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Fundamental theorem for totally complex submanifolds

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Abstract. The fundamental theorem (existence and uniqueness) for submanifolds of real space forms is well-known. We will discuss this theorem for some families of submanifolds in the framework of Grassmann geometries in a unified way. In particular, we show the fundamental theorem for half dimensional totally complex submanifolds of the quaternion projective space $\mathbb{H}P^n$ or the quaternion hyperbolic space $\mathbb{H}H^n$. This result is an affirmative answer to the conjecture by Alekseevsky and Marchiafava.

Key words: the fundamental theorem for submanifolds, totally complex submanifolds.

1. Introduction

For Riemannian submanifolds of a real space form, the fundamental theorem for submanifolds is well-known. We denote by $\bar{M}(c)$ an n-dimensional real space form, that is, a simply connected, complete Riemannian manifold of constant curvature c. The fundamental theorem tells us the following: Let M be an m-dimensional simply connected Riemannian manifold, $E \to M$ a Riemannian vector bundle of rank $(n-m)$ over M with a metric connection $\nabla^{\perp}$, and σ be an E-valued symmetric covariant tensor field of order 2 on M. If they satisfy the Gauss, Codazzi, and Ricci equations for the case of constant curvature c, there exists an isometric immersion $f\colon M \to \bar{M}(c)$ such that E is the normal bundle, $\nabla^{\perp}$ its normal connection, and σ is the second fundamental form. Moreover such an immersion f is unique up to the action by the group of isometries of $\bar{M}(c)$. For the precise statement and its proof, see Chapter 7 Part C in M. Spivak [12]. We will generalize this theorem for some families of submanifolds called $\mathcal{O}$-submanifolds in the framework of Grassmann geometries introduced by Harvey and Lawson [4].

We recall $\mathcal{O}$-submanifolds. Let $\bar{M}$ be an n-dimensional Riemannian manifold. We fix an integer m $(0 < m < n)$ and denote by $Gr_m(T\bar{M})$ the Grassmann bundle over $\bar{M}$ of all m-dimensional linear subspaces in the tangent spaces of $\bar{M}$. Let G be the identity component of the group of

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isometries of $\bar{M}$. Then G acts on $Gr_m(T\bar{M})$ through the differential of each isometry. We take an orbit $\mathcal{O}$ in $Gr_m(T\bar{M})$ by this action of G. Let M be an m-dimensional manifold and f be an immersion of M into $\bar{M}$. If $f_{*p}(T_pM) \in \mathcal{O}$ for any $p \in M$, then (M, f) is called an *$\mathcal{O}$-submanifold*. The collection of all $\mathcal{O}$-submanifolds forms a class of submanifolds, which is called an *$\mathcal{O}$-geometry*. Now we assume that for some, and hence for any, $V \in \mathcal{O}$ both V and its orthogonal complement $V^\perp$ are invariant under the curvature tensor $\bar{R}$ of $\bar{M}$, that is, $\bar{R}(V, V)V \subset V$ and $\bar{R}(V^\perp, V^\perp)V^\perp \subset V^\perp$. Then the orbit $\mathcal{O}$ is of *strongly curvature-invariant type* and its $\mathcal{O}$-geometry is also said to be of strongly curvature-invariant type. From now on we assume that $\bar{M}$ is a Riemannian symmetric space. Then the curvature-invariant subspaces of $\bar{M}$ are also known as Lie triple systems. If $p \in \bar{M}$ and $V \subset T_p\bar{M}$ is curvature-invariant, then there exists a unique connected, complete, totally geodesic submanifold M of $\bar{M}$ with $p \in M$ and $T_pM = V$. These totally geodesic submanifolds are $\mathcal{O}(V)$-submanifolds, where $\mathcal{O}(V)$ denotes the orbit in $Gr_m(T\bar{M})$ through V. H. Naitoh in a series of papers ([7], [8], [9], [10]) classified $\mathcal{O}$-geometries of strongly curvature-invariant type on Riemannian symmetric spaces and determined all $\mathcal{O}$-geometries containing non-totally geodesic submanifolds.

Theorem 1.1 (Naitoh) *Let $\bar{M}$ be a simply connected irreducible Riemannian symmetric space of compact type or of non-compact type and $\mathcal{O}$ be an orbit of strongly curvature-invariant type in $Gr_m(T\bar{M})$. All $\mathcal{O}$-geometries except the following ones have only totally geodesic submanifolds*:

(1) *the geometry of k-dimensional $(0 < k < n)$ submanifolds of the sphere S^n resp. of the real hyperbolic space $\mathbb{R}H^n$ $(n \geq 2)$;*

(2) *the geometry of k-dimensional $(0 < k < n)$ complex submanifolds of the complex projective space $\mathbb{C}P^n$ resp. of the complex hyperbolic space $\mathbb{C}H^n$ $(n \geq 2)$;*

(3) *the geometry of n-dimensional totally real submanifolds of the complex projective space $\mathbb{C}P^n$ resp. of the complex hyperbolic space $\mathbb{C}H^n$ $(n \geq 2)$;*

(4) *the geometry of $2n$-dimensional totally complex submanifolds of the quaternionic projective space $\mathbb{H}P^n$ resp. of the quaternionic hyperbolic space $\mathbb{H}H^n$ $(n \geq 2)$;*

(5) *the geometries associated with irreducible symmetric R-spaces and their noncompact dual geometries.*

In this paper, we will develop a fundamental theory of $\mathcal{O}$-submanifolds of strongly curvature-invariant type in a semi-simple Riemannian symmetric space and discuss the fundamental theorem for $\mathcal{O}$-submanifolds in a unified way (§2 Theorem 2.8). As an application, we show the fundamental theorem for the case (3) in the Theorem 1.1, that is, half dimensional totally real submanifolds of the complex projective space $\mathbb{C}P^n$ or the complex hyperbolic space $\mathbb{C}H^n$ (Theorem 2.9). Subsequently we study half dimensional totally complex submanifolds of the quaternion projective space $\mathbb{H}P^n$ or the quaternion hyperbolic space $\mathbb{H}H^n$ in detail (the case (4) in the Theorem 1.1) and show the fundamental theorem for them (§3 Theorem 3.5). This result is an affirmative answer to the conjecture by Alekseevsky and Marchiafava [1].

2. $\mathcal{O}$-geometry of strongly curvarture-invariant type

Let $\bar{M}$ be an n-dimensional semi-simple Riemannian symmetric space and (G, K) be a Riemannian symmetric pair associated with $\bar{M}$. Then $\bar{M}$ is described as the Riemannian symmetric homogeneous space G/K. We denote by $\bar{\pi}$ the projection of G onto $\bar{M}$ and by ρ the action of G on $\bar{M}$. We put $\bar{\pi}(K) = o$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the Lie subalgebra of $\mathfrak{g}$ which corresponds to K and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition associated with the Riemannian symmetric pair (G, K). The Maurer-Cartan form $\bar{\omega}$ on G satisfies the structure equation:

$$d\bar{\omega} + \frac{1}{2}[\bar{\omega}, \bar{\omega}] = 0. \tag{2.1}$$

Here for $\mathfrak{g}$-valued 1-forms ω_1, ω_2, we define a $\mathfrak{g}$-valued 2-form $[\omega_1, \omega_2]$ by

$$[\omega_1, \omega_2](X, Y) = [\omega_1(X), \omega_2(Y)] - [\omega_1(Y), \omega_2(X)].$$

We fix a linear isometry $\iota\colon \mathbb{R}^n \to \mathfrak{p}$ and identify $\mathfrak{p}$ with $\mathbb{R}^n$ via ι. Under this identification $\mathrm{Ad}_{\mathfrak{p}}(K)$ is a subgroup of $O(n)$, where $\mathrm{Ad}_{\mathfrak{p}}\colon K \to O(\mathfrak{p})$ denotes the adjoint representation of K on $\mathfrak{p}$. Then $u_o = \bar{\pi}_{*e} \circ \iota\colon \mathbb{R}^n \to T_o\bar{M}$ is a linear isometry and induces an orthonormal frame for $T_o\bar{M}$. Let $O(\bar{M})$ be the bundle of orthonormal frames over $\bar{M}$ and $\bar{\pi}\colon O(\bar{M}) \to \bar{M}$ be the projection. We define a smooth map $\phi\colon G \to O(\bar{M})$ by $\phi(g) = \rho(g)_{*o}u_o$. Then ϕ is a K-bundle homomorphism which corresponds to the Lie group homomorphism $\mathrm{Ad}_{\mathfrak{p}}\colon K \to O(\mathfrak{p}) = O(n)$. Let θ be the canonical form of $\bar{M}$, which is an $\mathbb{R}^n$-valued 1-form on $O(\bar{M})$. Then, via the identification

$\mathbb{R}^n \cong \mathfrak{p}$, we have $\phi^*\theta = \bar{\omega}_{\mathfrak{p}}$, where $\bar{\omega}_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-component of $\bar{\omega}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.

We fix an integer m $(0 < m < n)$. Let $Gr_m(T\bar{M})$ be the Grassmann bundle over $\bar{M}$ of all m-dimensional linear subspaces in the tangent spaces of $\bar{M}$, which is the fibre bundle associated with $O(\bar{M})$ with the standard fibre $Gr_m(\mathbb{R}^n) = O(n)/O(m) \times O(n-m)$. For an m-dimensional subspace $\mathfrak{m}$ in $\mathfrak{p}$, we define the orbit $\mathcal{O}(\mathfrak{m}) = \rho(G)\bar{\pi}_{*e}\mathfrak{m} \subset Gr_m(T\bar{M})$ and the group $K_+ = \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(\mathfrak{m}) = \mathfrak{m}\}$. The orbit $\mathcal{O}(\mathfrak{m})$ is a fibre bundle $G \times_K K/K_+$ associated with the principal fibre bundle $\bar{\pi}\colon G \to \bar{M}$. Let M be an m-dimensional manifold and f be an immersion of M into $\bar{M}$. If $f_{*p}(T_pM) \in \mathcal{O}(\mathfrak{m})$ for any $p \in M$, then (M, f) is called an $\mathcal{O}(\mathfrak{m})$-*submanifold.* The collection of all $\mathcal{O}(\mathfrak{m})$-submanifolds forms a class of submanifolds, which is called an $\mathcal{O}(\mathfrak{m})$-*geometry.*

From now on we assume that $\mathfrak{m} \subset \mathfrak{p}$ is a strongly curvature-invariant subspace, that is, $\big[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}\big] \subset \mathfrak{m}$, $\big[[\mathfrak{m}^\perp, \mathfrak{m}^\perp], \mathfrak{m}^\perp\big] \subset \mathfrak{m}^\perp$, where $\mathfrak{m}^\perp$ denotes the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{p}$. Then there exists an involutive automorphism τ of $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ such that $\tau(\mathfrak{k}) = \mathfrak{k}$, $\tau(\mathfrak{p}) = \mathfrak{p}$, $\tau = -\mathrm{Id}$ on $\mathfrak{m}$, and $\tau = \mathrm{Id}$ on $\mathfrak{m}^\perp$ (see e.g. Naitoh [6]). The automorphism τ induces ± 1-eigenspaces decompositions $\mathfrak{k} = \mathfrak{k}_+ + \mathfrak{k}_-$ and $\mathfrak{p} = \mathfrak{m} + \mathfrak{m}^\perp$ of $\mathfrak{k}$ and $\mathfrak{p}$. We note that $[\mathfrak{k}_+, \mathfrak{m}] \subset \mathfrak{m}$, $[\mathfrak{k}_+, \mathfrak{m}^\perp] \subset \mathfrak{m}^\perp$, $[\mathfrak{k}_-, \mathfrak{m}] \subset \mathfrak{m}^\perp$, and $[\mathfrak{k}_-, \mathfrak{m}^\perp] \subset \mathfrak{m}$. Moreover we have $\mathfrak{k}_+ = \{T \in \mathfrak{k} \,|\, [T, \mathfrak{m}] \subset \mathfrak{m}\}$. Let (M, f) be an $\mathcal{O}(\mathfrak{m})$-submanifold. Then we have the following two pull back bundles:

- the principal fibre bundle with the structure group K:

$$\begin{array}{ccc} f^*G & \xrightarrow{\tilde{f}} & G \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\bar{\pi}} \\ M & \xrightarrow[f]{} & \bar{M} \end{array} .$$

Here f^*G is given by

$$f^*G = \{(p, g) \in M \times G \,|\, f(p) = \bar{\pi}(g)\} \quad \subset M \times G,$$

and $\pi\colon f^*G \to M$ is the projection from $M \times G$ onto the first factor M which is restricted to f^*G.

- the associated fibre bundle with the standard fibre K/K_+

$$\begin{array}{ccc} f^*\mathcal{O}(\mathfrak{m}) & \longrightarrow & \mathcal{O}(\mathfrak{m}) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\bar\pi} \\ M & \underset{f}{\longrightarrow} & \bar{M} \end{array}.$$

By the definition of an $\mathcal{O}(\mathfrak{m})$-submanifold, there exists a section of the fibre bundle $f^*\mathcal{O}(\mathfrak{m}) \to M$. This implies that there exists a principal subbundle P of f^*G with the structure group K_+ such that the following diagram holds:

$$\begin{array}{ccc} P & \longrightarrow & f^*G \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ M & = & M \end{array}.$$

Here P is given by

$$P = \{(p,\, g) \in f^*G \mid f(p) = \bar\pi(g),\ \rho(g)_{*o}(\bar\pi_{*e}(\mathfrak{m})) = f_{*p}(T_pM)\}.$$

We restrict the pull back form $\tilde{f}^*\bar\omega$ of the Maurer-Cartan form $\bar\omega$ on G to P, which is denoted by ω. According to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{k}_+ + \mathfrak{k}_- + \mathfrak{m} + \mathfrak{m}^\perp$, we decompose

$$\omega = \omega_{\mathfrak{k}_+} + \omega_{\mathfrak{k}_-} + \omega_{\mathfrak{m}} + \omega_{\mathfrak{m}^\perp}.$$

Then $\omega_{\mathfrak{m}^\perp}$ vanishes. Moreover by (2.1) it follows that

$$d\omega + \frac{1}{2}[\omega,\, \omega] = 0. \tag{2.2}$$

Consequently for an $\mathcal{O}(\mathfrak{m})$-submanifold $(M,\, f)$ we obtained the pair $(P,\, \omega)$ of a principal fibre bundle P over M with the structure group K_+ and a $(\mathfrak{k}+\mathfrak{m})$-valued 1-form ω on P. Moreover the 1-form ω satisfies the following conditions:

(2.3.1) The map $\pi' \circ \omega\colon T_uP \to \mathfrak{k}_+ + \mathfrak{m}$ is surjective at each point $u \in P$ (in particular, $\omega\colon T_uP \to \mathfrak{k} + \mathfrak{m}$ is injective), where $\pi'\colon \mathfrak{k}_+ + \mathfrak{k}_- + \mathfrak{m} \to \mathfrak{k}_+ + \mathfrak{m}$ is the projection;

(2.3.2) $R_k^*\omega = \mathrm{Ad}(k^{-1})\omega$ for $k \in K_+$, where R_k denotes the right translation;

(2.3.3) $\omega(X^*) = X$ for $X \in \mathfrak{k}_+$, where X^* denotes the fundamental vector field on P which is generated by X.

In general we define a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry following Definition 5.2 in Sharpe [11] Chapter 6.

Definition 2.1 Let P be a principal fibre bundle over an m-dimensional manifold M ($m = \dim \mathfrak{m}$) with the structure group K_+ and ω be a $(\mathfrak{k}+\mathfrak{m})$-valued 1-form on P satisfying the conditions (2.3.1), (2.3.2), and (2.3.3) in the above. We call such the pair (P, ω) *a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry* on M.

By the definition, the pair (P, ω) which is constructed over an $\mathcal{O}(\mathfrak{m})$-submanifold (M, f) is a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry. Moreover in this case the 1-form ω satisfies (2.2). We consider the converse. The following result can be proved by a similar argument as for Proposition 5.8 in [11] Chapter 6.

Proposition 2.2 *Let M be an m-dimensional ($m = \dim \mathfrak{m}$) simply connected manifold and (P, ω) be a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry on M. If ω satisfies*

$$d\omega + \frac{1}{2}[\omega, \omega] = 0,$$

then there exists an immersion $f\colon M \to \bar{M}$ which is an $\mathcal{O}(\mathfrak{m})$-submanifold in $\bar{M}$ such that the locally ambient $\mathcal{O}(\mathfrak{m})$-geometry which corresponds to (M, f) is equivalent to (P, ω). Moreover, such an immersion f is unique up to the action by G.

Now let (P, ω) be a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry on M. We investigate the geometric properties induced by (P, ω) and the geometric meaning of the integrability condition (2.2). Firstly, we decompose the equation (2.2) according to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{k}_+ + \mathfrak{k}_- + \mathfrak{m} + \mathfrak{m}^\perp$. Then we have the following.

Proposition 2.3 *The equation (2.2) is equivalent to a quadruplet of the following equations*:

$$d\omega_{\mathfrak{k}_+} + \frac{1}{2}[\omega_{\mathfrak{k}_+}, \omega_{\mathfrak{k}_+}] + \frac{1}{2}[\omega_{\mathfrak{k}_-}, \omega_{\mathfrak{k}_-}] + \frac{1}{2}[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}] = 0; \tag{2.4.1}$$

$$d\omega_{\mathfrak{k}_-} + [\omega_{\mathfrak{k}_+}, \omega_{\mathfrak{k}_-}] = 0; \tag{2.4.2}$$

$$d\omega_{\mathfrak{m}} + [\omega_{\mathfrak{k}_+}, \omega_{\mathfrak{m}}] = 0; \tag{2.4.3}$$

$$[\omega_{\mathfrak{k}_-}, \omega_{\mathfrak{m}}] = 0. \tag{2.4.4}$$

We denote by $\rho'\colon K_+ \to O(\mathfrak{m})$ and $\rho''\colon K_+ \to O(\mathfrak{m}^\perp)$ the representations of K_+ which are obtained by restricting $\mathrm{Ad}_{\mathfrak{p}}(K_+)$ to $\mathfrak{m}$ and $\mathfrak{m}^\perp$, respectively. We put $K' = \rho'(K_+)$ and $K'' = \rho''(K_+)$ and denote by $\mathfrak{k}'$ and $\mathfrak{k}''$ their Lie algebras. We put $P' = P/\ker\rho'$ and denote by h' the projection of P onto P'. Then P' is a principal K'-bundle over M and h' is a bundle homomorphism. Similarly, the representation ρ'' induces a principal K''-bundle P'' over M and a bundle homomorphism $h''\colon P \to P''$.

Lemma 2.4 *The principal K'-bundle P' is a subbundle of the orthonormal frame bundle $O(M)$ over M. Let θ be the canonical form on $O(M)$ which is restricted to P'. Then we have $h'^*\theta = \omega_{\mathfrak{m}}$, where we identify $\mathfrak{m}$ with $\mathbb{R}^m$.*

Proof. We define a smooth map ϕ' of P into the linear frame bundle $L(M)$ over M as follows: at $u \in P$ we define a linear isomorphism $\phi'(u)\colon \mathfrak{m} \to T_{\pi(u)}M$ by

$$\phi'(u)(\xi) = \pi_{*u}(\pi' \circ \omega)_u^{-1}(\xi) \quad \text{for } \xi \in \mathfrak{m}.$$

Then $\phi'(uk)(\xi) = \phi'(u)(\mathrm{Ad}_{\mathfrak{p}}(k)(\xi)) = \phi'(u)(\rho'(k)(\xi))$ holds for $k \in K_+$. We introduce a Riemannian metric on M such that $\phi'(u)$ is a linear isometry of $\mathfrak{m}$ onto $T_{\pi(u)}M$ at any point $u \in P$. Thus ϕ' is a bundle homomorphism of P into $O(M)$ with a Lie group homomorphism $\rho'\colon K_+ \to O(\mathfrak{m})$. In particular it yields an injective homomorphism $\iota'\colon P' \to O(M)$. We view P' as a subbundle of $O(M)$ and omit the inclusion map ι' and hence $\phi' = h'$. By the definition of ϕ', it follows that $h'^*\theta = \omega_{\mathfrak{m}}$. □

By (2.3.2) and (2.3.3) it follows that $\omega_{\mathfrak{k}_+}$ is a connection form on P. We denote by ω' the connection form on P' which is the pushforward form of $\omega_{\mathfrak{k}_+}$; ω' is a $\mathfrak{k}'$-valued 1-form on P' such that $h'^*\omega' = \rho'\omega_{\mathfrak{k}_+}$. Then by Proposition 2.3 (2.4.3), we have $h'^*\{d\theta + \omega' \wedge \theta\} = 0$, which shows that ω' is a torsion-free connection. Hence we have proved

Lemma 2.5 *ω' is the Riemannian connection on M.*

Corollary 2.6 *The holonomy algebra of an $\mathcal{O}(\mathfrak{m})$-submanifold is a subalgebra of $\mathfrak{k}'$.*

Suppose that (P, ω) is a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry constructed on an $\mathcal{O}(\mathfrak{m})$-submanifold (M, f). By an argument similar to the case of P', we see that P'' is a subbundle of the orthonormal frame bundle $O(T^\perp M)$ of the normal bundle $T^\perp M$. We denote by ω'' the connection form on P'' which is the pushforward form of $\omega_{\mathfrak{k}_+}$; ω'' is a $\mathfrak{k}''$-valued 1-form on P'' such that $h''^*\omega'' = \rho''\omega_{\mathfrak{k}_+}$. The connection ω'' coincides with the normal connection in the normal bundle $T^{\perp}M$.

Let $\bar{R}$ be the curvature tensor of $\bar{M}$. Since $\mathfrak{m} \subset \mathfrak{p}$ is a strongly curvature-invariant subspace, we have $\bar{R}(\mathfrak{m}, \mathfrak{m})\mathfrak{m} \subset \mathfrak{m}$. From $\bar{R}$, we define a curvature-like tensor field $\bar{R}_M$ on M as follows: for a point $p \in M$, we choose $u \in P$ with $\pi(u) = p$, then $h'(u)$ is a linear isometry of $\mathbb{R}^m \cong \mathfrak{m}$ onto T_pM. We put

$$\bar{R}_M(X, Y)Z = h'(u)\bar{R}(h'(u)^{-1}X, h'(u)^{-1}Y)h'(u)^{-1}Z \qquad \text{for } X, Y, Z \in T_pM.$$

The right hand side in the equation above does not depend on the choice of $u \in P$ with $\pi(u) = p$. Therefore we can define a tensor field on M. Since $\bar{R}(\mathfrak{m}, \mathfrak{m})\mathfrak{m}^{\perp} \subset \mathfrak{m}^{\perp}$, similarly we can define a $\mathrm{End}(T^{\perp}M)$-valued 2-form $\bar{R}_M$ by

$$\bar{R}_M(X, Y)\xi = h''(u)\bar{R}(h'(u)^{-1}X, h'(u)^{-1}Y)h''(u)^{-1}\xi$$

for $X, Y \in T_pM$, $\xi \in T_p^{\perp}M$, and $u \in P$ with $\pi(u) = p$.

We introduce the second fundamental form. We define an $\mathfrak{m}^{\perp}$-valued bilinear form $\tilde{\sigma}$ on P as follows:

$$\tilde{\sigma}(X, Y) = [\omega_{\mathfrak{k}_-}(X), \omega_{\mathfrak{m}}(Y)] \quad \text{for } X, Y \in T_uP.$$

By Proposition 2.3 (2.4.4), $\tilde{\sigma}$ is symmetric, that is, $\tilde{\sigma}(X, Y) = \tilde{\sigma}(Y, X)$. The following lemma can be proved by an argument similar to that of Proposition 3.5 in Kobayashi and Nomizu [5], Chapter VII.

Lemma 2.7 *$\tilde{\sigma}$ is a tensorial form of type $(\rho'', \mathfrak{m}^{\perp})$ and defines a symmetric tensor field σ on M whose values are in the normal bundle $T^{\perp}M$. Moreover it coincides with the second fundamental form of an $\mathcal{O}(\mathfrak{m})$-submanifold (M, f).*

Now we will describe the geometric meaning of the equations (2.4.1) and (2.4.2). We view the $T^{\perp}M$-valued symmetric bilinear form σ on M in Lemma 2.7 as a $\mathrm{Hom}(TM, T^{\perp}M)$-valued 1-form and denote by $\hat{\sigma}$ a $\mathrm{Hom}(\mathfrak{m}, \mathfrak{m}^{\perp})$-valued 1-form on P which corresponds to such the $\mathrm{Hom}(TM, T^{\perp}M)$-valued 1-form σ. Then we have

$$\hat{\sigma}(X)(\xi) = [\omega_{\mathfrak{k}_-}(X), \xi] \quad \text{for } X \in T_uP \quad \text{and} \quad \xi \in \mathfrak{m}.$$

We define a linear map $\psi\colon \mathfrak{k}_- \to \mathrm{Hom}(\mathfrak{m}, \mathfrak{m}^{\perp})$ by $\psi(T)(\xi) = [T, \xi]$ for $T \in \mathfrak{k}_-$ and $\xi \in \mathfrak{m}$. Then ψ is injective. Since $\hat{\sigma} = \psi\omega_{\mathfrak{k}_-}$, the equation (2.4.2) implies

$$\begin{aligned}0 &= \psi\{d\omega_{\mathfrak{k}_-}(X, Y) + [\omega_{\mathfrak{k}_+}(X), \omega_{\mathfrak{k}_-}(Y)] - [\omega_{\mathfrak{k}_+}(Y), \omega_{\mathfrak{k}_-}(X)]\} \\ &= d\hat{\sigma}(X, Y) + [\omega_{\mathfrak{k}_+}(X), \hat{\sigma}(Y)] - [\omega_{\mathfrak{k}_+}(Y), \hat{\sigma}(X)],\end{aligned}$$

where $[\omega_{\mathfrak{k}_+}(X), \hat{\sigma}(Y)]$ means

$$[\omega_{\mathfrak{k}_+}(X), \hat{\sigma}(Y)](\xi) = \rho''(\omega_{\mathfrak{k}_+}(X))\hat{\sigma}(Y)(\xi) - \hat{\sigma}(Y)(\rho'(\omega_{\mathfrak{k}_+}(X))\xi).$$

Therefore the equation (2.4.2) corresponds to the Codazzi equation:

$$(\bar{\nabla}_X\sigma)(Y, Z) = (\bar{\nabla}_Y\sigma)(X, Z), \tag{2.4.2$'$}$$

for the tangent vectors X, Y, Z of M, where $\bar{\nabla}$ denotes the covariant differentiation with respect to the connection in $TM \oplus T^{\perp}M$.

Finally we will show that the equation (2.4.1) is nothing but the Gauss and Ricci equations. The form $\omega_{\mathfrak{k}_+}$ is the connection form on P and $\Omega = d\omega_{\mathfrak{k}_+} + (1/2)[\omega_{\mathfrak{k}_+}, \omega_{\mathfrak{k}_+}]$ is the curvature form of $\omega_{\mathfrak{k}_+}$. We denote by Ω' the curvature form of the Riemannian connection ω'. Since ω' is the pushforward connection of $\omega_{\mathfrak{k}_+}$, $\rho'\Omega = h'^*\Omega'$. Similarly, we have $\rho''\Omega = h''^*\Omega''$, where Ω'' is the curvature form of the connection ω'' of the normal bundle $T^{\perp}M$. On G, $\bar{\Omega} = d\bar{\omega}_{\mathfrak{k}} + (1/2)[\bar{\omega}_{\mathfrak{k}}, \bar{\omega}_{\mathfrak{k}}]$ is the curvature form of the canonical connection on the Riemannian symmetric space $\bar{M} = G/K$, which coincides with the Riemannian connection. By the structure equation (2.1) of the Maurer-Cartan form $\bar{\omega}$, we have

$$d\bar{\omega}_{\mathfrak{k}} + \frac{1}{2}[\bar{\omega}_{\mathfrak{k}}, \bar{\omega}_{\mathfrak{k}}] + \frac{1}{2}[\bar{\omega}_{\mathfrak{p}}, \bar{\omega}_{\mathfrak{p}}] = 0.$$

This implies $\bar{\Omega} = -(1/2)[\bar{\omega}_{\mathfrak{p}}, \bar{\omega}_{\mathfrak{p}}]$ and

$$\tilde{f}^*\bar{\Omega} = -\frac{1}{2}[\tilde{f}^*\bar{\omega}_{\mathfrak{p}}, \tilde{f}^*\bar{\omega}_{\mathfrak{p}}] = -\frac{1}{2}[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}] \in \mathfrak{k}_+.$$

Applying ρ' and ρ'' to (2.4.1), respectively, we have

$$h'^*\Omega' + \frac{1}{2}\rho'([\omega_{\mathfrak{k}_-}, \omega_{\mathfrak{k}_-}]) - \rho'\tilde{f}^*\bar{\Omega} = 0, \tag{2.4.1$'$}$$

$$h''^*\Omega'' + \frac{1}{2}\rho''([\omega_{\mathfrak{k}_-}, \omega_{\mathfrak{k}_-}]) - \rho''\tilde{f}^*\bar{\Omega} = 0. \tag{2.4.1$''$}$$

Let R be the Riemannian curvature tensor of M and $R^{\perp}$ be the curvature tensor of the normal bundle $T^{\perp}M$, respectively. Then (2.4.1)$'$ and

$(2.4.1)''$ are described as follows:

$$R(X, Y)Z = \bar{R}_M(X, Y)Z + S_{\sigma(Y, Z)}X - S_{\sigma(X, Z)}Y, \qquad (2.4.1)'$$
$$R^{\perp}(X, Y)\xi = \bar{R}_M(X, Y)\xi + \sigma(X, S_\xi Y) - \sigma(Y, S_\xi X), \qquad (2.4.1)''$$

for $X, Y, Z \in TM$ and $\xi \in T^{\perp}M$, where S_ξ denotes the shape operator which is defined by $\langle S_\xi X, Y\rangle = \langle \sigma(X, Y), \xi\rangle$.

Following the arguments above, we apply Proposition 2.2 to show fundamental theorem for $\mathcal{O}(\mathfrak{m})$-submanifolds. As before, $\bar{M}$ is an n-dimensional semi-simple Riemannian symmetric space and $\mathfrak{m} \subset \mathfrak{p}$ is a strongly curvature-invariant subspace ($\dim \mathfrak{m} = m$). Let $\iota_1 \colon \mathbb{R}^m \to \mathfrak{m}$ and $\iota_2 \colon \mathbb{R}^{n-m} \to \mathfrak{m}^{\perp}$ be linear isometries. We define Lie group homomorphisms $\rho_1 \colon K_+ \to O(m)$ and $\rho_2 \colon K_+ \to O(n-m)$ by $\rho_1(k) = \iota_1^{-1}\rho'(k)\iota_1$ and $\rho_2(k) = \iota_2^{-1}\rho''(k)\iota_2$, where $\rho' \colon K_+ \to O(\mathfrak{m})$ and $\rho'' \colon K_+ \to O(\mathfrak{m}^{\perp})$ denote the representations of K_+ as before. Going backward on the way of our arguments and applying Proposition 2.2, we can prove the following:

Theorem 2.8 (Fundamental theorem for $\mathcal{O}(\mathfrak{m})$-submanifolds)
Assumption: Let M be an m-dimensional simply connected Riemannian manifold with the curvature tensor R, $E \to M$ a Riemannian vector bundle of rank $(n-m)$ with a metric connection and its curvature tensor R^E and σ be an E-valued symmetric covariant tensor field of order 2 over M such that the Codazzi equation $(2.4.2)'$ holds with respect to the connection in $TM \oplus E$. Suppose that there exist a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry (P, ω) over M and bundle homomorphisms $h' \colon P \to O(M)$ and $h'' \colon P \to O(E)$ with the corresponding homomorphisms $\rho_1 \colon K_+ \to O(m)$ and $\rho_2 \colon K_+ \to O(n-m)$ such that the induced diffeomorphisms of M are identity, where $O(M)$ and $O(E)$ denote the bundles of orthonormal frames of M and E, respectively. We denote by P' and P'' the principal subbundles $h'(P)$ of $O(M)$ and $h''(P)$ of $O(E)$. Now we assume that they satisfy the following conditions:
(1) *For the canonical form θ on $O(M)$, $h'^*(\iota_1\theta) = \omega_{\mathfrak{m}}$,*
(2) *The Riemannian connection of M reduces to P' and its connection form ω' on P' satisfies $h'^*\omega' = \rho_1\omega_{\mathfrak{k}_+}$,*
(3) *The metric connection in E reduces to P'' and its connection form ω'' on P'' satisfies $h''^*\omega'' = \rho_2\omega_{\mathfrak{k}_+}$,*
(4) *At any point $u \in P$,*

$$[\omega_{\mathfrak{k}_-}(\tilde{X}), \omega_{\mathfrak{m}}(\tilde{Y})] = \iota_2 h''(u)^{-1}\sigma(\pi_*\tilde{X}, \pi_*\tilde{Y}),$$

for $\tilde{X}, \tilde{Y} \in T_u P$,

(5) *They satisfy Gauss and Ricci equations*: *at any point* $p \in M$,

$$R(X, Y)Z = \bar{R}_M(X, Y)Z + S_{\sigma(Y,Z)}X - S_{\sigma(X,Z)}Y,$$
$$R^E(X, Y)\xi = \bar{R}_M(X, Y)\xi + \sigma(X, S_\xi Y) - \sigma(Y, S_\xi X),$$

for $X, Y, Z \in T_pM$, $\xi \in E_p$.

Conclusion: *there exist an isometric immersion* $f\colon M \to \bar{M}$ *which is an* $\mathcal{O}(\mathfrak{m})$*-submanifold in* $\bar{M}$ *and a vector bundle isomorphism* $\tilde{f}\colon E \to T^\perp M$ *which preserves the metrics and the connections such that for every* $X, Y \in TM$,

$$\tilde{\sigma}(X, Y) = \tilde{f}\sigma(X, Y),$$

where $\tilde{\sigma}$ *is the second fundamental form of* f. *Moreover, such an immersion* f *is unique up to the action by* G.

After Theorem 2.8, it becomes a problem how to construct a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry (P, ω) from the geometric ingredients (a Riemannian manifold, a Riemannian vector bundle E, and an E-valued tensor field). In the rest of this section, we deal with the case (3) in Theorem 1.1 as an example.

Let $\bar{M}^n(\tilde{c})$ be a (complex) n-dimensional simply connected complete Kähler manifold of constant holomorphic sectional curvature $\tilde{c}$ ($\tilde{c} \neq 0$), that is, a complex projective space $\mathbb{C}P^n$ or a complex hyperbolic space $\mathbb{C}H^n$ according as $\tilde{c}$ is positive or negative. We denote by I and $\langle\ ,\ \rangle$ the complex structure and the Kähler metric on $\bar{M}^n(\tilde{c})$, respectively. Let M^n be a (real) n-dimensional Riemannian manifold isometrically immersed in $\bar{M}^n(\tilde{c})$ which satisfies $IT_pM = T_p^\perp M$ for all $p \in M$. Then M is called a *totally real submanifold*. Totally real submanifolds have the following remarkable properties (cf. Chen and Ogiue [2]): The complex structure I defines a bundle isomorphism of TM to $T^\perp M$ which preserves the metrics and the connections. For the second fundamental form σ, we define an $\mathrm{End}(TM)$-valued 1-form $\hat{\sigma}$ by $\hat{\sigma}(X)(Y) = -I\sigma(X, Y)$. Then it satisfies

$$\langle \hat{\sigma}(X)(Y), Z\rangle = \langle Y, \hat{\sigma}(X)(Z)\rangle, \quad \hat{\sigma}(X)(Y) = \hat{\sigma}(Y)(X). \tag{2.5}$$

Moreover we have $S_{IX}Y = \hat{\sigma}(Y)(X)$, where S_{IX} denotes the shape operator for the normal vector IX. Then the Gauss equation is given by

$$R(X,\,Y)Z = \frac{\tilde{c}}{4}\{\langle Y,\,Z\rangle X - \langle X,\,Z\rangle Y\} + [\hat{\sigma}(X),\,\hat{\sigma}(Y)](Z).$$

The Ricci equation coincides with the Gauss equation by the bundle isomorphism I.

Now we show the fundamental theorem for half dimensional totally real submanifolds of $\mathbb{C}P^n$ or $\mathbb{C}H^n$.

Theorem 2.9 *Let M^n be an n-dimensional simply connected Riemannian manifold and $\hat{\sigma}$ an* $\mathrm{End}(TM)$*-valued* 1*-form on M which satisfies the identities* (2.5)*. Suppose that $\hat{\sigma}$ satisfies the following equations of Gauss and Codazzi*

$$R(X,\,Y)Z = \frac{\tilde{c}}{4}\{\langle Y,\,Z\rangle X - \langle X,\,Z\rangle Y\} + [\hat{\sigma}(X),\,\hat{\sigma}(Y)](Z),$$
$$(\nabla_X\hat{\sigma})(Y) = (\nabla_Y\hat{\sigma})(X).$$

Then there exists an isometric immersion $f\colon M^n \to \bar{M}^n(\tilde{c})$ which is a totally real submanifold in $\bar{M}^n(\tilde{c}) = \mathbb{C}P^n$ or $\mathbb{C}H^n$ according as $\tilde{c}$ is positive or negative such that the second fundamental form of f coincides with $\hat{\sigma}$. Moreover, such an immersion f is unique up to the action by holomorphically isometries of $\bar{M}^n(\tilde{c})$.

Proof. First we will investigate the structure of Lie algebras of the ambient space. Let $(G,\,K)$ be the Riemannian symmetric pair associated with $\bar{M}^n(\tilde{c})$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ which corresponds to K and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition. Let $\mathbb{C}^n$ be the complex vector space of column n-tuples of complex numbers with the standard Hermitian inner product $\langle\ ,\ \rangle_{\mathbb{C}}$ and $U(n)$ be the unitary group. Then K is isomorphic to $U(n)$ and $\mathfrak{p}$ is isomorphic to $\mathbb{C}^n$ and the adjoint representation $\mathrm{Ad}_{\mathfrak{p}}(K)$ of K on $\mathfrak{p}$ is given by the canonical action of $U(n)$ on $\mathbb{C}^n$. We define a real linear endomorphism I of $\mathbb{C}^n$ by $I(\boldsymbol{x}) = i\boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{C}^n$. We denote by $\langle\ ,\ \rangle$ the real inner product on $\mathbb{C}^n$ defined by taking the real part of $\langle\ ,\ \rangle_{\mathbb{C}}$. The curvature tensor $\bar{R}$ of $\bar{M}^n(\tilde{c})$ on $\mathfrak{p}$ is of the form

$$\bar{R}(\boldsymbol{x},\,\boldsymbol{y})\boldsymbol{z} = \frac{\tilde{c}}{4}\big\{\langle \boldsymbol{y},\,\boldsymbol{z}\rangle\boldsymbol{x} - \langle \boldsymbol{x},\,\boldsymbol{z}\rangle\boldsymbol{y} + \langle I\boldsymbol{y},\,\boldsymbol{z}\rangle I\boldsymbol{x} - \langle I\boldsymbol{x},\,\boldsymbol{z}\rangle I\boldsymbol{y} - 2\langle I\boldsymbol{x},\,\boldsymbol{y}\rangle I\boldsymbol{z}\big\} \quad (2.6)$$

for $\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{z} \in \mathfrak{p} = \mathbb{C}^n$. We define real subspaces $\mathfrak{m}$ and $\mathfrak{m}^{\perp}$ of $\mathfrak{p} = \mathbb{C}^n$ by

$$\mathfrak{m} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n \;\middle|\; x_j \in \mathbb{R} \right\} = \mathbb{R}^n,$$
$$\mathfrak{m}^\perp = \left\{ i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n \;\middle|\; x_j \in \mathbb{R} \right\} \simeq \mathbb{R}^n.$$

Then $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are mutually orthogonal with respect to $\langle\ ,\ \rangle$. Since $I\mathfrak{m} = \mathfrak{m}^\perp$, both $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are totally real subspaces. By (2.6), we see that both $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are curvature-invariant and in particular $\mathfrak{m}$ is a strongly curvature-invariant subspace of $\mathfrak{p}$. It is evident that the class of n-dimensional totally real submanifolds in $\bar{M}^n(\tilde{c})$ coincides with that of $\mathcal{O}(\mathfrak{m})$-submanifolds, where $\mathfrak{m} \subset \mathfrak{p}$ is given above.

Let K_+ be the subgroup of $K = U(n)$ which leaves the subspace $\mathfrak{m}$ invariant. Then K_+ consists of unitary matrices whose entries are all real numbers and hence K_+ coincides with the orthogonal group $O(n)$. The representation $\rho' \colon K_+ \to O(\mathfrak{m})$ is the canonical action of $O(n)$ on $\mathbb{R}^n$ and $\rho'' \colon K_+ \to O(\mathfrak{m}^\perp)$ is equivalent to ρ' under the isomorphism by the complex structure I. The Lie algebra $\mathfrak{k}_+$ of K_+ coincides with the Lie algebra $\mathfrak{so}(n)$ consisting of real skew-symmetric matrices. We define the subspace $\mathfrak{k}_-$ of $\mathfrak{k}$ as follows

$$\mathfrak{k}_- = \{ iX \mid X \in \mathrm{Sym}(n, \mathbb{R}) \} \simeq \mathrm{Sym}(n, \mathbb{R}),$$

where $\mathrm{Sym}(n, \mathbb{R})$ denotes the space of real $n \times n$ symmetric matrices. Then we have the direct sum decomposition $\mathfrak{k} = \mathfrak{k}_+ + \mathfrak{k}_-$ and $\mathrm{ad}_{\mathfrak{p}}(\mathfrak{k}_-)(\mathfrak{m}) \subset \mathfrak{m}^\perp$, $\mathrm{ad}_{\mathfrak{p}}(\mathfrak{k}_-)(\mathfrak{m}^\perp) \subset \mathfrak{m}$. Let $\mathrm{Sym}(\mathfrak{m})$ be the space of symmetric transformations of $\mathfrak{m}$. We define a map $\psi \colon \mathfrak{k}_- \to \mathrm{Sym}(\mathfrak{m})$ by $\psi(X)(\boldsymbol{x}) = -iX\boldsymbol{x}$ for $X \in \mathfrak{k}_-$ and $\boldsymbol{x} \in \mathfrak{m} = \mathbb{R}^n$. Then ψ is a real linear isomorphism and we identify $\mathfrak{k}_-$ with $\mathrm{Sym}(\mathfrak{m})$ by ψ.

To apply Theorem 2.8, we will construct a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry (P, ω) which satisfies the assumptions in Theorem 2.8. Let $O(M)$ be the bundle of orthonormal frames over M with the Riemannian connection form ω' and the canonical 1-form θ. We put $P = O(M)$, which is the principal fibre bundle with the structure group $K_+ = O(n)$. We view forms ω' and θ on P as a $\mathfrak{k}_+$-valued 1-form and a $\mathfrak{m}$-valued 1-form, respectively. Under the identification of $\mathfrak{k}_-$ with $\mathrm{Sym}(\mathfrak{m})$, we put

$$\omega_{\mathfrak{k}_-}(\tilde{X}) = u^{-1}\hat{\sigma}(\pi_*\tilde{X})u \quad \text{for } \tilde{X} \in T_uP, \quad u \in P = O(M),$$

where $\pi\colon P \to M$ denotes the projection. Now putting $\omega = \omega' + \omega_{\mathfrak{k}_-} + \theta$, we define a $\mathfrak{k}_+ + \mathfrak{k}_- + \mathfrak{m}$-valued 1-form ω on P. It is easily seen that (P, ω) is a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry. If we view the tangent bundle TM as a Riemannian vector bundle E in Theorem 2.8, the conditions in Theorem 2.8 are satisfied. Therefore Theorem 2.9 has been proved. □

3. Totally complex submanifolds

In this section, applying the results in §2 we show the fundamental theorem for half dimensional totally complex submanifolds of the quaternion projective space $\mathbb{H}P^n$ or the quaternion hyperbolic space $\mathbb{H}H^n$.

First we recall the basic definitions and facts on totally complex submanifolds of a quaternionic Kähler manifold. Let $(\tilde{M}^{4n}, \tilde{g}, \tilde{Q})$ be a quaternionic Kähler manifold with the quaternionic Kähler structure $(\tilde{g}, \tilde{Q})$, that is, $\tilde{g}$ is the Riemannian metric on $\tilde{M}$ and $\tilde{Q}$ is a rank 3 subbundle of $\operatorname{End} T\tilde{M}$ which satisfies the following conditions:

(a) For each $p \in \tilde{M}$, there is a neighborhood U of p over which there exists a local frame field $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ of $\tilde{Q}$ satisfying

$$\tilde{I}^2 = \tilde{J}^2 = \tilde{K}^2 = -\operatorname{id}, \quad \tilde{I}\tilde{J} = -\tilde{J}\tilde{I} = \tilde{K},$$
$$\tilde{J}\tilde{K} = -\tilde{K}\tilde{J} = \tilde{I}, \quad \tilde{K}\tilde{I} = -\tilde{I}\tilde{K} = \tilde{J}.$$

(b) For any element $L \in \tilde{Q}_p$, $\tilde{g}_p$ is invariant by L, i.e., $\tilde{g}_p(LX, Y) + \tilde{g}_p(X, LY) = 0$ for $X, Y \in T_p\tilde{M}$, $p \in \tilde{M}$.

(c) The vector bundle $\tilde{Q}$ is parallel in $\operatorname{End} T\tilde{M}$ with respect to the Riemannian connection $\tilde{\nabla}$ associated with $\tilde{g}$.

In this paper we assume that the dimension of $\tilde{M}^{4n}$ is not less than 8 and that $\tilde{M}^{4n}$ has nonvanishing scalar curvature. A submanifold M^{2m} of $\tilde{M}$ is said to be *almost Hermitian* if there exists a section $\tilde{I}$ of the bundle $\tilde{Q}|_M$ such that (1) $\tilde{I}^2 = -\operatorname{id}$, (2) $\tilde{I}TM = TM$ (cf. D.V. Alekseevsky and S. Marchiafava [1]). We denote by I the almost complex structure on M induced from $\tilde{I}$. Evidently (M, I) with the induced metric g is an almost Hermitian manifold. If (M, g, I) is Kähler, we call it a *Kähler submanifold* of a quaternionic Kähler manifold $\tilde{M}$. An almost Hermitian submanifold M together with a section $\tilde{I}$ of $\tilde{Q}|_M$ is said to be *totally complex* if at each point $p \in M$ we have $LT_pM \perp T_pM$, for each $L \in \tilde{Q}_p$ with $\tilde{g}(L, \tilde{I}_p) = 0$ (cf. S. Funabashi [3]). It is known that a $2m$ $(m \geq 2)$-dimensional almost

Hermitian submanifold M^{2m} is Kähler if and only if it is totally complex ([1] Theorem 1.12).

Let M^{2m} be a $2m$ ($m \geq 2$)-dimensional totally complex submanifold of $(\tilde{M}^{4n}, \tilde{g}, \tilde{Q})$ together with a section $\tilde{I}$ of $\tilde{Q}|_M$. The bundle $\tilde{Q}|_M$ has the following decomposition:

$$\tilde{Q}|_M = \mathbb{R}\tilde{I} + Q', \tag{3.1}$$

where Q' is defined by $Q'_p = \{L \in \tilde{Q}_p | \tilde{g}(L, \tilde{I}_p) = 0\}$ at each point $p \in M$. Then the section $\tilde{I}$ of $\tilde{Q}|_M$ and the vector subbundle Q' are parallel with respect to the induced connection $\tilde{\nabla}$ on $\tilde{Q}|_M$ ([13] Lemma 2.10). At each point $p \in M$, we define a complex structure I on the fibre Q'_p by $IL = \tilde{I}L$ for $L \in Q'_p$. Hence Q' becomes a complex line bundle over M. Moreover the induced connection $\tilde{\nabla}$ is complex linear on Q'. The curvature form R' of the connection $\tilde{\nabla}$ on Q' is given by

$$R'(X, Y) = -\frac{\tilde{\tau}}{4n(n+2)}\Omega(X, Y)I,$$

where $\tilde{\tau}$ is the scalar curvature of $\tilde{M}$ and Ω is the Kähler form of M defined by $\Omega(X, Y) = g(IX, Y)$ for $X, Y \in T_pM$. Since the curvature R' is of degree (1, 1), there is a unique holomorphic line bundle structure in Q' such that a (local) holomorphic section L is defined by $\tilde{\nabla}_{IX}L = I\tilde{\nabla}_X L$ for any vector field X. The normal bundle $T^{\perp}M$ is a complex vector bundle with the complex strucutre I induced from $\tilde{I}$ which satisfies $\nabla^{\perp}_X I = 0$, where $\nabla^{\perp}$ denotes the connection of $T^{\perp}M$. Let σ be the second fundamental form of M in $\tilde{M}$. Then we have the following at each point $p \in M$ ([13] Proposition 2.11 and Lemma 2.13):

(1) $\sigma(IX, Y) = \sigma(X, IY) = I\sigma(X, Y)$ for $X, Y \in T_pM$,

(2) $\tilde{g}(\sigma(X, Y), LZ) = \tilde{g}(\sigma(X, Z), LY)$ for $L \in Q'_p$, $X, Y, Z \in T_pM$.

We study half dimensional totally complex submanifolds of the quaternion projective space or the quaternion hyperbolic space from the view point of a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry. For this purpose, we describe the structure of Lie algebras of the ambient space. Let (G, K) be a Riemannian symmetric pair associated with a (real) $4n$-dimensional quaternion projective space $\mathbb{H}P^n$ or a (real) $4n$-dimensional quaternion hyperbolic space $\mathbb{H}H^n$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the Lie subalgebra of $\mathfrak{g}$ which corresponds to K and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition. First we describe the adjoint representation of K on $\mathfrak{p}$. Let $\mathbb{H}^n$ be the space of column n-tuples with entries in the field $\mathbb{H}$ of quaternions. The space $\mathbb{H}^n$ is considered as a right $\mathbb{H}$-vector space, i.e., vectors are multiplied by quaternions from the right. We define a $\mathbb{H}$-Hermitian inner product $\langle\ ,\ \rangle_{\mathbb{H}}$ by

$$\langle \boldsymbol{x},\, \boldsymbol{y} \rangle_{\mathbb{H}} = \sum_{i=1}^{n} \overline{x_i} y_i, \quad \boldsymbol{x},\, \boldsymbol{y} \in \mathbb{H}^n$$

and its real inner product $\langle\ ,\ \rangle_{\mathbb{R}}$ by

$$\langle \boldsymbol{x},\, \boldsymbol{y} \rangle_{\mathbb{R}} = \text{the real part of } \langle \boldsymbol{x},\, \boldsymbol{y} \rangle_{\mathbb{H}}.$$

Let $Sp(1)$ be the Lie group of unit quaternions, i.e.,

$$Sp(1) = \big\{ \mu \in \mathbb{H} \bigm| \langle \mu,\, \mu \rangle_{\mathbb{R}} = 1 \big\}$$

and $Sp(n)$ be the Lie group of $\mathbb{H}$-linear transformations of $\mathbb{H}^n$ which leave the $\mathbb{H}$-Hermitian inner product invariant. The Lie algebra $\mathfrak{sp}(n)$ of $Sp(n)$ is the space of $\mathbb{H}$-linear transformations which are skew-Hermitian with respect to $\langle\ ,\ \rangle_{\mathbb{H}}$. The product Lie group $Sp(n) \times Sp(1)$ acts on $\mathbb{H}^n$ as $\mathbb{R}$-linear transformations which leave the real inner product $\langle\ ,\ \rangle_{\mathbb{R}}$ invariant by letting $Sp(n)$ act on the left and $Sp(1)$ act on the right:

$$(Sp(n) \times Sp(1)) \times \mathbb{H}^n \to \mathbb{H}^n \quad \big((A,\, \lambda),\, \boldsymbol{x}\big) \mapsto A\boldsymbol{x}\lambda^{-1} = A\boldsymbol{x}\bar{\lambda}.$$

We remark that the right multiplication by a quaternion $\lambda \in \mathbb{H}$ is real linear but not necessarily quaternion linear. We put $\mathfrak{p} = \mathbb{H}^n$ and $K = Sp(n) \times Sp(1)$. Then the action of K on $\mathfrak{p}$ is the adjoint representation of K on $\mathfrak{p}$ which corresponds to the Riemannian symmetric pair of $\mathbb{H}P^n$ or $\mathbb{H}H^n$. We put real linear transformations $\tilde{I}$, $\tilde{J}$, and $\tilde{K}$ as follows:

$$\tilde{I}\boldsymbol{x} = \boldsymbol{x}i,\ \tilde{J}\boldsymbol{x} = \boldsymbol{x}j, \quad \text{for } \boldsymbol{x} \in \mathbb{H}^n, \quad \text{and} \quad \tilde{K} = \tilde{I}\tilde{J}, \tag{3.2}$$

where $\{1,\, i,\, j,\, k\}$ denotes the standard basis of $\mathbb{H}$. Then it follows that

$$\tilde{I}^2 = \tilde{J}^2 = \tilde{K}^2 = -\operatorname{id}, \quad \tilde{I}\tilde{J} = -\tilde{J}\tilde{I} = \tilde{K},$$
$$\tilde{J}\tilde{K} = -\tilde{K}\tilde{J} = \tilde{I}, \quad \tilde{K}\tilde{I} = -\tilde{I}\tilde{K} = \tilde{J}.$$

The Lie group $Sp(1)$ is given by

$$\big\{ a_0 \operatorname{id} + a_1\tilde{I} + a_2\tilde{J} + a_3\tilde{K} \bigm| a_\alpha \in \mathbb{R},\ a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 \big\}$$

and its Lie algebra $\mathfrak{sp}(1)$ is spanned over $\mathbb{R}$ by $\tilde{I}$, $\tilde{J}$, and $\tilde{K}$. The curvature tensor $\bar{R}$ on $\mathfrak{p}$ of $\mathbb{H}P^n$ or $\mathbb{H}H^n$ is of the form

$$\begin{aligned}\bar{R}(\boldsymbol{x},\, \boldsymbol{y})\boldsymbol{z} = \frac{\tilde{c}}{4}\big\{&\langle \boldsymbol{y},\, \boldsymbol{z}\rangle \boldsymbol{x} - \langle \boldsymbol{x},\, \boldsymbol{z}\rangle \boldsymbol{y} + \langle \tilde{I}\boldsymbol{y},\, \boldsymbol{z}\rangle \tilde{I}\boldsymbol{x} - \langle \tilde{I}\boldsymbol{x},\, \boldsymbol{z}\rangle \tilde{I}\boldsymbol{y} + \\ &\langle \tilde{J}\boldsymbol{y},\, \boldsymbol{z}\rangle \tilde{J}\boldsymbol{x} - \langle \tilde{J}\boldsymbol{x},\, \boldsymbol{z}\rangle \tilde{J}\boldsymbol{y} + \langle \tilde{K}\boldsymbol{y},\, \boldsymbol{z}\rangle \tilde{K}\boldsymbol{x} - \langle \tilde{K}\boldsymbol{x},\, \boldsymbol{z}\rangle \tilde{K}\boldsymbol{y} \\ &- 2\langle \tilde{I}\boldsymbol{x},\, \boldsymbol{y}\rangle \tilde{I}\boldsymbol{z} - 2\langle \tilde{J}\boldsymbol{x},\, \boldsymbol{y}\rangle \tilde{J}\boldsymbol{z} - 2\langle \tilde{K}\boldsymbol{x},\, \boldsymbol{y}\rangle \tilde{K}\boldsymbol{z}\big\}, \end{aligned} \tag{3.3}$$

for $\boldsymbol{x},\, \boldsymbol{y},\, \boldsymbol{z} \in \mathfrak{p} = \mathbb{H}^n$, where $\tilde{c}$ is a positive or negative constant according as the space is $\mathbb{H}P^n$ or $\mathbb{H}H^n$ and we simply write $\langle\ ,\ \rangle$ for $\langle\ ,\ \rangle_{\mathbb{R}}$.

The quaternion vector space $\mathbb{H}^n$ can be considered as a complex vector space in a variety of natural ways. We choose a real linear transformation $\tilde{I}$ defined by (3.2) as a complex structure and define a complex scalar multiplication on $\mathbb{H}^n$ by $(a+bi)\boldsymbol{x} = (a\,\mathrm{id} + b\tilde{I})\boldsymbol{x}$ for $a,\, b \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{H}^n$. From now on we fix this complex structure $\tilde{I}$ on $\mathbb{H}^n$. We denote by $\mathrm{End}_{\mathbb{H}}(\mathbb{H}^n)$ the space of $\mathbb{H}$-linear transformations of $\mathbb{H}^n$ and by $\mathrm{End}_{\mathbb{C}}(\mathbb{H}^n,\, \tilde{I})$ the space of $\mathbb{C}$-linear transformations of $(\mathbb{H}^n,\, \tilde{I})$. Clearly $\mathrm{End}_{\mathbb{H}}(\mathbb{H}^n) \subset \mathrm{End}_{\mathbb{C}}(\mathbb{H}^n,\, \tilde{I})$. It is known that for $A \in \mathrm{End}_{\mathbb{H}}(\mathbb{H}^n)$, the complex determinant $\det_{\mathbb{C}} A$ viewed as a $\mathbb{C}$-linear transformation of $(\mathbb{H}^n,\, \tilde{I})$ is a non-negative real number. If $L = a\tilde{J} + b\tilde{K}$, $a,\, b \in \mathbb{R}$, then L is a semi-linear transformation of $(\mathbb{H}^n,\, \tilde{I})$, i.e.,

$$L(\lambda \boldsymbol{x}) = \bar{\lambda} L(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \mathbb{H}^n, \lambda \in \mathbb{C}.$$

Given $A \in \mathrm{End}_{\mathbb{C}}(\mathbb{H}^n,\, \tilde{I})$, we see that $A \in \mathrm{End}_{\mathbb{H}}(\mathbb{H}^n)$ if and only if $A\tilde{J} = \tilde{J}A$. As usual we use the subfield $\mathbb{C} \subset \mathbb{H}$ generated by 1 and i and put

$$\langle \boldsymbol{x},\, \boldsymbol{y}\rangle_{\mathbb{C}} = \text{the complex part of } \langle \boldsymbol{x},\, \boldsymbol{y}\rangle_{\mathbb{H}}.$$

Then $\langle\ ,\ \rangle_{\mathbb{C}}$ is a $\mathbb{C}$-Hermitian inner product on $(\mathbb{H}^n,\, \tilde{I})$. Since $\langle \boldsymbol{x},\, \boldsymbol{y}\rangle_{\mathbb{H}} = \langle \boldsymbol{x},\, \boldsymbol{y}\rangle_{\mathbb{C}} + j\langle \tilde{J}\boldsymbol{x},\, \boldsymbol{y}\rangle_{\mathbb{C}}$, we can show that

$$\begin{aligned} Sp(n) = \big\{A \in \mathrm{End}_{\mathbb{C}}(\mathbb{H}^n,\, \tilde{I}) \bigm| A\tilde{J} = \tilde{J}A,& \\ \langle A\boldsymbol{x},\, A\boldsymbol{y}\rangle_{\mathbb{C}} = \langle \boldsymbol{x},\, \boldsymbol{y}\rangle_{\mathbb{C}} \quad &\text{for } \boldsymbol{x},\, \boldsymbol{y} \in \mathbb{H}^n\big\}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \mathfrak{sp}(n) = \big\{X \in \mathrm{End}_{\mathbb{C}}(\mathbb{H}^n,\, \tilde{I}) \bigm| X\tilde{J} = \tilde{J}X,& \\ \langle X\boldsymbol{x},\, \boldsymbol{y}\rangle_{\mathbb{C}} + \langle \boldsymbol{x},\, X\boldsymbol{y}\rangle_{\mathbb{C}} = 0 \quad &\text{for } \boldsymbol{x},\, \boldsymbol{y} \in \mathbb{H}^n\big\}. \end{aligned} \tag{3.5}$$

Let $\{e_1,\, e_2,\, \ldots,\, e_n\}$ be the standard quaternion basis of $\mathbb{H}^n$ (i.e., e_i is the vector of $\mathbb{H}^n$ whose i-th component is 1 and the other components are zero). Then $\{e_1,\, e_2,\, \ldots,\, e_n,\, \tilde{J}e_1,\, \tilde{J}e_2,\, \ldots,\, \tilde{J}e_n\}$ is a unitary basis of $(\mathbb{H}^n,\, \tilde{I})$ with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$. Using this complex basis, we identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$: for $\boldsymbol{x} = {}^t(x_1, x_2, \ldots, x_n) \in \mathbb{H}^n$, we put $x_\alpha = v_\alpha + j w_\alpha$, $v_\alpha, w_\alpha \in \mathbb{C}$ ($\alpha = 1, 2, \ldots, n$) and $\boldsymbol{v} = {}^t(v_1, v_2, \ldots, v_n)$ and $\boldsymbol{w} = {}^t(w_1, w_2, \ldots, w_n)$. Then the identification of $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ is given by $\mathbb{H}^n \ni \boldsymbol{x} \mapsto \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} \in \mathbb{C}^{2n}$. Given $A \in \mathrm{End}_{\mathbb{C}}(\mathbb{H}^n, \tilde{I})$, we represent A by $2n \times 2n$-matrix with entries in $\mathbb{C}$ with respect to the basis $\{e_1, e_2, \ldots, e_n, \tilde{J}e_1, \tilde{J}e_2, \ldots, \tilde{J}e_n\}$. Then the Lie group $Sp(n)$ and its Lie algebra $\mathfrak{sp}(n)$ are given as follows:

$$Sp(n) = \left\{ \begin{pmatrix} A_{11} & -\overline{A_{21}} \\ A_{21} & \overline{A_{11}} \end{pmatrix} \middle| \; A_{11}, A_{21} \in M_n(\mathbb{C}) \right\} \cap U(2n) \tag{3.6}$$

$$\mathfrak{sp}(n) = \left\{ \begin{pmatrix} X_{11} & -\overline{X_{21}} \\ X_{21} & \overline{X_{11}} \end{pmatrix} \middle| \; \begin{matrix} X_{11}, X_{21} \in M_n(\mathbb{C}) \\ {}^t\overline{X_{11}} + X_{11} = O,\ {}^tX_{21} = X_{21} \end{matrix} \right\}, \tag{3.7}$$

where $M_n(\mathbb{C})$ denotes the space of $n \times n$-matrices with entries in $\mathbb{C}$. For $Sp(1)$, we have

$$\tilde{I}\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} i\boldsymbol{v} \\ i\boldsymbol{w} \end{pmatrix},\ \tilde{J}\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} -\overline{\boldsymbol{w}} \\ \overline{\boldsymbol{v}} \end{pmatrix},\ \tilde{K}\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} -i\overline{\boldsymbol{w}} \\ i\overline{\boldsymbol{v}} \end{pmatrix}.$$

We define complex subspaces $\mathfrak{m}$ and $\mathfrak{m}^\perp$ of $\mathfrak{p} = \mathbb{H}^n \simeq \mathbb{C}^{2n}$ by

$$\mathfrak{m} = \left\{ \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{0} \end{pmatrix} \in \mathbb{C}^{2n} \middle| \; \boldsymbol{v} \in \mathbb{C}^n \right\}, \quad \mathfrak{m}^\perp = \left\{ \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{w} \end{pmatrix} \in \mathbb{C}^{2n} \middle| \; \boldsymbol{w} \in \mathbb{C}^n \right\}.$$

Then $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are mutually orthogonal with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$ and hence $\langle\ ,\ \rangle_{\mathbb{R}}$. Since $\tilde{J}(\mathfrak{m}) = \mathfrak{m}^\perp$, both $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are totally complex subspaces. By (3.3), we see that $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are curvature-invariant. In particular $\mathfrak{m}$ is a strongly curvature-invariant subspace of $\mathfrak{p}$. We will describe the decomposition of the Lie algebra $\mathfrak{k} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ corresponding to the subspace $\mathfrak{m}$. We use the same notations as in §2. Let K_+ be the subgroup of $K = Sp(n) \times Sp(1)$ whose adjoint representation leaves the subspace $\mathfrak{m}$ invariant. Then K_+ is given as follows:

$$\begin{aligned} K_+ = &\left\{ \left(\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix},\ a\,\mathrm{id} + b\tilde{I} \right) \middle| \; A \in U(n),\ a, b \in \mathbb{R},\ a^2 + b^2 = 1 \right\} \\ \cup &\left\{ \left(\begin{pmatrix} 0 & -\overline{A} \\ A & 0 \end{pmatrix},\ a\tilde{J} + b\tilde{K} \right) \middle| \; A \in U(n),\ a, b \in \mathbb{R},\ a^2 + b^2 = 1 \right\} \end{aligned}$$

In particular the identity component $(K_+)_o$ of K_+ is given by

$$\left\{ \left(\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix},\ a\,\mathrm{id} + b\tilde{I} \right) \,\middle|\, A \in U(n),\ a,\, b \in \mathbb{R},\ a^2 + b^2 = 1 \right\}.$$

Therefore $(K_+)_o$ is identified with the product Lie group $U(n) \times U(1)$ by the isomorphism $\left(\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix},\ a\,\mathrm{id} + b\tilde{I} \right) \mapsto (A,\, a + bi)$. From now on we use this identification. The Lie algebra $\mathfrak{k}_+$ of $(K_+)_o$ is given by

$$\mathfrak{k}_+ = \left\{ \left(\begin{pmatrix} X & 0 \\ 0 & \overline{X} \end{pmatrix},\ x\tilde{I} \right) \,\middle|\, X \in \mathfrak{u}(n),\ x \in \mathbb{R} \right\}$$

and we denote the ideals of $\mathfrak{k}_+$ by

$$\mathfrak{k}_+^1 = \left\{ \begin{pmatrix} X & 0 \\ 0 & \overline{X} \end{pmatrix} \,\middle|\, X \in \mathfrak{u}(n) \right\} \quad \text{and} \quad \mathfrak{k}_+^2 = \left\{ x\tilde{I} \mid x \in \mathbb{R} \right\}.$$

Then $\mathfrak{k}_+^1$ and $\mathfrak{k}_+^2$ are naturally identified with $\mathfrak{u}(n)$ and $\mathfrak{u}(1)$ by the isomorphisms $\begin{pmatrix} X & 0 \\ 0 & \overline{X} \end{pmatrix} \mapsto X$ and $x\tilde{I} \mapsto xi$, respectively. The Lie group homomorphisms $\rho' \colon (K_+)_o \to O(\mathfrak{m})$ and $\rho'' \colon (K_+)_o \to O(\mathfrak{m}^\perp)$ are written as follows:

$$\rho'\big((A,\, \lambda)\big)(\boldsymbol{v}) = \lambda A \boldsymbol{v}, \tag{3.8}$$

$$\rho''\big((A,\, \lambda)\big)(\boldsymbol{v}) = \lambda \overline{A} \boldsymbol{v}, \tag{3.9}$$

for $(A,\, \lambda) \in U(n) \times U(1) \simeq (K_+)_o$ and $\boldsymbol{v} \in \mathbb{C}^n$, where we identify $\mathfrak{m}$ and $\mathfrak{m}^\perp$ with $\mathbb{C}^n$, respectively. The Lie algebra homomorphisms $\rho' \colon \mathfrak{k}_+ \to \mathfrak{so}(\mathfrak{m})$ and $\rho'' \colon \mathfrak{k}_+ \to \mathfrak{so}(\mathfrak{m}^\perp)$ are written as follows:

$$\rho'\big((X,\, xi)\big)(\boldsymbol{v}) = X\boldsymbol{v} + xi\boldsymbol{v}, \tag{3.10}$$

$$\rho''\big((X,\, xi)\big)(\boldsymbol{v}) = \overline{X}\boldsymbol{v} + xi\boldsymbol{v}, \tag{3.11}$$

for $(X,\, xi) \in \mathfrak{u}(n) \oplus \mathfrak{u}(1) \simeq \mathfrak{k}_+$.

The subspace $\mathfrak{k}_-$ of the Lie algebra $\mathfrak{k} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ is given by

$$\mathfrak{k}_- = \left\{ \left(\begin{pmatrix} 0 & -\overline{X} \\ X & 0 \end{pmatrix},\ x\tilde{J} + y\tilde{K} \right) \,\middle|\, X \in \mathrm{Sym}(n,\, \mathbb{C}),\ x,\, y \in \mathbb{R} \right\},$$

where $\mathrm{Sym}(n,\, \mathbb{C})$ denotes the space of complex $n \times n$ symmetric matrices.

We put the subspaces $\mathfrak{k}_-^1$ and $\mathfrak{k}_-^2$ as follows:

$$\mathfrak{k}_-^1 = \left\{ \begin{pmatrix} 0 & -\overline{X} \\ X & 0 \end{pmatrix} \,\middle|\, X \in \mathrm{Sym}(n,\,\mathbb{C}) \right\} \subset \mathfrak{sp}(n)$$
$$\mathfrak{k}_-^2 = \left\{ x\tilde{J} + y\tilde{K} \mid x,\, y \in \mathbb{R} \right\} \subset \mathfrak{sp}(1).$$

Then we have the direct sum decomposition $\mathfrak{k}_- = \mathfrak{k}_-^1 \oplus \mathfrak{k}_-^2$. We consider $\mathfrak{k}_-^2$ as a 1-dimensional complex vector space with the complex structure $\tilde{I}$, i.e., $i(x\tilde{J} + y\tilde{K}) = \tilde{I}(x\tilde{J} + y\tilde{K}) = -y\tilde{J} + x\tilde{K}$. Then the adjoint representation of $(K_+)_o$ on $\mathfrak{k}_-^2$ is given by

$$\mathrm{Ad}\big((A,\,\lambda)\big)(x\tilde{J} + y\tilde{K}) = \lambda^2(x\tilde{J} + y\tilde{K}),$$

for $(A,\,\lambda) \in U(n) \times U(1) \simeq (K_+)_o$.

Lemma 3.1 *The space $\mathfrak{m}^\perp$ is equivalent to the tensor product $\mathfrak{k}_-^2 \otimes_{\mathbb{C}} \overline{\mathfrak{m}}$ over $\mathbb{C}$ as the representation spaces by $(K_+)_o$, where $\overline{\mathfrak{m}}$ denotes the complex conjugate vector space of $\mathfrak{m}$.*

Proof. We define a map $\varphi\colon \mathfrak{k}_-^2 \times \mathfrak{m} \to \mathfrak{m}^\perp$ by $\varphi\big((L,\,\boldsymbol{v})\big) = L\boldsymbol{v}$, for $L \in \mathfrak{k}_-^2$, $\boldsymbol{v} \in \mathfrak{m}$. Then φ is a complex linear map for the first factor $\mathfrak{k}_-^2$ and a semi-linear map for the second factor $\mathfrak{m}$. For $(A,\,\lambda) \in U(n)\times U(1) \simeq (K_+)_o$, we have

$$\begin{aligned}\varphi\big((A,\,\lambda)\cdot(L,\,\boldsymbol{v})\big) &= \varphi\big((\mathrm{Ad}\big((A,\,\lambda)\big)(L),\,\rho'\big((A,\,\lambda)\big)(\boldsymbol{v}))\big)\\ &= \varphi(\lambda^2 L,\,\lambda A\boldsymbol{v})\\ &= \lambda^2 L(\lambda A\boldsymbol{v}) = \lambda^2\bar{\lambda}\overline{A}L(\boldsymbol{v})\\ &= \lambda\overline{A}L(\boldsymbol{v}) = \rho''\big((A,\,\lambda)\big)\varphi(L,\,\boldsymbol{v}).\end{aligned}$$

Therefore there exists a $(K_+)_o$-equivariant complex linear map $\tilde{\varphi}\colon \mathfrak{k}_-^2 \otimes_{\mathbb{C}} \overline{\mathfrak{m}} \to \mathfrak{m}^\perp$ which satisfies $\tilde{\varphi}(L \otimes \boldsymbol{v}) = \varphi\big((L,\,\boldsymbol{v})\big)$. Since $\tilde{J}$ is a semi-linear isomorphism of $\mathfrak{m}$ onto $\mathfrak{m}^\perp$, $\tilde{\varphi}$ is a linear isomorphism. □

Let $\psi\colon \mathfrak{k}_- \to \mathrm{Hom}(\mathfrak{m},\,\mathfrak{m}^\perp)$ be a linear map defined by the action of $\mathfrak{k}_-$ on $\mathfrak{m}$. It is injective as is stated in §2. The image $\psi(\mathfrak{k}_-^1)$ is characterized as follows:

Lemma 3.2 *For $C \in \mathrm{Hom}(\mathfrak{m},\,\mathfrak{m}^\perp)$, C is in $\psi(\mathfrak{k}_-^1)$ if and only if C is a complex linear map which satisfies $\langle C\boldsymbol{v},\,L\boldsymbol{w}\rangle_{\mathbb{R}} = \langle C\boldsymbol{w},\,L\boldsymbol{v}\rangle_{\mathbb{R}}$ for any $L \in \mathfrak{k}_-^2$ and $\boldsymbol{v},\,\boldsymbol{w} \in \mathfrak{m}$.*

Proof. For $T \in \mathfrak{k}^1_-$, T is a $\mathbb{H}$-linear transformation of $\mathfrak{p}$ and skew-symmetric with respect to $\langle\ ,\ \rangle_{\mathbb{H}}$ and hence $\langle\ ,\ \rangle_{\mathbb{R}}$. Therefore we have

$$\begin{aligned}\langle T\boldsymbol{v},\, L\boldsymbol{w}\rangle_{\mathbb{R}} = -\langle \boldsymbol{v},\, TL\boldsymbol{w}\rangle_{\mathbb{R}} &= -\langle \boldsymbol{v},\, LT\boldsymbol{w}\rangle_{\mathbb{R}} \\ &= \langle L\boldsymbol{v},\, T\boldsymbol{w}\rangle_{\mathbb{R}} = \langle T\boldsymbol{w},\, L\boldsymbol{v}\rangle_{\mathbb{R}}.\end{aligned}$$

This implies that $C = \psi(T)$ satisfies the requirements.

Conversely let C be a complex linear map which satisfies $\langle C\boldsymbol{v},\, L\boldsymbol{w}\rangle_{\mathbb{R}} = \langle C\boldsymbol{w},\, L\boldsymbol{v}\rangle_{\mathbb{R}}$. We define a linear map C^* of $\mathfrak{m}^\perp$ into $\mathfrak{m}$ by $\langle C^*\xi,\, \boldsymbol{v}\rangle_{\mathbb{R}} = -\langle \xi,\, C\boldsymbol{v}\rangle_{\mathbb{R}}$ for $\boldsymbol{v} \in \mathfrak{m}$, $\xi \in \mathfrak{m}^\perp$. Then C^* is a complex linear map. We define a complex linear transformation T of $\mathfrak{p}$ by $T(\boldsymbol{v} + \xi) = C\boldsymbol{v} + C^*\xi$. Then T is skew-symmetric with respect to $\langle\ ,\ \rangle_{\mathbb{R}}$. Using the equation $\langle C\boldsymbol{v},\, L\boldsymbol{w}\rangle_{\mathbb{R}} = \langle C\boldsymbol{w},\, L\boldsymbol{v}\rangle_{\mathbb{R}}$, we can easily prove $T\tilde{J} = \tilde{J}T$. Therefore T is a $\mathbb{H}$-linear transformation of $\mathfrak{p}$ and hence T is in $\mathfrak{sp}(n)$. Moreover since $T(\mathfrak{m}) \subset \mathfrak{m}^\perp$ and $T(\mathfrak{m}^\perp) \subset \mathfrak{m}$, T is in $\mathfrak{k}^1_-$. □

It is evident that a $2n$ ($n \geq 2$)-dimensional totally complex submanifold of $\bar{M} = \mathbb{H}P^n$ or $\mathbb{H}H^n$ is an $\mathcal{O}(\mathfrak{m})$-submanifold and vice versa, where $\mathfrak{m}$ is a totally complex subspace discussed above. Let M be a (real) $2n$ ($n \geq 2$)-dimensional totally complex submanifold of $\bar{M} = \mathbb{H}P^n$ or $\mathbb{H}H^n$. For simplicity we assume that M is simply connected. Let $(P,\, \omega)$ be the corresponding locally ambient $\mathcal{O}(\mathfrak{m})$-geometry on M. Now we may assume that the structure group of P is the identity component $(K_+)_o$ of K_+. Let $\rho_o\colon (K_+)_o \to \mathrm{End}(\mathfrak{sp}(1))$ be the representation of $(K_+)_o$ which is obtained by restricting the adjoint representation of $(K_+)_o$ to $\mathfrak{sp}(1)$ and $\mathfrak{sp}(1) = \mathfrak{k}^2_+ \oplus \mathfrak{k}^2_-$ be the decomposition to the invariant subspaces by this representation. Then we have

$$\ker \rho_o = \bigl\{(A,\, \pm 1) \bigm| A \in U(n)\bigr\} \subset U(n) \times U(1) \simeq (K_+)_o$$

and the identity component of $\ker \rho_o$ is isomorphic to $U(n)$. We put $P_o = P/U(n)$ and denote by h_o the projection of P onto P_o. Then P_o is a principal fibre bundle with the structure group $U(1)$. Let $\tilde{Q}|_M = \mathbb{R}\tilde{I} + Q'$ be the decomposition of the quaternionic Kähler structure $\tilde{Q}$ given by (3.1). Then Q' is the complex line bundle with the standard fibre $\mathfrak{k}^2_-$ associated with the principal fibre bundle P_o corresponding to the representation $\rho_o(\lambda) = \lambda^2\, \mathrm{id}_{\mathbb{C}}$ for $\lambda \in U(1)$. The 1-form ω on P is decomposed as follows:

$$\omega = \omega_{\mathfrak{k}_+} + \omega_{\mathfrak{k}_-} + \omega_{\mathfrak{m}} = \omega_{\mathfrak{k}^1_+} + \omega_{\mathfrak{k}^2_+} + \omega_{\mathfrak{k}^1_-} + \omega_{\mathfrak{k}^2_-} + \omega_{\mathfrak{m}}. \tag{3.12}$$

Let ω_o be the connection form on P_o which is the pushforward form of the connection form $\omega_{\mathfrak{k}_+}$ on P; ω_o is a $\mathfrak{u}(1)$-valued 1-form on P_o such that $h_o^*\omega_o = \omega_{\mathfrak{k}^2_+}$. We obtain the following for $\omega_{\mathfrak{k}^2_-}$ and ω_o.

Lemma 3.3 *The* 1*-form* $\omega_{\mathfrak{k}^2_-}$ *vanishes on* P*. The curvature form* $d\omega_o$ *of* ω_o *is given by* $d\omega_o = -(\tilde{c}/2)\Omega_{\tilde{I}} i$*, where* $\Omega_{\tilde{I}}$ *is the* $\mathbb{R}$*-valued* 2*-form on* P_o *defined by* $\Omega_{\tilde{I}}(X, Y) = g(\tilde{I}\pi_* X, \pi_* Y)$ *for* $X, Y \in T_u P_o$*, where* $\pi\colon P_o \to M$ *is the projection.*

Proof. We put $\omega_{\mathfrak{k}^2_+}(X) = \alpha(X)\tilde{I}$ and $\omega_{\mathfrak{k}^2_-}(X) = \beta(X)\tilde{J} + \gamma(X)\tilde{K}$ for $X \in T_u P$ at any point $u \in P$, where α, β, and γ are $\mathbb{R}$-valued 1-forms on P. By taking the $\mathfrak{k}^2_-$-component of the equation (2.4.2), we have $d\omega_{\mathfrak{k}^2_-} + [\omega_{\mathfrak{k}^2_+}, \omega_{\mathfrak{k}^2_-}] = 0$. Therefore

$$d\beta + 2\gamma \wedge \alpha = 0, \tag{3.13}$$

$$d\gamma + 2\alpha \wedge \beta = 0. \tag{3.14}$$

By taking the $\mathfrak{k}^2_+$-component of the equation (2.4.1), we have

$$d\omega_{\mathfrak{k}^2_+} + \frac{1}{2}[\omega_{\mathfrak{k}^2_-}, \omega_{\mathfrak{k}^2_-}] = (\tilde{f}^*\bar{\Omega})_{\mathfrak{k}^2_+}.$$

Here $\bar{\Omega}$ is the curvature form of $\bar{M} = \mathbb{H}P^n$ or $\mathbb{H}H^n$. From the form (3.3) of the curvature tensor, it follows that $(\tilde{f}^*\bar{\Omega})_{\mathfrak{k}^2_+} = -(\tilde{c}/2)\Omega_{\tilde{I}}\tilde{I}$. Therefore we have

$$d\alpha + 2\beta \wedge \gamma = -\frac{\tilde{c}}{2}\Omega_{\tilde{I}}. \tag{3.15}$$

Differentiating (3.13), we have $d\gamma \wedge \alpha - \gamma \wedge d\alpha = 0$. By (3.14) and (3.15), it follows $(\tilde{c}/2)\gamma \wedge \Omega_{\tilde{I}} = 0$. Similarly differentiating (3.14), we have $-(\tilde{c}/2)\Omega_{\tilde{I}} \wedge \beta = 0$. Since $\dim_{\mathbb{R}} M = 2n \geq 4$, we obtain $\beta = \gamma = 0$. This together with (3.15) implies that $d\alpha = -(\tilde{c}/2)\Omega_{\tilde{I}}$ and hence $d\omega_{\mathfrak{k}^2_+} = -(\tilde{c}/2)\Omega_{\tilde{I}}\tilde{I}$. Since $h_o^*(d\omega_o) = dh_o^*\omega_o = d\omega_{\mathfrak{k}^2_+} = -(\tilde{c}/2)\Omega_{\tilde{I}}\tilde{I}$, the curvature form $d\omega_o$ of ω_o is given by $-(\tilde{c}/2)\Omega_{\tilde{I}} i$. □

Let $\rho'\colon (K_+)_o \to O(\mathfrak{m})$ be the representation of $(K_+)_o$ on $\mathfrak{m}$. Under the identification of $\mathfrak{m}$ with $\mathbb{C}^n$, by (3.8) it follows that $\rho'\big((K_+)_o\big) = U(n)$. By Corollary 2.6, M is a Kähler manifold. We note that this property holds for totally complex submanifolds in a quaternionic Kähler manifold with nonvanishing scalar curvature as explained in the beginning of this section. The normal bundle $T^{\perp}M$ is the complex vector bundle with the standard fibre $\mathfrak{m}^\perp$ associated with the principal fibre bundle P corresponding to the Lie group homomorphism $\rho''\colon (K_+)_o \to O(\mathfrak{m}^\perp)$ given by (3.9). Then by Lemma 3.1 we obtain the following.

Lemma 3.4 *The normal bundle $T^\perp M$ is naturally isomorphic to the tensor product $Q' \otimes_{\mathbb{C}} \overline{TM}$, where $\overline{TM}$ denotes the complex conjugate bundle of the tangent bundle TM.*

As is shown in §2, the second fundamental form σ is obtained from the $\mathrm{Hom}(\mathfrak{m}, \mathfrak{m}^\perp)$-valued 1-form $\hat{\sigma} = \psi \circ \omega_{\mathfrak{k}_-}$ on P. By Lemma 3.3, $\omega_{\mathfrak{k}^2_-} = 0$. Therefore $\hat{\sigma}$ has its value in $\psi(\mathfrak{k}^1_-)$. Then for $X \in T_uP$, $\hat{\sigma}(X)$ is a complex linear map which satisfies $\langle \hat{\sigma}(X)\boldsymbol{v}, L\boldsymbol{w}\rangle_{\mathbb{R}} = \langle \hat{\sigma}(X)\boldsymbol{w}, L\boldsymbol{v}\rangle_{\mathbb{R}}$ for any $L \in \mathfrak{k}^2_-$ and $\boldsymbol{v}, \boldsymbol{w} \in \mathfrak{m}$.

Now we show the fundamental theorem for totally complex submanifolds of $\bar{M} = \mathbb{H}P^n$ or $\mathbb{H}H^n$. We assume that the scalar curvature of $\bar{M}$ is $4n(n+2)\tilde{c}$. First we prepare geometric objects and the assumptions which they need to satisfy so that we can apply Theorem 2.8. Let M be a (real) $2n$ $(n \geq 2)$-dimensional simply connected Kähler manifold with the complex structure I and the Kähler metric $\langle\ ,\ \rangle$. We denote by Ω its Kähler form. Let P_o be the principal $U(1)$-bundle over M with the connection ω_o whose curvature form is given by $-(\tilde{c}/2)(\pi^*\Omega)i$, where $\pi\colon P_o \to M$ is the projection. Let Q' be the complex line bundle over M associated with the principal fibre bundle P_o corresponding to the homomorphism $\rho_o\colon U(1) \to \mathrm{End}(\mathbb{C})$, $\lambda \mapsto \lambda^2\,\mathrm{id}_{\mathbb{C}}$. Then Q' has the Hermitian fibre metric $\langle\ ,\ \rangle$ and the metric connection induced from P_o. Here we mean by the Hermitian fibre metric a real inner product $\langle\ ,\ \rangle$ which satisfies $\langle ia, ib\rangle = \langle a, b\rangle$ for $a, b \in Q'_p$, $p \in M$. We denote by $\overline{TM}$ the complex conjugate vector bundle of the tangent bundle TM with the complex structure $\bar{I} = -I$. Let $E = Q' \otimes_{\mathbb{C}} \overline{TM}$ be the tensor product of Q' and $\overline{TM}$ over $\mathbb{C}$ with the complex structure $\tilde{I}$ and the Hermitian fibre metric $\langle\ ,\ \rangle$ (an $\tilde{I}$-invariant real inner product), where $\tilde{I}$ and $\langle\ ,\ \rangle$ are given by

$$\tilde{I}(a \otimes X) = (ia) \otimes X = a \otimes \bar{I}X = -a \otimes IX,$$

and

$$\begin{aligned}\langle a \otimes X, b \otimes Y\rangle &= \langle a, b\rangle\langle X, Y\rangle - \langle ia, b\rangle\langle \bar{I}X, Y\rangle \\ &= \langle a, b\rangle\langle X, Y\rangle + \langle ia, b\rangle\langle IX, Y\rangle,\end{aligned}$$

for a, $b \in Q'_p$, X, $Y \in T_pM$, $p \in M$. The connection in E is induced from those of Q' and $\overline{TM}$. We define a smooth section

$$\varphi \in \Gamma(\mathrm{Hom}(Q',\, \mathrm{End}_{\mathbb{R}}(TM + E)))$$

as follows: for a, $b \in Q'_p$, X, $Y \in T_pM$, $p \in M$

$$\varphi_a(X) = a \otimes X, \quad \varphi_a(b \otimes Y) = -\langle a,\, b\rangle Y + \langle ia,\, b\rangle IY.$$

Then φ_a satisfies the following properties:

(1) φ_a is a semi-linear map, i.e., $\varphi_a \circ I = -\tilde{I} \circ \varphi_a$ on T_pM and $\varphi_a \circ \tilde{I} = -I \circ \varphi_a$ on $E_p = (Q' \otimes_{\mathbb{C}} \overline{TM})_p$.

(2) $\varphi_a^2 = -\|a\|^2\, \mathrm{id}$.

(3) φ_a is skew-symmetric, i.e.,

$$\langle \varphi_a(X + b \otimes Y),\, X' + b' \otimes Y'\rangle + \langle X + b \otimes Y,\, \varphi_a(X' + b' \otimes Y')\rangle = 0.$$

We consider a $\mathrm{Hom}(TM,\, E)$-valued 1-form $\hat{\sigma}$ on M which satisfies the following conditions: for X, Y, $Z \in T_pM$, $a \in Q'_p$, $p \in M$

$$\hat{\sigma}(X)(IY) = \tilde{I}\hat{\sigma}(X)(Y), \tag{3.16.1}$$

$$\langle \hat{\sigma}(X)(Y),\, \varphi_a Z\rangle = \langle \hat{\sigma}(X)(Z),\, \varphi_a Y\rangle, \tag{3.16.2}$$

$$\hat{\sigma}(X)(Y) = \hat{\sigma}(Y)(X). \tag{3.16.3}$$

We define an $E = Q' \otimes_{\mathbb{C}} \overline{TM}$-valued covariant tensor field σ of order 2 on M by $\sigma(X,\, Y) = \hat{\sigma}(X)(Y)$ for X, $Y \in T_pM$, $p \in M$.

Theorem 3.5 (Fundamental theorem for totally complex submanifolds) *Let M be a (real) $2n$ $(n \geq 2)$-dimensional simply connected Kähler manifold with the Kähler form Ω and P_o be the principal $U(1)$-bundle over M with the connection whose curvature form is $-(\tilde{c}/2)(\pi^*\Omega)i$. We define the complex vector bundle $E = Q' \otimes_{\mathbb{C}} \overline{TM}$ as above. Let $\hat{\sigma}$ be a* $\mathrm{Hom}(TM,\, E)$*-valued* 1-*form on M which satisfies* (3.16.1), (3.16.2), (3.16.3). *In addition, suppose that the tensor field σ satisfies the following equations of Gauss and Codazzi*

$$\begin{aligned} R(X,\, Y)Z = \frac{\tilde{c}}{4}\big\{ &\langle Y,\, Z\rangle X - \langle X,\, Z\rangle Y + \langle IY,\, Z\rangle IX \\ &- \langle IX,\, Z\rangle IY - 2\langle IX,\, Y\rangle IZ\big\} \\ &+ S_{\sigma(Y,\, Z)}X - S_{\sigma(X,\, Z)}Y, \end{aligned}$$

$$(\bar{\nabla}_X \sigma)(Y,\, Z) = (\bar{\nabla}_Y \sigma)(X,\, Z)$$

for the tangent vectors X, Y, Z of M, where R denotes the curvature tensor of M and $\bar{\nabla}$ denotes the covariant differentiation with respect to the connection in $TM + E$. Then there exist an isometric immersion $f\colon M \to \bar{M}$ which is a totally complex submanifold in $\bar{M} = \mathbb{H}P^n$ or $\mathbb{H}H^n$ according as $\tilde{c}$ is positive or negative and a vector bundle isomorphism $\tilde{f}\colon E = Q' \otimes_{\mathbb{C}} \overline{TM} \to T^{\perp}M$ which preserves the complex structure, the metrics and the connections such that for every $X, Y \in TM$, $\tilde{\sigma}(X, Y) = \tilde{f}\sigma(X, Y)$, where $\tilde{\sigma}$ is the second fundamental form of f. Moreover, such an immersion f is unique up to the action by G, where G is the identity component of the isometry group of $\bar{M}$.

Remark 3.6 In Alekseevsky and Marchiafava ([1] p. 889) it was conjectured that the fundamental theorem of submanifold geometry holds for half-dimensional totally complex submanifolds in $\bar{M} = \mathbb{H}P^n$ or $\mathbb{H}H^n$. Theorem 3.5 gives an affirmative answer to this conjecture.

Proof of Theorem 3.5. To apply Theorem 2.8, we will construct a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry (P, ω) which satisfies the assumptions in Theorem 2.8. At first we construct the principal fibre bundle P with the structure group $U(n) \times U(1) \simeq (K_+)_o$. At each point $p \in M$, we view the tangent space T_pM as a $\mathbb{C}$-Hermitian vector space and consider a $\mathbb{C}$-linear isometry $u\colon \mathbb{C}^n \to T_pM$, which is called a unitary frame at $p \in M$. Let P' be the bundle of unitary frames over M. Then it is a principal fibre bundle with the structure group $U(n)$. We denote by $P = P' \times_M P_o$ the fibre product of two principal fibre bundles P' and P_o with the structure groups $U(n)$ and $U(1)$. Now we define a new right action $R_{(A,\,\lambda)}$ on $P = P' \times_M P_o$ for $(A, \lambda) \in U(n) \times U(1)$ by $R_{(A,\,\lambda)}(u, a) = (u(\lambda A), a\lambda)$. Let $h'\colon P \to P'$ be the projection from $P = P' \times_M P_o$ onto the first factor P'. Then h' is a bundle homomorphism corresponding to the Lie group homomorphism $\rho'\colon (K_+)_o \simeq U(n) \times U(1) \to U(n) \subset O(\mathfrak{m})$ given by (3.8). Let $h_o\colon P \to P_o$ is the projection from $P = P' \times_M P_o$ onto the second factor P_o. Then h_o is a bundle homomorphism corresponding to the homomorphism $\rho'_o\colon (K_+)_o \simeq U(n) \times U(1) \to U(1)$, $(A, \lambda) \mapsto \lambda$.

Let P'' be the bundle of unitary frames of the complex vector bundle $E = Q' \otimes_{\mathbb{C}} \overline{TM}$. We will construct a bundle homomorphism h'' of P onto P''. We recall the construction of the associated fibre bundle Q' from P_o (cf. Kobayashi and Nomizu [5] Vol. I, Chapter 1, §5). We define the right action of $U(1)$ on the product manifold $P_o \times \mathbb{C}$ as follows: an element $\lambda \in U(1)$

maps $(a, z) \in P_o \times \mathbb{C}$ into $(a\lambda, \lambda^{-2}z)$. Then Q' is the quotient space of $P_o \times \mathbb{C}$ by this group action. We denote by μ the projection of $P_o \times \mathbb{C}$ onto Q' and simply write $\mu(a)$ for the mapping $a \in P_o$ into $\mu(a, 1) \in Q'$. Then μ is the fibre-preserving immersion of P_o into Q' which satisfies $\mu(a\lambda) = \lambda^2\mu(a)$ for $\lambda \in U(1)$. Let τ be the complex conjugation of $\mathbb{C}^n$ defined by $\tau(\boldsymbol{v}) = \bar{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathbb{C}^n$. For $(u, a) \in P = P' \times_M P_o$, the mapping $\varphi_{\mu(a)} \circ u \circ \tau$ is a $\mathbb{C}$-linear isometry of $\mathbb{C}^n$ onto $E_p = (Q' \otimes_{\mathbb{C}} \overline{TM})_p$, $p = \pi(u, a)$, where $\varphi_{\mu(a)}$ is a semi-linear map of T_pM into E_p defined by $\mu(a) \in Q'_p$ and hence $\varphi_{\mu(a)} \circ u \circ \tau$ is a unitary frame of E_p. We define a mapping h'' of P into P'' by $h''(u, a) = \varphi_{\mu(a)} \circ u \circ \tau$. Then h'' is a bundle homomorphism corresponding to the homomorphism $\rho''\colon (K_+)_o \simeq U(n) \times U(1) \to U(n) \subset O(\mathfrak{m}^\perp)$ given by (3.9). We note that each $a \in P_o$ gives an identification of $\mathfrak{k}_-^2$ with Q'_p, $p = \pi(a)$ by the mapping $x\tilde{J} + y\tilde{K}$, $x, y \in \mathbb{R}$ into $\mu(a, x + iy) \in Q'_p$. Moreover the linear map $\psi\colon \mathfrak{k}_-^2 \to \mathrm{Hom}(\mathfrak{m}, \mathfrak{m}^\perp)$ is equivalent to the linear map $\varphi\colon Q'_p \to \mathrm{Hom}(T_pM, E_p)$. That is, we have

$$h''(u,a)^{-1} \circ \varphi_{\mu(a,\,x+iy)} \circ h'(u,\,a)(\boldsymbol{v}) = \psi(x\tilde{J} + y\tilde{K})(\boldsymbol{v}) \qquad (3.17)$$

for $\boldsymbol{v} \in \mathbb{C}^n \cong \mathfrak{m}$, $x, y \in \mathbb{R}$, at $(u, a) \in P$. Here as usual we identify $\mathfrak{m}$ and $\mathfrak{m}^\perp$ with $\mathbb{C}^n$, respectively and remark that under these identifications $\psi(\tilde{J})(\boldsymbol{v}) = \bar{\boldsymbol{v}}$ and $\psi(\tilde{K})(\boldsymbol{v}) = i\bar{\boldsymbol{v}}$.

Next we construct a $\mathfrak{k}_+ + \mathfrak{k}_- + \mathfrak{m}$-valued 1-form ω on P. Since M is Kählerian, the Riemannian connection of M is reduced to the bundle of unitary frames P', whose connection form is denoted by ω'. We denote by θ the canonical 1-form on P', i.e., a $\mathbb{C}^n$-valued 1-form which is defined by $\theta(X) = u^{-1}(\pi_* X)$ for $X \in T_uP'$. We define a $\mathfrak{k}_+^1 = \mathfrak{u}(n)$-valued 1-form $\omega_{\mathfrak{k}_+^1}$, a $\mathfrak{k}_+^2 = \mathfrak{u}(1)$-valued 1-form $\omega_{\mathfrak{k}_+^2}$, and a $\mathfrak{m} = \mathbb{C}^n$-valued 1-form $\omega_{\mathfrak{m}}$ on P as follows:

$$\begin{aligned} \omega_{\mathfrak{k}_+^1} &= h'^*\omega' - (h_o^*\omega_o)I_n \\ \omega_{\mathfrak{k}_+^2} &= h_o^*\omega_o \\ \omega_{\mathfrak{m}} &= h'^*\theta, \end{aligned}$$

where I_n denotes the $n \times n$-identity matrix and ω_o is the connection form on P_o. Then we have $h'^*\omega' = \rho'\omega_{\mathfrak{k}_+}$ and $h_o{}^*\omega_o = \rho'_o\omega_{\mathfrak{k}_+}$. Let ω'' be the connection form on P'' which corresponds to the connection on $E = Q' \otimes_{\mathbb{C}} \overline{TM}$ induced from those of Q' and $\overline{TM}$. By the straightforward computation, we see that $h''^*\omega'' = \rho''\omega_{\mathfrak{k}_+}$. Using $\hat{\sigma}$ we will define a $\mathfrak{k}_-^1$-valued 1-form $\omega_{\mathfrak{k}_-^1}$

on P. As previous arguments, we identify both $\mathfrak{m}$ and $\mathfrak{m}^{\perp}$ with $\mathbb{C}^n$, respectively and hence $\mathrm{Hom}(\mathfrak{m}, \mathfrak{m}^{\perp})$ with $\mathrm{End}(\mathbb{C}^n)$. At $(u, a) \in P$, we define an $\mathrm{End}(\mathbb{C}^n)$-valued 1-form $\tilde{\sigma}$ as follows:

$$\tilde{\sigma}(\tilde{X})(\boldsymbol{v}) = h''(u, a)^{-1}(\hat{\sigma}(\pi_* \tilde{X})(h'(u, a)(\boldsymbol{v}))) \qquad \text{for } \tilde{X} \in T_{(u,a)}P,\ \boldsymbol{v} \in \mathbb{C}^n.$$

Then by (3.16.1), $\tilde{\sigma}(\tilde{X})$ is a complex linear endomorphism of $\mathbb{C}^n$ and by (3.16.2) and (3.17) we have $\langle \tilde{\sigma}(\tilde{X})\boldsymbol{v}, L\boldsymbol{w}\rangle_{\mathbb{R}} = \langle \tilde{\sigma}(\tilde{X})\boldsymbol{w}, L\boldsymbol{v}\rangle_{\mathbb{R}}$ for any $L \in \mathfrak{k}_-^2$. By Lemma 3.2, it follows that $\tilde{\sigma}(\tilde{X}) \in \psi(\mathfrak{k}_-^1)$. Since $\psi\colon \mathfrak{k}_- \to \mathrm{Hom}(\mathfrak{m}, \mathfrak{m}^{\perp})$ is injective, we can define $\omega_{\mathfrak{k}_-^1}$ by $\psi(\omega_{\mathfrak{k}_-^1}(\tilde{X})) = \tilde{\sigma}(\tilde{X})$.

Now puting $\omega = \omega_{\mathfrak{k}_+^1} + \omega_{\mathfrak{k}_+^2} + \omega_{\mathfrak{k}_-^1} + \omega_{\mathfrak{m}}$, we define a $\mathfrak{k}_+ + \mathfrak{k}_- + \mathfrak{m}$-valued 1-form ω on P. By straightforward computation, we can show that ω satisfies (2.3.1), (2.3.2) and (2.3.3). Consequently we have constructed a locally ambient $\mathcal{O}(\mathfrak{m})$-geometry (P, ω). By the construction, it follows that the conditions (1) $\sim$ (4) in Theorem 2.8 are satisfied. The condition (5) in Theorem 2.8 is equivalent to that of (2.4.1) in Proposition 2.3. We denote by Ψ the $\mathfrak{k}_+$-valued 2-form defined by the left hand side of (2.4.1). We apply ρ_o' to Ψ, where $\rho_o'\colon \mathfrak{k}_+ \simeq \mathfrak{u}(n) \oplus \mathfrak{u}(1) \to \mathfrak{u}(1)$ is the projection. Then we have

$$\begin{aligned}\rho_o'\Psi &= d\rho_o'\omega_{\mathfrak{k}_+} + \frac{1}{2}\rho_o'[\omega_{\mathfrak{k}_+}, \omega_{\mathfrak{k}_+}] + \frac{1}{2}\rho_o'[\omega_{\mathfrak{k}_-}, \omega_{\mathfrak{k}_-}] + \frac{1}{2}\rho_o'[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}]\\ &= h_o^* d\omega_o - \bar{\Omega}_{\mathfrak{k}_+^2},\end{aligned}$$

where $\bar{\Omega}$ denotes the curvature form of $\bar{M}$. Since $d\omega_o = \bar{\Omega}_{\mathfrak{k}_+^2} = -(\tilde{c}/2)(\pi^*\Omega)i$, $\rho_o'\Psi = 0$. By the Gauss equation, it follows that $\rho'\Psi = 0$. These imply $\Psi = 0$. Thus the requirements of Theorem 2.8 are all saitisfied. Therefore Theorem 3.5 has been proved. □

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