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Large time behavior of solutions for parabolic equations with nonlinear gradient terms

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Abstract. In this paper we prove the global existence of mild solutions for the semilinear parabolic equation $u_t = \Delta u + a|\nabla u|^q + b|u|^{p-1}u$, $t > 0$, $x \in \mathbb{R}^n$, $n \geq 1$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, $p > 1 + (2/n)$, $(n+2)/(n+1) < q < 2$ and $q \leq p(n+2)/(n+p)$, with small initial data with respect to a norm related to the equation. We also prove that some of these global solutions are asymptotic to self–similar solutions of the equations $u_t = \Delta u + \nu a|\nabla u|^q + \mu b|u|^{p-1}u$, with ν, $\mu = 0$ or 1. The values of ν and μ depend on the decaying of the initial data and on the position of q with respect to $2p/(p+1)$.

Our results apply for the viscous Hamilton–Jacobi equation: $u_t = \Delta u + a|\nabla u|^q$ and hold without sign restriction neither on a nor on the initial data. We prove that if $(n+2)/(n+1) < q < 2$ and the initial data behaves, near infinity, like $c|x|^{-\alpha}$, $(2-q)/(q-1) \leq \alpha < n$, c is a small constant, then the resulting solution is global. Moreover, if $(2-q)/(q-1) < \alpha < n$, the solution is asymptotic to a self–similar solution of the linear heat equation. Whereas, if $(2-q)/(q-1) = \alpha < n$, the solution is asymptotic to a self–similar solution of the viscous Hamilton–Jacobi equation. The asymptotics are given, in particular, in $W^{1,\infty}(\mathbb{R}^n)$.

Key words: Semilinear parabolic equations, Global solutions, Large time behavior, Self–similar solutions, Nonlinear gradient term, Viscous Hamilton–Jacobi equation.

1. Introduction

In this paper we consider the semilinear parabolic equation

$$u_t = \Delta u + a|\nabla u|^q + b|u|^{p-1}u, \tag{1.1}$$

where $u = u(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, $n \geq 1$, $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $p > 1$, $1 < q < 2$. We are interested in the study of the global existence and the asymptotic behavior of solutions of the problem (1.1) with initial data $u(0, \cdot) = \varphi$ small with respect to the norm $\mathcal{N}$ defined below by (1.2). In particular, we consider initial values $\varphi \in C_0(\mathbb{R}^n)$ such that $\varphi(x) \sim c|x|^{-2/(P_1-1)}$ as $|x| \to \infty$, in some appropriate sense, $P_1 > 1 + (2/n)$ and c a small constant. These initial values are in $L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and are not in $L^1(\mathbb{R}^n)$.

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The aim of this paper is to prove the existence of global solutions which are asymptotic, as $t \to \infty$, to self–similar solutions of equations related to (1.1) and depending on P_1, p, q. We note that, in general, the equation (1.1) itself has no self–similar structure. The asymptotic equations are given by (1.4) and (1.5) below and have self–similar structures.

Equation (1.1) for $a \neq 0$ and $b \neq 0$ is introduced in [8], for a mathematical motivation. A physical meaning to this equation is given in [19]. A similar study to the present paper can be found in [17] where only the critical case $q = 2p/(p+1)$ is considered. In this paper we complete this study by considering the case $q \neq 2p/(p+1)$ and extend it, even for the case $q = 2p/(p+1)$ by considering other class of initial values. This allow us to obtain new asymptotic behavior. See Theorems 2, 3 and 4 below. In fact, in [17], the condition $q = 2p/(p+1)$ is needed in the norm which is used to prove the global existence. Here, we introduce an other norm $\mathcal{N}$ to measure the size of the initial values. See (1.2) below.

Equation (1.1) for $a = 0$ and $b \neq 0$ is the standard nonlinear heat equation. The global existence and the large time behavior of solutions to the nonlinear heat equation, in the spirit of the present study, is done in [7, 18]. The method used in this paper is inspired by the works in [10, 5, 7, 17, 18, 16]. We refer the reader to the introduction of [7] and to that of [18] for a more historical account of this method.

Equation (1.1) for $a \neq 0$ and $b = 0$ is typified as the viscous Hamilton–Jacobi equation in the stochastic control theory and in other physical situations. Also it is known as the Kardar–Parizi–Zhang equation (KPZ equation) from the theory of growth and roughening of surfaces. See [4, 1, 11, 12, 13, 14, 3] and references therein. See also [4, 2] for other applications of such an equation. The Large time behavior of solutions to this equation is studied in particular in the recent works [4, 1, 11, 17]. In [1, 11] the initial values are in $L^1(\mathbb{R}^n)$ and we do not cover this case in this work. We improve the results of [17] by exhibing new asymptotics, see Theorem 4 below. Also our result, if $a < 0$, can be compared to that of [4]. In fact, we give more refined asymptotics for large time. And unlike in [4], we didnot impose any restriction on the sign of the initial data, but we require the smallness of the initial value in some appropriate sense. See Theorem 4 below.

Let us present now an idea about the results obtained in this paper. We first prove the global existence of solution to the integral equation related to (1.1) for initial data φ small with respect to the norm

$$\mathcal{N}(\varphi) = \sup_{t>0}\big[t^{\beta_1}\|\mathrm{e}^{t\Delta}\varphi\|_{r_1},\, t^{\beta_1+1/2}\|\nabla \mathrm{e}^{t\Delta}\varphi\|_{r_1},\\ t^{\beta_2}\|\mathrm{e}^{t\Delta}\varphi\|_{r_2},\, t^{\beta_2+1/2}\|\nabla \mathrm{e}^{t\Delta}\varphi\|_{r_2}\big], \quad (1.2)$$

where here and in the rest of the paper, $\|\,.\,\|_r$ denotes the norm in $L^r(\mathbb{R}^n)$, $\mathrm{e}^{t\Delta}$ is the heat semigroup, P_1 and P_2 are two real numbers satisfying

$$1+\frac{2}{n} < P_1 \leq \min\Big(p,\, \frac{q}{2-q}\Big) \leq \max\Big(p,\, \frac{q}{2-q}\Big) \leq P_2.$$

If $n \geq 3$, $a \neq 0$, $b \neq 0$, $q \neq 2p/(p+1)$ and $p < 2$ we have to impose the condition $q \leq p(n+2)/(n+p)$, see Remark 3.1 below. r_1, r_2 are two Lebesgue numbers satisfying in particular

$$\frac{n(P_1-1)}{2} < r_1 = \frac{P_1-1}{P_2-1}r_2 < \infty$$

and specified below in Lemma 2.3. β_1, β_2 are given by

$$\beta_1 = \frac{1}{P_1-1} - \frac{n}{2r_1},\; \beta_2 = \frac{1}{P_2-1} - \frac{n}{2r_2}. \quad (1.3)$$

See Theorem 1 below.

Later, we show that if $\mathcal{N}(\varphi)$ is small enough and $\varphi \sim c|x|^{-2/(P_1-1)}$ as $|x| \to \infty$, $c \in \mathbb{R}$, then the resulting solution of (1.1) is asymptotic as $t \to \infty$ to a self–similar solution of the equation

$$w_t = \Delta w + a\nu|\nabla w|^q + b\mu|w|^{p-1}w, \quad (1.4)$$

where $w = w(t, x)$, $t > 0$, $x \in \mathbb{R}^n$; and $a \in \mathbb{R}$, $b \in \mathbb{R}$, $p > 1$, $1 < q < 2$ are the same parameters in Equation (1.1) and ν, μ are defined by

$$\begin{aligned}\nu &= \lim_{s\to\infty} s^{-(1/(P_1-1)-(2-q)/\{2(q-1)\})},\\ \mu &= \lim_{s\to\infty} s^{-(1/(P_1-1)-1/(p-1))}.\end{aligned} \quad (1.5)$$

Clearly since $1 < P_1 \leq \min(p, q/(2-q))$, the previous limits exist and are finite. More precisely, we have $\nu = 0$ or 1 and $\mu = 0$ or 1. One can verify that if w is a solution of (1.4) then

$$w_\lambda(t, x) = \lambda^{2/(P_1-1)}w(\lambda^2 t, \lambda x),\; \lambda > 0,\; (t, x) \in (0, \infty) \times \mathbb{R}^n$$

is also a solution of (1.4), for all $\lambda > 0$. A self–similar solution is a solution for which

$$w(t, x) = w_\lambda(t, x),\; \forall\lambda > 0,\; (t, x) \in (0, \infty) \times \mathbb{R}^n.$$

The self–similar solutions, to which the solutions of (1.1) are shown to converge, are constructed in [7, 17] and have in particular a slow spatial decay, see [7, Theorem 6.2] and [17, Theorem 2.4]. For spatially rapidly decaying self–similar solutions to the equation (1.4) see [4, 1, 3, 13, 25, 26] and references therein.

The asymptotic equation (1.4) depends on the values of P_1, p and q. If $1 < P_1 < \min(p,\, q/(2-q))$, then Equation (1.4) is the linear heat equation, that is (1.1) with $a = b = 0$. If $P_1 = \min(p,\, q/(2-q))$, we have the following three cases:

(i) if $1 < q < 2p/(p+1)$ (hence $P_1 = q/(2-q)$), then Equation (1.4) is the viscous Hamilton–Jacobi equation, that is (1.1) with $b = 0$,

(ii) if $q = 2p/(p+1)$ (hence $P_1 = q/(2-q) = p$), then Equation (1.4) is the equation (1.1),

(iii) if $2p/(p+1) < q < 2$ (hence $P_1 = p$), then Equation (1.4) is the nonlinear heat equation with one power nonlinearity, that is (1.1) with $a = 0$.

See Theorems 2–4 below, for the asymptotic behavior results. In particular, we give an estimate for the rate at which an asymptotically self-similar solution u converges, in $W^{1,r_1}(\mathbb{R}^n)$, to a self–similar solution w. For the viscous Hamilton-Jacobi equation, we show that $\|u(t) - w(t)\|_r$ and $\sqrt{t}\|\nabla u(t) - \nabla w(t)\|_r$, $r \in [r_1, \infty]$ decrease as a negative power of t faster than the decay of the self-similar solution w by itself. This gives more refined asymptotic results than those of [4, Corollaries 2.2, 2.3 and Theorem 2.11].

We remark that the large time behavior of global solutions of (1.1) depends on the position of q with respect to $2p/(p+1)$. If $q < 2p/(p+1)$, the term $|u|^{p-1}u$ has no effects on the large time behavior of solutions. Whereas if $q > 2p/(p+1)$, the term $|\nabla u|^q$ has no effects on the large time behavior of solutions. It is pointed out in some previous works that the position of q with respect to $2p/(p+1)$ has an influence on the behavior of blowing up solutions. For this, We refer the readers to [8, 9, 20, 21, 22, 23, 24] and references therein.

The rest of this paper is organized as follows. In Section 2, we establish some preliminaries which will be needed later in the proofs of the theorems. In Section 3, we state and prove Theorem 1 concerning the existence of global solutions. In Section 4, we state the asymptotic behavior results, Theorems 2–4, and give their proofs. In this paper, we sometimes denote $u(t,\,\cdot\,)$ by $u(t)$. C will designate a constant which may change from line to line and we also denote it by C_δ or $C(\delta)$ to indicate that it depends on a real number δ.

2. Preliminary lemmas

In this paper we need the following well-known smoothing properties of the heat semigroup:

$$\|\mathrm{e}^{t\Delta}\varphi\|_{s_2} \le (4\pi t)^{-(n/2)(1/s_1-1/s_2)}\|\varphi\|_{s_1}, \tag{2.1}$$
$$\|\nabla\mathrm{e}^{t\Delta}\varphi\|_{s_2} \le Ht^{-1/2-(n/2)(1/s_1-1/s_2)}\|\varphi\|_{s_1}, \tag{2.2}$$

for all $1 \le s_1 \le s_2 \le \infty$ and $t > 0$. H is a positive constant. Recall that $(\mathrm{e}^{t\Delta}\varphi)(x) = \big(E(t,\,\cdot\,)\star\varphi\big)(x)$, where $E(t,\,x) = (4\pi t)^{-n/2}\mathrm{e}^{-|x|^2/(4t)}$, $t > 0$, $x \in \mathbb{R}^n$, is the heat kernel and $\star$ denotes the convolution product. We also use the Gagliardo-Nirenberg inequality,

$$\|u\|_m \le G\|\nabla u\|_r^N\|u\|_r^{1-N}, \tag{2.3}$$

for $u \in W^{1,r}(\mathbb{R}^n)$, where $1/m = (1/r) - (N/n)$ and $0 < N < 1$, see [15]. The last condition on N is equivalent to: $r < m$ and $m < nr/(n-r)$ if $r < n$. We will use also the following interpolation inequality

$$\|u\|_s \le \|u\|_{r_1}^\theta\|u\|_{r_2}^{1-\theta}, \tag{2.4}$$

for $u \in L^{r_1}(\mathbb{R}^n)\cap L^{r_2}(\mathbb{R}^n)$, where $s \in [r_1,\, r_2]$, $\theta \in [0,\,1]$ with $1/s = (\theta/r_1) + (1-\theta)/r_2$.

In this section we establish some preliminary lemmas needed for the proofs of the main results of this paper. The proofs of some of the following lemmas is obvious and can be omitted.

Lemma 2.1 *Let $p > 1$ and $1 < q < 2$. Let P_1 and P_2 be two real numbers satisfying*

$$1 < P_1 \le \min\Big(p,\, \frac{q}{2-q}\Big) \le \max\Big(p,\, \frac{q}{2-q}\Big) \le P_2. \tag{2.5}$$

Then we have the following:

(i) $P_1/q - (P_1-1)/2 \le 1$, $P_1/p \le 1$,

(ii) $\{(P_1-1)/(P_2-1)\}P_2/q - (P_1-1)/2 \le 1$, $\{(P_1-1)/(P_2-1)\}P_2/p \le 1$,

(iii) $1 \le \{(P_2-1)/(P_1-1)\}P_1/q - (P_2-1)/2$, $1 \le \{(P_2-1)/(P_1-1)\}P_1/p$,

(iv) $1 \le P_2/q - (P_2-1)/2$, $1 \le P_2/p$.

Lemma 2.2 *Let n be a positive integer and the real numbers p and q be such that*

$$1+\frac{2}{n}<p \quad \textit{and} \quad \frac{n+2}{n+1}<q<2.$$

Let P_1 be a real number such that

$$P_1>1+\frac{2}{n}. \tag{2.6}$$

Then the following inequality

$$\frac{P_1}{n+p}<\frac{(P_1-1)}{2}\Big((P_1-1)\Big(1-\frac{q}{2}\Big)+1\Big), \tag{2.7}$$

holds for all real numbers P_1 satisfying (2.6) *if and only if*

$$q\le p\frac{n+2}{n+p}. \tag{2.8}$$

Proof. Equation (2.7) is equivalent to

$$f(P_1-1)>0, \quad \forall P_1>1+\frac{2}{n},$$

where f is a function given by

$$f(x)=x^2\Big(1-\frac{q}{2}\Big)(n+p)+x(n+p-2)-2, \quad x\in\mathbb{R}.$$

Clearly, since $q<2$, f is positive for large x. On the other hand, since f is negative at zero, it has a positive root, and a negative root. Then (2.7) is satisfied for all $P_1-1>2/n$ if and only if $f(2/n)\ge 0$ and then if and only if (2.8) is satisfied. This finishes the proof of the lemma. □

Lemma 2.3 *Let the positive integer n and the real numbers p and q be such that*

$$1+\frac{2}{n}<p,\ \frac{n+2}{n+1}<q\le p\frac{n+2}{n+p} \quad \textit{and} \quad q<2.$$

Let P_1, P_2 be two real numbers satisfying (2.5)*. Assume that*

$$\frac{n(P_1-1)}{2}>1.$$

Then there exist two real numbers r_1, $r_2=[(P_2-1)/(P_1-1)]r_1$ satisfying:
(i) $n(P_1-1)/2<r_1$, $n(P_1-1)/(p+1)<r_1$, $n(P_1-1)(1-q/2)<r_1$,

(ii) $(P_1 - 1)(1 - q/2) + 1 < r_1$,
(iii) $nP_1/(p + n) < r_1$,
(iv) $r_1 < \big(n(P_1 - 1)/2\big)\big((P_1 - 1)(1 - q/2) + 1\big)$,
(v) $r_1 < n(P_1 - 1)/(3 - P_1)$ *if* $P_1 < 3$,
(vi) $n(P_2 - 1)/2 < r_2$, $n(P_2 - 1)/(p + 1) < r_2$, $n(P_2 - 1)(1 - q/2) < r_2$
(vii) $(P_2 - 1)(1 - q/2) + 1 < r_2$,
(viii) $nP_2/(p + n) < r_2$,
(ix) $r_2 < \big(n(P_2 - 1)/2\big)\big((P_2 - 1)(1 - q/2) + 1\big)$,
(x) $r_2 < n(P_2 - 1)/(3 - P_2)$ *if* $P_2 < 3$.

We now state the following lemma.

Lemma 2.4 *Let the positive integer n and the real numbers p, q, P_1 and P_2 be as in* Lemma 2.3. *Let r_1, r_2 be the two real numbers given by* Lemma 2.3. *Then there exist two real numbers m_1 and m_2 such that*

$$p < m_1,\ \frac{pr_1}{P_1} < m_1,\ \frac{pr_1 n}{n + r_1} < m_1,$$
$$m_1 < r_1 p,\ m_1 < \frac{n(P_1 - 1)}{2}p;\ m_1 < \frac{npr_1}{nP_1 - pr_1} \quad \text{if} \quad \frac{pr_1}{P_1} < n,$$

and

$$p < m_2,\ \frac{p(P_2 - 1)r_1}{(P_1 - 1)P_2} < m_2,\ \frac{(P_2 - 1)pr_1 n}{n(P_1 - 1) + r_1(P_2 - 1)} < m_2,$$
$$m_2 < r_1 p\frac{P_2 - 1}{P_1 - 1},\ m_2 < \frac{n(P_2 - 1)}{2}p,$$
$$m_2 < \frac{npr_1(P_2 - 1)}{n(P_1 - 1)P_2 - pr_1(P_2 - 1)} \quad \text{if} \quad \frac{p(P_2 - 1)r_1}{(P_1 - 1)P_2} < n.$$

Let us define the real numbers r_{ij}, β_{ij} for i, $j \in \{1, 2\}$ by:

$$\frac{1}{r_{11}} = \frac{P_1 - 1}{r_1}\Big(\frac{P_1}{q(P_1 - 1)} - \frac{1}{2}\Big),$$
$$\beta_{11} = \Big(\frac{P_1}{q(P_1 - 1)} - \frac{1}{2}\Big) - \frac{n}{2r_{11}}, \tag{2.9}$$
$$\frac{1}{r_{12}} = \frac{1}{r_1}\frac{P_1}{p},$$
$$\beta_{12} = \frac{P_1}{p(P_1 - 1)} - \frac{n}{2r_{12}}, \tag{2.10}$$

$$\frac{1}{r_{21}} = \frac{P_1 - 1}{r_1}\Big(\frac{P_2}{q(P_2-1)} - \frac{1}{2}\Big),$$
$$\beta_{21} = \Big(\frac{P_2}{q(P_2-1)} - \frac{1}{2}\Big) - \frac{n}{2r_{21}}, \quad (2.11)$$
$$\frac{1}{r_{22}} = \frac{P_1 - 1}{r_1}\frac{P_2}{p(P_2-1)},$$
$$\beta_{22} = \frac{P_2}{p(P_2-1)} - \frac{n}{2r_{22}}. \quad (2.12)$$

We state now the following Lemma which contains all the needed tools for the proof of the global existence result.

Lemma 2.5 *Let the positive integer n and the real numbers p, q, P_1 and P_2 be as in* Lemma 2.3. *Let r_1, r_2 be the real numbers given by* Lemma 2.3. *Let β_1, β_2, r_{ij}, β_{ij}, i, $j \in \{1, 2\}$, be given respectively by* (1.3), (2.9)–(2.12). *Consider also the real numbers m_1 and m_2 given by* Lemma 2.4. *Then the real numbers r_1, r_2, β_1, β_2, m_1, m_2, r_{ij} and β_{ij} satisfy the following*:

(i) $\beta_1 > 0$, $\beta_2 > 0$, $r_{ij} \in [r_1, r_2]$, $\beta_{ij} \in [\beta_2, \beta_1]$, $\forall i, j \in \{1, 2\}$. *Also there exist* $0 \le \theta_{ij} \le 1$, *for* $i, j = 1, 2$ *such that*

$$\frac{1}{r_{ij}} = \frac{\theta_{ij}}{r_1} + \frac{1-\theta_{ij}}{r_2}; \quad \beta_{ij} = \theta_{ij}\beta_1 + (1-\theta_{ij})\beta_2,$$

(ii) $1 < r_{12} < m_1$, $1 < r_{22} < m_2$,
(iii) $m_1 < nr_{12}/(n - r_{12})$ *if* $r_{12} < n$ *and* $m_2 < nr_{22}/(n - r_{22})$ *if* $r_{22} < n$,
(iv) $1 < r_{11}/q < r_1$, $1 < r_{21}/q < r_2$, $1 < m_1/p < r_1$, $1 < m_2/p < r_2$,
(v) $\beta_1 - (n/2)(q/r_{11} - 1/r_1) - (\beta_{11} + 1/2)q + 1 = 0$,
(vi) $\beta_1 - (n/2)(p/r_{12} - 1/r_1) - p\beta_{12} + 1 = 0$,
(vii) $\beta_2 - (n/2)(q/r_{21} - 1/r_2) - (\beta_{21} + 1/2)q + 1 = 0$,
(viii) $\beta_2 - (n/2)(p/r_{22} - 1/r_2) - p\beta_{22} + 1 = 0$,
(ix) $n(q/r_{11} - 1/r_1) < 1$, $(\beta_{11} + 1/2)q < 1$,
(x) $n(q/r_{21} - 1/r_2) < 1$, $(\beta_{21} + 1/2)q < 1$,
(xi) $n(p/m_1 - 1/r_1) < 1$, $\beta_{12}p + (pn/2)(1/r_{12} - 1/m_1) < 1$,
(xii) $n(p/m_2 - 1/r_2) < 1$, $\beta_{22}p + (pn/2)(1/r_{22} - 1/m_2) < 1$.

Proof. Part (i) follows by Lemma 2.3 part (i) and Lemma 2.1. The proof of (ii)–(iv) and (ix)–(xii) follows by Lemmas 2.4 and 2.3. The properties (v)–(vii) follow by (1.3) and (2.9)–(2.12). □

In the proof of existence of global solutions to (1.1), in the next section, the property (i) is related to the interpolation inequality, the properties (ii)–(iii) are related to Gagliardo-Nirenberg inequality, the property (iv) in Lemma 2.5 represents compatibility conditions for the heat semigroup, that is $\mathrm{e}^{t\Delta}$ maps between the appropriate Lebesgue spaces; properties (ix)–(xii) are integrability conditions: to assure that the various integrals are convergent; finally, properties (v)–(viii) will allow the contraction mapping argument to be done on the time interval $(0, \infty)$ directly.

The following technical lemma will be needed in the proof of the asymptotic behavior results, Theorems 2 and 4 in Section 4.

Lemma 2.6 *Let the positive integer n and the real numbers p, q, P_1 and P_2 be as in* Lemma 2.3. *Let r_1, r_2 be the real numbers given by* Lemma 2.3. *Let β_1, β_2, r_{ij}, β_{ij}, i, $j \in \{1, 2\}$, be given respectively by* (1.3), (2.9)–(2.12). *Consider also the real numbers m_1 and m_2 given by* Lemma 2.4. *Let ν and μ be given by* (1.5). *Let $\delta > 0$ and define r'_{1j}, β'_{1j}, $j = 1, 2$ and m'_1 by*

$$\frac{1}{r'_{11}} = \frac{1}{r_{11}} + \delta(1-\nu)\Big(\frac{P_1}{P_1-1} - \frac{q}{2}\Big)^{-1}\frac{1}{r_{11}},$$
$$\beta'_{11} = \beta_{11} + \delta(1-\nu)\Big(\frac{P_1}{P_1-1} - \frac{q}{2}\Big)^{-1}\beta_{11}, \quad (2.13)$$

$$\frac{1}{r'_{12}} = \frac{1}{r_{12}} + \delta(1-\mu)\frac{P_1-1}{P_1}\frac{1}{r_{12}},$$
$$\beta'_{12} = \beta_{12} + \delta(1-\mu)\frac{P_1-1}{P_1}\beta_{12}, \quad (2.14)$$

and

$$\frac{1}{m'_1} = \frac{1}{m_1} + \delta(1-\mu).$$

Then there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ we have

(i) $r'_{1j} \in [r_1, r_2]$, $\beta'_{1j} \in [\beta_2, \beta_1]$, $\forall j \in \{1, 2\}$,
(ii) $1 < r'_{12} < m'_1$,
(iii) $m'_1 < nr'_{12}/(n - r'_{12})$ *if* $r'_{12} < n$,
(iv) $1 < r'_{11}/q < r_1$, $1 < m'_1/p < r_1$,
(v) $n(q/r'_{11} - 1/r_1) < 1$, $\quad (\beta'_{11} + 1/2)q + \nu\delta < 1$,
(vi) $n(p/m'_1 - 1/r_1) < 1$, $\quad \beta'_{12}p + (pn/2)(1/r'_{12} - 1/m'_1) + \mu\delta < 1$,
(vii) $\beta_1 - (n/2)(q/r'_{11} - 1/r_1) - (\beta'_{11} + 1/2)q + \delta(1-\nu) + 1 = 0$,
(viii) $\beta_1 - (n/2)(p/r'_{12} - 1/r_1) - p\beta'_{12} + \delta(1-\mu) + 1 = 0$,

Proof. Since conditions (i)–(vi) are satisfied for $\delta = 0$ by Lemma 2.5 they are still satisfied for $\delta > 0$ and small. δ_0 is the largest δ such that the previous conditions are satisfied for $0 < \delta < \delta_0$. (vii) follows by (2.9), (2.13) and (viii) follows by (2.10), (2.14). □

3. Global existence

In this section we prove the global existence of a mild solution of the equation (1.1). That is a solution of the integral equation associated to the equation (1.1) which is

$$u(t) = \mathrm{e}^{t\Delta}\varphi + a\int_0^t \mathrm{e}^{(t-\sigma)\Delta}(|\nabla u(\sigma)|^q)d\sigma \\ + b\int_0^t \mathrm{e}^{(t-\sigma)\Delta}(|u(\sigma)|^{p-1}u(\sigma))d\sigma. \tag{3.1}$$

We have obtained the following result.

Theorem 1 (Global existence) *Let the positive integer n and the real numbers p and q be such that*

$$1+\frac{2}{n} < p,\ \frac{n+2}{n+1} < q \le p\frac{n+2}{n+p} \quad \text{and} \quad q < 2.$$

Let P_1, P_2 be two real numbers satisfying (2.5). *Assume that*

$$\frac{n(P_1-1)}{2} > 1.$$

Let r_1, r_2 be the real numbers given by Lemma 2.3 *and consider the real numbers m_1 and m_2 given by* Lemma 2.4. *Consider also the real numbers β_1, β_2 and r_{ij}, β_{ij}, i, $j \in \{1, 2\}$, be given by* (1.3), (2.9)–(2.12) *respectively. Suppose further that $M > 0$ satisfies the inequality*

$$\mathcal{K}_1 M^{q-1} + \mathcal{K}_2 M^{p-1} < 1, \tag{3.2}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are positive constants given by (3.23) *and* (3.24) *below. Choose $R > 0$ such that*

$$R + \mathcal{K}_1 M^q + \mathcal{K}_2 M^p \le M. \tag{3.3}$$

Let φ be a tempered distribution such that

$$\mathcal{N}(\varphi) = \sup_{t>0}\big[t^{\beta_1}\|\mathrm{e}^{t\Delta}\varphi\|_{r_1},\ t^{\beta_1+1/2}\|\nabla \mathrm{e}^{t\Delta}\varphi\|_{r_1},$$

$$t^{\beta_2}\|\mathrm{e}^{t\Delta}\varphi\|_{r_2},\, t^{\beta_2+1/2}\|\nabla \mathrm{e}^{t\Delta}\varphi\|_{r_2}\big] \le R. \quad (3.4)$$

It follows that there exists a unique global solution u of (3.1) *such that*

$$\sup_{t>0}\big[t^{\beta_1}\|u(t)\|_{r_1},\, t^{\beta_1+1/2}\|\nabla u(t)\|_{r_1},$$
$$t^{\beta_2}\|u(t)\|_{r_2},\, t^{\beta_2+1/2}\|\nabla u(t)\|_{r_2}\big] \le M. \quad (3.5)$$

Furthermore,

(i) $\lim_{t\to 0} u(t) = \varphi$ *in the sense of tempered distributions.*

(ii) $u(t) - \mathrm{e}^{t\Delta}\varphi \in C([0,\,\infty),\, L^s(\mathbb{R}^n))$ *for* $\max(m_1/p,\, r_{11}/q) \le s < n(P_1-1)/2$ *and* $\max(m_2/p,\, r_{21}/q) \le s < n(P_2-1)/2$.

(iii) $u(t) - \mathrm{e}^{t\Delta}\varphi \in L^\infty\big((0,\,\infty); L^s(\mathbb{R}^n)\big)$, *for* $n(P_1-1)/2 \le s \le n(P_2-1)/2$.

In addition, if φ and ψ satisfy (3.4)*, and if u and v respectively are the solutions of* (3.1) *with initial values φ and ψ, then*

$$\sup_{t>0}\big[t^{\beta_1}\|u(t)-v(t)\|_{r_1},\, t^{\beta_1+1/2}\|\nabla u(t)-\nabla v(t)\|_{r_1},$$
$$t^{\beta_2}\|u(t)-v(t)\|_{r_2},\, t^{\beta_2+1/2}\|\nabla u(t)-\nabla v(t)\|_{r_2}\big]$$
$$\le \big(1-(\mathcal{K}_1 M^{q-1}+\mathcal{K}_2 M^{p-1})\big)^{-1}\mathcal{N}\big(\varphi-\psi\big). \quad (3.6)$$

Remark 3.1 Assume that

$$1+\frac{2}{n} < p \quad \text{and} \quad \frac{n+2}{n+1} < q < 2.$$

Then if $q = 2p/(p+1)$ or $p \ge 2$ the condition (2.8), i.e. $q \le p(n+2)/(n+p)$, is clearly satisfied. In particular, it does not appear for the critical case $q = 2p/(p+1)$, for $n = 1$ and for $n = 2$. Observe also that when $b = 0$ or $a = 0$ the condition (2.8) is not needed. We remark also that all the results of this paper still valid if we replace (2.8) by (2.7), see Lemma 2.2 above. Finally, we remark that the condition (2.8) seems to be technical, see Lemma 2.3 parts (iii)–(iv). Also see the first and the sixth inequality in Lemma 2.4 and Lemma 2.5 parts (iii)–(iv) and (ix).

Remark 3.2 As an example of initial values φ satisfying (3.4) we may take $\varphi \in L^{n(P_1-2)/2}(\mathbb{R}^n) \cap L^{n(P_2-2)/2}(\mathbb{R}^n)$ with $\|\varphi\|_{n(P_1-2)/2}$ and $\|\varphi\|_{n(P_2-2)/2}$ sufficiently small. See also Proposition 4.1 below for other examples.

Proof of Theorem 1. Throughout the proof, we use the notation established in Section 2. The proof is based on a contraction mapping argument on a suitable metric space. Let X be the set of Bochner measurable functions $u\colon (0,\,\infty) \longrightarrow W^{1,r_1}(\mathbb{R}^n) \cap W^{1,r_2}(\mathbb{R}^n)$ such that

$$\|u\|_X = \sup_{t>0}\big[t^{\beta_1}\|u(t)\|_{r_1},\, t^{\beta_1+1/2}\|\nabla u(t)\|_{r_1},\\ t^{\beta_2}\|u(t)\|_{r_2},\, t^{\beta_2+1/2}\|\nabla u(t)\|_{r_2}\big] < \infty, \quad (3.7)$$

where β_1, β_2 are given by (1.3) and r_1, r_2 are given by Lemma 2.3. Let $M > 0$ and define

$$X_M := \big\{u \in X \;\big|\; \|u\|_X \le M\big\}. \quad (3.8)$$

X_M endowed with the metric $d(u,\,v) = \|u-v\|_X$ is a complete metric space.

Consider the mapping $\mathcal{F}_\varphi$ defined by

$$\mathcal{F}_\varphi(u)(t) = \mathrm{e}^{t\Delta}\varphi + a\int_0^t \mathrm{e}^{(t-\sigma)\Delta}(|\nabla u(\sigma)|^q)d\sigma\\ + b\int_0^t \mathrm{e}^{(t-\sigma)\Delta}(|u(\sigma)|^{p-1}u(\sigma))d\sigma, \quad (3.9)$$

where φ is a tempered distribution satisfying (3.4). We will prove that $\mathcal{F}_\varphi$ is a strict contraction mapping on X_M. Let φ and ψ satisfy (3.4) and $u,\,v \in X_M$. For $i = 1$ or $i = 2$, we have

$$t^{\beta_i}\|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{r_i} \le t^{\beta_i}\|\mathrm{e}^{t\Delta}(\varphi-\psi)\|_{r_i}\\ + |a|t^{\beta_i}\int_0^t \Big\|\mathrm{e}^{(t-\sigma)\Delta}\big[|\nabla u(\sigma)|^q - |\nabla v(\sigma)|^q\big]\Big\|_{r_i} d\sigma\\ + |b|t^{\beta_i}\int_0^t \Big\|\mathrm{e}^{(t-\sigma)\Delta}\big[|u(\sigma)|^{p-1}u(\sigma) - |v(\sigma)|^{p-1}v(\sigma)\big]\Big\|_{r_i} d\sigma.$$

Now, we respectively use the smoothing property of the heat semigroup (2.1) with respectively $s_2 = r_i$, $s_1 = r_{i1}/q$ and $s_2 = r_i$, $s_1 = m_i/p$ on the second and the third term of the right-hand side of the last inequality, where the real numbers r_i, r_{i1} and m_i are as in Lemma 2.5, to obtain

$$t^{\beta_i}\|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{r_i} \le t^{\beta_i}\|\mathrm{e}^{t\Delta}(\varphi-\psi)\|_{r_i}\\ + |a|t^{\beta_i}\int_0^t \big(4\pi(t-\sigma)\big)^{-(n/2)(q/r_{i1}-1/r_i)}\\ \times \big\||\nabla u(\sigma)|^q - |\nabla v(\sigma)|^q\big\|_{r_{i1}/q} d\sigma$$

$$+|b|t^{\beta_i}\int_0^t \big(4\pi(t-\sigma)\big)^{-(n/2)(p/m_i-1/r_i)} \\ \times \big\||u(\sigma)|^{p-1}u(\sigma)-|v(\sigma)|^{p-1}v(\sigma)\big\|_{m_i/p}d\sigma. \quad (3.10)$$

By using the Hölder inequality, we have

$$\big\||\nabla u|^q-|\nabla v|^q\big\|_{r_{i1}/q} \\ \le q\big(\|\nabla u\|_{r_{i1}}^{q-1}+\|\nabla v\|_{r_{i1}}^{q-1}\big)\|\nabla u-\nabla v\|_{r_{i1}}, \quad (3.11)$$

and

$$\big\||u|^{p-1}u-|v|^{p-1}v\big\|_{m_i/p}\le p\big(\|u\|_{m_i}^{p-1}+\|v\|_{m_i}^{p-1}\big)\|u-v\|_{m_i}. \quad (3.12)$$

Also, by using the Gagliardo-Nirenberg inequality (2.3), with $m=m_i$, $r=r_{i2}$ in the right–hand side of the inequality (3.12) and by Lemma 2.5 parts (ii)–(iii) and (2.3), we obtain

$$\begin{aligned}&\big\||u|^{p-1}u-|v|^{p-1}v\big\|_{m_i/p}\\ &\le pG^p\big[\|\nabla u\|_{r_{i2}}^{n(1/r_{i2}-1/m_i)(p-1)}\|u\|_{r_{i2}}^{[1-n(1/r_{i2}-1/m_i)](p-1)}\\ &\qquad+\|\nabla v\|_{r_{i2}}^{n(1/r_{i2}-1/m_i)(p-1)}\|v\|_{r_{i2}}^{[1-n(1/r_{i2}-1/m_i)](p-1)}\big]\\ &\quad\times\|\nabla u-\nabla v\|_{r_{i2}}^{n(1/r_{i2}-1/m_i)}\|u-v\|_{r_{i2}}^{[1-n(1/r_{i2}-1/m_i)]}.\end{aligned} \quad (3.13)$$

Now by using (3.11) and (3.13), we deduce from (3.10) that

$$\begin{aligned}&t^{\beta_i}\|\mathcal{F}_\varphi(u)(t)-\mathcal{F}_\psi(v)(t)\|_{r_i}\le t^{\beta_i}\|\mathrm{e}^{t\Delta}(\varphi-\psi)\|_{r_i}\\ &+|a|qt^{\beta_i}\bigg[\int_0^t\big(4\pi(t-\sigma)\big)^{-(n/2)(q/r_{i1}-1/r_i)}\\ &\qquad\times\big(\|\nabla u(\sigma)\|_{r_{i1}}^{q-1}+\|\nabla v(\sigma)\|_{r_{i1}}^{q-1}\big)\|\nabla u(\sigma)-\nabla v(\sigma)\|_{r_{i1}}d\sigma\bigg]\\ &+|b|pG^pt^{\beta_i}\bigg[\int_0^t\big(4\pi(t-\sigma)\big)^{-(n/2)(p/m_i-1/r_i)}\\ &\qquad\times\big[\|\nabla u(\sigma)\|_{r_{i2}}^{n(1/r_{i2}-1/m_i)(p-1)}\|u(\sigma)\|_{r_{i2}}^{[1-n(1/r_{i2}-1/m_i)](p-1)}\\ &\qquad+\|\nabla v(\sigma)\|_{r_{i2}}^{n(1/r_{i2}-1/m_i)(p-1)}\|v(\sigma)\|_{r_{i2}}^{[1-n(1/r_{i2}-1/m_i)](p-1)}\big]\\ &\qquad\times\|\nabla u(\sigma)-\nabla v(\sigma)\|_{r_{i2}}^{n(1/r_{i2}-1/m_i)}\\ &\qquad\times\|u(\sigma)-v(\sigma)\|_{r_{i2}}^{[1-n(1/r_{i2}-1/m_i)]}d\sigma\bigg].\end{aligned} \quad (3.14)$$

Using the interpolation inequality (2.4) with $s=r_{i1}$ in the second term of the right hand-side of the last inequality and $s = r_{i2}$ in the third term, along with the fact that u, v belongs to X_M, we obtain

$$\begin{aligned}
&t^{\beta_i}\|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{r_i} \le t^{\beta_i}\|\mathrm{e}^{t\Delta}(\varphi-\psi)\|_{r_i}\\
&+ 2c_{i1}M^{q-1}t^{\beta_i-(n/2)(q/r_{i1}-1/r_i)-q(\beta_{i1}+1/2)+1}\\
&\times\left(\int_0^1(1-\sigma)^{-(n/2)(q/r_{i1}-1/r_i)}\sigma^{-q(\beta_{i1}+1/2)}d\sigma\right)\|u-v\|_X\\
&+ 2c_{i2}M^{p-1}t^{\beta_i-(n/2)(p/r_{i2}-1/r_i)-p\beta_{i2}+1}\\
&\times\left(\int_0^1(1-\sigma)^{-(n/2)(p/m_i-1/r_i)}\sigma^{-p\beta_{i2}-(np/2)(1/r_{i2}-1/m_i)}d\sigma\right)\|u-v\|_X.
\end{aligned}$$

where

$$c_{i1} = |a|q(4\pi)^{-(n/2)(q/r_{i1}-1/r_i)}, \tag{3.15}$$

and

$$c_{i2} = |b|pG^p(4\pi)^{-(n/2)(p/m_i-1/r_i)}, \tag{3.16}$$

where G is the constant appearing in (2.3). Then, due to Lemma 2.5 parts (v)–(viii), we have

$$\begin{aligned}
&t^{\beta_i}\|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{r_i}\\
&\le t^{\beta_i}\|\mathrm{e}^{t\Delta}(\varphi-\psi)\|_{r_i} + (C_{i1}M^{q-1} + C_{i2}M^{p-1})\|u-v\|_X,
\end{aligned} \tag{3.17}$$

where

$$C_{i1} = 2c_{i1}\int_0^1(1-\sigma)^{-(n/2)(q/r_{i1}-1/r_i)}\sigma^{-q(\beta_{i1}+1/2)}d\sigma \tag{3.18}$$

and

$$\begin{aligned}
C_{i2} = 2c_{i2}\int_0^1 &(1-\sigma)^{-(n/2)(p/m_i-1/r_i)}\\
&\times\sigma^{-p\beta_{i2}-(np/2)(1/r_{i2}-1/m_i)}d\sigma,
\end{aligned} \tag{3.19}$$

where the constants c_{i1} and c_{i2} are given respectively by (3.15) and (3.16). Due to Lemma 2.5, the positive constants C_{i1} and C_{i2} are finite for $i = 1, 2$.

Also, by the inequality (2.2) we obtain analogously for $i = 1, 2$

$$\begin{aligned}
t^{\beta_i+1/2}\|\nabla\mathcal{F}_\varphi(u)(t) - \nabla\mathcal{F}_\psi(v)(t)\|_{r_i} &\le t^{\beta_i+1/2}\|\nabla\mathrm{e}^{t\Delta}(\varphi-\psi)\|_{r_i}\\
&+2|a|qHM^{q-1}t^{\beta_i-(n/2)(q/r_{i1}-1/r_i)-q(\beta_{i1}+1/2)+1}
\end{aligned}$$

$$\times\Big(\int_0^1 (1-\sigma)^{-(n/2)(q/r_{i1}-1/r_i)-1/2}\sigma^{-q(\beta_{i1}+1/2)}d\sigma\Big)\|u-v\|_X$$
$$+2|b|pHG^pM^{p-1}t^{\beta_i-(n/2)(p/r_{i2}-1/r_i)-p\beta_{i2}+1}$$
$$\times\Big(\int_0^1 (1-\sigma)^{-(n/2)(p/m_i-1/r_i)-1/2}\sigma^{-p\beta_{i2}-(np/2)(1/r_{i2}-1/m_i)}d\sigma\Big)$$
$$\times\|u-v\|_X,$$

where G is the constant appearing in (2.3) and H is the constant appearing in (2.2). Then, due to Lemma 2.5 parts (v)–(viii), we have

$$t^{\beta_i+1/2}\|\nabla\mathcal{F}_\varphi(u)(t)-\nabla\mathcal{F}_\psi(v)(t)\|_{r_i}\le t^{\beta_i+1/2}\|\nabla \mathrm{e}^{t\Delta}(\varphi-\psi)\|_{r_i}$$
$$+(D_{i1}M^{q-1}+D_{i2}M^{p-1})\|u-v\|_X,\quad (3.20)$$

where

$$D_{i1}=2|a|qH\int_0^1 (1-\sigma)^{-(n/2)(q/r_{i1}-1/r_i)-1/2}\sigma^{-q(\beta_{i1}+1/2)}d\sigma \quad (3.21)$$

and

$$D_{i2}=2|b|pHG^p\int_0^1 (1-\sigma)^{-(n/2)(p/m_i-1/r_i)-1/2}$$
$$\times\,\sigma^{-p\beta_{i2}-(np/2)(1/r_{i2}-1/m_i)}d\sigma.\quad (3.22)$$

Due to Lemma 2.5, the positive constants D_{i1} and D_{i2} are finite for $i=1,\,2$.

Let

$$\mathcal{K}_1=\max_{i\in\{1,\,2\}}(C_{i1},\,D_{i1}) \quad (3.23)$$

and

$$\mathcal{K}_2=\max_{i\in\{1,\,2\}}(C_{i2},\,D_{i2}), \quad (3.24)$$

where the constants C_{i1}, C_{i2}, D_{i1} and D_{i2} are given respectively by (3.18), (3.19), (3.21) and (3.22). Now combining (3.17) with (3.20), we get by (3.4),

$$d\big(\mathcal{F}_\varphi(u),\,\mathcal{F}_\psi(v)\big)$$
$$\le\mathcal{N}(\varphi-\psi)+(\mathcal{K}_1M^{q-1}+\mathcal{K}_2M^{p-1})d(u,\,v).\quad (3.25)$$

Setting $\psi=0$ and $v\equiv 0$ in (3.25), we obtain

$$\|\mathcal{F}_\varphi(u)\|_X\le\mathcal{N}(\varphi)+(\mathcal{K}_1M^{q-1}+\mathcal{K}_2M^{p-1})\|u\|_X, \quad (3.26)$$

and so by (3.3) and (3.4) $\mathcal{F}_\varphi$ maps X_M into itself.

Letting $\varphi = \psi$ in (3.25), we get

$$d\big(\mathcal{F}_\varphi(u),\, \mathcal{F}_\varphi(v)\big) \le (\mathcal{K}_1 M^{q-1} + \mathcal{K}_2 M^{p-1}) d(u,\, v). \tag{3.27}$$

Hence inequality (3.2) gives that $\mathcal{F}_\varphi$ is a strict contraction mapping from X_M into itself, so $\mathcal{F}_\varphi$ has a unique fixed point u in X_M which is solution of (3.1).

We now prove that $u(t) - \mathrm{e}^{t\Delta}\varphi \in C\big([0,\, \infty); L^s(\mathbb{R}^n)\big)$ for

$$\max\Big(\frac{r_{i1}}{q},\, \frac{m_i}{p}\Big) \le s < \frac{n(P_i - 1)}{2},\ i = 1,\, 2. \tag{3.28}$$

First, the existence of a such s is insured by Lemma 2.3 parts (iv) and (ix), Lemma 2.4 and the expressions of r_{11} and r_{21} respectively given by (2.9) and (2.11). Now, since continuity for $t > 0$ can be handled by well known arguments, we only give the proof of (ii) at $t = 0$.

Let s be a positive real number satisfying (3.28), then

$$\begin{aligned}\|u(t) - \mathrm{e}^{t\Delta}\varphi\|_s \le |a| &\int_0^t \|\mathrm{e}^{(t-\sigma)\Delta}(|\nabla u(\sigma)|^q)\|_s d\sigma \\ &+ |b| \int_0^t \|\mathrm{e}^{(t-\sigma)\Delta}(|u(\sigma)|^{p-1}u(\sigma))\|_s d\sigma.\end{aligned}$$

Let $i = 1$ or 2. The using (2.1) in the right-hand side of the last inequality with $s_1 = r_{i1}/q;\ s_2 = s$ for the first term and $s_1 = m_i/p;\ s_2 = s$ for the second term, we obtain

$$\begin{aligned}\|u(t) - \mathrm{e}^{t\Delta}\varphi\|_s \le |a|&(4\pi)^{-(n/2)(q/r_{i1}-1/s)} \\ &\times \int_0^t (t-\sigma)^{-(n/2)(q/r_{i1}-1/s)} \|\nabla u(\sigma)\|_{r_{i1}}^q d\sigma \\ &+ |b|(4\pi)^{-(n/2)(p/m_i-1/s)} \\ &\times \int_0^t (t-\sigma)^{-(n/2)(p/m_i-1/s)} \|u(\sigma)\|_{m_i}^p d\sigma.\end{aligned}$$

We now use the Gagliardo-Nirenberg inequality (2.3) with $m = m_i$ and $r = r_{i2}$ in the third term of the last inequality. Then, by the interpolation inequality (2.4) and the inequality (3.5), we get

$$\|u(t) - \mathrm{e}^{t\Delta}\varphi\|_s \le |a|(4\pi)^{-(n/2)(q/r_{i1}-1/s)}$$

$$\times \int_0^t (t-\sigma)^{-(n/2)(q/r_{i1}-1/s)} \sigma^{-(\beta_{i1}+1/2)q} M^q d\sigma$$
$$+|b|(4\pi)^{-(n/2)(p/m_i-1/s)} G^p$$
$$\times \int_0^t (t-\sigma)^{-(n/2)(p/m_i-1/s)} \sigma^{-\beta_{i2}p-(pn/2)(1/r_{i2}-1/m_i)} M^p d\sigma,$$

which leads to

$$\|u(t)-\mathrm{e}^{t\Delta}\varphi\|_s \le \mathcal{C}_{i1} t^{-(n/2)(q/r_{i1}-1/s)-q(\beta_{i1}+1/2)+1} + \mathcal{C}_{i2} t^{-(n/2)(p/m_i-1/s)-p[\beta_{i2}+(n/2)(1/r_{i2}-1/m_i)]+1}, \quad (3.29)$$

where, for $i = 1, 2$;

$$\mathcal{C}_{i1} = |a|(4\pi)^{-(n/2)(q/r_{i1}-1/s)} M^q \times \int_0^1 (1-\sigma)^{-(n/2)(q/r_{i1}-1/s)} \sigma^{-(\beta_{i1}+1/2)q} d\sigma, \quad (3.30)$$

and

$$\mathcal{C}_{i2} = |b|(4\pi)^{-(n/2)(p/m_i-1/s)} G^p M^p \times \int_0^1 (1-\sigma)^{-(n/2)(p/m_i-1/s)} \sigma^{-\beta_{i2}p-(np/2)(1/r_{i2}-1/m_i)} d\sigma, \quad (3.31)$$

where G is the constant appearing in (2.3). One can easily see, owing to Lemma 2.5, that $\mathcal{C}_{i1}$ and $\mathcal{C}_{i2}$ for $i = 1, 2$ are finite constants.

Now, due to the expression of r_{ij} and β_{ij} given by (2.9)–(2.12), the inequality (3.29) becomes

$$\|u(t)-\mathrm{e}^{t\Delta}\varphi\|_s \le (\mathcal{C}_{i1}+\mathcal{C}_{i2}) t^{n/(2s)-1/(P_i-1)}. \quad (3.32)$$

Then, due to (3.28), the right-hand side of Inequality (3.32) converges to zero as $t \searrow 0$. This proves the statements (i) and (ii) of Theorem 1. Statement (iii) for the particular case $s = n(P_i - 1)/2$, $i = 1, 2$ follows from (3.32) which still holds if $s = n(P_i - 1)/2$. Statement (iii) follows then by interpolation.

Finally, the inequality (3.6) of Theorem 1 follows by considering $\mathcal{F}_\varphi(u) = u$ and $\mathcal{F}_\psi(v) = v$ in the estimate (3.25). This inequality expresses the continuous dependence of the solution on the initial data. This finishes the proof of Theorem 1. □

4. Asymptotic behavior

In this section we prove that some of the solutions of the equation (1.1) are asymptotic, as $t \to \infty$, to self-similar solutions of the integral equation associated to the equation (1.4), which is

$$w(t) = \mathrm{e}^{t\Delta}\varphi + a\nu \int_0^t \mathrm{e}^{(t-\sigma)\Delta}(|\nabla w(\sigma)|^q)d\sigma \\ + b\mu \int_0^t \mathrm{e}^{(t-\sigma)\Delta}(|w(\sigma)|^{p-1}w(\sigma))d\sigma, \quad (4.1)$$

where a, b, p and q are the same parameters appearing in Equation (1.1) and μ and ν are given by (1.5). We have obtained the following result.

Theorem 2 (Asymptotic behavior) *Let the positive integer n and the real numbers p and q be such that*

$$1+\frac{2}{n} < p, \; \frac{n+2}{n+1} < q \leq p\frac{n+2}{n+p} \quad \textit{and} \quad q < 2.$$

Let P_1, P_2 be two real numbers satisfying (2.5). *Assume that*

$$\frac{n(P_1-1)}{2} > 1.$$

Let r_1, r_2 be the real numbers given by Lemma 2.3 *and consider the real numbers m_1 and m_2 given by* Lemma 2.4. *Consider also the real numbers β_1, β_2 and r_{ij}, β_{ij}, i, $j \in \{1, 2\}$, be given by* (1.3), (2.9)–(2.12) *respectively.*

Let ψ be a tempered distribution satisfying (3.4), *where we also assume* (3.2) *and* (3.3), *let u be the solution of* (3.1) *with initial data ψ, constructed by* Theorem 1 *and let w be the solution of* (4.1) (*also constructed by* Theorem 1) *with initial data ψ. Then, for all δ, $0 < \delta < \delta_0$, and with M perhaps smaller, there exists $C_\delta > 0$ such that*

$$\|u(t) - w(t)\|_{r_1} \leq C_\delta t^{-\beta_1-\delta}, \quad \forall t > 0, \quad (4.2)$$

$$\|\nabla u(t) - \nabla w(t)\|_{r_1} \leq C_\delta t^{-\beta_1-1/2-\delta}, \quad \forall t > 0, \quad (4.3)$$

where δ_0 is given by Lemma 2.6.

Proof of Theorem 2. Throughout the proof, we use the notation established in Section 2 and Lemma 2.6. From the equations (3.1) and (4.1), we have

$$u(t)-w(t)=a\int_0^t \mathrm{e}^{(t-\sigma)\Delta}\big[|\nabla u(\sigma)|^q-\nu|\nabla w(\sigma)|^q\big]\,d\sigma$$
$$+\,b\int_0^t \mathrm{e}^{(t-\sigma)\Delta}\big[|u(\sigma)|^{p-1}u(\sigma)-\mu|w(\sigma)|^{p-1}w(\sigma)\big]\,d\sigma. \quad (4.4)$$

Then

$$\|u(t)-w(t)\|_{r_1}\le |a|\int_0^t \big\|\mathrm{e}^{(t-\sigma)\Delta}\big[|\nabla u(\sigma)|^q-\nu|\nabla w(\sigma)|^q\big]\big\|_{r_1}d\sigma$$
$$+\,|b|\int_0^t \big\|\mathrm{e}^{(t-\sigma)\Delta}\big[|u(\sigma)|^{p-1}u(\sigma)-\mu|w(\sigma)|^{p-1}w(\sigma)\big]\big\|_{r_1}d\sigma. \quad (4.5)$$

Now, we use the smoothing property of the heat semigroup (2.1) with $s_2 = r_1$, $s_1 = r'_{11}/q$ and $s_2 = r_1$, $s_1 = m'_1/p$ respectively in the first and the second term of the right-hand side of the last inequality, where the real numbers r_1, r'_{11} and m'_1 are as in Lemma 2.5 and Lemma 2.6, to obtain

$$\begin{aligned}
&t^{\beta_1+\delta}\|u(t)-w(t)\|_{r_1}\\
&\le |a|t^{\beta_1+\delta}\int_0^t\big(4\pi(t-\sigma)\big)^{-(n/2)(q/r'_{11}-1/r_1)}\\
&\qquad\qquad\times\big\||\nabla u(\sigma)|^q-\nu|\nabla w(\sigma)|^q\big\|_{r'_{11}/q}d\sigma\\
&\quad+|b|t^{\beta_1+\delta}\int_0^t\big(4\pi(t-\sigma)\big)^{-(n/2)(p/m'_1-1/r_1)}\\
&\qquad\qquad\times\big\||u(\sigma)|^{p-1}u(\sigma)-\mu|w(\sigma)|^{p-1}w(\sigma)\big\|_{m'_1/p}d\sigma, \qquad (4.6)
\end{aligned}$$

where $\delta>0$ is as in Lemma 2.6. Using the following inequality,

$$\big\||f|^{\gamma-1}f-\alpha|g|^{\gamma-1}g\big\|_{s/\gamma}\le\gamma\big(\|f\|_s^{\gamma-1}+\alpha\|g\|_s^{\gamma-1}\big)\|f-\alpha g\|_s, \quad (4.7)$$

where $1<\gamma<s$, $\alpha=1$ or $\alpha=0$, we deduce from the inequality (4.6) that

$$\begin{aligned}
&t^{\beta_1+\delta}\|u(t)-w(t)\|_{r_1}\\
&\le |a|qt^{\beta_1+\delta}\int_0^t\big(4\pi(t-\sigma)\big)^{-(n/2)(q/r'_{11}-1/r_1)}\\
&\qquad\times\ \big(\|\nabla u(\sigma)\|_{r'_{11}}^{q-1}+\nu\|\nabla w(\sigma)\|_{r'_{11}}^{q-1}\big)\|\nabla u(\sigma)-\nu\nabla w(\sigma)\|_{r'_{11}}d\sigma\\
&\quad+\ |b|pt^{\beta_1+\delta}\int_0^t\big(4\pi(t-\sigma)\big)^{-(n/2)(p/m'_1-1/r_1)}\\
&\qquad\times\ \big(\|u(\sigma)\|_{m'_1}^{p-1}+\mu\|w(\sigma)\|_{m'_1}^{p-1}\big)\|u(\sigma)-\mu w(\sigma)\|_{m'_1}d\sigma. \qquad (4.8)
\end{aligned}$$

Now, by using the Gagliardo-Nirenberg inequality (2.3), with $m = m_1'$ and $r = r_{12}'$ in the second term of the right-hand side of the inequality (4.8), the interpolation inequality (2.4) and inequality (3.5) we obtain

$$\begin{aligned}
&t^{\beta_1+\delta}\|u(t)-w(t)\|_{r_1} \\
&\le c_1 t^{\beta_1+\delta}\int_0^t (t-\sigma)^{-(n/2)(q/r_{11}'-1/r_1)} \\
&\qquad\qquad \times \sigma^{-(\beta_{11}'+1/2)(q-1)}\|\nabla u(\sigma)-\nu\nabla w(\sigma)\|_{r_{11}'}\,d\sigma \\
&\quad + c_2 t^{\beta_1+\delta}\int_0^t (t-\sigma)^{-(n/2)(p/m_1'-1/r_1)} \\
&\qquad\qquad \times \sigma^{-\beta_{12}'(p-1)-\{n(p-1)/2\}(1/r_{12}'-1/m_1')} \\
&\qquad\qquad \times \|u(\sigma)-\mu w(\sigma)\|_{r_{12}'}^{1-N}\|\nabla u(\sigma)-\mu\nabla w(\sigma)\|_{r_{12}'}^{N}\,d\sigma, \qquad (4.9)
\end{aligned}$$

where, $N = n(1/r_{12}' - 1/m_1')$ and

$$c_1 = |a|q(1+\nu)(4\pi)^{-(n/2)(q/r_{11}'-1/r_1)}M^{q-1} \qquad (4.10)$$

$$c_2 = |b|pG^{p-1}(1+\mu)(4\pi)^{-(n/2)(p/m_1'-1/r_1)}M^{p-1} \qquad (4.11)$$

where G is the constant appearing in (2.3).

Let $T > 0$ be an arbitrary real number. Then we have

$$\begin{aligned}
&t^{\beta_1+\delta}\|u(t)-w(t)\|_{r_1} \\
&\le c_1 t^{\beta_1+(1-\nu)\delta-(n/2)(q/r_{11}'-1/r_1)-q(\beta_{11}'+1/2)+1} \\
&\qquad \times \int_0^1 (1-\sigma)^{-(n/2)(q/r_{11}'-1/r_1)}\sigma^{-q(\beta_{11}'+1/2)-\nu\delta}\,d\sigma \\
&\qquad \times \sup_{t\in(0,T]}\big(t^{\beta_{11}'+\nu\delta}\|u(\sigma)-\nu w(\sigma)\|_{r_{11}'}, \\
&\qquad\qquad\qquad t^{\beta_{11}'+1/2+\nu\delta}\|\nabla u(\sigma)-\nu\nabla w(\sigma)\|_{r_{11}'}\big) \\
&\quad + c_2 t^{\beta_1+(1-\mu)\delta-p\beta_{12}'-(n/2)(p/r_{12}'-1/r_1)+1} \\
&\qquad \times \int_0^1 (1-\sigma)^{-(n/2)(p/m_1'-1/r_1)}\sigma^{-\beta_{12}'p-(np/2)(1/r_{12}'-1/m_1')-\mu\delta}\,d\sigma \\
&\qquad \times \sup_{t\in(0,T]}\big(t^{\beta_{12}'+\mu\delta}\|u(\sigma)-\mu w(\sigma)\|_{r_{12}'}, \\
&\qquad\qquad\qquad t^{\beta_{12}'+1/2+\mu\delta}\|\nabla u(\sigma)-\mu\nabla w(\sigma)\|_{r_{12}'}\big). \qquad (4.12)
\end{aligned}$$

Then by Lemma 2.5 and Lemma 2.6 we have

$$
\begin{aligned}
&t^{\beta_1+\delta}\|u(t)-w(t)\|_{r_1}\\
&\le C_1 \sup_{t\in(0,T]}\big(t^{\beta'_{11}+\nu\delta}\|u(\sigma)-\nu w(\sigma)\|_{r'_{11}},\\
&\qquad\qquad t^{\beta'_{11}+1/2+\nu\delta}\|\nabla u(\sigma)-\nu\nabla w(\sigma)\|_{r'_{11}}\big)\\
&\quad + C_2 \sup_{t\in(0,T]}\big(t^{\beta'_{12}+\mu\delta}\|u(\sigma)-\mu w(\sigma)\|_{r'_{12}},\\
&\qquad\qquad t^{\beta'_{12}+1/2+\mu\delta}\|\nabla u(\sigma)-\mu\nabla w(\sigma)\|_{r'_{12}}\big),
\end{aligned}
\tag{4.13}
$$

where

$$
C_1 = c_1\int_0^1 (1-\sigma)^{-(n/2)(q/r'_{11}-1/r_1)}\sigma^{-q(\beta'_{11}+1/2)-\nu\delta}d\sigma, \tag{4.14}
$$

$$
\begin{aligned}
C_2 = c_2\int_0^1 &(1-\sigma)^{-(n/2)(p/m'_1-1/r_1)}\\
&\times \sigma^{-\beta'_{12}p-(np/2)(1/r'_{12}-1/m'_1)-\mu\delta}d\sigma,
\end{aligned}
\tag{4.15}
$$

where c_1 and c_2 are given respectively by (4.10) and (4.11). By Lemma 2.6, C_1, C_2 are finite positive constants. By similar calculations as above, but by using (2.2), we obtain

$$
\begin{aligned}
&t^{\beta_1+1/2+\delta}\|\nabla u(t)-\nabla w(t)\|_{r_1}\\
&\le D_1 \sup_{t\in(0,T]}\big(t^{\beta'_{11}+\nu\delta}\|u(\sigma)-\nu w(\sigma)\|_{r'_{11}},\\
&\qquad\qquad t^{\beta'_{11}+1/2+\nu\delta}\|\nabla u(\sigma)-\nu\nabla w(\sigma)\|_{r'_{11}}\big)\\
&\quad + D_2 \sup_{t\in(0,T]}\big(t^{\beta'_{12}+\mu\delta}\|u(\sigma)-\mu w(\sigma)\|_{r'_{12}},\\
&\qquad\qquad t^{\beta'_{12}+1/2+\mu\delta}\|\nabla u(\sigma)-\mu\nabla w(\sigma)\|_{r'_{12}}\big),
\end{aligned}
\tag{4.16}
$$

where

$$
\begin{aligned}
D_1 = |a|qH(1+\nu)M^{q-1}\int_0^1 &(1-\sigma)^{-(n/2)(q/r'_{11}-1/r_1)-1/2}\\
&\times \sigma^{-q(\beta'_{11}+1/2)-\nu\delta}d\sigma,
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
D_2 = |b|pHG^{p-1}(1+\mu)M^{p-1}\int_0^1 &(1-\sigma)^{-(n/2)(p/m'_1-1/r_1)-1/2}\\
&\times \sigma^{-\beta'_{12}p-(np/2)(1/r'_{12}-1/m'_1)-\mu\delta}d\sigma.
\end{aligned}
\tag{4.18}
$$

By Lemma 2.6, D_1 and D_2 are finite positive constants. Now, by using

(4.13) and (4.16) we have

$$
\begin{aligned}
&\sup_{t\in(0,T]}\big(t^{\beta_1+\delta}\|u(t)-w(t)\|_{r_1},\, t^{\beta_1+1/2+\delta}\|\nabla u(t)-\nabla w(t)\|_{r_1}\big)\\
&\le \max(C_1,\, D_1)\sup_{t\in(0,T]}\big(t^{\beta'_{11}+\nu\delta}\|u(\sigma)-\nu w(\sigma)\|_{r'_{11}},\\
&\qquad\qquad t^{\beta'_{11}+1/2+\nu\delta}\|\nabla u(\sigma)-\nu\nabla w(\sigma)\|_{r'_{11}}\big)\\
&\quad+\max(C_2,\, D_2)\sup_{t\in(0,T]}\big(t^{\beta'_{12}+\mu\delta}\|u(\sigma)-\mu w(\sigma)\|_{r'_{12}},\\
&\qquad\qquad t^{\beta'_{12}+1/2+\mu\delta}\|\nabla u(\sigma)-\mu\nabla w(\sigma)\|_{r'_{12}}\big).
\end{aligned}
\tag{4.19}
$$

We have to distinguish the cases ν, $\mu = 0$ or 1. As one can remark if ν or $\mu = 0$, then the corresponding term in the right-hand side of the inequality (4.19) is bounded by $M > 0$. Otherwise, if $\nu = 1$ then $r'_{11} = r_{11} = r_1$ and $\beta'_{11} = \beta_{11} = \beta_1$ and the corresponding term in the right-hand side of the last inequality is the term in left-hand side, up to a constant. If $\mu = 1$ then $r'_{12} = r_{12} = r_1$ and $\beta'_{12} = \beta_{12} = \beta_1$ and the corresponding term in the right-hand side of the last inequality is the term in left-hand side, up to a constant. See (1.5), (2.9)–(2.10) and (2.13)–(2.14). Then, for M perhaps smaller, we obtain

$$
\sup_{t\in(0,T]}\big(t^{\beta_1+\delta}\|u(t)-w(t)\|_{r_1},\, t^{\beta_1+1/2+\delta}\|\nabla u(t)-\nabla w(t)\|_{r_1}\big)\le C(\delta),
$$

where $C(\delta)$ is a positive constant not depending on T. Then the previous inequality is valid for any $T > 0$ and we have

$$
\|u(t)-w(t)\|_{r_1}\le C(\delta)t^{-\beta_1-\delta},\quad \forall t>0, \tag{4.20}
$$

$$
\|\nabla u(t)-\nabla w(t)\|_{r_1}\le C(\delta)t^{-\beta_1-1/2-\delta},\quad \forall t>0, \tag{4.21}
$$

for $0 < \delta < \delta_0$. This finishes the proof of the theorem. □

We now establish the following result.

Proposition 4.1 *Let the positive integer n and the real numbers p and q be such that*

$$
1+\frac{2}{n}<p,\ \frac{n+2}{n+1}<q\le p\frac{n+2}{n+p}\quad \textit{and}\quad q<2.
$$

Let P_1, P_2 be two real numbers satisfying (2.5). *Assume that*

$$\frac{n(P_1-1)}{2} > 1.$$

Let r_1, r_2 *be the real numbers given by* Lemma 2.3 *and the real numbers* β_1, β_2 *be given by* (1.3).

Let φ *be a tempered distribution which is also homogeneous of degree* $-2/(P_1-1)$ *and such that*

$$\varphi(x) = \omega(x)|x|^{-2/(P_1-1)}, \tag{4.22}$$

where $\omega \in L^{r_2}(S^{n-1})$ *is homogeneous of degree* 0*. Then*

$$\sup_{t>0}\bigl(t^{\beta_1}\|\mathrm{e}^{t\Delta}\varphi\|_{r_1},\, t^{\beta_1+1/2}\|\nabla \mathrm{e}^{t\Delta}\varphi\|_{r_1}\bigr) < \infty. \tag{4.23}$$

Also, for any L^∞ *cut-off function* η (*identically equal to* 1 *near the origin and with compact support*), *we have*

(i) $\sup_{t>0}\bigl(t^{\beta_1+\delta}\|\mathrm{e}^{t\Delta}(\eta\varphi)\|_{r_1},\, t^{\beta_1+1/2+\delta}\|\nabla \mathrm{e}^{t\Delta}(\eta\varphi)\|_{r_1}\bigr) < \infty,$ *for* $0<\delta<n/2-1/(P_1-1)$,

(ii) $\mathcal{N}\bigl[(1-\eta)\varphi\bigr] < \infty.$

The proof of the previous proposition is similar to that of [7, Lemma 4.2], [17, Theorem 2.7] and [18, Proposition 4.2] and so is omitted. We note that for a tempered distribution ϕ, by the smoothing properties of the heat semigroup, $\mathcal{N}(\phi)$ is equivalent to $\sup_{t>0}\bigl(t^{\beta_1}\|\mathrm{e}^{t\Delta}\phi\|_{r_1},\, t^{\beta_2}\|\mathrm{e}^{t\Delta}\phi\|_{r_2}\bigr)$.

We give now the self-similar asymptotic behavior.

Theorem 3 (Asymptotically self-similar solutions) *Let the positive integer* n *and the real numbers* p *and* q *be such that*

$$1+\frac{2}{n} < p,\ \frac{n+2}{n+1} < q \le p\frac{n+2}{n+p} \quad \text{and} \quad q<2.$$

Let P_1, P_2 *be two real numbers satisfying* (2.5)*. Assume that*

$$\frac{n(P_1-1)}{2} > 1.$$

Let r_1, r_2 *be the real numbers given by* Lemma 2.3 *and consider the real numbers* m_1 *and* m_2 *given by* Lemma 2.4*. Consider also the real numbers* β_1, β_2 *and* r_{ij}, β_{ij}, $i,\, j \in \{1,\, 2\}$, *be given by* (1.3), (2.9)–(2.12) *respectively.*

Let φ *be a tempered distribution which is also homogeneous of degree* $-2/(P_1-1)$ *and such that*

$$\varphi(x) = \omega(x)|x|^{-2/(P_1-1)}, \tag{4.24}$$

where $\omega \in L^{r_2}(S^{n-1})$ *is homogeneous of degree* 0. *Let* $\psi = (1-\eta)\varphi$ *where* η *is any* L^∞ *cut-off function* (*identically equal to* 1 *near the origin and with compact support*). *If necessary, we multiply* φ *by some constant such that* $\sup_{t>0}\big(t^{\beta_1}\|e^{t\Delta}\varphi\|_{r_1},\, t^{\beta_1+1/2}\|\nabla e^{t\Delta}\varphi\|_{r_1}\big)$ *and* $\mathcal{N}(\psi)$ *are smaller.*

Let u *be the solution of* (3.1) *with initial data* ψ, *constructed by* Theorem 1. *Let* v *be the self-similar solution of* (4.1) *with initial data* φ *constructed by* Theorem 2.7 *in* [17]. *Then, for all* δ, $0 < \delta < \delta_0'$, *and with* M *perhaps smaller, there exists* $C_\delta > 0$ *such that*

$$\|u(t) - v(t)\|_{r_1} \le C_\delta t^{-\beta_1-\delta}, \quad \forall t > 0, \tag{4.25}$$

$$\|\nabla u(t) - \nabla v(t)\|_{r_1} \le C_\delta t^{-\beta_1-1/2-\delta}, \quad \forall t > 0, \tag{4.26}$$

and

$$\|t^{1/(P_1-1)}u(t,\,\cdot\sqrt{t}) - v(1,\,\cdot\,)\|_{r_1} \le C_\delta t^{-\delta}, \quad \forall t > 0, \tag{4.27}$$

$$\|t^{1/(P_1-1)+1/2}\nabla u(t,\,\cdot\sqrt{t}) - \nabla v(1,\,\cdot\,)\|_{r_1} \le C_\delta t^{-\delta}, \quad \forall t > 0, \tag{4.28}$$

where δ_0' *can explicitly be computed.*

In particular, there exist $d_1 > 0$, $d_2 > 0$ *two constants, such that*

$$d_1 t^{-\beta_1} \le \|u(t)\|_{r_1} \le d_2 t^{-\beta_1},$$
$$d_1 t^{-\beta_1-1/2} \le \|\nabla u(t)\|_{r_1} \le d_2 t^{-\beta_1-1/2},$$

for large t.

Remark 4.2 For the particular case where $\nu = \mu = 1$ in (4.1), that is $q = 2p/(p+1)$ and $P_1 = p$, the previous result is established in [17, Theorem 2.8]. If $\nu = \mu = 0$ in (4.1), that is $P_1 < \min(p,\, q/(2-q))$, then (4.1) is the linear heat equation and in this case the self-similar solution v in the previous theorem is given by

$$v(t,\,x) = e^{t\Delta}\big(\omega(x)|x|^{-2/(P_1-1)}\big),\ P_1 > 1 + \frac{2}{n}.$$

Proof of Theorem 3. We begin by remarking that the existence of u is insured by Theorem 1 and Proposition 4.1 part (ii). The existence of v is insured by [17, Theorem 2.7]. We remark that one can prove the existence of v using the first two terms of the norm (1.2), that is using the norm: $\sup_{t>0}\big(t^{\beta_1}\|e^{t\Delta}\varphi\|_{r_1},\, t^{\beta_1+1/2}\|\nabla e^{t\Delta}\varphi\|_{r_1}\big)$, as in the proof of Theorem 1 and using similar idea as in [17]. We note that if $\nu = 1, \mu = 0$ or if $\nu = 0, \mu = 1$, we do not need all the conditions on the Lebesgue number r_1

in [17, Lemma 2.1, p. 1294]. Precisely, we only need to require the conditions related to the corresponding nonlinear term. See [17, Lemma 2.2, Corollary 2.3].

Let now w be the solution of (4.1) with initial data ψ. If we write

$$\|u(t) - v(t)\|_{r_1} \le \|u(t) - w(t)\|_{r_1} + \|w(t) - v(t)\|_{r_1},$$

we obtain by (4.2) in Theorem 2 and by [17, Theorem 2.8]

$$\|u(t) - v(t)\|_{r_1} \le C_\delta t^{-\beta_1-\delta} + C'_\delta t^{-\beta_1-\delta} \tag{4.29}$$

where $0 < \delta < \delta_0$ and $0 < \delta < \delta_1$. $\delta_0 > 0$ is given by Lemma 2.6 and $\delta_1 = \delta_1(\nu, \mu) > 0$ is given by:

$$\delta_1(0, 0) = \frac{n}{2} - \frac{1}{P_1 - 1},\ \delta_1(0, 1) = \frac{np}{2m_1} - \frac{1}{P_1 - 1}$$
$$\delta_1(1, 0) = \frac{n}{2r_1}\Big[(P_1 - 1)\Big(1 - \frac{q}{2}\Big) + 1\Big] - \frac{1}{P_1 - 1}$$

and

$$\delta_1(1, 1) = \min\bigg(\frac{np}{2m_1} - \frac{1}{P_1 - 1}, \frac{n}{2r_1}\Big[(P_1 - 1)\Big(1 - \frac{q}{2}\Big) + 1\Big] - \frac{1}{P_1 - 1}\bigg).$$

See [17, Theorem 2.8]. Hence (4.29) gives (4.25) for $0 < \delta < \min(\delta_0, \delta_1) := \delta'_0$. (4.26) is obtained by similar arguments. (4.27) and (4.28) follow by dilation arguments. This finishes the proof of Theorem 3. $\square$

We now turn to prove the $W^{1,\infty}$ result for the viscous Hamilton-Jacobi equation.

Theorem 4 ($W^{1,\infty}$-Asymptotic) *Let the positive integer n and the real number q be such that*

$$\frac{n+2}{n+1} < q < 2. \tag{4.30}$$

Let P_1, P_2 be two real numbers satisfying

$$1 < P_1 \le \frac{q}{2-q} \le P_2. \tag{4.31}$$

Assume that

$$\frac{n(P_1 - 1)}{2} > 1.$$

Assume $b = 0$ *in* (1.1). *Let* r_1, r_2 *be the real numbers given by* Lemma 2.3 *and consider the real numbers* m_1 *and* m_2 *given by* Lemma 2.4. *Consider also the real numbers* β_1, β_2 *and* r_{ij}, β_{ij}, $i, j \in \{1, 2\}$, *be given by* (1.3), (2.9)–(2.12) *respectively* (*we may take* $p = q/(2-q)$). *Define* $\beta_1(r)$, $\beta_2(r)$ *by*

$$\beta_1(r) := \frac{1}{P_1 - 1} - \frac{n}{2r}, \ \beta_2(r) := \frac{1}{P_2 - 1} - \frac{n}{2r}, \quad \forall r > 1. \tag{4.32}$$

Assume (3.2)–(3.4). *Let* u *be the solution of* (3.1) *with* $b = 0$ *constructed by* Theorem 1. *Then we have*

$$\sup_{t>0} \big[t^{\beta_1(r)} \|u(t)\|_r, \, t^{\beta_1(r)+1/2} \|\nabla u(t)\|_r \big] < \infty, \quad \forall r \in [r_1, \infty], \tag{4.33}$$

$$\sup_{t>0} \big[t^{\beta_2(r)} \|u(t)\|_r, \, t^{\beta_2(r)+1/2} \|\nabla u(t)\|_r \big] < \infty, \quad \forall r \in [r_2, \infty]. \tag{4.34}$$

Let φ *be a tempered distribution which is also homogeneous of degree* $-2/(P_1 - 1)$ *and such that*

$$\varphi(x) = \omega(x) |x|^{-2/(P_1 - 1)}, \tag{4.35}$$

where $\omega \in L^{r_2}(S^{n-1})$ *is homogeneous of degree* 0. *Let* $\psi = (1-\eta)\varphi$ *where* η *is any* L^∞ *cut-off function* (*identically equal to* 1 *near the origin and with compact support*). *If necessary, we multiply* φ *by some constant such that* $\sup_{t>0} \big(t^{\beta_1} \|\mathrm{e}^{t\Delta}\varphi\|_{r_1}, \, t^{\beta_1 + 1/2} \|\nabla \mathrm{e}^{t\Delta}\varphi\|_{r_1} \big)$ *and* $\mathcal{N}(\psi)$ *are smaller.*

Let v *be the solution of* (3.1) *with* $b = 0$ *and with initial data* ψ, *constructed by* Theorem 1. *Let* w *be the self-similar solution of* (4.1) *with* $b = 0$ *and with initial data* φ *constructed by* Theorem 2.7 *in* [17]. *Then, for all* δ, $0 < \delta < \delta_0''$, *and with* M *perhaps smaller, there exists* $C_\delta > 0$ *such that*

$$\|v(t) - w(t)\|_r \le C_\delta t^{-\beta_1(r) - \delta}, \quad \forall t > 0, \tag{4.36}$$

$$\|\nabla v(t) - \nabla w(t)\|_r \le C_\delta t^{-\beta_1(r) - 1/2 - \delta}, \quad \forall t > 0, \tag{4.37}$$

and

$$\|t^{1/(P_1 - 1)} v(t, \cdot \sqrt{t}) - w(1, \cdot)\|_r \le C_\delta t^{-\delta}, \quad \forall t > 0, \tag{4.38}$$

$$\|t^{1/(P_1 - 1) + 1/2} \nabla v(t, \cdot \sqrt{t}) - \nabla w(1, \cdot)\|_r \le C_\delta t^{-\delta}, \quad \forall t > 0, \tag{4.39}$$

$\forall r \in [r_1, \infty]$, *where* δ_0'' *can explicitly be computed.*

In particular, there exist $d_1' > 0$, $d_2' > 0$ *two constants, such that*

$$d_1' t^{-\beta_1(r)} \le \|v(t)\|_r \le d_2' t^{-\beta_1(r)},$$

$$d_1' t^{-\beta_1(r)-1/2} \le \|\nabla v(t)\|_r \le d_2' t^{-\beta_1(r)-1/2},$$

for large t and for all $r_1 \le r \le \infty$.

Proof of Theorem 4. The result of Theorem 1 is established for all $b \in \mathbb{R}$ and then in particular for $b = 0$. In the particular case where $b = 0$, we may take an arbitrary value of p and in particular $p = q/(2-q)$. With this choice of p, we remark that if $q > (n+2)/(n+1)$, then $p > 1 + 2/n$ and the condition (2.5), satisfied by P_1 and P_2, becomes the condition (4.31). On the other hand the condition $q \le p(n+2)/(n+p)$ is satisfied, since $q > (n+2)/(n+1)$, $q = 2p/(p+1)$. See Remark 3.1. Thus by (3.5) we have (4.33) for $r = r_1$ and (4.34) for $r = r_2$.

We turn now to prove the $W^{1,\infty}$-estimates. We apply an iterative argument as in [18]. This iterative argument was used in [3, Proposition 2.3, p. 253] for the KPZ equation. Here we prove other estimates. Let us denote the real numbers r_1, r_2, r_{11}, r_{21}, β_{11}, β_{21} respectively by s_0, s_0', s_{01}, s_{01}', β_{01}, β_{01}'. Choose the real numbers s_1, s_1' such that

$$\begin{aligned} &s_1' = \frac{P_2 - 1}{P_1 - 1} s_1 \\ &s_0 < s_1,\ 1 < \frac{s_{01}}{q} < s_1,\ n\Big(\frac{q}{s_{01}} - \frac{1}{s_1}\Big) < 1 \end{aligned}$$

and

$$s_0' < s_1',\ 1 < \frac{s_{01}'}{q} < s_1',\ n\Big(\frac{q}{s_{01}'} - \frac{1}{s_1'}\Big) < 1.$$

Precisely a choice of such s_1, s_1' is possible thanks to Lemma 2.5. With these notations, we have from (3.5) that

$$\begin{aligned} \sup_{t>0} \big[& t^{\beta_1(s_0)} \|u(t)\|_{s_0},\ t^{\beta_1(s_0)+1/2} \|\nabla u(t)\|_{s_0}, \\ & t^{\beta_2(s_0')} \|u(t)\|_{s_0'},\ t^{\beta_2(s_0')+1/2} \|\nabla u(t)\|_{s_0'} \big] \le M. \end{aligned}$$

In the sequel of the proof, C denotes a constant which may vary form line to line. Sometimes we make precise its dependence on the parameters. We write now

$$u(t) = \mathrm{e}^{(t/2)\Delta} u\Big(\frac{t}{2}\Big) + a \int_{t/2}^{t} \mathrm{e}^{(t-\sigma)\Delta} (|\nabla u(\sigma)|^q) d\sigma.$$

It follows from the smoothing properties of the heat semigroup (2.1), that

$$\begin{aligned}\|u(t)\|_{s_1} \leq & C\Big(\frac{t}{2}\Big)^{-(n/2)(1/s_0-1/s_1)}\|u(t/2)\|_{s_0}\\ & + C\int_{t/2}^{t}(t-\sigma)^{-(n/2)(q/s_{01}-1/s_1)}\|\nabla u(\sigma)\|_{s_{01}}^{q}d\sigma.\end{aligned}$$

By using the interpolation inequality (2.4) and the inequality (3.5) we have

$$\begin{aligned}t^{\beta_1(s_1)}\|u(t)\|_{s_1} \leq & C\Big(\frac{t}{2}\Big)^{\beta_1(s_0)}\Big\|u\Big(t/2\Big)\Big\|_{s_0}\\ & + M^q C t^{\beta_1(s_1)-(n/2)(q/s_{01}-1/s_1)-(\beta_{01}+1/2)q+1}\\ & \times \int_{1/2}^{1}(1-\sigma)^{-(n/2)(q/s_{01}-1/s_1)}\sigma^{-(\beta_{01}+1/2)q}d\sigma.\end{aligned}$$

By the hypotheses on s_1, the previous integral is finite. Note that the integral does not present a singularity at $\sigma = 0$. By Lemma 2.5 part (v) and the definition of β_{01} given by (2.9), we have that

$$\begin{aligned}&\beta_1(s_1) - \frac{n}{2}\Big(\frac{q}{s_{01}} - \frac{1}{s_1}\Big) - \Big(\beta_{01} + \frac{1}{2}\Big)q + 1\\ &= \beta_1(s_0) - \frac{n}{2}\Big(\frac{q}{s_{01}} - \frac{1}{s_0}\Big) - \Big(\beta_{01} + \frac{1}{2}\Big)q + 1\\ &= 0.\end{aligned}$$

Then, we obtain

$$t^{\beta_1(s_1)}\|u(t)\|_{s_1} \leq C(M),$$

where $C(M)$ is a constant depending on M.

On the other hand, we have that

$$\nabla u(t) = \mathrm{e}^{(t/2)\Delta}\nabla u\Big(\frac{t}{2}\Big) + a\int_{t/2}^{t}\nabla \mathrm{e}^{(t-\sigma)\Delta}(|\nabla u(\sigma)|^q)d\sigma.$$

By using the smoothing properties of the heat semigroup (2.1) on the first term of the right-hand side of the previous equality and (2.2) on the second term followed by an interpolation argument, we obtain

$$\begin{aligned}&t^{\beta_1(s_1)+1/2}\|\nabla u(t)\|_{s_1}\\ &\leq C\Big(\frac{t}{2}\Big)^{\beta_1(s_1)+1/2-(n/2)(1/s_0-1/s_1)}\Big\|\nabla u\Big(\frac{t}{2}\Big)\Big\|_{s_0}\end{aligned}$$

$$
\begin{aligned}
&+ Ct^{\beta_1(s_1)+1/2}\int_{t/2}^{t}(t-\sigma)^{-(n/2)(q/s_{01}-1/s_1)-1/2}\|\nabla u(\sigma)\|_{s_{01}}^{q}\,d\sigma\\
\le{}& C\Big(\frac{t}{2}\Big)^{\beta_1(s_0)+1/2}\Big\|\nabla u\Big(\frac{t}{2}\Big)\Big\|_{s_0}\\
&+ M^qCt^{\beta_1(s_1)-(n/2)(q/s_{01}-1/s_1)-(\beta_{01}+1/2)q+1}\\
&\quad\times\int_{1/2}^{1}(1-\sigma)^{-(n/2)(q/s_{01}-1/s_1)-1/2}\sigma^{-(\beta_{01}+1/2)q}\,d\sigma.
\end{aligned}
$$

By the hypotheses on s_1, the last integral is finite. We deduce by Lemma 2.5 and the estimates (3.5) that

$$t^{\beta_1(s_1)+1/2}\|\nabla u(t)\|_{s_1}\le C(M),$$

where $C(M)$ is a constant depending on M. By a similar argument we obtain

$$t^{\beta_2(s_1')}\|u(t)\|_{s_1'}\le C(M),\; t^{\beta_1'(s_1')}\|\nabla u(t)\|_{s_1'}\le C(M),$$

where $C(M)$ is a constant depending on M. Finally,

$$
\begin{aligned}
\sup_{t>0}\big[t^{\beta_1(s_1)}\|u(t)\|_{s_1},\, t^{\beta_1(s_1)+1/2}\|\nabla u(t)\|_{s_1},&\\
t^{\beta_2(s_1')}\|u(t)\|_{s_1'},\, t^{\beta_2(s_1')+1/2}\|\nabla u(t)\|_{s_1'}\big]&\le C(M).
\end{aligned}
$$

We iterate this procedure and define, for all positive integer k, the sequences s_k, s_k' such that

$$
\begin{aligned}
&s_k'=\frac{P_2-1}{P_1-1}s_k\\
&s_k<s_{k+1},\; 1<\frac{s_{k1}}{q}<s_{k+1},\; n\Big(\frac{q}{s_{k1}}-\frac{1}{s_{k+1}}\Big)<1
\end{aligned}
$$

and

$$s_k'<s_{k+1}',\; 1<\frac{s_{k1}'}{q}<s_{k+1}',\; n\Big(\frac{q}{s_{k1}'}-\frac{1}{s_{k+1}'}\Big)<1,$$

where

$$
\begin{aligned}
\frac{1}{s_{k1}}&=\frac{P_1-1}{s_k}\Big(\frac{P_1}{q(P_1-1)}-\frac{1}{2}\Big),\\
\beta_{k1}&=\Big(\frac{P_1}{q(P_1-1)}-\frac{1}{2}\Big)-\frac{n}{2s_{k1}},
\end{aligned}
\tag{4.40}
$$

$$\frac{1}{s'_{k1}} = \frac{P_2-1}{s'_k}\Big(\frac{P_2}{q(P_2-1)} - \frac{1}{2}\Big),$$
$$\beta'_{k1} = \Big(\frac{P_2}{q(P_2-1)} - \frac{1}{2}\Big) - \frac{n}{2s_{k1}}. \tag{4.41}$$

One can check that we can construct a suitable sequence $(s_k)_k$ such that we may take $s_{k_0+1} = \infty$ for some finite k_0. This proves (4.33) and (4.34) for $r = \infty$. For the other values of r, (4.33) and (4.34) follow by an interpolation argument.

We turn now to prove the asymptotic behavior results. Let v be the solution of (3.1) with $b = 0$ and with initial data ψ. Then v satisfies (4.33)–(4.34). Let w be the self-similar solution of (4.1) with $b = 0$ and with initial data φ. We remark also that one can prove (4.33) for the self-similar solution w by iterative argument as for v but only by using the two first terms of the norm (1.2), i.e.: $\sup_{t>0}\big[t^{\beta_1}\|e^{t\Delta}\varphi\|_{r_1},\, t^{\beta_1+1/2}\|\nabla e^{t\Delta}\varphi\|_{r_1}\big]$ as for the proof of its existence.

We first prove the $W^{1,\infty}$-asymptotic. Let $T > 0$ be an arbitrary real number and let $\delta > 0$ be sufficiently small. Define θ'_{11} by

$$\theta'_{11} = \theta_{11} + \delta(1-\nu)\theta_{11}\Big(\frac{P_1}{P_1-1} - \frac{q}{2}\Big)^{-1}.$$

We note that

$$\frac{\theta_{11}}{P_1-1} + \frac{1-\theta_{11}}{P_2-1} = \frac{P_1}{q(P_1-1)} - \frac{1}{2}.$$

Also, we remark that if $\nu = 1$, then by (1.5) we have $\theta'_{11} = \theta_{11} = 1$. Using an interpolation argument combined with (4.33)–(4.34) we have that

$$\sup_{t>0}\big(t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+1/2}\|\nabla v(t)\|_\infty\big) \le C(M). \tag{4.42}$$

Write

$$v(t) - w(t) = e^{(t/2)\Delta}\left(v\Big(\frac{t}{2}\Big) - w\Big(\frac{t}{2}\Big)\right) + a\int_{t/2}^{t} e^{(t-\sigma)\Delta}(|\nabla v(\sigma)|^q - \nu|\nabla w(\sigma)|^q)d\sigma.$$

Then, by using the smoothing properties of the heat semigroup (2.1), also by using (4.7) with $s = \infty$ and $\gamma = q$ and (4.42), we obtain

$$t^{1/(P_1-1)+\delta}\|v(t)-w(t)\|_\infty \le t^{1/(P_1-1)+\delta}\Big\|\mathrm{e}^{(t/2)\Delta}\Big(v\Big(\frac{t}{2}\Big)-w\Big(\frac{t}{2}\Big)\Big)\Big\|_\infty$$
$$+\ |a|t^{1/(P_1-1)+\delta}\int_{t/2}^{t}\big\|\mathrm{e}^{(t-\sigma)\Delta}\big(|\nabla v(\sigma)|^q-\nu|\nabla w(\sigma)|^q\big)\big\|_\infty d\sigma,$$

hence

$$t^{1/(P_1-1)+\delta}\|v(t)-w(t)\|_\infty \le Ct^{\beta_1+\delta}\Big\|v\Big(\frac{t}{2}\Big)-w\Big(\frac{t}{2}\Big)\Big\|_{r_1}$$
$$+C(M)|a|t^{1/(P_1-1)+\delta}\Big(\int_{t/2}^{t}\sigma^{-q(\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1))-q/2-\nu\delta}d\sigma\Big)$$
$$\times\Big(\sup_{t\in(0,T]}\big[t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+\nu\delta}\|v(t)-\nu w(t)\|_\infty,$$
$$t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+1/2+\nu\delta}\|\nabla v(t)-\nu\nabla w(t)\|_\infty\big]\Big)$$

and

$$t^{1/(P_1-1)+\delta}\|v(t)-w(t)\|_\infty \le Ct^{\beta_1+\delta}\Big\|v\Big(\frac{t}{2}\Big)-w\Big(\frac{t}{2}\Big)\Big\|_{r_1}$$
$$+C(M)|a|t^{1/(P_1-1)-q(\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1))-q/2+(1-\nu)\delta+1}$$
$$\times\Big(\int_{1/2}^{1}\sigma^{-q(\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1))-q/2-\nu\delta}d\sigma\Big)$$
$$\times\Big(\sup_{t\in(0,T]}\big[t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+\nu\delta}\|v(t)-\nu w(t)\|_\infty,$$
$$t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+1/2+\nu\delta}\|\nabla v(t)-\nu\nabla w(t)\|_\infty\big]\Big).$$

We note that if $\nu=1$ hence $\theta'_{11}=1$ then (4.42) is verified by w. Otherwise, if $\nu=0$, clearly we do not need it for w.

Now, by using (4.25) and the definition of θ'_{11} we get

$$t^{1/(P_1-1)+\delta}\|v(t)-w(t)\|_\infty \le C(\delta)+C(M,\,\delta)$$
$$\times\Big(\sup_{t\in(0,T]}\big[t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+\nu\delta}\|v(t)-\nu w(t)\|_\infty,$$
$$t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+1/2+\nu\delta}\|\nabla v(t)-\nu\nabla w(t)\|_\infty\big]\Big). \quad (4.43)$$

Write

$$\nabla v(t)-\nabla w(t)=\mathrm{e}^{(t/2)\Delta}\Big(\nabla v\Big(\frac{t}{2}\Big)-\nabla w\Big(\frac{t}{2}\Big)\Big)$$

$$+ a \int_{t/2}^{t} \nabla \mathrm{e}^{(t-\sigma)\Delta}(|\nabla v(\sigma)|^q - \nu |\nabla w(\sigma)|^q) d\sigma.$$

Then, by using the smoothing properties of the heat semigroup (2.1)–(2.2) and inequality (4.26), we get

$$\begin{aligned} &t^{1/(P_1-1)+1/2+\delta}\|\nabla v(t) - \nabla w(t)\|_\infty \le C(\delta) + C(M,\, \delta) \\ &\times \Big(\sup_{t\in(0,\,T]} \big[t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+\nu\delta}\|v(t) - \nu w(t)\|_\infty, \\ &\qquad t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+1/2+\nu\delta}\|\nabla v(t) - \nu \nabla w(t)\|_\infty \big] \Big). \end{aligned} \tag{4.44}$$

Now, if $\nu = 1$ then $\theta'_{11} = \theta_{11} = 1$, we obtain

$$\begin{aligned} &\sup_{t\in(0,\,T]} \big(t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+\nu\delta}\|v(t) - \nu w(t)\|_\infty, \\ &\qquad t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+1/2+\nu\delta}\|\nabla v(t) - \nu \nabla w(t)\|_\infty \big) \\ &= \sup_{t\in(0,\,T]} \big(t^{1/(P_1-1)+\delta}\|v(t) - w(t)\|_\infty, \\ &\qquad t^{1/(P_1-1)+1/2+\delta}\|\nabla v(t) - \nabla w(t)\|_\infty \big). \end{aligned}$$

On the other hand, if $\nu = 0$ then by using (4.33)–(4.34) and an interpolation argument, we obtain

$$\begin{aligned} &\sup_{t\in(0,\,T]} \big(t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+\nu\delta}\|v(t) - \nu w(t)\|_\infty, \\ &\qquad t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+1/2+\nu\delta}\|\nabla v(t) - \nu \nabla w(t)\|_\infty \big) \\ &= \sup_{t\in(0,\,T]} \big(t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)}\|v(t)\|_\infty, \\ &\qquad t^{\theta'_{11}/(P_1-1)+(1-\theta'_{11})/(P_2-1)+1/2}\|\nabla v(t)\|_\infty \big) \le C(M). \end{aligned}$$

Then by using (4.43)–(4.44) and for M perhaps smaller we get, for arbitrary $T > 0$,

$$\begin{aligned} &\sup_{t\in(0,\,T]} \big(t^{1/(P_1-1)+\delta}\|v(t) - w(t)\|_\infty, \\ &\qquad t^{1/(P_1-1)+1/2+\delta}\|\nabla v(t) - \nabla w(t)\|_\infty \big) \le C(\delta,\, M). \end{aligned}$$

Since the constant C does not depend on T, one can take the supremum over $(0, \infty)$. Hence we obtain (4.36) and (4.37) for $r = \infty$. For the other values of r, the result follows by using (4.25)–(4.26) and an interpolation argument. The estimates (4.38) and (4.39) follow by a dilation argument. This finishes the proof of the theorem. □

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