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Modelling minimal foliated spaces with positive entropy

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Abstract. Using methods and results of decomposition theory we construct minimal actions of groups of homeomorphisms of some classical fractals (the Sierpiński carpet and its generalizations, and the Menger curve) with positive entropy. Suspending these group actions we get minimal foliated spaces which have positive geometric entropy and are modelled on these fractals.

Key words: group action, Sierpiński sets, foliation.

1. Introduction

A *foliated space* or *lamination* (see, for instance, [4], Chapter 11) is, roughly speaking, a locally compact, separable, metrizable space M which is locally homeomorphic to the product $D \times V$, where V is an open subset of another locally compact, separable, metrizable space Z while D is an open ball in $\mathbb{R}^p$. The local homeomorphisms should be organized to form a *foliated atlas* $\mathcal{A}$, i.e. in such a way that for any two of them $\varphi_i : U_i \to D_i \times V_i$, $i = 1, 2$, the composition $\varphi_{12} = \varphi_2 \circ \varphi_1^{-1}$ is of the form

$$\varphi_{12}(x, z) = (f(x, z), h(z)). \tag{1}$$

The elements of the foliated atlas are called *foliated charts*, sets of the form $\varphi^{-1}(D \times \{z\})$, where $z \in Z$ and φ is a foliated chart, are called *plaques* and the connected components of M equipped with the (new) topology generated by all the plaques are called *leaves*. The family $\mathcal{F}$ of all the leaves forms a *foliation* of M and the pair $(M, \mathcal{F})$ becomes our foliated space.

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Usually, one assumes that leaves have structures of smooth manifolds which varies continuously in the transverse direction. That is, the maps f and h in (1) are continuous with respect to both variables, f is C^∞ differentiable with respect to x and all its partial derivatives are continuous (with respect to both variables, again).

We shall assume that our atlas $\mathcal{A}$ is *maximal*, i.e. that any chart φ_2 satisfying (1) (with some f and h fulfilling the smoothness conditions above) for all charts $\varphi_1 \in \mathcal{A}$ belongs to $\mathcal{A}$. Also, we assume that our space Z contains no superfluous points, i.e. that any $z \in Z$ satisfies $(x, z) = \varphi(y)$ for some $y \in M$, $\varphi \in \mathcal{A}$ and $x \in \mathbb{R}^p$. In this situation, we shall say that $(M, \mathcal{F})$ is *modelled transversely* on Z.

All the local homeomorphisms h of Z which appear in (1) for arbitrary foliated charts φ_1 and $\varphi_2 \in \mathcal{A}$ generate a pseudogroup $\mathcal{H}$ called the *holonomy pseudogroup* of $\mathcal{F}$.

The foliation $\mathcal{F}$ (or, a foliated space $(M, \mathcal{F})$) is called *minimal* whenever M itself is the only non-empty closed set saturated by the leaves of $\mathcal{F}$; equivalently, whenever all the leaves of $\mathcal{F}$ (resp., all the $\mathcal{H}$-orbits of points of Z) are dense in M (resp., in Z).

Certainly, standard foliations of differentiable manifolds as well as closed saturated subsets of foliated manifolds provide most natural examples of foliated spaces. However, there exist many foliated spaces which arise from different constructions. Some of them cannot be embedded into foliated manifolds. One of the constructions providing foliated spaces is that of *suspending* homomorphisms of fundamental groups of manifolds into the group $\mathrm{Homeo}(Z)$ of homeomorphisms of Z. More precisely, if B is a manifold and $h\colon \pi_1(B) \to \mathrm{Homeo}(Z)$ a group homomorphism, then $\pi_1(B)$ acts in a natural way on $\tilde{B}$, the universal covering space of B, and one can put $M = (\tilde{B} \times Z)/\equiv_h$, where the equivalence relation $\equiv_h$ is defined in the following way: $(x, z) \equiv_h (x', z')$ whenever there exists $g \in \pi_1(B)$ for which $g^{-1}(x) = x'$ and $h(g)(z) = z'$. The space M fibres over B with fibre Z and can be equipped with a foliation $\mathcal{F}$ which consists of the leaves of the form $L = \pi(\tilde{B} \times \{z\})$, where $z \in Z$ and $\pi\colon \tilde{B} \times Z \to M$ is the canonical projection. The foliated space $(M, \mathcal{F})$ is called the *suspension* of h.

In [12], the geometric entropy of a foliation was defined in the case of regular foliations. One of its equivalent definitions there was given in terms of points separated by elements of a holonomy pseudogroup and (as was observed in [4]) can be generalized to arbitrary foliated spaces (see

Section 2.1 below). Therefore, given Z, one can ask for examples of foliated spaces which are modelled on Z and have positive entropy. Such a problem is relatively easy if without any restriction on M and $\mathcal{F}$. Therefore, we shall formulate and discuss here the following, a bit more restrictive, problem.

Problem 1 Given a compact metrizable space Z, find a compact minimal foliated space $(M, \mathcal{F})$ which is modelled on Z and has positive entropy.

This article is organized as follows.

In Section 2, we recall the notion of geometric entropy for foliations and foliated spaces and, for the convenience of the reader, provide some results of the decomposition theory (see [8]) which will be used in our constructions. In Section 3, we show that our problem has an affirmative solution when Z is either the Sierpiński carpet or one of its generalizations (called here Sierpiński sets). First, using the classical Smale horseshoe we construct a homeomorphism of the Sierpiński carpet which has positive entropy. Then, on several Sierpiński sets, we construct finitely generated groups of global homeomorphisms which have all the orbits dense. These two constructions produce a group of homeomorphisms of the Sierpiński carpet with positive entropy and all the orbits dense. Suspending such a group action we obtain a foliated space as wanted. Also, we observe that the situation becomes quite different when one replaces the Sierpiński carpet by the Sierpiński gasket: the suspension construction cannot provide solutions to Problem 1 for $Z = S$, the Sierpiński gasket in the plane. In last two sections (Sections 4 and 5) we get some related results for Menger curves and list some appropriate questions.

2. Preliminaries

In this Section, we recall the notion of entropy for groups, pseudogroups and foliated spaces, and provide (after [8]) some results of decomposition theory to be used later.

2.1. Entropy

Let us take a topological space X and denote by $\mathrm{HomeoLoc}(X)$ the family of all homeomorphisms between open subsets of X. If $g \in \mathrm{HomeoLoc}(X)$, then D_g is its domain and $R_g = g(D_g)$.

Definition 1 A subfamily $\mathcal{G}$ of HomeoLoc(X) is said to be a *pseudogroup* if it is closed under composition, inversion, restriction to open subdomains and amalgamation. More precisely, $\mathcal{G}$ should satisfy the following conditions:
(i) $g \circ h \in \mathcal{G}$ whenever g and $h \in \mathcal{G}$,
(ii) $g^{-1} \in \mathcal{G}$ whenever $g \in \mathcal{G}$,
(iii) $g|U \in \mathcal{G}$ whenever $g \in \mathcal{G}$ and $U \subset D_g$ is open,
(iv) if $g \in \text{HomeoLoc}(X)$, $\mathcal{U}$ is an open cover of D_g and $g|U \in \mathcal{G}$ for any $U \in \mathcal{U}$, then $g \in \mathcal{G}$.
Moreover, we shall always assume that
(v) $\text{id}_X \in \mathcal{G}$ (or, equivalently, $\cup\{D_g; g \in \mathcal{G}\} = X$).

Now, let $\mathcal{G}$ be a pseudogroup on X and $\mathcal{G}_1$ be a good, finite, symmetric set generating $\mathcal{G}$. This means that $\text{id}_X \in \mathcal{G}_1$, $\mathcal{G}_1^{-1} \subset \mathcal{G}_1$, any element $g \in \mathcal{G}_1$ can be extended to $\tilde{g} \in \mathcal{G}$ in such a way that the domain of $\tilde{g}$ contains the closure of the domain of g which is assumed to be compact, and for any $h \in \mathcal{G}$ and x in the domain of h there exist generators $g_1, \ldots, g_k \in \mathcal{G}_1$ such that $h = g_1 \circ \cdots \circ g_k$ in a neighbourhood of x. The set of all the compositions $g_1 \circ \cdots \circ g_n$ with $g_i \in \mathcal{G}_1$ is denoted here by $\mathcal{G}_n$.

In the following, we assume (X, d) is a compact metric space.

Definition 2 Points x and y of X are *(n, ε)-separated* ($n \in \mathbb{N}$, $\varepsilon > 0$) if there exists $g \in \mathcal{G}_n$ such that

$$\{x, y\} \subset D_g \quad \text{and} \quad d(g(x), g(y)) \geq \varepsilon.$$

A set $A \subset X$ is *(n, ε)-separated* when any two points x and y of A, $x \neq y$, have this property.

Since X is compact, any (n, ε)-separated set is finite and we may put

$$s(n, \varepsilon, \mathcal{G}_1) = \max\{\sharp A;\ A \subset X \quad \text{is } (n, \varepsilon)\text{-separated}\}.$$

Next, let

$$s(\varepsilon, \mathcal{G}_1) = \limsup_{n\to\infty} \frac{1}{n} \log s(n, \varepsilon, \mathcal{G}_1) \tag{2}$$

and

$$h(\mathcal{G}, \mathcal{G}_1) = \lim_{\varepsilon\to 0} s(\varepsilon, \mathcal{G}_1). \tag{3}$$

Obviously, the limit (either finite or infinite) in (3) exists.

Definition 3 The number $h(\mathcal{G}, \mathcal{G}_1)$ is called the (topological) *entropy* of $\mathcal{G}$ with respect to $\mathcal{G}_1$.

It is easy to see that if $\mathcal{G}_1'$ is another good, finite, symmetric set generating $\mathcal{G}$, then either both entropies $h(\mathcal{G}, \mathcal{G}_1)$ and $h(\mathcal{G}, \mathcal{G}_1')$ are positive or vanish simultaneously. Therefore, one can talk about pseudogroups of vanishing (or, positive) entropy without referring to a particular generating set.

Now, if $(M, \mathcal{F})$ is a compact foliated space, $\mathcal{A}$ is a finite foliated atlas on $(M, \mathcal{F})$, $\mathcal{H}_1$ is a finite set of holonomy maps h (see (1)) corresponding to all pairs of overlapping charts φ_1 and $\varphi_2 \in \mathcal{A}$, then – restricting the domains of the charts of $\mathcal{A}$ if necessary – we can observe that $\mathcal{H}_1$ is finite, symmetric and good, and generates a pseudogroup $\mathcal{H}_{\mathcal{A}}$, the holonomy pseudogroup of $\mathcal{F}$. Finally, the smoothness conditions posed on foliated charts in Introduction allow to equip the leaves of $\mathcal{F}$ with a collection g of Riemannian structures varying continuously all over M and to calculate diameters of plaques of charts $\varphi \in \mathcal{A}$ with respect to these metrics. Let $\Delta(\mathcal{A})$ be the smallest upper bound for these diameters.

Definition 4 The *geometric entropy* $h(\mathcal{F}, g)$ of a foliated space $(M, \mathcal{F})$ (with respect to a continuous family g of Riemannian structures on the leaves) is defined by

$$h(\mathcal{F}, g) = \sup_{\mathcal{A}} \frac{h(\mathcal{H}_{\mathcal{A}}, \mathcal{H}_1)}{\Delta(\mathcal{A})},$$

where $\mathcal{A}$ ranges over all finite foliated atlases on $(M, \mathcal{F})$.

As in the case of pseudogroups and groups, the property of having positive (or, zero) geometric entropy is independent of the choice of the family g of Riemannian structures on $(M, \mathcal{F})$.

2.2. From decomposition theory

Let us recall that a *decomposition* $\mathcal{D}$ of a topological space S is just a partition of S, that is, a family of pairwise disjoint nonempty sets that cover S. With any decomposition $\mathcal{D}$ of a space S we associate a *decomposition space* $S/\mathcal{D}$. The map $\pi\colon S \to S/\mathcal{D}$ sending each $s \in S$ to the unique element of $\mathcal{D}$ containing s determines the topology of $S/\mathcal{D}$, the richest topology for which π is continuous. A subset X of S is *saturated* (or, *$\mathcal{D}$-saturated*) if $\pi^{-1}(\pi(X)) = X$. For any decomposition $\mathcal{D}$ of a space S let us put

$$H_{\mathcal{D}} = \{A \in \mathcal{D}; A \text{ contains more than one point}\}$$

and

$$N_{\mathcal{D}} = \bigcup_{A \in H_{\mathcal{D}}} A.$$

A decomposition $\mathcal{D}$ of a space S is said to be *upper semicontinuous* (usc) if

1) each $A \in \mathcal{D}$ is compact in S,
2) for each $A \in \mathcal{D}$ and each open subset U of S containing A, there exists another open subset V of S containing A such that every $A' \in \mathcal{D}$ intersecting V is contained in U.

An upper semicontinuous decomposition $\mathcal{D}$ of a space S is said to be *shrinkable* if and only if for each $\mathcal{D}$-saturated open cover $\mathcal{U}$ of S and each arbitrary open cover $\mathcal{V}$ of S, there is a homeomorphism $h\colon S \to S$ satisfying

a) for each $s \in S$ there exists $U \in \mathcal{U}$ such that s, $h(s) \in U$, and
b) for each $A \in \mathcal{D}$ there exists $V \in \mathcal{V}$ such that $h(A) \subset V$.

Such a $\mathcal{D}$ is *strongly shrinkable* if, for every open set W containing $N_{\mathcal{D}}$, $\mathcal{D}$ is shrinkable fixing $S \setminus W$, that is a homeomorphism h in the previous definition can be chosen in such a way that $h(x) = x$ whenever $x \notin W$. Obviously, every strongly shrinkable decomposition is shrinkable.

Let $f\colon S \to X$ be a surjective map and $\mathcal{W}$ an open cover of X. Then a map $F\colon S \to X$ is said to be $\mathcal{W}$-*close* to f if for each $s \in S$ there exists $W \in \mathcal{W}$ such that $f(s)$, $F(s) \in W$. The map f is said to be a *near homeomorphism* if for each open cover $\mathcal{W}$ of X there exists a homeomorphism F of S onto X that is $\mathcal{W}$-close to f.

Finally, recall that a countable collection $\{A_i\}$ of subsets of a metric space is said to form a *null sequence* if, for each $\varepsilon > 0$, only finitely many of sets A_i have diameter greater than ε.

Now, let us formulate the result which will be used later.

Theorem 1 *If (D_n) is a null sequence of mutually disjoint closed discs in a compact manifold M, and $M' = M/\equiv$, where $x \equiv y \iff$ either $x = y$ or there exists $n \in \mathbb{N}$ such that $\{x, y\} \subset D_n$, then M' is homeomorphic to M.*

Here, by a *closed disc* in M we mean the inverse image $\varphi^{-1}(B)$, where B is a closed ball in the Euclidean space while φ is a chart on M.

Let us provide a proof of Theorem 1 for the convenience of the reader. The proof is based on a number of results extracted from [8].

Proof. Let M^n be a compact manifold with a metric d, and $(D_k,\, k \in \mathbb{N})$ a null sequence of closed discs on M.

Fix $k_0 \in \mathbb{N}$ and $\varepsilon > 0$. Choose a chart $\varphi\colon U \to \mathbb{R}^n$ such that $D_{k_0} \subset U$ and $D'_{k_0} = \varphi(D_{k_0})$ is a Euclidean ball. Without loosing generality we may assume that $\overline{U}$ is compact and φ can be extended to another open subset V of M for which $\overline{U} \subset V$. Let $D'_k = \varphi(D_k)$ whenever $D_k \subset U$. All the sets D'_k form a null sequence and are compact, therefore the decomposition of $\mathbb{R}^n$ determined by these sets is usc (by [8], Proposition 3, p. 14). Since $\varphi^{-1}\colon W = \varphi(U) \to M$ is uniformly continuous, there exists $\delta > 0$ such that $N(D'_{k_0}, \delta)$, where $N(A, \delta)$ denotes the δ-neighbourhood of a set A, is contained in $\varphi(U)$ and

$$d(z_1, z_2) < \delta \quad \Longrightarrow \quad d(\varphi^{-1}(z_1), \varphi^{-1}(z_2)) < \varepsilon$$

for all z_1 and $z_2 \in W$. Now, since D'_{k_0} is convex (therefore, starlike), there exists a homeomorphism $F\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $F|\mathbb{R}^n \setminus N(D'_{k_0}, \delta) = \mathrm{id}$, $\operatorname{diam} F(D'_{k_0}) < \delta$ and for all k either $\operatorname{diam} F(D'_k) < \delta$ or $F(D'_k) = D'_k$ (by [8], Lemma 5, p. 55).

Let δ_0 be the distance between ∂U and D_{k_0} while δ_1 the minimum distance between D_{k_0} and these discs D_k for which $\operatorname{diam}(D_k) > \delta_0/3$. If $\varepsilon < \min\{\delta_0, \delta_1\}/3$, then no disc D_k meets both ∂U and $N(D_{k_0}, \varepsilon)$. Assume that our ε is such small and define a homeomorphism $h\colon M \to M$ by $h|U = \varphi^{-1} \circ F \circ \varphi$ and $h|M \setminus U = \mathrm{id}$. Then, $\operatorname{diam} h(D_{k_0}) < \varepsilon$, $h = \mathrm{id}$ outside $N(D_{k_0}, \varepsilon)$ and, for all k, either $\operatorname{diam} h(D_k) < \varepsilon$ or $h(D_k) \subset N(D_k, \varepsilon)$. Again, the decomposition $\mathcal{D}$ of M determined by our null sequence (D_k) is countable and usc, therefore it is (strongly) shrinkable (by [8], Theorem 5, p. 47).

Finally, by Theorem 6 in [8] (p. 28), $\pi\colon M \to M/\mathcal{D}$ is a near homeomorphism and can be approximated by homeomorphisms. In particular, M and $M/\mathcal{D}$ are homeomorphic. □

3. Sierpiński sets

Recall that the Sierpiński carpet is obtained from a rectangle by removing infinitely many open subrectangles with pairwise disjoint closures (see Fig. 2 below). In 1958, Whyburn [29] showed that it is homeomorphic to any subset of S^2 obtained by removing the interiors of mutually disjoint closed discs $\{D_i\}_{i=1,2,\dots}$ such that $\bigcup_{i \geqq 1} D_i$ is dense in S^2 and $\{D_i\}$

is a null sequence. Kato ([17]) and Aarts-Oversteegen ([1]) showed that the Sierpiński carpet does not admit an expansive homeomorphism. However, there exist its homeomorphisms with positive entropy.

Theorem 2 *The Sierpiński carpet admits a homeomorphism with positive topological entropy.*

Proof. First, let us recall the Smale's construction of a horseshoe f on the 2-dimensional sphere S^2 (Fig. 1). In the following, S^2 is regarded as $\mathbb{R}^2 \cup \{\infty\}$.

Let R be a rectangle in S^2, and Δ_1 and Δ_2 be two half discs attached to the opposite edges of R respectively. Then the image of R by f has the shape of a horseshoe intersecting R in two rectangles. Let A denote the union $\Delta_1 \cup R \cup \Delta_2$. We assume that $f(A)$ contains A in its interior, and $f^n(f(A) \setminus A)$ converges uniformly to infinity as $n \to \infty$. Furthermore, f is assumed to be linear near $R \cap f(R)$. In particular, f expands horizontally and shrinks vertically near $R \cap f(R)$. In Δ_1, f is assumed to be expanding from a repellor p. Then, the sets $\bigcap_{n \leqq 0} f^n(R)$ and $\bigcap_{n \geqq 0} f^n(R)$ are homeomorphic to the product of the Cantor set and a closed interval and their intersection $\bigcap_{n \in \mathbb{Z}} f^n(R)$ is a closed invariant set homeomorphic to the Cantor set.

In the following, starting from a horseshoe, we will construct a homeo-

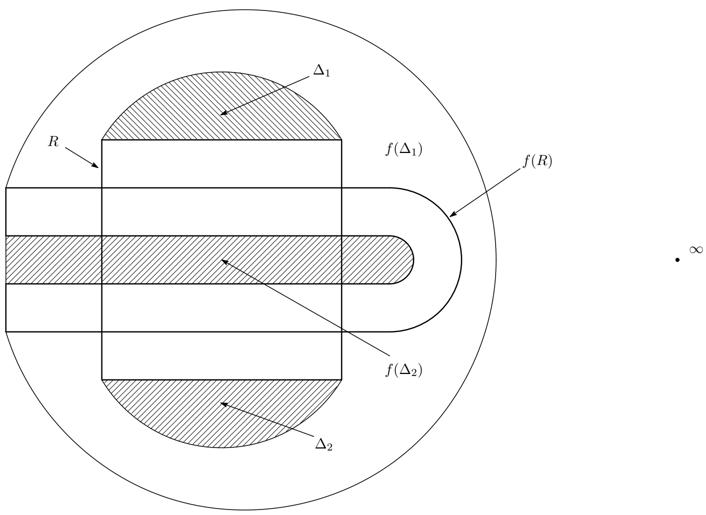

Fig. 1. A horseshoe

morphism f of the Sierpiński carpet by removing interiors of a suitable null sequence of mutually disjoint closed discs such that their union is dense in S^2 and disjoint from the closed invariant set $\bigcap_{n\in\mathbb{Z}} f^n(R)$. Then the restriction of f to the Cantor set $\bigcap_{n\in\mathbb{Z}} f^n(R)$ becomes topologically conjugate to the shift with two symbols. As it is well known from symbolic dynamics that this shift has positive topological entropy (see, for instance, [28], p. 177), the resulting homeomorphism of the Sierpiński carpet has positive entropy too.

Let R_C be a rectangle in which the typical Sierpiński carpet is contained. Let $\{H_i\}_{i=1,2,\ldots}$ denote the subrectangles (holes) in R_C such that $R_C \setminus \bigcup_{i\geqq 1} \operatorname{int} H_i$ is the Sierpiński carpet. Then $\{H_i\}$ are mutually disjoint closed topological discs whose union is dense in R_C. Certainly, they form a null sequence.

The set $\operatorname{int} f(A) \setminus (A \cup f(R))$ consists of three components B_1, B_2 and B_3 homeomorphic to open discs. Let $\varphi_j\colon R_C \to \overline{B_j}$ $(j = 1, 2, 3)$ be a homeomorphism. Then $\{\varphi_j(H_i)\}_{i=1,2,\ldots,j=1,2,3}$ are mutually disjoint closed discs whose union is dense in $B_1 \cup B_2 \cup B_3$ (Fig. 2). Since $f^n(f(A) \setminus A)$ tends towards ∞ as $n \to \infty$, the sets $f^n(\varphi_j(H_i))$ also tend towards infinity as $n \to \infty$. In particular, $\operatorname{diam} f^n(\varphi_j(H_i))$ converges to 0 as $n \to \infty$. On the other hand, $f^{-1}(\varphi_j(H_i))$ is contained in $A \setminus R$, and, in particular, in $\Delta_1 \cup \Delta_2$.

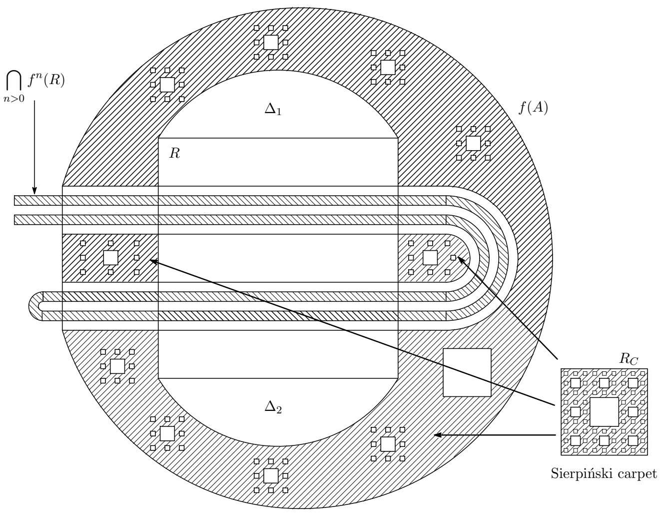

Fig. 2. Removing mutually disjoint closed discs

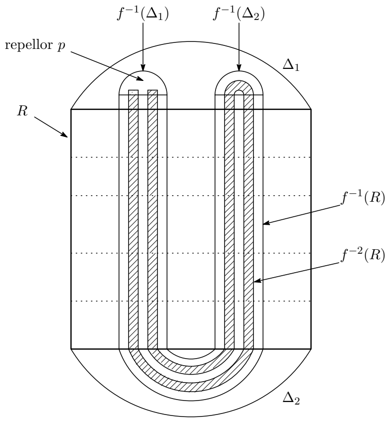

Fig. 3. The set $\bigcap_{n\leqq 0} f^n(A)$ is nowhere dense.

Since $f^{-1}(\Delta_1 \cup \Delta_2)$ is contained in Δ_1 (Fig. 3), the sets $f^{-n}(\varphi_j(H_i))$ tend towards the repellor p as $n \to \infty$, and $\operatorname{diam} f^{-n}(\varphi_j(H_i)) \to 0$ as $n \to \infty$.

Next we insert countably many Sierpiński carpets into $(\operatorname{int} f(R) \setminus R) \setminus \bigcap_{n>0} f^n(R)$ (see Fig. 2 again). Let $\{U_j\}_{j=1,2,\dots}$ denote all the components of $(\operatorname{int} f(R) \setminus R) \setminus \bigcap_{n>0} f^n(R)$. Then $\overline{U_j}$ are mutually disjoint closed discs because, in each component of $\operatorname{int} f(R) \setminus R$, the subset $\bigcap_{n>0} f^n(R)$ is homeomorphic to the product of the Cantor set and an open interval. However $\operatorname{diam} f^{-n}(U_j)$ does not converge to 0 as $n \to \infty$ because $\overline{U_j}$ intersects $\bigcap_{n>0} f^n(R)$ and, for some point z of U_j, $f^{-n}(z)$ converges to p. Thus we have to insert a Sierpiński carpet into each such component U_j. Let $\psi_j : R_C \to \overline{U_j}$ be a homeomorphism. Then $\{\psi_j(H_i)\}_{i,j=1,2,\dots}$ are mutually disjoint closed discs whose union is dense in $(\operatorname{int} f(R) \setminus R) \setminus \bigcap_{n>0} f^n(R)$. Furthermore, the union is also dense in $\operatorname{int} f(R) \setminus R$ because the restriction of $\bigcap_{n>0} f^n(R)$ to each component of $\operatorname{int} f(R) \setminus R$ is homeomorphic to the product of the Cantor set and an open interval.

By the same argument as for $\{\varphi_j(H_i)\}$, we obtain that all the sets $f^n(\psi_j(H_i))$ tend uniformly to infinity as $n \to \infty$.

Let $C_n = \operatorname{int} f(R) \setminus (R \cup f^n(R))$. Then $\{C_n\}_{n>0}$ are increasing sets converging to $\bigcup_{n>0} C_n = (\operatorname{int} f(R) \setminus R) \setminus \bigcap_{n>0} f^n(R)$. Thus, for any i and j, the hole $\psi_j(H_i)$ is contained in some C_n $(n > 0)$. By definition of C_n, $\psi_j(H_i)$ is disjoint from $f^n(R)$. In particular, $f^{-n}(\psi_j(H_i)) \cap R = \emptyset$. On the other hand, $f^{-n}(\psi_j(H_i))$ is contained in A (just because $\psi_j(H_i)$ itself is contained in A). Thus $f^{-n}(\psi_j(H_i))$ is contained either in Δ_1 or in Δ_2, and hence $f^{-n-1}(\psi_j(H_i))$ is contained in Δ_1.

Let $a = \max\{\|df^{-1}(x)\|,\ x \in S^2\}$. (Certainly, $a > 1$.) Since Sierpiński carpets can be obtained by removing from R_C arbitrary null sequences of open discs with mutually disjoint closures and dense union, we may assume without loss of generality that each $\psi_j(H_i)$ is disjoint from the union of boundaries of C_n and furthermore, if $\psi_j(H_i) \subset C_n \setminus C_{n-1}$, then

$$\operatorname{diam} \psi_j(H_i) \le \frac{1}{i^2 + j^2 + n^2} \cdot a^{-(n+1)}$$

for any n. Then,

$$\operatorname{diam} f^{-(n+1)}(\psi_j(H_i)) \le \frac{1}{i^2 + j^2 + n^2}$$

and $f^{-n-1}(\psi_j(H_i))$ is contained in Δ_1 as above. Moreover, since $f^{-1}\colon \Delta_1 \to \Delta_1$ is a contraction with a constant λ, $0 < \lambda < 1$, $\operatorname{diam} f^{-1}(D) \le \lambda \cdot \operatorname{diam} D$ for any disc $D \subset \Delta_1$ and

$$\operatorname{diam} f^{-m}(\psi_j(H_i)) \le \frac{1}{i^2 + j^2 + n^2} \cdot \lambda^{m-n-1}$$

when $m > n$. From these inequalities and similar ones for $\operatorname{diam} f^n(\psi_j(H_i))$, $n > 0$, it follows that all the discs $f^n(\varphi_k(H_i))$ and $f^n(\psi_j(H_i))$, where $k = 1, 2, 3$, $i, j \in \mathbb{N}$ and $n \in \mathbb{Z}$, form a null sequence.

It remains to show that the union of all the discs removed is dense in S^2. However we have already shown that their union is dense in $f(A) \setminus A$. Thus we have only to show that the set $A_\infty = \bigcup_{n\in\mathbb{Z}} f^n(f(A) \setminus A)$ is dense in S^2. Since $\{f^n(A)\}$ is a sequence increasing with n, the complement of $\bigcup_{n\ge k} f^n(f(A) \setminus A) = \bigcup_{n\ge k}(f^{n+1}(A) \setminus f^n(A))$ coincides with $f^k(A) \cup \{\infty\}$ for any $k \in \mathbb{Z}$. Hence the complement of $\bigcup_{n\in\mathbb{Z}} f^n(f(A) \setminus A) = \bigcup_{k\le 0}(\bigcup_{n\ge k} f^n(f(A) \setminus A))$ is equal to $\{\infty\} \cup \bigcap_{n\le 0} f^n(A)$. By construction, $\bigcap_{n\le 0} f^n(A)$ is nowhere dense (Fig. 3 again). This implies that our set A_∞ is dense in S^2 indeed. □

Aarts and Oversteegen [1] showed also that the Sierpiński carpet does not admit a minimal homeomorphism. Thus, *a fortiori*, there are no minimal homeomorphisms of the Sierpiński carpet with positive topological entropy. However there exists a minimal group action on the Sierpiński carpet with positive entropy. To establish this fact we shall show first that minimal group actions exist on several more general spaces called Sierpiński sets hereafter.

Let M, $\dim M = m$, be a compact differentiable manifold and S a subspace of M obtained by removing a null sequence of open balls B_i, $i = 1, 2, \ldots$, with pairwise disjoint closures $\overline{B}_i$ (being closed balls) and $\overline{\bigcup B_i} = M$. (By a ball in M we mean the inverse image $\varphi^{-1}(B)$ of an Euclidean ball $B \subset \mathbb{R}^m$ obtained *via* a chart φ on M. Saying "a null sequence" we mean that $\mathrm{diam}(B_i) \to 0$ as $i \to \infty$ where "diam" is calculated with the distance function on M coming from an arbitrary but fixed Riemannian metric.) In case when M is 2-dimensional, Whyburn [29] showed that sets S and S' obtained from M and M', a metric space homeomorphic to M, by removing two different null sequences with pairwise disjoint closures and dense union are homeomorphic. If $M = S^2$, S becomes the classical Sierpiński carpet, therefore we call such a set S a *Sierpiński M-set* here.

Theorem 3 *Let G be a finitely generated group of diffeomorphisms of M such that for any $x \in M$ the orbit $G(x)$ is dense and G acts freely on $G(x_0)$ for some point x_0. Then there exists a group $\hat{G}$ of homeomorphisms of M which is isomorphic to G and admits a Sierpiński M-set as a minimal set.*

Our proof is analogous to the construction of Aarts and Oversteegen in [1]. They used foliations emanating from the orbit. In order to generalize their results to groups, we use the derivatives of diffeomorphisms.

Proof. Fix a finite symmetric set G_1 of generators of G (i.e., $e \in G_1$ and $G_1^{-1} = G_1$), equip G with the word metric determined by G_1 and denote by G_n, $n \in \mathbb{N}$, the ball in G of radius n and centre e. Set also $G_0 = \{e\}$.

First we construct an inverse limit $\hat{M}$ by inserting a family of balls at points $h(x_0)$ for $h \in G$. Then the complement of the interiors of the inserted balls will become homeomorphic to a Sierpiński M-set. Let $\exp_z \colon T_zM \to M$ denote the exponential map at z ($z \in M$). For an integer n ($n \geqq 0$), we choose $\varepsilon_n > 0$ such that, for any $h \in G_n$, the exponential map $\exp_{h(x_0)} | B(0, \varepsilon_n)$ is injective and $\{\exp_{h(x_0)}(B(0, \varepsilon_n))\}_{h \in G_n}$ is pairwise disjoint. Let $D_h = \{w \in T_{h(x_0)}M; |w| < 1 + \varepsilon_n\}$ for $h \in G_n$. We take the disjoint union of $M \setminus G_n(x_0)$ and $\{D_h\}_{h \in G_n}$, and identify a point w in $\{w \in D_h; 1 < |w| < 1 + \varepsilon_n\}$ with $\exp_{h(x_0)}\big((|w| - 1)/|w|\big)w$ in $M \setminus G_n(x_0)$ (see Fig. 4). We equip the given set, M_n, with an arbitrary metric d_n so that (M_n, d_n) is homeomorphic to M. Here we remark that M_n is independent of the choice of ε_n if ε_n is small enough. Let $B_h = \{w \in D_h; |w| \leqq 1\}$. Then

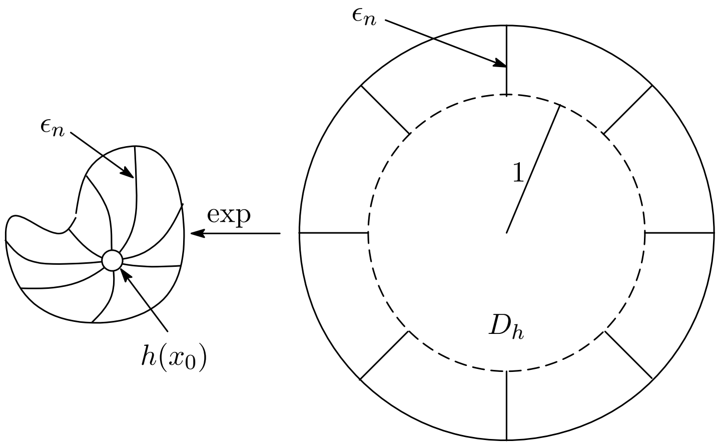

Fig. 4. $M_n = \big((M \smallsetminus G_n(x_0)) \cup (\bigsqcup_{h \in G_n} D_h)\big) / \sim$

the inverse system $\psi_{kl} \colon M_l \to M_k$ $(k < l)$ is defined by collapsing each B_h $(h \in G_l \smallsetminus G_k)$ to a point. Let $\hat{M}$ denote the inverse limit (M_n, ψ_{kl}), i. e. $\hat{M} = \{(z_n) \in \prod_{n \geqq 0} M_n; \psi_{kl}(z_l) = z_k \ (k < l)\}$, where the topology is given by the product metric

$$d\big((z_n),\, (z'_n)\big) = \sum_{n \geqq 0} \frac{1}{2^n} \, \frac{d_n(z_n,\, z'_n)}{1 + d_n(z_n,\, z'_n)}$$

of $\prod_{n \geqq 0} M_n$. In particular, $\hat{M}$ is compact.

Let $\pi_n \colon M_n \to M$ denote the projection obtained by collapsing each B_h $(h \in G_n)$ to a point and let $\pi_M \colon \hat{M} \to M$ denote the projection defined by $\pi_M((z_n)) = z_0$. For $h \in G$, $\pi_M^{-1}(h(x_0))$ is denoted by B'_h. Then it is homeomorphic to B_h in M_j $(j \geqq k)$ if $h \in G_k$. Thus $\pi_M^{-1}(G(x_0))$ consists of pairwise disjoint closed discs. Now $\{B'_h\}$ is a null sequence because, if (z_n) and (z'_n) are contained in $\pi_M^{-1}(h(x_0))$ for $h \in G_k$, then $z_j = z'_j$ for $j = 0, 1, \ldots, k-1$, and hence $d((z_n), (z'_n)) \leqq \sum_{n \geqq k} 1/2^n = 1/2^{k-1}$. Therefore the decomposition $\{B'_h; h \in G\}$ is shrinkable, and hence π_M is a near homeomorphism ([8]). In particular, $\hat{M}$ is homeomorphic to M (see [8] and Section 2.2 of this paper).

Let $S = \hat{M} \smallsetminus (\bigcup_{h \in G} \operatorname{Int} B'_h)$. Since $G(x_0)$ is dense in M and π_M is an open map, S has no interior points. Moreover, $\{B'_h\}$ is a null-sequence. Thus we conclude that S is homeomorphic to a Sierpiński M-set.

For $h \in G$, we will define a homeomorphism $\hat{h}\colon \hat{M} \to \hat{M}$, where the collection $\hat{G}$ of all $\hat{h}$ is a group and will satisfy the condition of our Theorem. We choose a positive integer l so that h is contained in G_l. Let (z_n) be an element of $\hat{M}$. If $\pi_M((z_n)) \notin G(x_0)$, then $h(\pi_n(z_n))$ is not contained in $G(x_0)$ either, and hence we can define $\hat{h}((z_n))$ by $\hat{h}((z_n)) = (\pi_n^{-1} h \pi_n(z_n))$. Next we consider the case when $\pi_M((z_n))$ is contained in $G(x_0)$. We will determine $(w_n) = \hat{h}((z_n))$. Let g be an element of some G_k such that $\pi_M((z_n)) = g(x_0)$. Then g and hg are contained in G_{k+l}. For $j \geqq k+l$, the element z_j of M_j is contained in B_g $(\subset M_j)$, and thus we define an element w_j of B_{hg} $(\subset M_j)$ by $w_j = (|z_j|/|Dh(z_j)|)Dh(z_j)$, where z_j and w_j are regarded as elements of the tangent spaces $T_{g(x_0)}M$ and $T_{hg(x_0)}M$ respectively and Dh is the derivative of h. For $j < k+l$, we define w_j by $w_j = \psi_{j,k+l}(w_{k+l})$. Then (w_n) is an element of $\hat{M}$ and $\pi_M \hat{h} = h \pi_M$. If w_j is given for a sufficiently large j, then $w_0, w_1, w_2, \ldots, w_{j-1}$ are automatically determined by ψ_{kl}. Let h' be another element of G. For a sufficiently large j,

$$\begin{aligned}\frac{|z_j|}{|D(h'h)(z_j)|} D(h'h)(z_j) &= \frac{|z_j|}{|Dh'(Dh(z_j))|} Dh'(Dh(z_j)) \\ &= \frac{|z_j|}{|Dh'(w_j)|} Dh'(w_j) \\ &= \frac{|w_j|}{|Dh'(w_j)|} Dh'(w_j).\end{aligned}$$

Thus $h \mapsto \hat{h}$ is a homomorphism (i.e. $\hat{h}'\hat{h} = \widehat{h'h}$). Furthermore, $h \mapsto \hat{h}$ is an isomorphism because $\pi_M \hat{h} = h \pi_M$. We define a continuous map

$$h_n\colon (M_n \setminus \pi_n^{-1}(G_{n+l}(x_0))) \cup \pi_n^{-1}(G_{n-l}(x_0)) \to M_n$$

by

$$h_n(z) = \begin{cases} \pi_n^{-1} h \pi_n(z) & \text{if } z \notin \pi_n^{-1}(G_{n+l}(x_0)) \\ \dfrac{|z|}{|Dh(z)|} Dh(z) & \text{if } z \in \pi_n^{-1}(G_{n-l}(x_0)). \end{cases}$$

When $(w_n) = \hat{h}((z_n))$, then $w_n = h_n(z_n)$ if $z_n \in (M_n \setminus \pi_n^{-1}(G_{n+l}(x_0))) \cup \pi_n^{-1}(G_{n-l}(x_0))$.

Next we will show that $\hat{h}$ is continuous at (z_n). Let $(w_n) = \hat{h}((z_n))$. For an arbitrary number $\varepsilon > 0$, we choose a positive number $N > 0$ so that $\sum_{n \geqq N} 1/2^n < \varepsilon/2$. Suppose first that $\pi_M((z_n)) \notin G(x_0)$. For $0 \leqq n \leqq N-1$, the map h_n is continuous at z_n in M_n. Thus, if (z'_n) is sufficiently near (z_n), then $w'_n = h_n(z'_n)$ satisfies $\sum_{n=0}^{N-1}(1/2^n)\big(d_n(w_n, w'_n)/(1+d_n(w_n,w'_n))\big) < \varepsilon/2$, and hence $d((w_n),(w'_n)) < \varepsilon$. Therefore $\hat{h}$ is continuous at (z_n). Assume now that $\pi_M((z_n)) \in G(x_0)$. Then there is an element g of some G_k such that $\pi_M((z_n)) = g(x_0)$. We take an integer K greater than $k+l$ and N. Since $\pi_K(z_K) = g(x_0)$ and $g \in G_k \subset G_{K-l}$, the map h_K is well defined and continuous at z_K. For $j < N$, $\psi_{j,K}$ is also continuous. Thus $w'_j = \psi_{jK} h_K(z'_K)$ is near w_j if (z'_n) is near (z_n). Since $\sum_{n \geqq N}(1/2^n)\big(d_n(w_n, w'_n)/(1+d_n(w_n,w'_n))\big) < \varepsilon/2$, $\hat{h}$ is continuous at (z_n).

Furthermore, $\hat{h}$ is a homeomorphism because $\widehat{(h^{-1})}$ is an inverse map of $\hat{h}$.

Finally we will show that S is a minimal set. Let (z_n) be a point of S and let U be an open set such that $U \cap S \neq \emptyset$. Since π_M is an open map, the open set $\pi_M(U)$ contains a point $g(\pi_M((z_n)))$ for some $g \in G_k$. Here we can take an arbitrarily large k. Then $\pi_M^{-1}(g\pi_M((z_n)))$ is small enough. Thus the orbit passing through (z_n) intersects U. This implies that S is a minimal set. □

Now, let $M = S^2$ be the unit sphere (equipped with the standard Riemannian metric). Consider two rotations R_1 and R_2 with non-parallel rotation axes l_1 and l_2 intersecting at the origin 0 and rotation angles $2\pi\alpha_1$ and $2\pi\alpha_2$ with α_1 and α_2 being irrational. The group H generated by R_1 and R_2 satisfies assumptions of Theorem 3. In fact, all the orbits of H are dense and, since any $g \in H \setminus \{e\}$ is a nontrivial rotation, the set S^2_g of points fixed by g consists of two points, so the union $\cup_{g \in G \setminus \{e\}} S^2_g$ is at most countable. Therefore, orbits with free action of H exist (and are uncountably many).

By the Theorem mentioned above, H induces $\hat{H}$, a group of homeomorphisms of S^2 which admits a Sierpiński carpet C as a minimal set. Let f be a diffeomorphism of S^2 constructed in the proof of Theorem 2 and $C' \subset S^2$ be the corresponding f-invariant Sierpiński carpet. Since C and C' are homeomorphic, we can assume without loss of generality that $C' = C$. The group $\hat{G}$ of homeomorphisms of C generated by $f|C$ and the corresponding to R_1 and R_2 homeomorphisms $\hat{R}_1|C$ and $\hat{R}_2|C$ has positive entropy and all its orbits are dense in C. Denote by Σ_3 a closed oriented surface of genus 3. There exists a homomorphism $h\colon \pi_1(\Sigma_3) \to \mathrm{Homeo}(C)$ which maps $\pi_1(\Sigma_3)$

onto $\hat{G}$. Indeed,

$$\pi(\Sigma_3) = \langle a_1, a_2, a_3, b_1, b_2, b_3 \mid \Pi_{i=1}^3 a_i b_i a_i^{-1} b_i^{-1} = e \rangle$$

and we may put $h(a_3) = f|C$, $h(a_j) = \hat{R}_j|C$ and $h(b_i) = \mathrm{id}_C$ for $j = 1, 2$ and $i = 1, 2, 3$.

Suspending this homomorphism we get the following.

Corollary 1 *There exists a minimal foliated space of positive entropy modelled on a Sierpiński carpet.*

Let us complete this section by a discussion of similar problems for the Sierpiński gasket S which – as it is well known – can be defined as the unique non-empty compact subset of the complex plane $\mathbb{C}$ that satisfies the condition

$$S = f_1(S) \cup f_2(S) \cup f_3(S),$$

where

$$f_i(z) = \frac{z - P_i}{2} + P_i \quad \text{for } i = 1, 2, 3,$$

while $\{P_1, P_2, P_3\}$ is the set of vertices of an equilateral triangle in the complex plane $\mathbb{C}$ (see Fig. 5). It is well known that the topology of S is quite different from that of the Sierpiński carpet C even if both of the sets, S and C, can be obtained from S^2 by removing a null sequence of open discs D_n. However, in the case of S, some intersections of closures of D_n's are not disjoint. The mentioned difference in topology may be seen easily with the aid of ramification index defined (compare [11]) as follows. A metric space X has *ramification index* m at a point x whenever

(1) for any number $\varepsilon > 0$ there exists an open neighbourhood of x in X which has diameter less than ε and whose boundary has cardinality at most m, and

(2) there exists $\varepsilon_0 > 0$ such that the boundary of any open neighbourhood of x in X whose diameter is less than ε_0 has cardinality at least m.

Below, we list some observations which show in particular that the sets S and C are different also from the point of view of dynamics.

Let f be any homeomorphism of S. First, notice that the set $A = \{P_{13}, P_{32}, P_{12}\}$ consisting of the centres of the segments $\overline{P_1P_3}$, $\overline{P_3P_2}$ and $\overline{P_1P_2}$, is the unique subset of S such that it contains exactly three points

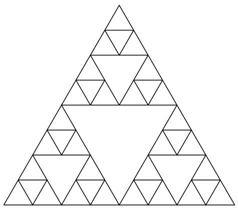

Fig. 5. The Sierpiński gasket

while the complement $S \setminus A$ splits into three connected components. Therefore, the set A is f-invariant. We may assume that any point of A is fixed by f. Otherwise, we shall replace f by the homeomorphism $f' = f \circ I$, where I is the isometry of S such that $(f \circ I)(P_{ij}) = P_{ij}$, $i, j = 1, 2, 3$.

Denote by S_1, S_2 and S_3 the connected components of $S \setminus A$. Since $f|A = \mathrm{id}$, $f(S_i) = S_i$. For each i, the closure $\bar{S}_i$ of S_i is homeomorphic (*via* f_i) to S. It follows that the sets $A_i = f_i(A)$ are f-invariant. Consequently, $f|A \cup A_1 \cup A_2 \cup A_3 = \mathrm{id}$. By induction, $f = \mathrm{id}$ on the set S_0 of all the points of the form $(f_{i_1} \circ \cdots \circ f_{i_n})(P_j)$ where i_k, $j = 1, 2, 3$ and $n \in \mathbb{N}$. Since S_0 is dense in S, $f = \mathrm{id}$ on S.

This shows that

Observation 1 *Any homeomorphism f of S is a periodic isometry* (*the restriction to S of an isometry of the triangle $\triangle P_1P_2P_3$*), *therefore any group of global homeomorphisms of S has zero entropy.*

Certainly, the system (f_1, f_2, f_3) of homotheties defining S satisfies so called *Moran's open set condition* ([10], p. 160): There exists a non-empty open subset U of S for which $f_i(U) \subset U$ and $f_i(U) \cap f_j(U) = \emptyset$ whenever $i, j \in \{1, 2, 3\}$ and $i \neq j$. Moreover, the system (h_i), $h_i = f_i^2$, satisfies the following *strong Moran's open set condition*: for all $i \neq j$, $d(h_i(U), h_j(U)) > 0$, where U is a certain non-empty open set such that $h_i(U) \subset U$ for all i's. Choose U like this and set $\varepsilon_0 = \min\{d(h_i(U), h_j(U)); i \neq j\}$. Fix $x_0 \in U$ and put $B_n = \{(h_{i_1} \circ \cdots \circ h_{i_n})(x_0); i_1, \ldots, i_n = 1, 2, 3\}$. If $x = (h_{i_1} \circ \cdots \circ h_{i_n})(x_0)$, $y = (h_{j_1} \circ \cdots \circ h_{j_n})(x_0) \in B_n$ and $x \neq y$, then there exists $m < n$ such that $i_l = j_l$ for $l \leq m$ and $i_{m+1} \neq j_{m+1}$. If $f = h_{i_1} \circ \cdots \circ h_{i_m}$, then $d(f^{-1}(x), f^{-1}(y)) \geq d(h_{i_{m+1}}(U), h_{j_{m+1}}(U)) \geq \varepsilon_0$. In other words, the set B_n is $(2n, \varepsilon_0)$-separated with respect to the system $\mathcal{G}_1 = \{\mathrm{id}, f_1^{\pm 1}, f_2^{\pm 1}, f_3^{\pm 1}\}$ generating a pseudogroup $\mathcal{G}$ of local homeomorphisms of S. Since $\sharp B_n = 3^n$, the entropy of $\mathcal{G}$ is positive. Clearly, $\mathcal{G}$-orbits of all points of S are dense. Therefore,

Observation 2 *There exists a pseudo-group $\mathcal{G}$ on the Sierpiński gasket S with all the orbits dense and positive entropy.*

Certainly, most of the above argument can be applied to any system of maps of compact metric spaces satisfying strong Moran's open set condition. Hence,

Observation 3 *Any pseudogroup $\mathcal{G}$ on a compact metric space X which is generated by a finite set $\mathcal{G}_1$ containing a subsystem $\mathcal{G}_0$ of cardinality > 1 and satisfying the strong Moran's open set condition has positive entropy.*

4. Menger curve

Let I denote the unit interval $[0, 1]$. Split I into three equal segments and the cube I^3 into 27 corresponding pieces. The Menger curve μ can be obtained from I^3 by removing the part which consists of 7 pieces: the central one and six containing the centres of the faces of the cube, and repeating this procedure infinitely many times in each of the remaining cube of size $(1/3)^n$. The restriction of the Menger curve to the face $I^2 \times \{0\}$ is homeomorphic to the Sierpiński carpet.

Recall after Bestvina [2] that a subset A of μ is called a *Z-set* whenever for any $\delta > 0$ there exists a continuous map $g\colon \mu \to \mu \setminus A$ which is uniformly δ-close to identity. Given $\delta > 0$ one can find $\varepsilon \in (0, \delta)$ such that the cross-section $I^2 \times \{\varepsilon\} \cap \mu$ equals C, the Sierpiński carpet (see Fig. 6). Other cross-sections of μ by planes parallel to the bottom face B of I^3 are smaller than C. Therefore, the map $g\colon \mu \to \mu \setminus B$ defined by $g(t_1, t_2, t_3) = (t_1, t_2, \varepsilon)$ when $t_3 \leq \varepsilon$ and $g(t_1, t_2, t_3) = (t_1, t_2, t_3)$ when $t_3 > \varepsilon$ is well defined and — obviously — δ-close to id. That is, B is a Z-set.

By Corollary 3.1.5 of [2], every homeomorphism f of B onto itself extends to a homeomorphism $\tilde{f}$ of μ. If f has positive topological entropy, its extension $\tilde{f}$ has positive entropy as well. Therefore, Theorem 2 yields immediately the following.

Corollary 2 *The Menger curve μ admits a homeomorphism with positive topological entropy.*

The question about the existence of a single minimal homeomorphism

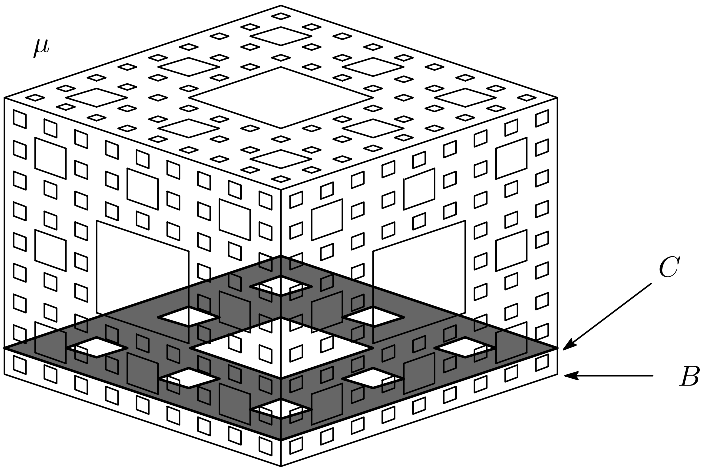

Fig. 6. B is a Z-subset of μ

of μ with positive entropy remains, as far as we know, open. However, Stark [23] constructed on μ a minimal action of a free group F_2 generated by two homeomorphisms f_1 and f_2. The action of G, the group generated by f_1, f_2 and $\tilde{f}$, the homeomorphism considered above, is minimal and has positive entropy. Suspending this action of G over Σ_3 as in Section 3 we end up with the following.

Corollary 3 *There exists a minimal foliated space of positive geometric entropy modelled on the Menger curve μ.*

Since minimal actions of finitely generated groups exist ([23], Corollary 4.2) on Menger sets μ^n of arbitrary dimension as well as on several Menger manifolds, one should be able to generalize the above to these cases too.

To complete this Section, let us recall that the Menger curve μ can be also constructed as a Pasynkov [22] partial topological product which is defined as follows.

Given X, a compact connected metric space equipped with the distance function d, $\mathcal{U} = (U_i)$, a countable null basis for the topology of X (that is, $\mathcal{U}$ is a basis such that $\mathrm{diam}(U_i) \to 0$ as $i \to \infty$) and $A = (A_i)$, a countable family of finite sets, each containing at least two elements, set $P_0 = X$ and, for $r \geq 1$,

$$P_r = \Big(X \times \prod_{i=1}^{r} A_i\Big) \Big/ \sim,$$

where the equivalence relation "$\sim$" is generated by

$$(x, a_1, a_2, \ldots, a_i, \ldots, a_r) \sim (x, a_1, a_2, \ldots, a_i', \ldots, a_r),$$

whenever $x \notin U_i$ and $a_i, a_i' \in A_i$. The natural projection $\prod_{i=1}^{n+1} A_i \to \prod_{i=1}^{n} A_i$ induces a projection $\varphi_{r+1} \colon P_{r+1} \to P_r$ (Fig. 7). The *Pasynkov partial product* P of the (A_i) over X with respect to $\mathcal{U}$ equals the inverse limit of the system (P_r, φ_r); the induced map $\pi \colon P \to P_0 = X$ is called the *Pasynkov projection*. For any r, one has also natural projections $\pi_r \colon P \to P_r$ and $\hat{\pi}_r \colon P_r \to X$ (such that $\pi = \hat{\pi}_r \circ \pi_r$).

Assume also that the metric space X is nontrivial (i.e. contains more than one point) and geodesic (i.e. any two points x and $y \in X$ can be joined by a curve (*geodesic*) $\gamma \colon [0, 1] \to X$ such that the length $l(\gamma \mid [t_1, t_2])$ of $\gamma \mid [t_1, t_2]$ equals $d(\gamma(t_1), \gamma(t_2))$ for all t_1 and t_2 in $[0, 1]$), and $\partial U_i \neq \emptyset$ for all $i \in \mathbb{N}$; also – just to simplify some estimates – that $\operatorname{diam} X = 1$. Then, all the spaces P_r are arcwise connected. Indeed, given r and two points $x = [(u, a_1, \ldots, a_r)]_\sim$ and $y = [(w, b_1, \ldots, b_r)]_\sim$ of P_r ($u, w \in X$, $a_i, b_i \in A_i$) one can join u to w by a curve $\gamma \colon [0, 1] \to X$ which intersects all the boundaries ∂U_i, $i \leq r$. Choose points $t_i \in [0, 1]$ for which $\gamma(t_i) \in \partial U_i$ and define the curve $\tilde{\gamma} \colon [0, 1] \to P_r$ by

$$\tilde{\gamma}(t) = [(\gamma(t), \alpha_1(t), \ldots, \alpha_r(t))]_\sim,$$

where

$$\alpha_i(t) = a_i \text{ when } t \leq t_i \quad \text{and} \quad \alpha_i(t) = b_i \text{ when } t \geq t_i.$$

Certainly, $\tilde{\gamma}$ is well defined, continuous and connects x to y. Therefore, we can consider the path metric d_r on P_r, where the length of a curve γ in P_r is defined as the length (in X) of its projection $\hat{\pi}_r \circ \gamma$. Certainly, d_r is symmetric and satisfies the triangle inequality. Also, if $x = [(u, a_1, \ldots, a_r)]_\sim$ and $y = [(w, b_1, \ldots, b_r)]_\sim$ are points of P_r and $u \neq w$, then any curve joining x to y in P_r has length at least $d(u, w)$ and $d_r(x, y) \geq d(u, w) > 0$; if $w = u$ but $x \neq y$, then the set $I_r = \{i \leq r; u \in U_i\}$ is nonempty and for any curve γ joining x to y in P_r there exists $i \in I_r$ for which $\hat{\pi}_r \circ \gamma$ quits U_i, therefore

$$d_r(x, y) \geq \min\{d(u, X \setminus U_i); i \in I_r\} > 0.$$

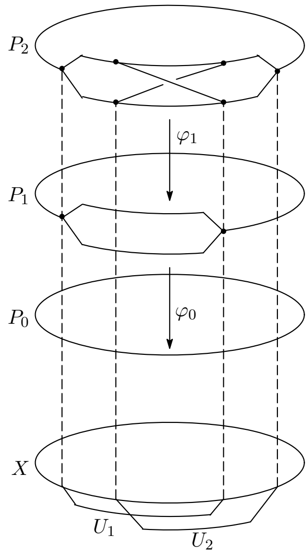

Fig. 7. Pasynkov construction

Finally, put

$$d_\infty(x, y) = \sum_{r=0}^{\infty} 2^{-(r+1)} d_r(\pi_r(x), \pi_r(y)) \tag{4}$$

for $x, y \in P$, where $P_0 = X$, $\pi_0 = \pi$ and $d_0 = d$. Clearly, d_∞ is a metric on P. In the special case, when $X = S^1$ is a circle of radius r, say $r = 1/2\pi$, and $A_i = \{0, 1\}$ for each $i \in \mathbb{N}$, P is homeomorphic to the Menger curve μ (see [2]). Therefore, analogously to the classical result on entropies of homeomorphisms of a circle, we can get – in a similar way – the following fact which could evoke the reader's impression that the negative answer to the question on existence of minimal homeomorphisms of μ with positive topological entropy is better-founded than the positive one.

Proposition 1 *Let f be a homeomorphism of μ. If there exists a homeomorphism φ of S^1 such that $\pi \circ f = \varphi \circ \pi$, then the topological entropy of f equals* 0.

To simplify redaction of the proof let us introduce some terminology and notation.

An arc $c : I = [0, 1] \to \mu$ is called *horizontal* (or, for short, an *h-arc*) if and only if $\pi \circ c \colon I \to S^1$ is monotone. Now, given a homeomorphism f of μ, we say that a subset E of μ is (n, ε)-*horizontally spanning* (*h-spanning*, for short) if and only if for any $x \in \mu$ there exist $y \in E$ and h-arcs $c_i \colon I \to \mu$ such that $c_i(0) = f^i(x)$, $c_i(1) = f^i(y)$ and $l(c_i) \le \varepsilon$ for any $i = 0, 1, \ldots, n-1$. We denote by $r(n, \varepsilon)$ the minimum cardinality of (n, ε)-h-spanning subsets of μ.

By classical results in dynamical systems (see, for example, [28], p. 174), the topological entropy $h_{\mathrm{top}}(f)$ of f can be defined by formulae analogous to (2) and (3) with $s(n, \varepsilon, \mathcal{G}_1)$ replaced by $r(n, \varepsilon)$. That is,

$$h_{\mathrm{top}}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \varepsilon).$$

To begin the proof of our Proposition let us choose $\varepsilon < 0.25$ and such small that the inequality $d_\infty(x, y) < \varepsilon$ implies that

$$d_\infty(f(x), f(y)) < 0.25 \quad \text{and} \quad d_\infty(f^{-1}(x), f^{-1}(y)) < 0.25.$$

Also, let us choose r_0 such that $\mathrm{diam}(U_r) < \varepsilon/2$ for any $r \ge r_0$ and $\sum_{r \ge r_0} 2^{-r} < \varepsilon/2$. Let E be an $(\varepsilon/2)$-dense subset of S^1 of cardinality $\le [2/\varepsilon] + 1$ and $F = \hat{\pi}_{r_0}^{-1}(E)$. Certainly, $\sharp F \le 2^{r_0}([2/\varepsilon] + 1)$. From (4), it follows directly that the set F is $(1, \varepsilon)$-h-spanning.

Let now F be an $(n-1, \varepsilon)$-h-spanning subset of μ with minimum cardinality $r(n-1, \varepsilon)$ and E an $(1, \varepsilon)$-h-spanning subset of μ with cardinality $\le 2^{r_0}([2/\varepsilon] + 1)$. Put

$$F' = F \cup f^{-(n-1)}(E).$$

The statement of our Proposition results directly from the following.

Lemma 1 *The set F' is (n, ε)-h-spanning.*

Proof. Fix $x \in \mu$. There exist $y \in F'$ and h-arcs $c_i \colon I \to \mu$ such that $c_i(0) = f^i(x)$, $c_i(1) = f^i(y)$ and $l(c_i) \le \varepsilon$ for any $i = 0, 1, \ldots, n-2$.

If there exists $c_{n-1} \colon I \to \mu$ such that $c_{n-1}(0) = f^{n-1}(x)$, $c_{n-1}(1) = f^{n-1}(y)$ and $l(c_{n-1}) \le \varepsilon$, then our claim is proved.

If not, since E is $(1, \varepsilon)$-h-spanning, there exists a point $v \in E$ and an h-arc $\xi \colon I \to \mu$ of length $l(\xi) \le \varepsilon$ connecting $f^{n-1}(x)$ with v.

Put

$$z = f^{-(n-1)}(v), \quad \eta = f^{-(n-1)}(\xi) \quad \text{and} \quad \tau'_j = f^j(c_0)$$

where $j = 0, 1, \ldots, n-1$. Without loss of generality we may assume that the arcs $\pi\circ\xi$ and $\pi\circ\tau'_{n-1}$ have the same orientation in S^1. Since $l(\tau'_{n-1}) > \varepsilon$, there exists a point $u \in \tau'_{n-1}$ such that $\pi(u) = \pi(v)$.

Let $z' = f^{-(n-1)}(u)$; then $f^j(z') \in \tau'_j$ for all $j = 0, 1, \ldots, n-1$.

Observe that $\pi(f^{n-1}(z)) = \pi(f^{n-1}(z'))$. Denote by α_j a subarc of τ'_j which connects $f^j(x)$ with $f^j(z')$. We shall show that $l(\alpha_j) \leq \varepsilon$ for all $j = 0, 1, \ldots, n-2$.

For $j = 0$, this is clear because $z' \in c_0$ and $l(c_0) \leq \varepsilon$. Take now some $j \geq 0$ such that $l(\alpha_j) \leq \varepsilon$. Then

$$l(\alpha_{j+1}) \leq l(\tau'_{j+1}) = l(c_{j+1}) \leq \varepsilon \tag{5}$$

(for $j+1 \leq n-2$) because the h-arcs τ'_{j+1} and c_{j+1} have the same end points and – by our choice of ε – lengths ≤ 0.25. Moreover,

$$\pi(v) = \pi(f^{n-1}(z'))$$

and

$$l(\alpha_{n-1}) = l(\xi) \leq \varepsilon.$$

Thus, inequality (5) holds for all $j = 0, 1, \ldots, n-1$.

The h-arcs $f^j(\eta)$, $j = 0, 1, \ldots, n-1$, join $f^j(x)$ with $f^j(z)$. We claim that for any such j the inequality

$$l(f^j(\eta)) \leq \varepsilon \tag{6}$$

holds. Indeed, if $j = n-1$, then

$$l(f^{n-1}(\eta)) = l(\xi) \leq \varepsilon.$$

Consider the case $j = n-2$. Both $f^{n-2}(\eta)$ and α_{n-2} are h-arcs with a common origin $f^{n-2}(x)$ and such that $\pi(f^{n-2}(z)) = \pi(f^{n-2}(z'))$. Moreover, again by our choice of ε, both have length ≤ 0.25. Thus,

$$l(f^{n-2}(\eta)) = l(\alpha_{n-2}) \leq \varepsilon.$$

Using the same argument inductively we can show that (6) holds for any j. This completes the proof of the Lemma (and of the Proposition). □

5. Final remarks

First, in the final part of Section 3 we observed that any group of homeomorphisms of the Sierpiński gasket S has zero entropy. It seems that the reason for the proof works is that S has finite ramification index. So, one could consider the following.

Problem 2 Are the observations concerning the Sierpiński gasket valid for any (planar) continuum with finite ramification index?

Next, Cantwell and Conlon defined in [5] an interesting class of pseudogroups (called *Markov pseudogroups*) and have shown in [6] that any minimal set of a Markov pseudogroup on the real line $\mathbb{R}$ can be realized by a codimension-1 foliation on a compact 3-dimensional manifold (compare [16] and [24] for other methods of construction of such foliations). The pseudogroup on the Sierpiński gasket S which was constructed to get Observation 2 is not Markov: the images $f_i(S)$ are not disjoint in S. However, it is not too far from being Markov: $f_i(S) \cap f_j(S)$ is just a singleton when $i \neq j$. So, one could consider also the following.

Problem 3 Does there exist a foliation (or, a foliated space) of codimension 2 on a compact manifold with a minimal set which is transversely locally homeomorphic to the Sierpiński gasket S?

The "global version" of this problem ("Does there exist minimal foliated spaces modelled transversely on S ?") makes no sense. The leaves of any such foliated space which correspond to the vertices of the triangle $\triangle P_1P_2P_3$ cannot be dense since – due to different ramification indexes – no neighbourhoods of P_i are homeomorphic to neighbourhoods of points $P \in S \setminus \{P_1, P_2, P_3\}$.

By results of [19], there exists a diffeomorphism f of T^2 of class $C^{3-\varepsilon}$ (where $\varepsilon > 0$ can be arbitrarily small) which leaves a Sierpiński T^2-set S invariant and has all the orbits of points of S dense in S. Suspending this single homeomorphism we will get a $C^{3-\varepsilon}$-smooth 1-dimensional foliation $\mathcal{F}$ of a compact 3-manifold M. Certainly, M contains a minimal set transversely homeomorphic to S. The diffeomorphism f of [19] is semi-conjugated to a translation $\tau\colon T^2 \to T^2$ which obviously has zero topological entropy. At the moment, it is not clear for us what is the entropy of f. If the entropy of f were positive, our foliation $\mathcal{F}$ would have positive entropy too. McSwiggen [20] performed similar constructions of diffeomorphisms preserving Sierpiński sets on tori T^k of arbitrary dimension $k \geq 1$. All of that motivates the following.

Problem 4 Given $k \geq 2$ and $\varepsilon > 0$, construct a codimension-k foliation $\mathcal{F}$ of class $C^{k+1-\varepsilon}$ on a closed manifold M with positive geometric entropy and an exceptional minimal set homeomorphic transversely to a Sierpiński T^k-set.

Finally, let us recall that Kawamura [18] proved that every finitely generated group G acts on the Menger curve μ. His proof consists in a construction of a suitable μ-manifold X built from the thickened Cayley graph Γ of G. Since the Freudenthal compactification $\overline{X}$ of X is known to be homeomorphic to μ (see [7]) and G acts in a natural way on Γ, hence on X and $\overline{X}$ as well, G acts on μ. On the other hand, if G is hyperbolic (in the Gromov's [13] sense), then G acts, again in a natural way, on the ideal boundary $\partial\Gamma$ of Γ. In many cases, this action has positive entropy (see [3]). Therefore, one could compare these results to study the following.

Problem 5 Describe the class of all Gromov hyperbolic groups which act on μ in such a way that the action is minimal and has positive entropy.

The argument proving our Corollary 3 shows that the free group F_3 belongs to this class.

The positive answer to our Problem 3 can be derived from the following example which was kindly brought to our attention by Takashi Tsuboi who described a minimal action of $PSL(2, \mathbb{Z})$ on an Apollonian gasket (see [26]).

The Apollonian gasket is a fractal which can be obtained by the following procedure. Consider three mutually externally tangent balls with maximal radius on a unit 2-sphere and delete the interiors of the balls. Inscribe in the remaining set, which consists of two spherical triangles, touching only in vertex points, two balls and delete its interiors. Repeating the inscribing and deleting procedure infinitely many times we get the Apollonian gasket as a limit. It is easy to notice that the Apollonian gasket is homeomorphic to two copies of Sierpiński gaskets inserted on a sphere and glued at vertex points. It is known (see [21], p. 197) that the Apollonian gasket is a limit set of a Schottky group made by pairing tangent circles. Notice that by the following theorem (called sometimes [27] the Ping-Pong Lemma),

Theorem 4 (Klein's Criterion (see [15], p. 130)) *Let G be a group acting on a set S, let Γ_1, Γ_2 be two subgroups of G and let Γ be the subgroup they generate; assume that Γ_1 contains at least three elements. Assume that there exist two non empty subsets S_1, S_2 in S with S_2 not included in S_1 such* *that* $\gamma(S_2) \subset S_1$ *for all* $\gamma \in \Gamma_1 - \{1\}$ *and* $\gamma(S_1) \subset S_2$ *for all* $\gamma \in \Gamma_2 - \{1\}$. *Then* Γ *is isomorphic to the free product* $\Gamma_1 * \Gamma_2$.

we get that the group F generated by matrices

$$g = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

is a free subgroup of $PSL(2, \mathbb{Z})$. Therefore, the growth grow(F) of F is equal to $\log 3$ (compare [3], p. 209, or [14], p. 70). By the results of [3] we conclude that the topological entropies $h(PSL(2, \mathbb{Z}))$, $h(F)$ of the hyperbolic group $PSL(2, \mathbb{Z})$ and its subgroup F, acting on Apollonian gasket, satisfy the following inequality

$$h(PSL(2, \mathbb{Z})) \geqslant h(F) = \log 3 > 0$$

Finally, suspending the natural action of $PSL(2, \mathbb{Z})$ on the sphere S^2 (regarded as the ideal boundary of the hyperbolic 3-space) over a compact surface Σ_g of genus $g \geq 2$ we get a codimension two foliation $\mathcal{F}$. The total space of the suspension is a fibration with fibre S^2 such that the intersection of a minimal subset $\mathcal{F}_0$ of $\mathcal{F}$ with the transversal is the Apollonian gasket. Therefore, the minimal subset $\mathcal{F}_0$ of $\mathcal{F}$ is a foliated space with positive geometric entropy and transversal which is locally homeomorphic to the Sierpiński gasket.

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