger}^2 = -\mathbf{1}_{\mathfrak{m}(\mu)}, \quad \widehat{\nu}^\dagger P^\dagger = -P^\dagger \widehat{\nu}^\dagger = Q^\dagger,$$
$$Q^\dagger \widehat{\nu}^\dagger = -\widehat{\nu}^\dagger Q^\dagger = P^\dagger, \quad P^\dagger Q^\dagger = -Q^\dagger P^\dagger = \widehat{\nu}^\dagger.$$

For the proof, we prepare the following

Lemma 6.8 *Let $\alpha \in \Delta(\mu)$. Then:*
(1) $[Z_\nu, [Z_{-\nu}, Z_\alpha]] + [Z_{-\nu}, [Z_\nu, Z_\alpha]] = -Z_\alpha$.
(2) $[\widehat{\nu}, [Z_{\pm\nu}, Z_\alpha]] + [Z_{\pm\nu}, [\widehat{\nu}, Z_\alpha]] = 0$.

Proof. Assume that $\alpha - \nu \in \Delta$. Then we have $A_{\alpha,\nu} = 1$ and $\alpha + \nu \notin \Delta \cup \{0\}$ (see Lemma 6.6). By Lemma 6.1 we have

$$[Z_\nu, [Z_{-\nu}, Z_\alpha]] = -A_{\alpha,\nu} Z_\alpha = -Z_\alpha,$$
$$[\widehat{\nu}, [Z_{-\nu}, Z_\alpha]] = \sqrt{-1} A_{\alpha-\nu,\nu} [Z_{-\nu}, Z_\alpha] = -\sqrt{-1}[Z_{-\nu}, Z_\alpha],$$
$$[Z_{-\nu}, [\widehat{\nu}, Z_\alpha]] = \sqrt{-1} A_{\alpha,\nu} [Z_{-\nu}, Z_\alpha] = \sqrt{-1}[Z_{-\nu}, Z_\alpha].$$

By these equalities and $[Z_\nu, Z_\alpha] = 0$, we get the assertions (1) and (2). Similarly, in the case $\alpha + \nu \in \Delta$ we can prove (1) and (2). □

We now prove Proposition 6.7. We first note that since $\theta\nu = \tau\nu = \nu$, the endomorphism $\mathrm{ad}\, \widehat{\nu}$ commutes with θ and τ. Similarly, since $\theta P = \tau P = P$ and $\theta Q = \tau Q = Q$, we know that $\mathrm{ad}\, P$ and $\mathrm{ad}\, Q$ commute with θ and τ.

Let $\alpha \in \Delta(\mu)$. By Lemmas 6.8, 6.6 and a direct calculation, we have

$$\begin{aligned}(\mathrm{ad}\,\widehat{\nu})^2(Z_\alpha) &= \big[\widehat{\nu}, \sqrt{-1}A_{\alpha,\nu}Z_\alpha\big] = -A_{\alpha,\nu}^2 Z_\alpha = -Z_\alpha,\\ (\mathrm{ad}\,P)^2(Z_\alpha) &= (\mathrm{ad}\,Q)^2(Z_\alpha) = \big[Z_\nu, \big[Z_{-\nu}, Z_\alpha\big]\big]\\ &\qquad + \big[Z_{-\nu}, \big[Z_\nu, Z_\alpha\big]\big] = -Z_\alpha.\end{aligned}$$

Therefore, by the same reason stated in the proof of (2) of Proposition 6.4 we can conclude $(\widehat{\nu}^\dagger)^2 = {P^\dagger}^2 = {Q^\dagger}^2 = -\mathbf{1}_{\mathfrak{m}(\mu)}$. Further, by Lemmas 6.8, 6.6 and by a direct calculation, we can prove $\big[P, \big[Q, Z_\alpha\big]\big] = -\big[Q, \big[P, Z_\alpha\big]\big]$, $\big[\widehat{\nu}, \big[P, Z_\alpha\big]\big] = -\big[P, \big[\widehat{\nu}, Z_\alpha\big]\big]$ and $\big[\widehat{\nu}, \big[Q, Z_\alpha\big]\big] = -\big[Q, \big[\widehat{\nu}, Z_\alpha\big]\big]$. By the same reason as above, we have $P^\dagger Q^\dagger = -Q^\dagger P^\dagger$, $\widehat{\nu}^\dagger P^\dagger = -P^\dagger \widehat{\nu}^\dagger$ and $\widehat{\nu}^\dagger Q^\dagger = -Q^\dagger \widehat{\nu}^\dagger$. From these equalities, it follows

$$\begin{aligned}\widehat{\nu}^\dagger P^\dagger &= (1/2)(\widehat{\nu}^\dagger P^\dagger - P^\dagger \widehat{\nu}^\dagger) = (1/2)\big[\widehat{\nu}, P\big]^\dagger = Q^\dagger,\\ Q^\dagger \widehat{\nu}^\dagger &= (1/2)(Q^\dagger \widehat{\nu}^\dagger - \widehat{\nu}^\dagger Q^\dagger) = (1/2)\big[Q, \widehat{\nu}\big]^\dagger = P^\dagger,\\ P^\dagger Q^\dagger &= (1/2)(P^\dagger Q^\dagger - Q^\dagger P^\dagger) = (1/2)\big[P, Q\big]^\dagger = \widehat{\nu}^\dagger.\end{aligned}$$

This completes the proof of the proposition. □

Finally, we prove

Proposition 6.9 *$I^\dagger = \varepsilon\,\widehat{\nu}^\dagger$, where $\varepsilon \in \boldsymbol{R}$ and $\varepsilon^2 = 1$.*

If the above proposition is true, we can get Theorem 3.5. In fact, set $J = -(1/2)\big[I, Q\big]$, $K = (1/2)\big[I, P\big]$. Then we have $J,\ K \in \mathfrak{k}(2\mu)$ and

$$\begin{aligned}J^\dagger &= -(1/2)\big[I^\dagger, Q^\dagger\big] = -(\varepsilon/2)\big[\widehat{\nu}, Q\big]^\dagger = \varepsilon\,P^\dagger,\\ K^\dagger &= (1/2)\big[I^\dagger, P^\dagger\big] = (\varepsilon/2)\big[\widehat{\nu}, P\big]^\dagger = \varepsilon\,Q^\dagger.\end{aligned}$$

Consequently, by Proposition 6.7 it is shown that the triplet $\{\varepsilon I^\dagger, \varepsilon J^\dagger, \varepsilon K^\dagger\}$ ($\subset \mathfrak{k}(2\mu)^\dagger$) determines a quaternion structure of $\mathfrak{m}(\mu)$.

Now we show Proposition 6.9. For each $\alpha \in \Delta(\mu)$ let us define a complex number ρ_α by

$$\big[Z_{-2\mu}, Z_\alpha\big] = \sqrt{-1}\rho_\alpha Z_{\theta\alpha}. \tag{6.1}$$

ρ_α is well-defined, because $-2\mu + \alpha = \theta\alpha \in \Delta$ and hence $\big[Z_{-2\mu}, Z_\alpha\big] \in \mathfrak{g}_{\theta\alpha}$.

Lemma 6.10 (1) *${\rho_\alpha}^2 = 1$, $\rho_{-\theta\alpha} = -\rho_\alpha$.*
(2) *$\big[I, Y(\alpha)^\pm\big] = \pm\rho_\alpha Y(\alpha)^\mp$ holds for each $\alpha \in \Delta(\mu)$.*

Proof. From (6.1), we easily get

$$[I, Z_\alpha] = \sqrt{-1}\rho_\alpha Z_{\theta\alpha}. \tag{6.2}$$

Applying θ to the both sides of (6.2), we have $[I, Z_{\theta\alpha}] = \sqrt{-1}\rho_\alpha Z_\alpha$. Hence we have

$$[I, [I, Z_\alpha]] = [I, \sqrt{-1}\rho_\alpha Z_{\theta\alpha}] = -{\rho_\alpha}^2 Z_\alpha.$$

Since $(\operatorname{ad} I)^2(Z_\alpha) = -Z_\alpha$, the above equality implies ${\rho_\alpha}^2 = 1$.

Applying τ and $\theta\tau$ to the both sides of (6.2), we have $[I, Z_{-\alpha}] = -\sqrt{-1}\rho_\alpha Z_{-\theta\alpha}$ and $[I, Z_{-\theta\alpha}] = -\sqrt{-1}\rho_\alpha Z_{-\alpha}$. From the latter equality, it follows that $\rho_{-\theta\alpha} = -\rho_\alpha$. Moreover, by an easy calculation we obtain the assertion (2). □

We need two more lemmas concerning the values ρ_α ($\alpha \in \Delta(\mu)$).

Lemma 6.11 *Let $\zeta \in \Delta_0$, $\alpha \in \Delta(\mu)$. Assume that $\alpha + \zeta \in \Delta(\mu)$. Then:*

$$\rho_{\alpha+\zeta} = \begin{cases} \rho_\alpha, & \text{if } \zeta \neq \pm\nu, \\ -\rho_\alpha, & \text{if } \zeta = \nu \text{ or } -\nu. \end{cases}$$

Proof. First note that $[Z_\zeta, Z_\alpha] \in \mathfrak{g}_{\alpha+\zeta}$ and $[Z_\zeta, Z_\alpha] \neq 0$. We also note

$$\begin{aligned}[][[Z_{-2\mu}, Z_\zeta], Z_\alpha] &= [Z_{-2\mu}, [Z_\zeta, Z_\alpha]] - [Z_\zeta, [Z_{-2\mu}, Z_\alpha]] \\ &= \sqrt{-1}(\rho_{\alpha+\zeta} - \rho_\alpha)\theta[Z_\zeta, Z_\alpha].\end{aligned}$$

Assume that $\zeta \neq \pm\nu$. Then we have $-2\mu + \zeta \notin \Delta \cup \{0\}$ (see (1) of Lemma 6.5). Since $[Z_{-2\mu}, Z_\zeta] = 0$, we have $\rho_{\alpha+\zeta} = \rho_\alpha$. On the contrary, assume that $\zeta = \nu$ or $-\nu$. Then we have $-2\mu + \zeta \in \Delta$ and $-2\mu + \zeta + \alpha = \theta(\alpha + \zeta) \in \Delta$. Hence, $[[Z_{-2\mu}, Z_\zeta], Z_\alpha] \neq 0$. Consequently, we have $\rho_{\alpha+\zeta} - \rho_\alpha \neq 0$. Since ${\rho_{\alpha+\zeta}}^2 = {\rho_\alpha}^2 = 1$, it follows that $\rho_{\alpha+\zeta} = -\rho_\alpha$. □

Lemma 6.12 *Let α, $\alpha' \in \Delta(\mu)$. Then, $A_{\alpha',\nu}/\rho_{\alpha'} = A_{\alpha,\nu}/\rho_\alpha$ holds.*

Proof. If $\alpha' = \alpha$, then there is nothing to prove. Next consider the case $\alpha' = -\theta\alpha$. By Lemma 6.10 (1), we have $\rho_{-\theta\alpha} = -\rho_\alpha$. On the other hand, we have

$$A_{-\theta\alpha,\nu} = \frac{2(-\theta\alpha, \nu)}{(\nu, \nu)} = -\frac{2(\alpha, \theta\nu)}{(\nu, \nu)} = -\frac{2(\alpha, \nu)}{(\nu, \nu)} = -A_{\alpha,\nu}.$$

This shows that the lemma is true for the case $\alpha' = -\theta\alpha$.

Now assume that $\alpha' \neq \alpha$, $-\theta\alpha$. Replacing α by $-\theta\alpha$ if necessary, we may assume that $A_{\alpha',\alpha} = 1$ (see Lemma 6.5 (3)). Then, setting $\zeta = \alpha' - \alpha$, we have $\zeta \in \Delta_0$. In view of Lemma 6.6, we have $A_{\alpha,\nu} = \pm 1$, $A_{\alpha',\nu} = \pm 1$. First consider the case $A_{\alpha',\nu} = A_{\alpha,\nu}$. Then we have $A_{\zeta,\nu} = A_{\alpha',\nu} - A_{\alpha,\nu} = 0$ and hence $\zeta \neq \pm\nu$. Therefore, by Lemma 6.11, we have $\rho_{\alpha'} = \rho_{\alpha+\zeta} = \rho_\alpha$. This implies that $A_{\alpha',\nu}/\rho_{\alpha'} = A_{\alpha,\nu}/\rho_\alpha$.

Next consider the case $A_{\alpha',\nu} = -A_{\alpha,\nu}$. Then we have $A_{\zeta,\nu} = A_{\alpha',\nu} - A_{\alpha,\nu} = -2A_{\alpha,\nu} = \pm 2$, which implies $(\zeta, \nu) = \pm(\nu, \nu)$. Since $(\alpha, \alpha) = (\alpha', \alpha') = (\nu, \nu)$ (see Lemma 6.5), we have

$$(\zeta, \zeta) = (\alpha' - \alpha, \alpha' - \alpha) = (\alpha, \alpha)\left(2 - A_{\alpha',\alpha}\right) = (\nu, \nu).$$

By these equalities $(\zeta, \nu) = \pm(\nu, \nu)$ and $(\zeta, \zeta) = (\nu, \nu)$, we have $\zeta = \nu$ or $-\nu$. Therefore, by Lemma 6.11 we have $\rho_{\alpha'} = \rho_{\alpha+\zeta} = -\rho_\alpha$. Hence, in this case, we get $A_{\alpha',\nu}/\rho_{\alpha'} = A_{\alpha,\nu}/\rho_\alpha$. □

We are now in the final stage of the proof of Theorem 3.5. By a simple calculation, we have

$$\left[\widehat{\nu}, Y(\alpha)^{\pm}\right] = \mp A_{\alpha,\nu} Y(\alpha)^{\mp}.$$

Compare this equality with (2) of Lemma 6.10. Then, we know that Proposition 6.9 immediately follows from Lemma 6.12. Thus, we complete the proof of Theorem 3.5.

Table 3. Multiplicities and local pseudo-nullities.

Type	G/K	Σ	$\mathcal{M}(G/K)$	$\mathcal{N}(G/K)$
AI_n	$SU(n+1)/SO(n+1)\ (n \geq 1)$	A_n	$\{1,0,0\}$	$\{1,0,0\}$
AII_n	$SU(2(n+1))/Sp(n+1)\ (n \geq 1)$	A_n	$\{4,0,0\}$	$\{4,0,0\}$
$AIII_{p,q}$	$SU(p+q)/S(U(p) \times U(q))\ (p \geq q \geq 1, p \geq 2)$			
	$(p > q > 1)$	BC_q	$\{2,2(p-q),1\}$	$\{2,p-q,1\}$
	$(p = q > 1)$	C_q	$\{1,2,0\}$	$\{1,2,0\}$
	$(p > q = 1)$	BC_1	$\{0,2(p-1),1\}$	$\{0,p-1,1\}$
$BI_{p,q}$	$SO(p+q)/SO(p) \times SO(q)\ (p+q = odd,\ p > q \geq 2)$			
		B_q	$\{1,p-q,0\}$	$\{1,p-q,0\}$
BII_p	$SO(p+1)/SO(p)\ (p = even \geq 2)$	A_1	$\{p-1,0,0\}$	$\{p-1,0,0\}$
CI_n	$Sp(n)/U(n)\ (n \geq 2)$	C_n	$\{1,1,0\}$	$\{1,1,0\}$
$CII_{p,q}$	$Sp(p+q)/Sp(p) \times Sp(q)\ (p \geq q \geq 1)$			
	$(p > q > 1)$	BC_q	$\{4,4(p-q),3\}$	$\{4,p-q,3\}$
	$(p = q > 1)$	C_q	$\{3,4,0\}$	$\{3,4,0\}$
	$(p > q = 1)$	BC_1	$\{0,4(p-1),3\}$	$\{0,p-1,3\}$
	$(p = q = 1)$	A_1	$\{3,0,0\}$	$\{3,0,0\}$
$DI_{p,q}$	$SO(p+q)/SO(p) \times SO(q)\ (p+q = even,\ p \geq q \geq 2,\ (p,q) \neq (2,2))$			
	$(p \geq q+2)$	B_q	$\{1,p-q,0\}$	$\{1,p-q,0\}$
	$(p = q)$	D_q	$\{1,0,0\}$	$\{1,0,0\}$
DII_p	$SO(p+1)/SO(p)\ (p = odd \geq 3)$	A_1	$\{p-1,0,0\}$	$\{p-1,0,0\}$
$DIII_n$	$SO(2n)/U(n)\ (n \geq 4)$			
	$(n = 2m)$	C_m	$\{1,4,0\}$	$\{1,4,0\}$
	$(n = 2m+1)$	BC_m	$\{4,4,1\}$	$\{4,2,1\}$
EI	$E_6/Sp(4)$	E_6	$\{1,0,0\}$	$\{1,0,0\}$
EII	$E_6/SU(6) \cdot SU(2)$	F_4	$\{1,2,0\}$	$\{1,2,0\}$
$EIII$	$E_6/Spin(10) \cdot SO(2)$	BC_2	$\{6,8,1\}$	$\{6,4,1\}$
EIV	E_6/F_4	A_2	$\{8,0,0\}$	$\{8,0,0\}$
EV	$E_7/SU(8)$	E_7	$\{1,0,0\}$	$\{1,0,0\}$
EVI	$E_7/Spin(12) \cdot SU(2)$	F_4	$\{1,4,0\}$	$\{1,4,0\}$
$EVII$	$E_7/E_6 \cdot SO(2)$	C_3	$\{1,8,0\}$	$\{1,8,0\}$
$EVIII$	$E_8/Spin(16)$	E_8	$\{1,0,0\}$	$\{1,0,0\}$
EIX	$E_8/E_7 \cdot SU(2)$	F_4	$\{1,8,0\}$	$\{1,8,0\}$
FI	$F_4/Sp(3) \cdot SU(2)$	F_4	$\{1,1,0\}$	$\{1,1,0\}$
FII	$F_4/Spin(9)$	BC_1	$\{0,8,7\}$	$\{0,1,7\}$
G	$G_2/SO(4)$	G_2	$\{1,1,0\}$	$\{1,1,0\}$

Table 4. Categorical pseudo-nullities $p_{cat}(G/K)$ (Classical type).

Type	Maximal SOS	$\#\Gamma$	$b(\Gamma)$	$p_{cat}(G/K)$
AI_n	$\Gamma(A_n)^0$	$[(n+1)/2]$	n	n
AII_n	$\Gamma(A_n)^0$	$[(n+1)/2]$	$n+3[(n+1)/2]$	$n+3[(n+1)/2]$
$AIII_{p,q}$				
$(p>q>1)$	$\Gamma(BC_q)^{0,s}$	$q-s$	$q+s$	
	$(0\leq s\leq[q/2])$			
	$\Gamma(BC_q)^{1,t}$	$q-t$	$p+t-1$	
	$(0\leq t\leq[(q-1)/2])$			$\max\{[3q/2],$
$(p=q>1)$	$\Gamma(C_q)^s$	$q-s$	$q+s$	$[(2p+q-3)/2]\}$
	$(0\leq s\leq[q/2])$			
$(p>q=1)$	$\Gamma(BC_1)^{0,0}$	1	1	
	$\Gamma(BC_1)^{1,0}$	1	$p-1$	
$BI_{p,q}$	$\Gamma(B_q)^0$ ($q=even$)	q	q	$p-1$
	$\Gamma(B_q)^1$	$2[(q-1)/2]+1$	$p-1$	
BII_p	$\Gamma(A_1)^0$	1	$p-1$	$p-1$
CI_n	$\Gamma(C_n)^s$	$n-s$	n	n
	$(0\leq s\leq[n/2])$			
$CII_{p,q}$				
$(p>q>1)$	$\Gamma(BC_q)^{0,s}$	$q-s$	$3q-s$	
	$(0\leq s\leq[q/2])$			
	$\Gamma(BC_q)^{1,t}$	$q-t$	$p+2q-t-3$	
	$(0\leq t\leq[(q-1)/2])$			$\max\{3q,p+2q-3\}$
$(p=q>1)$	$\Gamma(C_q)^s$	$q-s$	$3q-s$	
	$(0\leq s\leq[q/2])$			
$(p>q=1)$	$\Gamma(BC_1)^{0,0}$	1	3	
	$\Gamma(BC_1)^{1,0}$	1	$p-1$	
$(p=q=1)$	$\Gamma(A_1)^0$	1	3	
$DI_{p,q}$				
$(p\geq q+2)$	$\Gamma(B_q)^0$ ($q=even$)	q	q	
	$\Gamma(B_q)^1$	$2[(q-1)/2]+1$	$p-1$	$\max\{p-1,q\}$
$(p=q)$	$\Gamma(D_q)^0$	$2[q/2]$	q	
DII_p	$\Gamma(A_1)^0$	1	$p-1$	$p-1$
$DIII_n$				
$(n=2m)$	$\Gamma(C_m)^s$	$m-s$	$m+3s$	$m+3[m/2]$
	$(0\leq s\leq[m/2])$			
$(n=2m+1)$	$\Gamma(BC_m)^{0,s}$	$m-s$	$m+3s$	$[5m/2]$
	$(0\leq s\leq[m/2])$			
	$\Gamma(BC_m)^{1,t}$	$m-t$	$m+3t+1$	
	$(0\leq t\leq[(m-1)/2])$			

Table 5. Categorical pseudo-nullities $p_{cat}(G/K)$ (Exceptional type).

Type	Maximal SOS	$\#\Gamma$	$b(\Gamma)$	$p_{cat}(G/K)$
EI	$\Gamma(E_6)^0$	4	6	6
EII	$\Gamma(F_4)^0$	4	4	5
	$\Gamma(F_4)^1$	3	5	
EIII	$\Gamma(BC_2)^{0,s}$ $(s=0,1)$	$2-s$	$2+5s$	7
	$\Gamma(BC_2)^{1,0}$	2	5	
EIV	$\Gamma(A_2)^0$	1	9	9
EV	$\Gamma(E_7)^0$	7	7	7
EVI	$\Gamma(F_4)^0$	4	4	7
	$\Gamma(F_4)^1$	3	7	
EVII	$\Gamma(C_3)^s$ $(s=0,1)$	$3-s$	$3+7s$	10
EVIII	$\Gamma(E_8)^0$	8	8	8
EIX	$\Gamma(F_4)^0$	4	4	11
	$\Gamma(F_4)^1$	3	11	
FI	$\Gamma(F_4)^s$ $(s=0,1)$	$4-s$	4	4
FII	$\Gamma(BC_1)^{0,0}$	1	7	7
	$\Gamma(BC_1)^{1,0}$	1	1	
G	$\Gamma(G_2)^1$	2	2	2

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