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The Haar wavelets and the Haar scaling function in weighted L^p spaces with $A_p^{\mathrm{dy},m}$ weights

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Abstract. The new class of weights called $A_p^{\mathrm{dy},m}$ weights is introduced. We prove that a characterization and an unconditional basis of the weighted L^p space $L^p(\mathbb{R}^n, w(x)dx)$ with $w \in A_p^{\mathrm{dy},m}$ $(1 < p < \infty)$ are given by the Haar wavelets and the Haar scaling function. As an application of these results, we establish a greedy basis by using the Haar wavelets and the Haar scaling function again.

Key words: the Haar wavelets, the Haar scaling function, weighted L^p space, $A_p^{\mathrm{dy},m}$ weight, greedy basis.

1. Introduction

The relations between wavelets and weighted L^p spaces

$$L^p(w) := L^p(\mathbb{R}^n, w(x)dx) \quad (1 < p < \infty)$$

have been considered in [ABM], [GK], [Ka] and [L]. In particular, H.A. Aimar, A.L. Bernardis and F.J. Martín-Reyes proved that some characterizations of $L^p(w)$ with $w \in A_p$ and an unconditional basis for it are given by 1-regular wavelets (e.g., the Meyer wavelets, the Daubechies wavelets etc.) ([ABM]). They also proved that the similar results followed for the case of the Haar wavelets replacing A_p weights by A_p^{dy} weights. On the other hand, in 1994, P.G. Lemarié-Rieusset considered for the case of Daubechies'. He proved that a characterization of $L^p(w)$ and an unconditional basis for it were given by using not only the wavelet but also the scaling function which constructed the wavelet for the case of $w \in A_p^{\mathrm{loc}}$ ([L]). Let us remark that the class of A_p^{loc} weights was first defined by V.S. Rychkov in 2001 ([R]). We don't explain in detail, however, we shall point out that he gives some interesting results about A_p^{loc} weights and weighted function spaces with them. We prove that the similar results to P.G. Lemarié-Rieusset's hold for the case of Haar's replacing A_p^{loc} weights by $A_p^{\mathrm{dy},m}$ weights, first defined in

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this article, on the basis of the idea of [ABM].

Let us explain the outline of this article. We describe briefly the basic concept of wavelets associated with an MRA in Section 2. In Section 3, we introduce four classes of weights, namely, A_p, A_p^{dy}, A_p^{loc} and $A_p^{\mathrm{dy},m}$. Also we give some examples of them respectively. Section 4 consists of the main results and the proof of them. We show that a characterization and an unconditional basis of the weighted L^p space with $A_p^{\mathrm{dy},m}$ weights are given by the Haar wavelets and the Haar scaling function. Using these results, we establish the greedy basis in the weighted L^p space with $A_p^{\mathrm{dy},m}$ weights by the Haar wavelets and the Haar scaling function again in Section 5. So to speak, our weighted wavelet (and scaling function) method is valid for the results of [ABM] and [L], too. We explain them in Section 6.

Lastly, we would like to remark on the studies of greedy bases briefly. [CDH], [GH], [KT] and [Ky] give the remarkable results respectively. We shall point out, however, that they consider only for non-weighted cases. This article studies greedy bases in the weighted L^p space with four kinds of weights.

2. Wavelets

Definition 2.1 (wavelet set, wavelet basis and wavelet) Let $\{\psi^e\}_{e=1}^{2^n-1}$ be a sequence of functions belong to $L^2(dx) := L^2(\mathbb{R}^n, dx)$. We define

$$\psi_{j,k}^e(x) := 2^{jn/2}\psi^e(2^j x - k) = 2^{jn/2}\psi^e(2^j x_1 - k_1, \dots, 2^j x_n - k_n) \quad (x = (x_1, \dots, x_n) \in \mathbb{R}^n)$$

for each $e = 1, 2, \dots, 2^n - 1$, $j \in \mathbb{Z}$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$. The sequence $\{\psi^e\}_{e=1}^{2^n-1}$ is called a wavelet set if $\{\psi_{j,k}^e : e = 1, 2, \dots, 2^n - 1, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is an orthonormal basis in $L^2(dx)$. Then we say that $\{\psi_{j,k}^e : e = 1, 2, \dots, 2^n - 1, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is a wavelet basis in $L^2(dx)$ and that any ψ^e is a wavelet.

Let $f \in L^2(dx)$ and $\{\psi^e\}_{e=1}^{2^n-1}$ be a wavelet set. We obtain the wavelet expansion of f

$$f = \sum_{e=1}^{2^n-1} \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \langle f, \psi_{j,k}^e\rangle \psi_{j,k}^e \quad \text{in} \quad L^2(dx)$$

and Parseval's equality

$$\|f\|_{L^2(dx)} = \Big(\sum_{e=1}^{2^n-1}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} |\langle f,\, \psi^e_{j,k}\rangle|^2\Big)^{1/2}.$$

Here $\langle f, \psi^e_{j,k}\rangle := \int_{\mathbb{R}^n} f(x)\overline{\psi^e_{j,k}(x)}dx$. By above two equalities, we see that the L^2-norm of f is characterized by the wavelet coefficients $\langle f,\, \psi^e_{j,k}\rangle$ $(e = 1,\, 2,\, \ldots,\, 2^n - 1,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^n)$.

We can construct wavelets which have various properties by the proper way of constructions. The remarkable feature of wavelets, which have a proper smoothness, a proper decay or a compact support, give characterizations and unconditional bases of various function spaces, such as Hardy spaces, Sobolev spaces etc. ([W], [HW], [M], [G], [GM], [D]).

Additionally, we shall point out that a sequence of closed subspaces of $L^2(dx)$ called MRA gives wavelets.

Definition 2.2 (MRA) An MRA (multiresolution analysis) is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(dx)$ such that

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$.
(b) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(dx)$.
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
(d) $f \in V_j$ holds if and only if $f(2^{-j}x) \in V_0$ for all $j \in \mathbb{Z}$.
(e) $f \in V_0$ holds if and only if $f(x-k) \in V_0$ for every $k \in \mathbb{Z}^n$.
(f) There exists a function $\varphi \in V_0$, called a scaling function of $\{V_j\}_{j\in\mathbb{Z}}$, such that the system $\{\varphi(x-m)\}_{m\in\mathbb{Z}^n}$ is an orthonormal basis in V_0.

Given an MRA $\{V_j\}_{j\in\mathbb{Z}}$, there exists a wavelet set $\{\psi^e\}_{e=1}^{2^n-1}$ such that $\{\psi^e_{j,k}\colon e = 1,\, 2,\, \ldots,\, 2^n - 1,\, k \in \mathbb{Z}^n\}$ is an orthonormal basis in W_j for all $j \in \mathbb{Z}$, where W_j means the orthogonal complement of V_j in V_{j+1} (cf. [M, Chapter 3] or [W, Chapter 5]). Taking suitable MRAs enables us to obtain wavelets which have various properties. Let us introduce the Haar wavelets and the Haar scaling function which play an important role in this paper.

Definition 2.3 (the Haar wavelet set, the Haar wavelet and the Haar scaling function) Let $E := \{0,\, 1\}^n \setminus \{(0,\, \ldots,\, 0)\}$, $\psi^1 := \chi_{[0,\,1/2)} - \chi_{[1/2,\,1)}$, $\psi^0 := \chi_{[0,\,1)}$ and

$$\psi^e(x) := \prod_{i=1}^{n} \psi^{e_i}(x_i)$$

$$(e = (e_1, \dots, e_n) \in E,\ x = (x_1, \dots, x_n) \in \mathbb{R}^n),$$

where χ_F means the characteristic function of a measurable set F. We say that the sequence $\{\psi^e\}_{e\in E}$ is the Haar wavelet set and that any ψ^e is the Haar wavelet. Additionally we call the function $\chi_{[0,1)^n}$ the Haar scaling function.

On the other hand, we shall also remark that there are well-known examples of smooth wavelets such as the Meyer wavelets which belong to Schwartz class, the Daubechies wavelets which belong to $C^r(\mathbb{R}^n)$ for some $r \in \mathbb{N}$ and have compact support, and so forth.

3. Weights

A weight on $\mathbb{R}^n$ is a function w defined on $\mathbb{R}^n$ such that $w \geq 0$ a.e. $\mathbb{R}^n$ and $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let w be a weight on $\mathbb{R}^n$ and $1 < p < \infty$ without notices in the following of this paper. We shall explain some notations here in order to introduce four classes of weights.

Notation 3.1

(a) We define a dyadic cube $Q_{j,k}$ for $j \in \mathbb{Z}$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ by

$$Q_{j,k} := \prod_{i=1}^{n} \left[2^{-j}k_i, 2^{-j}(k_i+1)\right).$$

(b) We write $w(E) := \int_E w(x)dx$ for a measurable set $E \subset \mathbb{R}^n$. And $|E|$ means the Lebesgue measure of E.

Definition 3.2 (four classes of weights)

(a) We define the class of weights A_p which consists of all weights w satisfying

$$A_p(w) := \sup_{Q:\text{cube}} \frac{1}{|Q|} w(Q) \Big(\frac{1}{|Q|} \int_Q w(y)^{-1/(p-1)} dy \Big)^{p-1} < \infty,$$

and say that $w \in A_p$ is an A_p weight.

(b) We define the class of weights A_p^{dy} which consists of all weights w satisfying

$$A_p^{\mathrm{dy}}(w) := \sup_{Q_{j,k}} \frac{1}{|Q_{j,k}|} w(Q_{j,k}) \Big(\frac{1}{|Q_{j,k}|} \int_{Q_{j,k}} w(y)^{-1/(p-1)} dy \Big)^{p-1} < \infty,$$

and say that $w \in A_p^{\mathrm{dy}}$ is an A_p^{dy} weight.

(c) We define the class of weights A_p^{loc} which consists of all weights w satisfying

$$A_p^{\mathrm{loc}}(w) := \sup_{\substack{|Q|\le 1,\\ Q:\text{cube}}} \frac{1}{|Q|} w(Q) \Big(\frac{1}{|Q|} \int_Q w(y)^{-1/(p-1)} dy \Big)^{p-1} < \infty,$$

and say that $w \in A_p^{\mathrm{loc}}$ is an A_p^{loc} weight.

(d) Let $m \in \mathbb{Z}$. We define the class of weights $A_p^{\mathrm{dy},m}$ which consists of all weights w satisfying

$$A_p^{\mathrm{dy},m}(w) := \sup_{|Q_{j,k}| \le 2^{-mn}} \frac{1}{|Q_{j,k}|} w(Q_{j,k}) \times \Big(\frac{1}{|Q_{j,k}|} \int_{Q_{j,k}} w(y)^{-1/(p-1)} dy \Big)^{p-1} < \infty,$$

and say that $w \in A_p^{\mathrm{dy},m}$ is an $A_p^{\mathrm{dy},m}$ weight.

The class of A_p^{loc} weights is defined by V.S. Rychkov. This class is independent of the upper bound for the cube size used in its definitions. Namely, we can replace $|Q| \le 1$ by $|Q| \le r$ in Definition 3.2 (c) for any $0 < r < \infty$ ([R]). Thus, we obtain the following inclusion relations between above four classes of weights:

- $A_p^{\mathrm{dy},m-1} \subset A_p^{\mathrm{dy},m}$ $(m \in \mathbb{Z})$.
- $A_p \subset A_p^{\mathrm{dy}},\ A_p^{\mathrm{loc}}$.
- $A_p^{\mathrm{dy}},\ A_p^{\mathrm{loc}} \subset A_p^{\mathrm{dy},m}$ $(m \in \mathbb{Z})$.

However, notice that the signs of equality are not satisfied for each inclusion relations, i.e., strictly speaking, above all three "$\subset$" mean "$\subsetneq$". These facts become clear by giving some examples in the following Example 3.3:

Example 3.3 We consider only in the case of $n = 1$ for convenience.

(a) Let $-1 < \alpha < p - 1$. Then, we have $|x|^\alpha \in A_p$ (cf. [To]).

(b) Let $-1 < \alpha,\ \beta < p - 1$ $(\alpha \ne \beta)$. We define a function w_0 as follows:

$$w_0(x) := \begin{cases} |x|^\alpha & (x > 0) \\ 0 & (x = 0) \\ |x|^\beta & (x < 0) \end{cases}$$

Then, it follows that $w_0 \in A_p^{\mathrm{dy}} \setminus A_p$.

(c) Let $r \in \mathbb{R} \setminus \{0\}$. Then, we obtain $e^{r|x|} \in A_p^{\mathrm{loc}} \setminus A_p$.

(d) Let $m \in \mathbb{Z}$, $s > 1$ and $r \in \mathbb{R} \setminus \{0\}$. Then it follows that

$$\sum_{k\in\mathbb{Z}} e^{rk^s} \chi_{Q_{m,k}}(x) \in A_p^{\mathrm{dy},m} \setminus A_p^{\mathrm{dy},m-1}.$$

4. The Haar wavelets and the Haar scaling function in weighted L^p spaces

To state the main result Theorem 4.2, we shall introduce some notations.

Notation 4.1

(a) We denote $\varphi_k(x) := \varphi_{0,k}(x) = \varphi(x-k)$ and $Q_k := Q_{0,k} = \prod_{i=1}^n [k_i, k_i+1)$ for each $k \in \mathbb{Z}^n$.
(b) We write $\chi_{j,k} := 2^{(jn)/2} \chi_{Q_{j,k}}$ for each $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$.
(c) $\mathbb{N}$ means the set of natural numbers. And we denote $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.
(d) p' means the conjugate exponent of p, that is, p' satisfies $p^{-1} + p'^{-1} = 1$.

Theorem 4.2 (Main Result) *Let $\{\psi^e\}_{e\in E}$ be the Haar wavelet set, $\varphi := \chi_{[0,1)^n}$ (i.e., the Haar scaling function), μ be a positive Borel measure on $\mathbb{R}^n$, finite on compact sets, $m \in \mathbb{Z}$ and $1 < p < \infty$. Then, the following three conditions are equivalent:*

(I1) *μ is absolutely continuous with regard to the Lebesgue measure. And there exists a $w \in A_p^{\mathrm{dy},m}$ such that $d\mu(x) = w(x)dx$.*

(I2) *We define*

$$\mathcal{M}_{p,\mu,m}(f) := \Big(\sum_{k\in\mathbb{Z}^n} \big| \langle f, \varphi_{m,k} \rangle \|\varphi_{m,k}\|_{L^p(d\mu)} \big|^p \Big)^{1/p} + \Big\| \Big(\sum_{e\in E} \sum_{j=m}^{\infty} \sum_{k\in\mathbb{Z}^n} \big| \langle f, \psi_{j,k}^e \rangle \chi_{j,k} \big|^2 \Big)^{1/2} \Big\|_{L^p(d\mu)}.$$

Then, there exist two constants $0 < c \le C < \infty$ independent of f such that $c\|f\|_{L^p(d\mu)} \le \mathcal{M}_{p,\mu,m}(f) \le C\|f\|_{L^p(d\mu)}$ for all $f \in L^p(d\mu) := L^p\big(\mathbb{R}^n, d\mu(x)\big)$. Namely $\mathcal{M}_{p,\mu,m}(\cdot)$ defines the equivalent norm to $\|\cdot\|_{L^p(d\mu)}$ on $L^p(d\mu)$. And $\mu(Q_{j,k}) > 0$ for all $j \ge m$ and $k \in \mathbb{Z}^n$.

(I3) *The sequence*

$$\{\varphi_{m,k} : k \in \mathbb{Z}^n\} \cup \{\psi_{j,k}^e : e \in E,\ j \ge m,\ k \in \mathbb{Z}^n\}$$

forms an unconditional basis for $L^p(d\mu)$. And

$$\{(\varphi_{m,k})^* : k \in \mathbb{Z}^n\} \cup \{(\psi^e_{j,k})^* : e \in E,\ j \geq m,\ k \in \mathbb{Z}^n\} \subset L^p(d\mu)^*.$$

Here $L^p(d\mu)^*$ means the dual space of $L^p(d\mu)$ and

$$(\varphi_{m,k})^*(f) := \int_{\mathbb{R}^n} f(x)\overline{\varphi_{m,k}(x)}dx \quad (f \in L^p(d\mu)).$$

It is known that there are several equivalent definitions of an unconditional basis in a Banach space ([KS], [LT]). We shall remark that we adopt the definition of an unconditional basis by [W, Chapter 7] in this paper.

Definition 4.3 (unconditional convergence, unconditional basis) Let A be a countable index set, $\{x_m\}_{m\in A}$ be a sequence of elements in a Banach space X and $\{\tilde{x}_k\}_{k\in A}$ be a sequence of elements in X^*, where X^* is the dual space of X.

(a) We say that the series $\sum_{m\in A} x_m$ is unconditionally convergent in X if the series $\sum_{i=1}^{\infty} x_{\sigma(i)}$ converges in X for all $\sigma : \mathbb{N} \to A$, a 1 to 1 and onto map.

(b) We call $\{x_m, \tilde{x}_m\}_{m\in A}$ an unconditional basis in X if the following three conditions are satisfied:

(i) $\{x_m, \tilde{x}_m\}_{m\in A}$ is a biorthogonal system, i.e., $\tilde{x}_k(x_m) = \delta_{m,k}$. Here $\delta_{m,k}$ means Kronecker's delta, that is, $\delta_{m,m} = 1$ and $\delta_{m,k} = 0$ if $m \neq k$.

(ii) $\overline{\mathrm{span}\{x_m\}_{m\in A}}^X = X$, where $\mathrm{span}\{x_m\}_{m\in A}$ means the set of finite linear combinations of elements in $\{x_m\}_{m\in A}$.

(iii) There exists a constant $0 < C < \infty$ such that $\left\|\sum_{m\in B} \tilde{x}_m(x)x_m\right\|_X \leq C\|x\|_X$ for every $x \in X$ and every finite subset $B \subset A$.

Remark 4.4 Let A be a countable index set and $\{x_m, \tilde{x}_m\}_{m\in A}$ be an unconditional basis in a Banach space X.

(a) ([W, Theorem 7.7 (i)]) The series $\sum_{m\in A} \tilde{x}_m(x)x_m$ converges unconditionally in X to x for every $x \in X$.

(b) ([W, Remark 7.2]) We see that the functionals $\{\tilde{x}_k\}_{k\in A} \subset X^*$ are determined by the vectors $\{x_m\}_{m\in A} \subset X$ from two conditions (i) and (ii) in Definition 4.3 (b). Thus we often say that $\{x_m\}_{m\in A}$ is an unconditional basis in X.

We will prove the equivalence of three conditions (I1), (I2) and (I3) in Theorem 4.2 in the rest of this section. Before starting the proof, let us remark the following facts:

- $\varphi_{m,k}(x) = 2^{(mn)/2}\varphi_k(2^m x)$ for all $m \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$.
- Let w be a weight on $\mathbb{R}^n$. Then, $w \in A_p^{\mathrm{dy},m}$ if and only if $w(2^{-m}x) \in A_p^{\mathrm{dy},0}$ for all $m \in \mathbb{Z}$.
- $\langle 2^{mn/2} f(2^m \cdot),\, \varphi_{m,k}\rangle = \langle f,\, \varphi_k\rangle$ and $\langle 2^{mn/2} f(2^m \cdot),\, \psi^e_{j,k}\rangle = \langle f,\, \psi^e_{j-m,k}\rangle$ for every $e \in E$, $m \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ and $j \geq m$.

Thus we see that we have only to prove Theorem 4.2 for the case of $m = 0$. We will give the proof referring to the statements of [ABM] and partly using the results of [V].

4.1. Proof of (I1) ⇒ (I2)

First of all, we assume (I1). It is clear that $\mu(Q_{j,k}) = w(Q_{j,k}) > 0$ for all $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$.

Lemma 4.5 *It follows that*

$$\Big(\sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \varphi_k\rangle \|\varphi_k\|_{L^p(w)}\big|^p\Big)^{1/p} \leq A_p^{\mathrm{dy},0}(w)^{1/p}\|f\|_{L^p(w)}$$

for all $f \in L^p(w)$.

Proof of Lemma 4.5. By straightforward calculations, Hölder's inequality and $|Q_k| = 1$ for all $k \in \mathbb{Z}^n$, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \varphi_k\rangle \|\varphi_k\|_{L^p(w)}\big|^p \\
&= \sum_{k\in\mathbb{Z}^n} |\langle f,\, \varphi_k\rangle|^p \cdot \|\varphi_k\|^p_{L^p(w)} \\
&= \sum_{k\in\mathbb{Z}^n} \Big|\int_{Q_k} f(x)w(x)^{1/p}\cdot \varphi_k(x)w(x)^{-1/p}dx\Big|^p \cdot \int_{Q_k} |\varphi_k(x)|^p w(x)dx \\
&\leq \sum_{k\in\mathbb{Z}^n} \int_{Q_k} |f(x)|^p w(x)dx \cdot \Big(\int_{Q_k} w(x)^{-1/(p-1)}dx\Big)^{p-1}\cdot w(Q_k) \\
&\leq A_p^{\mathrm{dy},0}(w) \sum_{k\in\mathbb{Z}^n} \int_{Q_k} |f(x)|^p w(x)dx \\
&= A_p^{\mathrm{dy},0}(w)\|f\|^p_{L^p(w)}.
\end{aligned}$$

□

Lemma 4.6 *There exists a constant* $0 < C_1 < \infty$ *independent of* f *such that for all* $f \in L^p(w)$,

$$\Big\| \Big(\sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \big| \langle f,\, \psi_{j,k}^e \rangle \chi_{j,k} \big|^2 \Big)^{1/2} \Big\|_{L^p(w)} \le C_1 \|f\|_{L^p(w)}.$$

Proof of Lemma 4.6. We shall follow the lines of [M] using Khintchine's inequality.

Proposition 4.7 (Khintchine's inequality, cf. [Z]) *Let Ω be the product set $\{-1,\, 1\}^\Lambda$ and $d\mu(\omega)$ be the Bernoulli probability measure on Ω for $\omega = \big\{\{\omega(\lambda)\}_{\lambda\in\Lambda} \colon \omega(\lambda) = \pm 1\big\} \in \Omega$, obtained by taking the product of the measures on each factor which give a mass of $1/2$ to each of the points -1 and 1. Then, for all $1 < p < \infty$, there exist two constants $0 < C_{p,1} \le C_{p,2} < \infty$ such that for all $\{\alpha(\lambda)\}_{\lambda\in\Lambda} \subset \mathbb{C}$,*

$$\begin{aligned} C_{p,1} \Big(\sum_{\lambda\in\Lambda} |\alpha(\lambda)|^2 \Big)^{1/2} &\le \Big(\int_\Omega \Big| \sum_{\lambda\in\Lambda} \alpha(\lambda)\omega(\lambda) \Big|^p d\mu(\omega) \Big)^{1/p} \\ &\le C_{p,2} \Big(\sum_{\lambda\in\Lambda} |\alpha(\lambda)|^2 \Big)^{1/2}. \end{aligned}$$

Now we define

$$T_\varepsilon f(x) := \sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \varepsilon_{j,k}^e \langle f,\, \psi_{j,k}^e \rangle \psi_{j,k}^e(x)$$

for each $\varepsilon = \{\varepsilon_{j,k}^e \colon e \in E,\, j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n\} \in \Omega := \big\{\varepsilon = \{\varepsilon_{j,k}^e\}_{e,j,k} \colon \varepsilon_{j,k}^e = \pm 1\big\}$. Denoting $b_{j,k}^e := \langle f,\, \psi_{j,k}^e \rangle \psi_{j,k}^e$, we can express

$$T_\varepsilon f(x) = \sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \varepsilon_{j,k}^e b_{j,k}^e(x).$$

By Khintchine's inequality, there exists a constant $0 < a_0 < \infty$ such that

$$\begin{aligned} a_0 \Big(\sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} |b_{j,k}^e(x)|^2 \Big)^{p/2} \\ \le \int_\Omega \Big| \sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \varepsilon_{j,k}^e b_{j,k}^e(x) \Big|^p d\nu(\varepsilon) \quad \text{a.e. } x \in \mathbb{R}^n \end{aligned}$$

for all $f \in L^p(w)$, where $d\nu(\varepsilon)$ denotes the Bernoulli probability measure

on Ω. Since ${\psi^e_{j,k}}^2 = {\chi_{j,k}}^2$, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\big|\langle f,\,\psi^e_{j,k}\rangle\chi_{j,k}\big|^2\Big)^{1/2}\Big\|^p_{L^p(w)}\\
&=\int_{\mathbb{R}^n}\Big(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}|b^e_{j,k}(x)|^2\Big)^{p/2}w(x)dx\\
&\le {a_0}^{-1}\int_{\mathbb{R}^n}\Big\{\int_\Omega\Big|\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\varepsilon^e_{j,k}b^e_{j,k}(x)\Big|^p d\nu(\varepsilon)\Big\}w(x)dx\\
&={a_0}^{-1}\int_\Omega\Big\{\int_{\mathbb{R}^n}|T_\varepsilon f(x)|^p w(x)dx\Big\}d\nu(\varepsilon)\\
&={a_0}^{-1}\int_\Omega\|T_\varepsilon f\|^p_{L^p(w)}d\nu(\varepsilon).
\end{aligned}$$

On the other hand, $\operatorname{supp} T_\varepsilon(f\cdot\chi_{Q_l})\subset\prod_{i=1}^n[l_i,\,l_i+1]$ for any $l\in\mathbb{Z}^n$. Besides there exists a unique $l(x)\in\mathbb{Z}^n$ such that $x\in Q_{l(x)}$ for all $x\in\mathbb{R}^n$. Hence we have

$$\begin{aligned}
|T_\varepsilon f(x)|^p&=\Big|\sum_{l\in\mathbb{Z}^n}T_\varepsilon(f\cdot\chi_{Q_l})(x)\Big|^p\\
&=\big|T_\varepsilon(f\cdot\chi_{Q_{l(x)}})(x)\big|^p\le\sum_{l\in\mathbb{Z}^n}\big|T_\varepsilon(f\cdot\chi_{Q_l})(x)\big|^p.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\|T_\varepsilon f\|^p_{L^p(w)}&\le\int_{\mathbb{R}^n}\sum_{l\in\mathbb{Z}^n}|T_\varepsilon(f\cdot\chi_{Q_l})(x)|^p w(x)dx\\
&=\sum_{l\in\mathbb{Z}^n}\int_{Q_l}|T_\varepsilon(f\cdot\chi_{Q_l})(x)|^p w(x)dx\\
&=\sum_{l\in\mathbb{Z}^n}\|T_\varepsilon(f\cdot\chi_{Q_l})\|^p_{L^p(w)}.
\end{aligned}$$

Now we define w_l ($l\in\mathbb{Z}^n$) to fulfill the following two conditions:

- $w_l(x)=w(\tau_{l_1}(x_1),\,\dots,\,\tau_{l_n}(x_n))$ if $x\in\prod_{i=1}^n[l_i,\,l_i+2)$, where for $\nu\in\mathbb{Z}$ and $t\in[\nu,\,\nu+2)$,

$$\tau_\nu(t):=\begin{cases}t & (t\in[\nu,\,\nu+1))\\ 2(\nu+1)-t & (t\in[\nu+1,\,\nu+2))\end{cases}$$

- Each w_l is a $2\mathbb{Z}^n$-periodic function on $\mathbb{R}^n$.

Then we have $A_p^{\mathrm{dy}}(w_l) \le 3^{np} A_p^{\mathrm{dy},0}(w)$ for all $l \in \mathbb{Z}^n$ ([R, Proof of Lemma 1.1] or [L, Proof of Proposition 2 (ii)]). To estimate $\|T_\varepsilon(f \cdot \chi_{Q_l})\|_{L^p(w)}$ we apply the following theorem:

Theorem 4.8 ([V, Theorem 4.1]) *Let $v \in A_p^{\mathrm{dy}}$. Then, there exist two constants $0 < b_0 \le b_1 < \infty$ depending only on $A_p^{\mathrm{dy}}(v)$ and p such that*

$$\begin{aligned} b_0\|g\|_{L^p(v)} &\le \Big\| \Big(\sum_{e\in E} \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \big|\langle g,\, \psi_{j,k}^e\rangle A_{j,k}\chi_{j,k}\big|^2 \Big)^{1/2} \Big\|_{L^p(dx)} \\ &\le b_1\|g\|_{L^p(v)} \end{aligned}$$

for all $g \in L^p(v)$, where $A_{j,k} := \big((1/|Q_{j,k}|)v(Q_{j,k})\big)^{1/p}$.

Remark 4.9

(a) A. Volberg gives the above result for the case of $v \in A_p$ and one-variable. We can extend it to the case of $v \in A_p^{\mathrm{dy}}$ and several-variables by the exactly same arguments in [V].

(b) Applying Theorem 4.8 and Khintchine's inequality gives another proof of [ABM, Theorem 6 (H2) $\Rightarrow$ (H3)] (Theorem 6.3 in this paper).

By Theorem 4.8, there exist two constants $0 < b_i = b_i(A_p^{\mathrm{dy},0}(w),\, p) < \infty$ $(i = 0,\, 1)$ such that

$$\begin{aligned} &\|T_\varepsilon(f \cdot \chi_{Q_l})\|_{L^p(w)} = \|T_\varepsilon(f \cdot \chi_{Q_l})\|_{L^p(w_l)} \\ &\le {b_0}^{-1} \Big\| \Big(\sum_{e\in E} \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \big|\langle T_\varepsilon(f \cdot \chi_{Q_l}),\, \psi_{j,k}^e\rangle A_{j,k}\chi_{j,k}\big|^2 \Big)^{1/2} \Big\|_{L^p(dx)} \\ &= {b_0}^{-1} \Big\| \Big(\sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \big|\langle f \cdot \chi_{Q_l},\, \psi_{j,k}^e\rangle A_{j,k}\chi_{j,k}\big|^2 \Big)^{1/2} \Big\|_{L^p(dx)} \\ &\le {b_0}^{-1} \Big\| \Big(\sum_{e\in E} \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \big|\langle f \cdot \chi_{Q_l},\, \psi_{j,k}^e\rangle A_{j,k}\chi_{j,k}\big|^2 \Big)^{1/2} \Big\|_{L^p(dx)} \\ &\le {b_0}^{-1} b_1 \|f \cdot \chi_{Q_l}\|_{L^p(w_l)} \\ &= {b_0}^{-1} b_1 \|f \cdot \chi_{Q_l}\|_{L^p(w)} \end{aligned}$$

for all $l \in \mathbb{Z}^n$, where $A_{j,k} := \big((1/|Q_{j,k}|)w(Q_{j,k})\big)^{1/p}$. Therefore we obtain

$$\Big\| \Big(\sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \psi_{j,k}^e\rangle \chi_{j,k}\big|^2 \Big)^{1/2} \Big\|_{L^p(w)}^p$$

$$\le {a_0}^{-1} \int_\Omega \sum_{l\in\mathbb{Z}^n} {b_0}^{-p} b_1^p \|f\cdot\chi_{Q_l}\|_{L^p(w)}^p d\nu(\varepsilon)$$
$$= {a_0}^{-1}{b_0}^{-p} b_1^p \int_\Omega d\nu(\varepsilon)\cdot \sum_{l\in\mathbb{Z}^n} \|f\cdot\chi_{Q_l}\|_{L^p(w)}^p$$
$$= {a_0}^{-1}{b_0}^{-p} b_1^p \|f\|_{L^p(w)}^p. \qquad \square$$

Lemma 4.10 *There exists a constant $0 < C_2 < \infty$ independent of f such that for all $f \in L^p(w)$,*

$$C_2\|f\|_{L^p(w)} \le \Big(\sum_{k\in\mathbb{Z}^n} \big|\langle f,\,\varphi_k\rangle\|\varphi_k\|_{L^p(w)}\big|^p\Big)^{1/p} + \Big\|\Big(\sum_{e\in E}\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^n} \big|\langle f,\,\psi_{j,k}^e\rangle\chi_{j,k}\big|^2\Big)^{1/2}\Big\|_{L^p(w)}$$
$$\le \big(A_p^{\mathrm{dy},0}(w)^{1/p} + C_1\big)\|f\|_{L^p(w)}.$$

Proof of Lemma 4.10. The right hand side inequality follows clearly from Lemma 4.5 and Lemma 4.6. So we have only to prove the left hand side inequality. By the duality, we have

$$\|f\|_{L^p(w)} = \sup\Big\{|\langle f,\,g\rangle|\colon g\in L^2(dx) \text{ with } \|g\|_{L^{p'}(w^{-1/(p-1)})}\le 1\Big\}$$

for all $f \in L^p(w)\cap L^2(dx)$. On the other hand, f can be expanded as follows:

$$f = \sum_{k\in\mathbb{Z}^n}\langle f,\,\varphi_k\rangle\varphi_k + \sum_{e\in E}\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^n}\langle f,\,\psi_{j,k}^e\rangle\psi_{j,k}^e \quad \text{in} \quad L^2(dx)$$

because $\{\varphi_k\colon k\in\mathbb{Z}^n\}\cup\{\psi_{j,k}^e\colon e\in E,\, j\in\mathbb{Z}_+,\, k\in\mathbb{Z}^n\}$ forms an orthonormal basis for $L^2(dx)$. Thus, for all $g\in L^2(dx)$ with $\|g\|_{L^{p'}(w^{-1/(p-1)})}\le 1$, we have

$$|\langle f,\,g\rangle| \le \Big|\sum_{k\in\mathbb{Z}^n}\langle f,\,\varphi_k\rangle\langle\varphi_k,\,g\rangle\Big| + \Big|\sum_{e\in E}\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^n}\langle f,\,\psi_{j,k}^e\rangle\langle\psi_{j,k}^e,\,g\rangle\Big|.$$

To begin with, we consider the first sum in the right hand side. By $\int_{\mathbb{R}^n}\varphi_k(x)^2dx = 1$ and Hölder's inequality, we get

$$\Big|\sum_{k\in\mathbb{Z}^n}\langle f,\varphi_k\rangle\langle\varphi_k,g\rangle\Big| = \Big|\sum_{k\in\mathbb{Z}^n}\langle f,\varphi_k\rangle\langle g,\varphi_k\rangle\Big|\cdot\int_{\mathbb{R}^n}\varphi_k(x)^2dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \Big| \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_k\rangle \varphi_k(x) w(x)^{1/p} \cdot \langle g, \varphi_k\rangle \varphi_k(x) w(x)^{-1/p} \Big| dx \\
&\le \Big\| \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_k\rangle \varphi_k w^{1/p} \Big\|_{L^p(dx)} \cdot \Big\| \sum_{k\in\mathbb{Z}^n} \langle g, \varphi_k\rangle \varphi_k w^{-1/p} \Big\|_{L^{p'}(dx)} \\
&\le \Big(\sum_{k\in\mathbb{Z}^n} \|\langle f, \varphi_k\rangle \varphi_k w^{1/p}\|_{L^p(dx)}^p \Big)^{1/p} \cdot \Big(\sum_{k\in\mathbb{Z}^n} \|\langle g, \varphi_k\rangle \varphi_k w^{-1/p}\|_{L^{p'}(dx)}^{p'} \Big)^{1/p'} \\
&= \Big(\sum_{k\in\mathbb{Z}^n} \big|\langle f, \varphi_k\rangle \|\varphi_k\|_{L^p(w)}\big|^p \Big)^{1/p} \cdot \Big(\sum_{k\in\mathbb{Z}^n} \big|\langle g, \varphi_k\rangle \|\varphi_k\|_{L^{p'}(w^{-1/(p-1)})}\Big|^{p'} \Big)^{1/p'}.
\end{aligned}
$$

Now we have $w^{-1/(p-1)} \in A_{p'}^{\mathrm{dy},0}$ because of $w \in A_p^{\mathrm{dy},0}$. Therefore by the proof of Lemma 4.5, we obtain

$$
\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}^n} \Big|\langle g, \varphi_k\rangle \|\varphi_k\|_{L^{p'}(w^{-1/(p-1)})}\Big|^{p'} \Big)^{1/p'} \\
&\qquad\qquad \le A_{p'}^{\mathrm{dy},0} (w^{-1/(p-1)})^{1/p'} \cdot \|g\|_{L^{p'}(w^{-1/(p-1)})} \\
&\qquad\qquad \le A_{p'}^{\mathrm{dy},0} (w^{-1/(p-1)})^{1/p'}.
\end{aligned}
$$

Thus we have the estimate

$$
\begin{aligned}
&\Big| \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_k\rangle \langle \varphi_k, g\rangle \Big| \\
&\qquad\qquad \le A_{p'}^{\mathrm{dy},0} (w^{-1/(p-1)})^{1/p'} \Big(\sum_{k\in\mathbb{Z}^n} \big|\langle f, \varphi_k\rangle \|\varphi_k\|_{L^p(w)}\big|^p \Big)^{1/p}.
\end{aligned}
$$

Next we consider the second sum. By Schwarz's inequality and Hölder's inequality, it follows that

$$
\begin{aligned}
&\Big| \sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \langle f, \psi_{j,k}^e\rangle \langle \psi_{j,k}^e, g\rangle \Big| \\
&= \Big| \sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \langle f, \psi_{j,k}^e\rangle \langle g, \psi_{j,k}^e\rangle \cdot \int_{\mathbb{R}^n} \chi_{j,k}(x)^2 dx \Big| \\
&\le \int_{\mathbb{R}^n} \Big| \sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \langle f, \psi_{j,k}^e\rangle \chi_{j,k}(x) \cdot \langle g, \psi_{j,k}^e\rangle \chi_{j,k}(x) \Big| dx \\
&\le \int_{\mathbb{R}^n} \Big(\sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \big|\langle f, \psi_{j,k}^e\rangle \chi_{j,k}(x)\big|^2 \Big)^{1/2}
\end{aligned}
$$

$$\times \Big(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \big|\langle g,\, \psi_{j,k}^e\rangle \chi_{j,k}(x)\big|^2\Big)^{1/2} dx$$

$$\le \Big\|\Big(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \psi_{j,k}^e\rangle \chi_{j,k}\big|^2\Big)^{1/2} w^{1/p}\Big\|_{L^p(dx)}$$

$$\times \Big\|\Big(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \big|\langle g,\, \psi_{j,k}^e\rangle \chi_{j,k}\big|^2\Big)^{1/2} w^{-1/p}\Big\|_{L^{p'}(dx)}$$

$$= \Big\|\Big(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \psi_{j,k}^e\rangle \chi_{j,k}\big|^2\Big)^{1/2}\Big\|_{L^p(w)}$$

$$\times \Big\|\Big(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \big|\langle g,\, \psi_{j,k}^e\rangle \chi_{j,k}\big|^2\Big)^{1/2}\Big\|_{L^{p'}(w^{-1/(p-1)})}.$$

Note that $w^{-1/(p-1)} \in A_{p'}^{\mathrm{dy},0}$ because of $w \in A_p^{\mathrm{dy},0}$. Hence by Lemma 4.6, there exists a constant $0 < a_1 < \infty$ independent of f and g such that

$$\Big\|\Big(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \big|\langle g,\, \psi_{j,k}^e\rangle \chi_{j,k}\big|^2\Big)^{1/2}\Big\|_{L^{p'}(w^{-1/(p-1)})}$$

$$\le a_1 \|g\|_{L^{p'}(w^{-1/(p-1)})} \le a_1.$$

Thus we obtain

$$\Big|\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \langle f,\, \psi_{j,k}^e\rangle\langle \psi_{j,k}^e,\, g\rangle\Big|$$

$$\le a_1 \Big\|\Big(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \psi_{j,k}^e\rangle \chi_{j,k}\big|^2\Big)^{1/2}\Big\|_{L^p(w)}.$$

Consequently it follows that

$$\|f\|_{L^p(w)} \le \max\{A_{p'}^{\mathrm{dy},0}(w^{-1/(p-1)})^{1/p'},\, a_1\}$$

$$\times \Big\{\Big(\sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \varphi_k\rangle \|\varphi_k\|_{L^p(w)}\big|^p\Big)^{1/p}$$

$$+ \Big\|\Big(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \psi_{j,k}^e\rangle \chi_{j,k}\big|^2\Big)^{1/2}\Big\|_{L^p(w)}\Big\}.$$

Following the density of $L^p(w) \cap L^2(dx)$ in $L^p(w)$, the above inequality is

satisfied for all $f \in L^p(w)$. □

4.2. Proof of (I2) ⇒ (I3)

In this subsection, we give the proof of (I2) ⇒ (I3) of Theorem 4.2. First of all, we show that $(\varphi_k)^*$, $(\psi^e_{j,k})^* \in L^p(d\mu)^*$ for all $e \in E$, $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$. Following (I2), there exists a constant $0 < C_3 < \infty$ such that $\mathcal{M}_{p,\mu,0}(f) \le C_3\|f\|_{L^p(d\mu)}$ for all $f \in L^p(d\mu)$. Besides we see that $0 < \mu(Q_k)$, $\mu(Q_{j,k}) < \infty$. Then we have

$$\begin{aligned}
\big|(\varphi_k)^*(f)\big| &= \big|\langle f,\, \varphi_k\rangle\|\varphi_k\|_{L^p(d\mu)}\big| \cdot \big\|\varphi_k\big\|^{-1}_{L^p(d\mu)} \\
&\le \Big(\sum_{K\in\mathbb{Z}^n} \big|\langle f,\, \varphi_K\rangle\|\varphi_K\|_{L^p(d\mu)}\big|^p\Big)^{1/p} \cdot \mu(Q_k)^{-1/p} \\
&\le \mathcal{M}_{p,\mu,0}(f) \cdot \mu(Q_k)^{-1/p} \\
&\le C_3\mu(Q_k)^{-1/p}\|f\|_{L^p(d\mu)}.
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
\big|(\psi^e_{j,k})^*(f)\big| &= \big|\langle f,\, \psi^e_{j,k}\rangle\|\chi_{j,k}\|_{L^p(d\mu)}\big| \cdot \|\chi_{j,k}\|^{-1}_{L^p(d\mu)} \\
&= \Big(\int_{\mathbb{R}^n} \big|\langle f,\, \psi^e_{j,k}\rangle\chi_{j,k}(x)\big|^p d\mu(x)\Big)^{1/p} \cdot \big(2^{jn/2}\mu(Q_{j,k})^{1/p}\big)^{-1} \\
&\le \left(\int_{\mathbb{R}^n} \Big(\sum_{\lambda\in E}\sum_{J=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \psi^\lambda_{J,K}\rangle\chi_{J,K}(x)\big|^2\Big)^{1/2\cdot p} d\mu(x)\right)^{1/p} \\
&\qquad\qquad \cdot 2^{-jn/2}\mu(Q_{j,k})^{-1/p} \\
&= \left\|\Big(\sum_{\lambda\in E}\sum_{J=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \psi^\lambda_{J,K}\rangle\chi_{J,K}\big|^2\Big)^{1/2}\right\|_{L^p(d\mu)} \cdot 2^{-jn/2}\mu(Q_{j,k})^{-1/p} \\
&\le \mathcal{M}_{p,\mu,0}(f) \cdot 2^{-jn/2}\mu(Q_{j,k})^{-1/p} \\
&\le C_3 2^{-jn/2}\mu(Q_{j,k})^{-1/p}\|f\|_{L^p(d\mu)}.
\end{aligned}$$

Consequently we have proved $(\varphi_k)^*$, $(\psi^e_{j,k})^* \in L^p(d\mu)^*$.

In order to prove that the sequence $\{\varphi_k\colon k \in \mathbb{Z}^n\} \cup \{\psi^e_{j,k}\colon e \in E,\, j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n\}$ forms an unconditional basis for $L^p(d\mu)$, we shall check the following two conditions:

(I) There exists a constant $0 < C_4 < \infty$ independent of f, A and B such that $\|T_{A,B}f\|_{L^p(d\mu)} \le C_4\|f\|_{L^p(d\mu)}$ for all $f \in L^p(d\mu)$ and all finite subsets $A \subset \mathbb{Z}^n$ and $B \subset E \times \mathbb{Z}_+ \times \mathbb{Z}^n$, where $T_{A,B}f := \sum_{k\in A}\langle f,\, \varphi_k\rangle\varphi_k + \sum_{(e,j,k)\in B}\langle f,\, \psi^e_{j,k}\rangle\psi^e_{j,k}$.

(II) $L^p(d\mu) = \overline{\operatorname{span}\{\varphi_k \colon k \in \mathbb{Z}^n\} \cup \operatorname{span}\{\psi_{j,k}^e \colon e \in E,\, j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n\}}^{L^p(d\mu)}$.

We show the condition (I) first. By the assumption (I2), there exists a constant $0 < C_5 < \infty$ independent of f, A and B such that $C_5\|T_{A,B}f\|_{L^p(d\mu)} \le \mathcal{M}_{p,\mu,0}(T_{A,B}f)$. Since $\{\varphi_k \colon k \in \mathbb{Z}^n\} \cup \{\psi_{j,k}^e \colon e \in E, j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n\}$ forms an orthonormal system for $L^2(dx)$, we obtain

$$\begin{aligned}
\mathcal{M}_{p,\mu,0}(T_{A,B}f) &= \Big(\sum_{k\in A} \big|\langle f,\, \varphi_k\rangle \|\varphi_k\|_{L^p(d\mu)}\big|^p\Big)^{1/p} \\
&\quad + \Big\| \Big(\sum_{(e,j,k)\in B} \big|\langle f,\, \psi_{j,k}^e\rangle \chi_{j,k}\big|^2 \Big)^{1/2} \Big\|_{L^p(d\mu)} \\
&\le \Big(\sum_{k\in A} \big|\langle f,\, \varphi_k\rangle \|\varphi_k\|_{L^p(d\mu)}\big|^p\Big)^{1/p} \\
&\quad + \Big\| \Big(\sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \psi_{j,k}^e\rangle \chi_{j,k}\big|^2 \Big)^{1/2} \Big\|_{L^p(d\mu)} \\
&= \mathcal{M}_{p,\mu,0}(f) \\
&\le C_3\|f\|_{L^p(d\mu)}.
\end{aligned}$$

Consequently, we have showed $\|T_{A,B}f\|_{L^p(d\mu)} \le {C_5}^{-1} C_3 \|f\|_{L^p(d\mu)}$.

Lastly we check (II). We shall prove that

$$\lim_{A\nearrow\mathbb{Z}^n,\, B\nearrow E\times\mathbb{Z}_+\times\mathbb{Z}^n} \mathcal{M}_{p,\mu,0}(f - T_{A,B}f) = 0,$$

since $C_5\|f - T_{A,B}f\|_{L^p(d\mu)} \le \mathcal{M}_{p,\mu,0}(f - T_{A,B}f)$ for all $f \in L^p(d\mu)$ from the hypothesis (I2). Now we define

$$\mathcal{M}_1(f) := \Big(\sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \varphi_k\rangle \|\varphi_k\|_{L^p(d\mu)}\big|^p\Big)^{1/p}$$

$$\text{and} \quad \mathcal{M}_2(f) := \Big\| \Big(\sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \psi_{j,k}^e\rangle \chi_{j,k}\big|^2 \Big)^{1/2} \Big\|_{L^p(d\mu)}.$$

Then we have $\mathcal{M}_{p,\mu,0}(f - T_{A,B}f) = \mathcal{M}_1(f - T_{A,B}f) + \mathcal{M}_2(f - T_{A,B}f)$. We omit the detail because it is obvious that the orthonormality of the system $\{\varphi_k \colon k \in \mathbb{Z}^n\} \cup \{\psi_{j,k}^e \colon e \in E,\, j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n\}$ with regard to the L^2-product, the boundedness of $\mathcal{M}_i(f - T_{A,B}f)$ $(i = 1, 2)$ and Lebesgue's dominated convergence theorem give us the desired result. □

4.3. Proof of (I3) ⇒ (I1)

In the end of this section, we prove (I3) ⇒ (I1) of Theorem 4.2. Since the hypothesis (I3) states that $\{\varphi_k : k \in \mathbb{Z}^n\} \cup \{\psi^e_{j,k} : e \in E,\ j \in \mathbb{Z}_+,\ k \in \mathbb{Z}^n\}$ forms an unconditional basis for $L^p(d\mu)$, we have the following Lemma from the arguments in [ABM]:

Lemma 4.11 (cf. [ABM, Proof of Theorem 4 (D1) ⇒ (D2), Step a and Step b]) *μ is equivalent to the Lebesgue measure. Namely, $\mu(E) = 0$ if and only if $|E| = 0$ for all bounded Borel set $E \subset \mathbb{R}^n$.*

Using Lemma 4.11 and Radon-Nikodym theorem, there exists a unique $w \in L^1_{\mathrm{loc}}(dx)$ such that $d\mu(x) = w(x)dx$.

Finally we prove $w \in A_p^{\mathrm{dy},0}$. We define

$$S_j f := \sum_{k\in\mathbb{Z}^n} \langle f,\, \varphi_k\rangle \varphi_k + \sum_{e\in E}\sum_{l=0}^{j-1}\sum_{k\in\mathbb{Z}^n} \langle f,\, \psi^e_{l,k}\rangle \psi^e_{l,k} \quad (j \in \mathbb{N}),$$
$$P_j f := \sum_{k\in\mathbb{Z}^n} \langle f,\, \varphi_{j,k}\rangle \varphi_{j,k} \quad (j \in \mathbb{Z}).$$

Lemma 4.12 *It follows that $S_j f = P_j f$ a.e. $\mathbb{R}^n$ for all $j \in \mathbb{N}$ and $f \in L^2(dx)$.*

Proof of Lemma 4.12. We see that for every $j \in \mathbb{N}$,

$$\overline{\mathrm{span}\{\varphi_k : k \in \mathbb{Z}^n\}}^{L^2(dx)} \oplus \overline{\mathrm{span}\{\psi^e_{l,k} : e \in E,\ l = 0,\ 1,\ \ldots,\ j-1,\ k \in \mathbb{Z}^n\}}^{L^2(dx)} = \overline{\mathrm{span}\{\varphi_{j,k} : k \in \mathbb{Z}^n\}}^{L^2(dx)},$$

in the sense of subspaces in $L^2(dx)$. Hence we have $S_j f = P_j f$ in $L^2(dx)$ for all $f \in L^2(dx)$. Thus we get $|S_j f(x) - P_j f(x)|^2 = 0$ a.e. $x \in \mathbb{R}^n$ since $\int_{\mathbb{R}^n} |S_j f(x) - P_j f(x)|^2 dx = 0$. Namely we have showed $S_j f(x) = P_j f(x)$ a.e. $x \in \mathbb{R}^n$. □

Lemma 4.13 ([HW, Lemma 2.7 in Chapter 5]) *Let X be a Banach space and $\{x_j\}_{j\in\mathbb{N}}$ be an unconditional basis for X. Then, there exists a unique sequence $\{f_j(x)\}_{j\in\mathbb{N}} \subset \mathbb{C}$ for every $x \in X$ such that $x = \sum_{j\in\mathbb{N}} f_j(x)x_j$ converges unconditionally in X. Additionally we define $S_\beta(x) := \sum_{j\in\mathbb{N}} \beta_j f_j(x) x_j$ for every $\beta = \{\beta_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ with $|\beta_j| \le 1$ $(j \in \mathbb{N})$. Then, $S_\beta(x)$ converges unconditionally in X, and $\{S_\beta\}_\beta$ is uniformly bounded on X.*

Now we define

$$\beta_l^j := \begin{cases} 1 & (0 \le l \le j-1) \\ 0 & (l \ge j) \end{cases}$$

for each $j \in \mathbb{N}$. Then we can denote

$$S_j f = \sum_{k\in\mathbb{Z}^n} \langle f,\, \varphi_k\rangle \varphi_k + \sum_{e\in E}\sum_{l=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \beta_l^j \langle f,\, \psi_{l,k}^e\rangle \psi_{l,k}^e \quad (f \in L^p(w)).$$

Hence, using Lemma 4.13, $S_j(f)$ converges unconditionally in $L^p(w)$ for all $f \in L^p(w)$ and $j \in \mathbb{N}$. We also obtain that $\{S_j\}_{j\in\mathbb{N}}$ is uniformly bounded on $L^p(w)$. On the other hand, we have

$$P_0 f = \sum_{k\in\mathbb{Z}^n} \langle f,\, \varphi_k\rangle \varphi_k + \sum_{e\in E}\sum_{l=0}^{\infty}\sum_{k\in\mathbb{Z}^n} 0 \cdot \langle f,\, \psi_{l,k}^e\rangle \psi_{l,k}^e \quad (f \in L^p(w)).$$

Thus, it follows that $P_0(f)$ converges unconditionally in $L^p(w)$ for all $f \in L^p(w)$ and that P_0 is bounded on $L^p(w)$ by using Lemma 4.13 similarly. Consequently we obtain the uniformly weak type $(p,\, p)$ inequality with respect to $w(x)dx$ for $\{P_j\}_{j\in\mathbb{Z}_+}$. Namely there exists a constant $0 < C_5 < \infty$ such that

$$s \cdot w\big(\{x \in \mathbb{R}^n \colon |P_j f(x)| > s\}\big)^{1/p} \le C_5 \|f\|_{L^p(w)}$$

for all $j \in \mathbb{Z}_+$, $f \in L^p(w) \cap L^2(dx)$ and $0 < s < \infty$. We consider a dyadic cube Q with $|Q| = 2^{-jn}$ for a fixed $j \in \mathbb{Z}_+$. We define $\sigma_\varepsilon(x) := \big(w(x) + \varepsilon\big)^{-1/(p-1)}$ and $\sigma_\varepsilon(Q) := \int_Q \sigma_\varepsilon(y)dy$ for every $\varepsilon > 0$. Then we get

$$\begin{aligned} P_j\left(\sigma_\varepsilon \chi_Q\right)(x) &= \int_{\mathbb{R}^n} 2^{jn} \sum_{k\in\mathbb{Z}^n} \chi_{Q_{j,k}}(x)\chi_{Q_{j,k}}(y) \cdot \left(\sigma_\varepsilon \chi_Q\right)(y)dy \\ &= 2^{jn} \int_Q \sigma_\varepsilon(y)dy \\ &= |Q|^{-1}\sigma_\varepsilon(Q) \quad \text{a.e. } x \in Q. \end{aligned}$$

Since $\sigma_\varepsilon \chi_I \in L^p(w) \cap L^2(dx)$, we have

$$s \cdot w\big(\big\{x \in \mathbb{R}^n \colon |P_j(\sigma_\varepsilon\chi_Q)(x)| > s\big\}\big)^{1/p} \le C_5 \|\sigma_\varepsilon \chi_Q\|_{L^p(w)}.$$

Now notice that

$$w\big(\{x \in \mathbb{R}^n \colon |P_j(\sigma_\varepsilon\chi_Q)(x)| > s\}\big) \ge w\big(\{x \in Q \colon |Q|^{-1}\sigma_\varepsilon(Q) > s\}\big)$$

and $\|\sigma_\varepsilon \chi_Q\|_{L^p(w)} = \big(\int_Q |\sigma_\varepsilon(y)|^p w(y)dy\big)^{1/p}$. Therefore defining $s_\delta := \sigma_\varepsilon(Q) \times(|Q|+\delta)^{-1}$ for any fixed $\delta > 0$, we have $|Q|^{-1}\sigma_\varepsilon(Q) > s_\delta$ and

$$\begin{aligned} w(Q) &= w\big(\{x \in Q\colon |Q|^{-1}\sigma_\varepsilon(Q) > s_\delta\}\big) \\ &\le s_\delta^{-p} {C_5}^p \int_Q |\sigma_\varepsilon(y)|^p w(y)dy \\ &= s_\delta^{-p} {C_5}^p \int_Q \big(w(y)+\varepsilon\big)^{-p/(p-1)} w(y)dy \\ &\le s_\delta^{-p} {C_5}^p \int_Q \big(w(y)+\varepsilon\big)^{-1/(p-1)} dy \\ &= {C_5}^p (|Q|+\delta)^p \sigma_\varepsilon(Q)^{1-p}. \end{aligned}$$

Hence it follows that $w(Q) \le {C_5}^p |Q|^p \sigma_\varepsilon(Q)^{1-p}$ when $\delta \to 0$. Additionally letting $\varepsilon \to 0$, we have $w(Q) \le {C_5}^p |Q|^p \big(\int_Q w(y)^{-1/(p-1)}dy\big)^{1-p}$, i.e., we obtain

$$\frac{1}{|Q|} w(Q) \Big(\frac{1}{|Q|}\int_Q w(y)^{-1/(p-1)}dy\Big)^{p-1} \le {C_5}^p.$$

Consequently we have proved $w \in A_p^{\mathrm{dy},0}$. □

5. The greedy bases in $L^p(w)$ with $w \in A_p^{\mathrm{dy},m}$

The theory of greedy approximations on several non-weighted function spaces has been studied so far ([CDH], [GH], [KT], [Ky]). In this section, we give the greedy basis of $L^p(w)$ with $w \in A_p^{\mathrm{dy},m}$ using our result Theorem 4.2.

5.1. Definitions and statement of the result

Let us begin with introducing two kinds of bases.

Let X be a Banach space and $\{x_k\}_{k=1}^\infty$ be a Schauder basis in X such that $\|x_k\|_X = 1$ for all $k \in \mathbb{N}$. Then there exists a unique sequence $\{c_k(x)\}_{k=1}^\infty \subset \mathbb{C}$ such that $x = \sum_{k=1}^\infty c_k(x)x_k$ in X for all $x \in X$.

Definition 5.1 (greedy basis) We call $\{x_k\}_{k=1}^\infty$ a greedy basis for X if there exists a constant $0 < C < \infty$ such that for every $x \in X$ there exists a permutation ρ of $\mathbb{N}$ which satisfies

$$|c_{\rho(1)}(x)| \ge |c_{\rho(2)}(x)| \ge \cdots \ge |c_{\rho(N)}(x)|$$

$$\text{and} \quad \Big\|x-\sum_{k=1}^{N} c_{\rho(k)}(x)x_{\rho(k)}\Big\|_X \le C \inf_{y\in\Sigma_N}\|x-y\|_X,$$

for every $N\in\mathbb{N}$, where $\Sigma_N := \big\{\sum_{i\in\Lambda}\alpha_i x_i\colon \alpha_i\in\mathbb{C},\ \sharp\Lambda\le N,\ \Lambda\subset\mathbb{N}\big\}$.

Definition 5.2 (democratic basis) We say that $\{x_k\}_{k=1}^{\infty}$ is a democratic basis for X if there exists a constant $0<D<\infty$ independent of P and Q such that $\big\|\sum_{k\in P}x_k\big\|_X \le D\big\|\sum_{k\in Q}x_k\big\|_X$ for any finite subsets $P,Q\subset\mathbb{N}$ with the same cardinality $\sharp P=\sharp Q$.

Theorem 5.3 we describe next is the key to the proof of Theorem 5.5 which is the main result of this section:

Theorem 5.3 ([KT, Theorem 1]) *$\{x_k\}_{k=1}^{\infty}$ is a greedy basis if and only if it is an unconditional and democratic basis.*

Remark 5.4 ([KT, Section 3]) S.V. Konyagin and V.N. Temlyakov give some examples of bases, which are not democratic but unconditional, or which are not unconditional but democratic.

The following is the conclusion of this section:

Theorem 5.5 *Let $\{\psi^e\}_{e\in E}$ be the Haar wavelet set, $\varphi := \chi_{[0,1)^n}$ (i.e., the Haar scaling function), $m\in\mathbb{Z}$, $1<p<\infty$ and $w\in A_p^{\mathrm{dy},m}$. We define*

$$\tilde{\varphi}_{m,k} := \frac{\varphi_{m,k}}{\|\varphi_{m,k}\|_{L^p(w)}} = 2^{-mn/2}w(Q_{m,k})^{-1/p}\varphi_{m,k}$$

$$\text{and} \quad \widetilde{\psi^e}_{j,k} := \frac{\psi^e_{j,k}}{\|\psi^e_{j,k}\|_{L^p(w)}} = 2^{-jn/2}w(Q_{j,k})^{-1/p}\psi^e_{j,k}.$$

Then, the sequence $\{\tilde{\varphi}_{m,k}\colon k\in\mathbb{Z}^n\}\cup\{\widetilde{\psi^e}_{j,k}\colon e\in E,\ j\ge m,\ k\in\mathbb{Z}^n\}$ forms a greedy basis for $L^p(w)$.

5.2. Two lemmas

We prepare the following two Lemmas which we will need in the next subsection.

Lemma 5.6 *Let $m\in\mathbb{Z}$, $1<p<\infty$ and $w\in A_p^{\mathrm{dy},m}$. Then, w satisfies the dyadic reverse doubling condition, i.e., there exists a constant $1<d<\infty$ independent of I and I' such that $dw(I')\le w(I)$ for all dyadic cubes I, I' satisfying $I'\subsetneqq I$.*

The proof of Lemma 5.6 is found in [GR, Section II. 1, p. 141] or [Ta, Proof of Corollary 1.1]. Additionally, we can get the following Lemma by easy calculations.

Lemma 5.7 *Given $0 \le a, b < \infty$, it follows that $2^{(1-p)/p}(a^{1/p} + b^{1/p}) \le (a+b)^{1/p} \le a^{1/p} + b^{1/p}$. In particular, the right hand side equality sign holds if and only if $a = 0$ or $b = 0$. On the other hand, the left hand side equality sign holds if and only if $a = b$.*

5.3. Proof of Theorem 5.5

The proof of Theorem 5.5 we give here is based on [CDH, the proof of Lemma 4.1]. Following Theorem 4.2 (I3) and Theorem 5.3, it is enough to prove that $\{\tilde{\varphi}_{m,k} \colon k \in \mathbb{Z}^n\} \cup \{\widetilde{\psi^e}_{j,k} \colon e \in E,\, j \ge m,\, k \in \mathbb{Z}^n\}$ is democratic.

We see that $\{\varphi_{m,k} \colon k \in \mathbb{Z}^n\} \cup \{\psi^e_{j,k} \colon e \in E,\, j \ge m,\, k \in \mathbb{Z}^n\}$ forms an unconditional basis for $L^p(w)$ by Theorem 4.2 (I3). Thus we can write

$$f = \sum_{k\in\mathbb{Z}^n} a_{m,k}(f)\tilde{\varphi}_{m,k} + \sum_{e\in E}\sum_{j=m}^{\infty}\sum_{k\in\mathbb{Z}^n} b^e_{j,k}(f)\widetilde{\psi^e}_{j,k}$$

for all $f \in L^p(w)$, where

$$a_{m,k}(f) := \langle f,\, \varphi_{m,k}\rangle \|\varphi_{m,k}\|_{L^p(w)}, \quad b^e_{j,k}(f) := \langle f,\, \psi^e_{j,k}\rangle \|\psi^e_{j,k}\|_{L^p(w)}.$$

Using Theorem 4.2 (I2), we have

$$\begin{aligned} c\|f\|_{L^p(w)} &\le \Big(\sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \varphi_{m,k}\rangle \|\varphi_{m,k}\|_{L^p(w)}\big|^p\Big)^{1/p} \\ &\quad + \Big\| \Big(\sum_{e\in E}\sum_{j=m}^{\infty}\sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \psi^e_{j,k}\rangle \chi_{j,k}\big|^2\Big)^{1/2} \Big\|_{L^p(w)} \\ &= \Big(\sum_{k\in\mathbb{Z}^n} |a_{m,k}(f)|^p\Big)^{1/p} \\ &\quad + \Big\| \Big(\sum_{e\in E}\sum_{j=m}^{\infty}\sum_{k\in\mathbb{Z}^n} \big|w(Q_{j,k})^{-1/p} b^e_{j,k}(f)\chi_{Q_{j,k}}\big|^2\Big)^{1/2} \Big\|_{L^p(w)} \\ &\le C\|f\|_{L^p(w)}. \end{aligned} \tag{1}$$

Let us denote $\tilde{\varphi}_Q := \tilde{\varphi}_{j,k}$ and $\widetilde{\psi^e}_Q := \widetilde{\psi^e}_{j,k}$ for a dyadic cube $Q = Q_{j,k}$. Now we take finite subsets $A_i \subset \{Q_{m,k} \colon k \in \mathbb{Z}^n\}$, $E_i \subset E$ and $B_i \subset \{Q_{j,k} \colon j \ge m,\, k \in \mathbb{Z}^n\}$ $(i = 1, 2)$ satisfying $\sharp A_1 + \sharp(E_1 \times B_1) = \sharp A_2 +$

$\sharp(E_2 \times B_2)$ arbitrarily, and set $g := \sum_{I \in A_1} \tilde{\varphi}_I + \sum_{e \in E_1} \sum_{J \in B_1} \widetilde{\psi^e}_J$ and $h := \sum_{I \in A_2} \tilde{\varphi}_I + \sum_{e \in E_2} \sum_{J \in B_2} \widetilde{\psi^e}_J$. Using (1) and $1 \le \sharp E_1 \le \sharp E = 2^n - 1$, we obtain

$$\begin{aligned}
c\|g\|_{L^p(w)} &\le (\sharp A_1)^{1/p} + \Big\| \Big(\sum_{e \in E_1} \sum_{J \in B_1} |w(J)^{-1/p} \chi_J|^2 \Big)^{1/2} \Big\|_{L^p(w)} \\
&= (\sharp A_1)^{1/p} + (\sharp E_1)^{1/2} \\
&\quad \cdot \bigg(\int_{\bigcup_{J' \in B_1} J'} \Big(\sum_{J \in B_1} w(J)^{-2/p} \chi_J(x) \Big)^{p/2} w(x) dx \bigg)^{1/p} \\
&\le (\sharp A_1)^{1/p} + (2^n - 1)^{1/2} (\sharp E_1)^{1/p} \\
&\quad \cdot \bigg(\int_{\bigcup_{J' \in B_1} J'} \Big(\sum_{J \in B_1} w(J)^{-2/p} \chi_J(x) \Big)^{p/2} w(x) dx \bigg)^{1/p}. \qquad (2)
\end{aligned}$$

For each $x \in \bigcup_{J \in B_1} J$, $J_1(x)$ denotes the minimal dyadic cube in B_1 with regard to the inclusion relation that contains x. Then we get

$$\sum_{J \in B_1} w(J)^{-2/p} \chi_J(x) \le \sum_{r=0}^{\infty} w(J_r)^{-2/p}, \qquad (3)$$

where $J_0 := J_1(x)$, J_r is a dyadic cube satisfying $J_{r-1} \subset J_r$ and $2^n |J_{r-1}| = |J_r|$ for every $r \in \mathbb{N}$. By Lemma 5.6, we obtain

$$w(J_r) \ge d w(J_{r-1}) \ge \cdots \ge d^r w(J_0) = d^r w\big(J_1(x)\big)$$

for all $r \in \mathbb{N}$. Thus we have

$$\sum_{r=0}^{\infty} w(J_r)^{-2/p} \le \sum_{r=0}^{\infty} \big(d^r w\big(J_1(x)\big) \big)^{-2/p} = C_0 w\big(J_1(x)\big)^{-2/p}, \qquad (4)$$

where $C_0 := \big(1 - d^{-2/p}\big)^{-1}$. Following (3) and (4), we obtain

$$\begin{aligned}
&\int_{\bigcup_{J' \in B_1} J'} \Big(\sum_{J \in B_1} w(J)^{-2/p} \chi_J(x) \Big)^{p/2} w(x) dx \\
&\le \int_{\bigcup_{J \in B_1} J} \big(C_0 w\big(J_1(x)\big)^{-2/p} \big)^{p/2} w(x) dx \\
&= C_0{}^{p/2} \int_{\bigcup_{J \in B_1} J} w\big(J_1(x)\big)^{-1} w(x) dx. \qquad (5)
\end{aligned}$$

Now we set $\tilde{J} := \{x \in \bigcup_{J\in B_1} J \colon J_1(x) = J\}$ for each $J \in B_1$. Then, since $\tilde{J} \subset J$ and $\bigcup_{J\in B_1} J = \bigcup_{J\in B_1} \tilde{J}$, it follows that

$$\begin{aligned}\int_{\bigcup_{J\in B_1} J} w\big(J_1(x)\big)^{-1} w(x)dx &= \int_{\bigcup_{J\in B_1} \tilde{J}} w\big(J_1(x)\big)^{-1} w(x)dx \\ &= \int_{\bigcup_{J\in B_1} \tilde{J}} w\big(J\big)^{-1} w(x)dx \\ &\le \sum_{J\in B_1} \int_J w(J)^{-1} w(x)dx \\ &= \sharp B_1. \end{aligned} \tag{6}$$

Following (2)–(6), we have

$$c\|g\|_{L^p(w)} \le (\sharp A_1)^{1/p} + {C_0}^{1/2}(2^n-1)^{1/2}(\sharp E_1)^{1/p}(\sharp B_1)^{1/p}.$$

In addition, using Lemma 5.7, there exists a constant $0 < C_1 < \infty$ independent of g, A_1, E_1 and B_1 such that

$$\|g\|_{L^p(w)} \le C_1\big\{\sharp A_1 + \sharp(E_1 \times B_1)\big\}^{1/p}. \tag{7}$$

On the other hand, applying (1) to $h = \sum_{I\in A_2} \tilde{\varphi}_I + \sum_{e\in E_2}\sum_{J\in B_2} \widetilde{\psi^e}_J$ and using $1 \le \sharp E_2 \le \sharp E = 2^n - 1$, we have

$$\begin{aligned} C\|h\|_{L^p(w)} &\ge (\sharp A_2)^{1/p} + \Big\| \Big(\sum_{e\in E_2}\sum_{J\in B_2} \big|w(J)^{-1/p}\chi_J\big|^2 \Big)^{1/2} \Big\|_{L^p(w)} \\ &= (\sharp A_2)^{1/p} + (\sharp E_2)^{1/2} \\ &\quad \cdot \bigg(\int_{\bigcup_{J'\in B_2} J'} \Big(\sum_{J\in B_2} w(J)^{-2/p}\chi_J(y) \Big)^{p/2} w(y)dy \bigg)^{1/p} \\ &\ge (\sharp A_2)^{1/p} + (2^n-1)^{-1/p}(\sharp E_2)^{1/p} \\ &\quad \cdot \bigg(\int_{\bigcup_{J'\in B_2} J'} \Big(\sum_{J\in B_2} w(J)^{-2/p}\chi_J(y) \Big)^{p/2} w(y)dy \bigg)^{1/p}. \end{aligned} \tag{8}$$

For each $y \in \bigcup_{J\in B_2} J$, $J_2(y)$ denotes the minimal dyadic cube in B_2 with regard to the inclusion relation that contains y. Then we have

$$\Big(\sum_{J\in B_2} w(J)^{-2/p}\chi_J(y) \Big)^{p/2} \ge w\big(J_2(y)\big)^{-1}. \tag{9}$$

Now using the same argument as (3)–(4), replacing "B_1, $-2/p$ and $J_1(x)$" by "B_2, -1 and $J_2(y)$" respectively, we get

$$\sum_{J\in B_2} w(J)^{-1}\chi_J(y) \le C_0' w\big(J_2(y)\big)^{-1}, \tag{10}$$

where C_0' is a constant depending on only p and d. Following (8)–(10), we obtain

$$\begin{aligned}
C\|h\|_{L^p(w)} &\ge (\sharp A_2)^{1/p} + (2^n-1)^{-1/p}(\sharp E_2)^{1/p} \\
&\quad\cdot\left(\int_{\bigcup_{J'\in B_2} J'} {C_0'}^{-1} \sum_{J\in B_2} w(J)^{-1}\chi_J(y)w(y)dy\right)^{1/p} \\
&= (\sharp A_2)^{1/p} + (2^n-1)^{-1/p}(\sharp E_2)^{1/p} \\
&\qquad\qquad\cdot\Big({C_0'}^{-1} \sum_{J\in B_2} w(J)^{-1}\int_J w(y)dy\Big)^{1/p} \\
&= (\sharp A_2)^{1/p} + {C_0'}^{-1/p}(2^n-1)^{-1/p}(\sharp E_2)^{1/p}(\sharp B_2)^{1/p}.
\end{aligned}$$

By Lemma 5.7, there exists a constant $0 < C_2 < \infty$ independent of h, A_2, E_2 and B_2 such that

$$C_2\|h\|_{L^p(w)} \ge \{\sharp A_2 + \sharp(E_2\times B_2)\}^{1/p}. \tag{11}$$

Following $\sharp A_1 + \sharp(E_1\times B_1) = \sharp A_2 + \sharp(E_2\times B_2)$, (7) and (11), we get $\|g\|_{L^p(w)} \le C_1C_2\|h\|_{L^p(w)}$. Consequently we have proved that $\{\tilde{\varphi}_{m,k}\colon k\in\mathbb{Z}^n\}\cup\{\widetilde{\psi^e}_{j,k}\colon e\in E,\, j\ge m,\, k\in\mathbb{Z}^n\}$ is democratic. □

6. Further results

As we described in Section 1, H.A. Aimar et al. and P.G. Lemarié-Rieusset give the important results on characterizations and unconditional bases of weighted L^p spaces ($1 < p < \infty$) respectively ([ABM], [L]). We will introduce their results and state that we can obtain three conclusions about greedy bases in weighted L^p spaces by applying the same arguments as Subsection 5.5 to them respectively.

6.1. The greedy bases in $L^p(w)$ with $w \in A_p$ or $w \in A_p^{\mathrm{dy}}$

Let us begin with the definition of 1-regular functions.

Definition 6.1 (1-regular function) A function f on $\mathbb{R}^n$ is 1-regular if for every $m\in\mathbb{N}$ there exists a constant $0 < C_m < \infty$ such that $|\partial^\alpha f(x)| \le$

$C_m(1+|x|)^{-m}$ for all $x \in \mathbb{R}^n$ and for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in {\mathbb{Z}_+}^n$ with $|\alpha| \le 1$. Here $\partial^\alpha := \prod_{i=1}^n \partial^{\alpha_i}/\partial x_i^{\alpha_i}$ and $|\alpha| := \sum_{i=1}^n \alpha_i$.

We shall remark that we can construct a wavelet set which consists of 1-regular wavelets for a given MRA which has a 1-regular scaling function (cf. [M, Chapter 3] or [W, Chapter 5]).

Now we are ready to describe two results of [ABM]:

Theorem 6.2 ([ABM, Theorem 4]) *Let $1 < p < \infty$, μ be a positive Borel measure on $\mathbb{R}^n$, finite on compact sets and $\{\psi^e\}_{e=1}^{2^n-1}$ be a wavelet set such that each ψ^e is 1-regular. Then, the following conditions are equivalent*:

(D1) *The sequence $\{\psi_{j,k}^e : e = 1, 2, \ldots, 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ forms an unconditional basis for $L^p(d\mu)$, and $\{(\psi_{j,k}^e)^* : e = 1, 2, \ldots, 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\} \subset L^p(d\mu)^*$.*

(D2) *μ is absolutely continuous with regard to the Lebesgue measure. In addition, there exists a $w \in A_p$ such that $d\mu(x) = w(x)dx$.*

(D3) *$\|\psi_{j,k}^e\|_{L^p(d\mu)} > 0$ for all $e = 1, 2, \ldots, 2^n - 1$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. Additionally there exist two constants $0 < c \le C < \infty$ independent of f such that for every $f \in L^p(d\mu)$,*

$$c\|f\|_{L^p(d\mu)} \le \left\| \Big(\sum_{e=1}^{2^n-1} \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \psi_{j,k}^e\rangle \psi_{j,k}^e\big|^2 \Big)^{1/2} \right\|_{L^p(d\mu)} \le C\|f\|_{L^p(d\mu)}.$$

(D4) *$\mu(Q_{j,k}) > 0$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. And there exist two constants $0 < c \le C < \infty$ independent of f such that for every $f \in L^p(d\mu)$,*

$$c\|f\|_{L^p(d\mu)} \le \left\| \Big(\sum_{e=1}^{2^n-1} \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \big|\langle f,\, \psi_{j,k}^e\rangle \chi_{j,k}\big|^2 \Big)^{1/2} \right\|_{L^p(d\mu)} \le C\|f\|_{L^p(d\mu)}.$$

Theorem 6.3 ([ABM, Theorem 6]) *Let $1 < p < \infty$, μ be a positive Borel measure on $\mathbb{R}^n$, finite on compact sets and $\{\psi^e\}_{e\in E}$ be the Haar wavelet set. Then, the following conditions are equivalent*:

(H1) *The sequence $\{\psi_{j,k}^e : e \in E,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ forms an unconditional basis for $L^p(d\mu)$. Additionally, $\{(\psi_{j,k}^e)^* : e \in E,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\} \subset L^p(d\mu)^*$.*

(H2) *μ is absolutely continuous with regard to the Lebesgue measure. In*

addition, there exists a $w \in A_p^{\mathrm{dy}}$ *such that* $d\mu(x) = w(x)dx$.

(H3) $\mu(Q_{j,k}) > 0$ *for all* $j \in \mathbb{Z}$ *and* $k \in \mathbb{Z}^n$. *Additionally, there exist two constants* $0 < c \le C < \infty$ *independent of* f *such that for every* $f \in L^p(d\mu)$,

$$c\|f\|_{L^p(d\mu)} \le \Big\| \Big(\sum_{e\in E} \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \big| \langle f,\, \psi_{j,k}^e \rangle \chi_{j,k} \big|^2 \Big)^{1/2} \Big\|_{L^p(d\mu)} \le C\|f\|_{L^p(d\mu)}.$$

Applying the same arguments as Subsection 5.5 to above two theorems respectively, we can obtain the following two results about greedy bases in $L^p(w)$. Here let us mention that both of Theorem 6.2 and Theorem 6.3 state that one sequence given by the wavelets, which forms an orthonormal basis in $L^2(dx)$, becomes an unconditional basis for $L^p(w)$. Hence we don't need Lemma 5.7 in order to obtain the following two conclusions.

Corollary 6.4 *Let* $1 < p < \infty$, $w \in A_p$ *and* $\{\psi^e\}_{e=1}^{2^n-1}$ *be a wavelet set such that each* ψ^e *is* 1*-regular. Then, the sequence* $\{\widetilde{\psi^e}_{j,k} \colon e = 1, 2, \dots, 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ *forms a greedy basis for* $L^p(w)$.

Corollary 6.5 *Let* $1 < p < \infty$, $w \in A_p^{\mathrm{dy}}$ *and* $\{\psi^e\}_{e\in E}$ *be the Haar wavelet set. Then, the sequence* $\{\widetilde{\psi^e}_{j,k} \colon e \in E,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ *forms a greedy basis for* $L^p(w)$.

6.2. The greedy bases in $L^p(w)$ with $w \in A_p^{\mathrm{loc}}$

P.G. Lemarié-Rieusset gives the next result. He proved it in the case of one-variable, however, it is true in the case of several-variables with obvious modifications.

Theorem 6.6 (cf. [L, Proposition 2 (ii)]) *Let* $1 < p < \infty$, w *be a weight on* $\mathbb{R}^n$, $m \in \mathbb{Z}$, φ *be the Daubechies scaling function and* $\{\psi^e\}_{e=1}^{2^n-1}$ *be the Daubechies wavelet set. Then, the following conditions are equivalent*:

(E1) *The sequence* $\{\varphi_{m,k} \colon k \in \mathbb{Z}^n\} \cup \{\psi_{j,k}^e \colon e = 1, 2, \dots, 2^n - 1,\ j \ge m,\ k \in \mathbb{Z}^n\}$ *forms an unconditional basis for* $L^p(w)$.

(E2) *We define*

$$\mathcal{M}_{p,w,m}(f) := \Big(\sum_{k\in\mathbb{Z}^n} \big| \langle f,\, \varphi_{m,k} \rangle \|\varphi_{m,k}\|_{L^p(w)} \big|^p \Big)^{1/p}$$

$$+\left\|\Big(\sum_{e=1}^{2^n-1}\sum_{j=m}^{\infty}\sum_{k\in\mathbb{Z}^n}\big|\langle f,\,\psi_{j,k}^e\rangle\chi_{j,k}\big|^2\Big)^{1/2}\right\|_{L^p(w)}.$$

Then, there exist two constants $0 < c \le C < \infty$ *such that* $c\|f\|_{L^p(w)} \le \mathcal{M}_{p,w,m}(f) \le C\|f\|_{L^p(w)}$ *for all* $f \in L^p(w)$.

(E3) $w \in A_p^{\mathrm{loc}}$.

We obtain the following conclusion by applying the arguments in Subsection 5.5 to Theorem 6.6 directly:

Corollary 6.7 *Let* $1 < p < \infty$, $w \in A_p^{\mathrm{loc}}$, $m \in \mathbb{Z}$, φ *be the Daubechies scaling function and* $\{\psi^e\}_{e=1}^{2^n-1}$ *be the Daubechies wavelet set. Then, the sequence* $\{\tilde{\varphi}_{m,k}\colon k \in \mathbb{Z}^n\} \cup \{\widetilde{\psi^e}_{j,k}\colon e = 1, 2, \ldots, 2^n - 1,\ j \ge m,\ k \in \mathbb{Z}^n\}$ *forms a greedy basis for* $L^p(w)$.

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