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Endpoint estimates for commutators of a class of Littlewood-Paley operators

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Abstract. In this paper, the weak $L\log L$ estimates for the commutators of a class of Littlewood-Paley operators with real parameter are established by using a technique of the sharp function.

Key words: parameterized Littlewood-Paley operators, commutators, sharp function, Young function, Luxemburg norm, weak $L\log L$ estimates.

1. Introduction

Let b be a locally integrable function on $\mathbb{R}^n$ and let T be a bounded linear (or sublinear) operator on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. Then the commutator $[b,\,T]$ is defined by

$$[b,\,T](f)(x)=b(x)T(f)(x)-T(bf)(x).$$

The commutators connect closely with the problem of the second order linear elliptic equations. A famous result of Coifman, Rochberg and Weiss [CRW] states that if $b\in\mathrm{BMO}(\mathbb{R}^n)$ and T is the Calderón-Zygmund singular integral operator, then the commutator $[b,\,T]$ is bounded operator on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. For the endpoint case, a simple example (see [P]) shows that $[b,\,T]$ is not weak type (1, 1) for $b\in\mathrm{BMO}(\mathbb{R}^n)$. As its replacement, in 1995, Perez gave the following result:

Theorem A ([P]) *Let $m=0,\,1,\,2,\,\ldots$. If $b\in\mathrm{BMO}(\mathbb{R}^n)$ and T is the Calderón-Zygmund singular integral operator, then there exists $C>0$ such that for any $\beta>0$ and smooth function with compact support f,*

$$\left|\{y\in\mathbb{R}^n:\,|T_b^m(f)(y)|>\beta\}\right|\le C_{\|b\|_*^m}\int_{\mathbb{R}^n}\frac{|f(y)|}{\beta}\left(1+\log^+\left(\frac{|f(y)|}{\beta}\right)\right)^m dy,$$

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where $\|b\|_*$ *denotes the BMO norm of* b *and* $T_b^0 = T$, $T_b^m = [b, T_b^{m-1}]$ *denotes* m *order commutator of* T.

On the other hand, it is well known that Littlewood-Paley operators, such as the Littlewood-Paley g-function, the Lusin area integral and Littlewood-Paley g_λ^* function play very important roles in harmonic analysis and PDE (for example, see [St3], [K] and [CWW]). Therefore, it is a very interesting problem to discuss the boundedness of the commutators for the Littlewood-Paley operators. The first result about the commutators of the Marcinkiewicz integral μ_Ω appeared in the paper [TW] by Torchinsky and Wang in 1990. The Marcinkiewicz integral operator μ_Ω of higher dimension was defined first by Stein [St1], which is a generalized Littlewood-Paley g-function. We refer to see [St1], [BCP], [DFP1], [DFP2] and [FSa] for the properties of μ_Ω. Torchinsky and Wang [TW] proved that if $b \in \mathrm{BMO}(\mathbb{R}^n)$, then the commutator $[b, \mu_\Omega]$ is bounded operator on the weighted space $L^p(\mathbb{R}^n, w)$ for $1 < p < \infty$ and $w \in A_p$ (see Definition 3 below). In 2002, Ding, Lu and Yabuta [DLY] gave the weighted L^p-boundedness of the higher order commutator $\mu_{\Omega,b}^m$ for rough Marcinkiewicz integral μ_Ω. Recently, Ding, Lu and Zhang [DLZ] gave the endpoint weighted estimates for the higher order commutator $\mu_{\Omega,b}^m$, which is similar to the conclusion of Theorem A.

Naturally, it is an important and interesting problem to study the endpoint properties for the commutators of the Lusin area integral and Littlewood-Paley g_λ^* function. The purpose of this paper is to provide an endpoint estimates for these commutators of a class of Parameterized Littlewood-Paley operators. Because in the proofs of the main theorems in this paper, we will view these Parameterized Littlewood-Paley operators as Hilbert space valued operators, we therefore give the definitions of some Hilbert spaces.

Definition 1 Suppose that $u(y, t)$ is a measurable function on $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty)$ $(n \geq 2)$, then the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ on $\mathbb{R}_+^{n+1}$ are defined by

$$\mathcal{H}_1 = \left\{ u \colon \|u\|_{\mathcal{H}_1} = \left(\iint_{\mathbb{R}_+^{n+1}} |u(y, t)|^2 \chi_{\{|y|<1\}}(y) \frac{dydt}{t} \right)^{1/2} < \infty \right\},$$

and

$$\mathcal{H}_2=\left\{u\colon \|u\|_{\mathcal{H}_2}=\left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{1}{1+|y|}\right)^{\lambda n}|u(y,t)|^2\frac{dydt}{t}\right)^{1/2}<\infty,\ \lambda>1\right\},$$

respectively.

Take $\phi(x) = \Omega(x)|x|^{-(n-\rho)}\chi_{\{|x|<1\}}$, where $o < \rho < n$ and Ω always satisfies the following conditions in this paper:
(a) $\Omega(\lambda x) = \Omega(x)$ for all $\lambda > 0$;
(b) $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$;
(c) $\Omega \in L^1(S^{n-1})$.
Here S^{n-1} denotes the unit sphere of $\mathbb{R}^n$ equipped with Lebesgue measure $d\sigma(x')$. Let

$$F(f)(x,\, y,\, t) = \int_{\mathbb{R}^n} t^{-n}\phi\left(\frac{x-z}{t} - y\right)f(z)dz. \tag{1.1}$$

Then the Parameterized area integral μ_S^ρ and the Parameterized Littlewood-Paley g_λ^* function $\mu_\lambda^{*,\rho}$ are defined by

$$\begin{aligned}\mu_S^\rho(f)(x) &:= \|F(f)(x,\,\cdot\,,\,\cdot\,)\|_{\mathcal{H}_1}\\ &= \left(\iint_{\Gamma(x)}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2},\end{aligned}$$

where $\Gamma(x) = \{(y,\, t) \in \mathbb{R}^{n+1}_+\colon |x-y| < t\}$, and

$$\begin{aligned}\mu_\lambda^{*,\rho}(f)(x) &:= \|F(f)(x,\,\cdot\,,\,\cdot\,)\|_{\mathcal{H}_2}\\ &= \left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.\\ &\qquad\left.\times\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2},\end{aligned}$$

respectively. In 1990, Torchinsky and Wang [TW] gave the weighted $L^2(\mathbb{R}^n)$ boundedness of μ_S^ρ and $\mu_\lambda^{*,\rho}$ for $\rho = 1$ and $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \le 1)$. For general ρ, in 1999, Sakamoto and Yabuta [SY] gave L^p boundedness for μ_S^ρ and $\mu_\lambda^{*,\rho}$. Recently, we extended Torchinsky and Wang's weighted results with general p and ρ under a more weaker condition, which will be applied to prove the main theorem in this paper. Before stating some results, let us recall the definitions of integral modulus of continuity and the L^q-Dini condition.

Definition 2 Suppose that $\Omega(x') \in L^q(S^{n-1})$, $q \geq 1$. Then the *integral modulus* $\omega_q(\delta)$ *of continuity of order* q of Ω is defined by

$$\omega_q(\delta) = \sup_{\|\gamma\| \leq \delta} \left(\int_{S^{n-1}} |\Omega(\gamma x') - \Omega(x')|^q d\sigma(x') \right)^{1/q},$$

where γ denotes a rotation on S^{n-1} and $\|\gamma\| = \sup_{x' \in S^{n-1}} |\gamma x' - x'|$. The function Ω is said to satisfy the L^q-Dini condition, if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta < \infty.$$

Definition 3 A nonnegative locally integrable function $w(x)$ on $\mathbb{R}^n$ is said to be in A_p $(1 \leq p < \infty)$, if there exists a constant $C > 0$ such that for every cube $Q \subset \mathbb{R}^n$

$$\left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C \qquad \text{for } 1 < p < \infty,$$

and for a.e., $x \in \mathbb{R}^n$ and $Q \ni x$

$$\frac{1}{|Q|} \int_Q w(y) dy \leq C w(x), \quad \text{for } p = 1.$$

Recently, we obtained the following weighted results and weak (1, 1) estimates about μ_S^ρ and $\mu_\lambda^{*,\rho}$:

Theorem B ([DX2]) *Suppose that* $\rho > n/2$, $\lambda > 2$ *and* $\Omega \in L^2(S^{n-1})$ *satisfies*

$$\int_0^1 \frac{\omega_2(\delta)}{\delta} (1 + |\log \delta|)^\sigma d\delta < \infty, \quad \sigma > 1. \tag{1.2}$$

If $1 < p < \infty$ *and* $w \in A_p$, *then both of* μ_S^ρ *and* $\mu_\lambda^{*,\rho}$ *are bounded operators on the weighted space* $L^p(\mathbb{R}^n, w)$.

Theorem C ([DX1]) *Let* $\Omega \in L^2(S^{n-1})$ *satisfies* (1.2). *Then for* $\rho > n/2$ *and* $\lambda > 2$, *both of* μ_S^ρ *and* $\mu_\lambda^{*,\rho}$ *are of weak type* (1, 1).

Remark 1.1 The condition (1.2) is weaker than the $\mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \leq 1)$ condition, see Remark 2 in [DLX] for the details. On the other hand, the L^p $(1 < p < \infty)$ and the weak (1, 1) boundedness of $\mu_{\Omega,S}^\rho$ and $\mu_\lambda^{*,\rho}$ don't hold for $0 < \rho \leq n/2$ and $n > 2$.

Now let us turn to the definition of commutators of μ_S^ρ and $\mu_\lambda^{*,\rho}$. For $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $m = 1, 2, \ldots$, the higher order commutators $\mu_{S,b}^{\rho,m}$ of Parameterized area integral μ_S^ρ are defined by

$$\begin{aligned}\mu_{S,b}^{\rho}(f)(x) &:= [b, \mu_S^\rho](f)(x) = \|b(x)F(f)(x,\cdot,\cdot) - F(bf)(x,\cdot,\cdot)\|_{\mathcal{H}_1}\\ &= \bigg(\iint_{\Gamma(x)} \bigg|\frac{1}{t^\rho}\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}}\\ &\qquad \times \big(b(x)-b(z)\big) f(z)dz\bigg|^2 \frac{dydt}{t^{n+1}}\bigg)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\mu_{S,b}^{\rho,m}(f)(x) &:= [b, \mu_{S,b}^{\rho,m-1}](f)(x)\\ &= \bigg(\iint_{\Gamma(x)} \bigg|\frac{1}{t^\rho}\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}}\\ &\qquad \times \big(b(x)-b(z)\big)^m f(z)dz\bigg|^2 \frac{dydt}{t^{n+1}}\bigg)^{1/2}.\end{aligned}$$

We denote simply $\mu_{S,b}^{\rho,1}(f) = \mu_{S,b}^{\rho}(f)$ and $\mu_{S,b}^{\rho,0}(f) = \mu_S^\rho(f)$, respectively. Similarly, the higher order commutators $\mu_{\lambda,b}^{*,\rho,m}$ of the operator $\mu_\lambda^{*,\rho}$ are defined as follows.

$$\begin{aligned}\mu_{\lambda,b}^{*,\rho}(f)(x) &:= [b, \mu_\lambda^{*,\rho}](f)(x) = \|b(x)F(f)(x,\cdot,\cdot) - F(bf)(x,\cdot,\cdot)\|_{\mathcal{H}_2}\\ &= \bigg(\iint_{\mathbb{R}_+^{n+1}} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\\ &\quad \times \bigg|\frac{1}{t^\rho}\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}}\big(b(x)-b(z)\big) f(z)dz\bigg|^2 \frac{dydt}{t^{n+1}}\bigg)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\mu_{\lambda,b}^{*,\rho,m}(f)(x) &:= [b, \mu_{\lambda,b}^{*,\rho,m-1}](f)(x)\\ &= \bigg(\iint_{\mathbb{R}_+^{n+1}} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\\ &\quad \times \bigg|\frac{1}{t^\rho}\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}}\big(b(x)-b(z)\big)^m f(z)dz\bigg|^2 \frac{dydt}{t^{n+1}}\bigg)^{1/2}.\end{aligned}$$

We denote simply $\mu_{\lambda,b}^{*,\rho,1}(f) = \mu_{\lambda,b}^{*,\rho}(f)$ and $\mu_{\lambda,b}^{*,\rho,0}(f) = \mu_\lambda^{*,\rho}(f)$, respectively.

In this paper, we will show that the higher order commutators $\mu_{S,b}^{\rho,m}$ and $\mu_{\lambda,b}^{*,\rho,m}$ have the same endpoint estimates as the conmmutator T_b^m of the Calderón-Zygmund singular integral operator T shown in Theorem A.

Theorem 1 *Suppose that $\rho > n/2$, $\lambda > 2$ and $\Omega \in L^2(S^{n-1})$ satisfying (1.2) for $\sigma > 2$. If $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $m = 1, 2, \ldots$, then there exists a constant $C > 0$, such that for any $\beta > 0$ and each smooth function f with compact support,*

(i) $$\left|\{x\in\mathbb{R}^n \colon \mu_{S,b}^{\rho,m}(f)(x)>\beta\}\right|\le C_{\|b\|_*^m}\int_{\mathbb{R}^n}\frac{|f(x)|}{\beta}\left(1+\log^+\left(\frac{|f(x)|}{\beta}\right)\right)^m dx;$$

(ii) $$\left|\{x\in\mathbb{R}^n \colon \mu_{\lambda,b}^{*,\rho,m}(f)(x)>\beta\}\right|\le C_{\|b\|_*^m}\int_{\mathbb{R}^n}\frac{|f(x)|}{\beta}\left(1+\log^+\left(\frac{|f(x)|}{\beta}\right)\right)^m dx.$$

Using the method of proving Theorem 1 and combining with some idea in [P], we may get the following weighted endpoint estimates for the commutators $\mu_{S,b}^{\rho,m}$ and $\mu_{\lambda,b}^{*,\rho,m}$. Here we omit the details of the proof of the following Theorem 2.

Theorem 2 *If $w \in A_1$, then under the same conditions as one in* Theorem 1*, the following inequalities hold:*

(i) $$w\big(\{x \in \mathbb{R}^n \colon \mu_{S,b}^{\rho,m}(f)(x) > \beta\}\big) \le C_{\|b\|_*^m}\int_{\mathbb{R}^n}\frac{|f(x)|}{\beta}\left(1+\log^+\left(\frac{|f(x)|}{\beta}\right)\right)^m w(x)dx;$$

(ii) $$w\big(\{x \in \mathbb{R}^n \colon \mu_{\lambda,b}^{*,\rho,m}(f)(x) > \beta\}\big) \le C_{\|b\|_*^m}\int_{\mathbb{R}^n}\frac{|f(x)|}{\beta}\left(1+\log^+\left(\frac{|f(x)|}{\beta}\right)\right)^m w(x)dx,$$

where $\beta > 0$ and f is smooth function with compact support. Moreover, the constant $C > 0$ is independent of β and f.

In the proofs of Theorems 1 and 2, we need the weighted L^p boundedness $(1 < p < \infty)$ of the commutators $\mu_{S,b}^{\rho,m}(f)(x)$ and $\mu_{\lambda,b}^{*,\rho,m}(f)(x)$. Of course, these results are also of interest independently.

Theorem 3 *Suppose that $\rho > n/2$, $\lambda > 2$ and $\Omega \in L^2(S^{n-1})$ satisfying (1.2) for $\sigma > 2$. If $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $m = 0, 1, 2, \ldots$, then for $1 < p < \infty$ and $w \in A_p$ there exists a constant $C > 0$ such that for any $f \in L^p(\mathbb{R}^n, w)$*

(i) $\|\mu_{S,b}^{\rho,m}(f)\|_{L^p(w)} \le C\|f\|_{L^p(w)}$;

(ii) $\|\mu_{\lambda,b}^{*,\rho,m}(f)\|_{L^p(w)} \le C\|f\|_{L^p(w)}$.

Remark 1.2 Note that the commutators discussed in [TW] and [DLZ] are only formed by the Marcinkiewicz integral μ_Ω, so the results in this paper can be regarded as an extension of the conclusions in [TW] and [DLZ]. On the other hand, in [TW] and [DLZ] the kernel function Ω needs to satisfy $\mathrm{Lip}_\alpha(S^{n-1})$ condition for $0 < \alpha \le 1$. However, in the conclusions of this paper the conditions (1.2) assumed on Ω are weaker than the $\mathrm{Lip}_\alpha(S^{n-1})$ condition (see Remark 1.1). Therefore, our results in this paper are also an improvement of the conclusions in [TW] and [DLZ].

Remark 1.3 It is easy to check that $\mu_{S,b}^{\rho,m}(f)(x) \le 2^{\lambda n}\mu_{\lambda,b}^{*,\rho,m}(f)(x)$ for $m = 0, 1, 2, \ldots$, (see the proof of (19) in [St2, p. 89], for example), we therefore give only the proofs of Theorem 1 and Theorem 3 for $\mu_{\lambda,b}^{*,\rho,m}$, respectively.

Remark 1.4 The L^2 condition assumed on Ω in Theorem 1–3 comes from the Minkowski inequality we used, and can't be replaced by any $L^q(1 < q < 2)$ if one use this inequality.

2. Proof of Theorem 3

By Remark 1.3, we only prove that the following inequality holds under the conditions of Theorem 3,

$$\int_{\mathbb{R}^n} [\mu_{\lambda,b}^{*,\rho,m}(f)(x)]^p w(x)dx \le C \int_{\mathbb{R}^n} |f(x)|^p w(x)dx. \tag{2.1}$$

The idea of proving (2.1) is taken from [DLY]. The proof will be finished by induction on m. For $m = 0$, it is just the conclusion of Theorem B. For $m \ge 1$, we assume (2.1) holds for $m - 1$, and we need to prove that (2.1) holds also for m. Replace the operator $\mu_{\Omega,b}^{m-1}$ and the weight class $A_{p/q'}$ in [DLY, pp. 65–66] by $\mu_{\lambda,b}^{*,\rho,m-1}$ and A_p, respectively. Follow the same steps of proving Theorem 1 in [DLY], by Theorem B and the Stein-Weiss interpolation theorem with change of measure we may prove that for any $\theta \in [0, 2\pi]$ and any $\phi \in L^p(we^{pb\cos\theta})$,

$$\|\mu_{\lambda,b}^{*,\rho,m-1}(\phi)\|_{p,we^{pb(x)\cos\theta}} \le C\|\phi\|_{p,we^{pb(x)\cos\theta}}, \tag{2.2}$$

where C depends on n, p, b, w, but not on θ and ϕ. (See [DLY, pp. 65–66] for the detail). Now denote $F(y) = e^{y(b(x)-b(z))}$, $y \in \mathbb{C}$. Then by the analyticity of $F(y)$ on $\mathbb{C}$ and the Cauchy integration formula, we have

$$b(x)-b(z)=\frac{1}{2\pi i}\int_{|y|=1}\frac{F(y)}{y^2}dy$$
$$=\frac{1}{2\pi}\int_0^{2\pi}e^{e^{i\theta}(b(x)-b(z))}e^{-i\theta}d\theta. \tag{2.3}$$

By (2.3) and the Minkowski inequality we have

$$\begin{aligned}
&\mu_{\lambda,b}^{*,\rho,m}(f)(x)\\
&=\bigg(\iint_{\mathbb{R}_+^{n+1}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\\
&\qquad\times\bigg|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\big(b(x)-b(z)\big)^m f(z)dz\bigg|^2\frac{dydt}{t^{n+1}}\bigg)^{1/2}\\
&=\bigg(\iint_{\mathbb{R}_+^{n+1}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\bigg|\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\\
&\qquad\times\big(b(x)-b(z)\big)^{m-1}e^{e^{i\theta}\big(b(x)-b(z)\big)}e^{-i\theta}f(z)dzd\theta\bigg|^2\frac{dydt}{t^{n+1}}\bigg)^{1/2}\\
&\le\frac{1}{2\pi}\int_0^{2\pi}\bigg(\iint_{\mathbb{R}_+^{n+1}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\bigg|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\\
&\qquad\times\big(b(x)-b(z)\big)^{m-1}e^{-e^{i\theta}b(z)}f(z)dz\bigg|^2\frac{dydt}{t^{n+1}}\bigg)^{1/2}e^{b(x)\cos\theta}d\theta\\
&=\frac{1}{2\pi}\int_0^{2\pi}\mu_{\lambda,b}^{*,\rho,m-1}(f^\theta)(x)e^{b(x)\cos\theta}d\theta,
\end{aligned}$$

where $f^\theta(z)=f(z)e^{-e^{i\theta}b(z)}$ for $\theta\in[0,\,2\pi]$. Then by the Minkowski inequality and inequality (2.2), we obtain

$$\begin{aligned}
&\|\mu_{\lambda,b}^{*,\rho,m}(f)\|_{L^p(w)}\\
&\le\bigg(\int_{\mathbb{R}^n}\bigg|\frac{1}{2\pi}\int_0^{2\pi}\mu_{\lambda,b^{m-1}}^{*,\rho}(f^\theta)(x)e^{b(x)\cos\theta}d\theta\bigg|^p w(x)dx\bigg)^{1/p}\\
&\le C\frac{1}{2\pi}\int_0^{2\pi}\bigg(\int_{\mathbb{R}^n}[\mu_{\lambda,b^{m-1}}^{*,\rho}(f^\theta)(x)]^p w(x)e^{b(x)\cos\theta}dx\bigg)^{1/p}d\theta\\
&\le C\frac{1}{2\pi}\int_0^{2\pi}\bigg(\int_{\mathbb{R}^n}|f^\theta(x)|^p w(x)e^{b(x)\cos\theta}dx\bigg)^{1/p}d\theta=C\|f\|_{L^p(w)}.
\end{aligned}$$

The last step we used the fact $f^\theta \in L^p(we^{pb\cos\theta})$ and $\|f^\theta\|_{L^p(we^{pb\cos\theta})} = \|f\|_{L^p(w)}$. Thus we complete the proof of Theorem 3.

3. Some preliminary lemmas

Let us begin by recalling the Kolmogorov Lemma (see [GR, p. 485]).

Lemma 3.1 *Let $0 < r < \ell < \infty$. For each function f, define*

$$\|f\|_{WL^\ell} = \sup_{t>0} t|\{x\colon |f(x)| > t\}|^{1/\ell}$$

and

$$N_{\ell,r}(f) = \sup_E \frac{\|f\chi_E\|_r}{\|\chi_E\|_s}, \quad \frac{1}{s} = \frac{1}{r} - \frac{1}{\ell},$$

where the supremum is taken over all the measurable sets E with $0 < |E| < \infty$. Then

$$\|f\|_{WL^\ell} \le N_{\ell,r}(f) \le \Big(\frac{\ell}{\ell - r}\Big)^{1/r} \|f\|_{WL^\ell}.$$

To state the following lemmas, let us give some definitions and notations. For $\delta > 0$, we define $M_\delta(f) = \big[M(|f|^\delta)\big]^{1/\delta}$, and $M^\sharp_\delta(f) = \big[M^\sharp(|f|^\delta)\big]^{1/\delta}$, where M and $M^\sharp$ denote the Hardy-Littlewood maximal operator and the Fefferman-Stein's sharp function, respectively, the latter is defined by

$$M^\sharp(f)(x) = \sup_{Q\ni x} \inf_c \frac{1}{|Q|} \int_Q \big|f(y) - c\big| dy \approx \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where $f_Q = (1/|Q|)\int_Q f(y)dy$. The corresponding *dyadic maximal operators* are denoted by M^Δ_δ and $M^{\sharp,\Delta}_\delta$, respectively. For simpleness, we will denote M_1, $M^\sharp_1$, M^Δ_1, $M^{\sharp,\Delta}_1$ by M, $M^\sharp$, M^Δ, $M^{\sharp,\Delta}$, respectively. A function $A\colon [0, \infty) \to [0, \infty)$ is said to be a *Young function* if it is continuous, convex and increasing satisfying $A(0) = 0$ and $A(t) \to \infty$ as $t \to \infty$. The *complementary Young function* $\overline{A}(t)$ of the Young function $A(t)$ is defined by

$$\overline{A}(s) = \sup_{0\le t<\infty} [st - A(t)], \quad 0 \le s < \infty.$$

As an example, $\Phi_m(t) = t(1 + \log^+ t)^m$ $(1 \le m < \infty)$ is a Young function with it's complementary $\overline{\Phi}_m(t) \approx e^{t^{1/m}}$ (see [P]). If A is a Young function, then the *Luxemburg norm* of f on a cube $Q \subset \mathbb{R}^n$ is defined by

$$\|f\|_{A,Q} = \inf\Big\{\lambda > 0\colon \frac{1}{|Q|}\int_Q A\Big(\frac{|f(y)|}{\lambda}\Big)dy \le 1\Big\}.$$

If $A(t) = \Phi_1(t)$, we denote $\|f\|_{L\log L,Q} = \|f\|_{\Phi_1,Q}$, $\|f\|_{\exp L,Q} = \|f\|_{\overline{\Phi}_1,Q}$ and $M_{L\log L}f(x) = \sup_{Q\ni x}\|f\|_{L\log L,Q}$. For the Luxemburg norm, there is the following generalized Hölder's inequality:

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \le \|f\|_{A,Q}\|g\|_{\bar{A},Q}. \tag{3.1}$$

Now we state some known-results which will be used in the proofs of theorems in this paper.

Lemma 3.2 ([FSt]) *For the dyadic maximal operators $M^{\triangle}$ and $M^{\triangle,\sharp}$, the following results hold*:

(i) $|\{y \in \mathbb{R}^n\colon M^{\triangle}(f)(y) > \lambda,\ M^{\triangle,\sharp}(f)(y) \le \lambda\varepsilon\}|$
$$\le C\varepsilon|\{y \in \mathbb{R}^n\colon M^{\triangle}(f)(y) > \lambda/2\}|,$$
where $\lambda > 0$, $\varepsilon > 0$ and C is independent of λ, ε and f.

(ii) *Let $\varphi\colon (0,\infty) \to (0,\infty)$ be a doubling function. then there exists a positive constant C, such that*

$$\sup_{\lambda>0}\varphi(\lambda)|\{y \in \mathbb{R}^n\colon M_\delta^{\triangle}(f)(y) > \lambda\}| \le C\sup_{\lambda>0}\varphi(\lambda)|\{y \in \mathbb{R}^n\colon M_\delta^{\triangle,\sharp}(f)(y) > \lambda\}|$$

for all function f such that the left side is finite.

If denote $M^2 = M \circ M$, then (see [P, p.170])

$$M(f)(x) \le M_{L\log L}(f)(x) \quad\text{and}\quad M^2(f) \sim M_{L\log L}(f). \tag{3.2}$$

Moreover, Pérez [P] gave the following weak type estimate for M^2:

Lemma 3.3 ([P]) *There exists $C > 0$ such that for any $\beta > 0$ and $f \in L\log^+ L(\mathbb{R}^n)$*

$$|\{x \in \mathbb{R}^n\colon M^2(f)(x) > \beta\}| \le C\int_{\mathbb{R}^n}\frac{|f(x)|}{\beta}\Big(1 + \log^+\Big(\frac{|f(x)|}{\beta}\Big)\Big)dx.$$

Lemma 3.4 ([DL]) *Suppose that $0 < \rho < n$, Ω is homogeneous of degree zero and satisfies the L^2-Dini condition. If there exists a constant $0 < \theta < 1/2$ such that $|x| < \theta R$, then we have the following inequality*

$$\left(\int_{R<|y|<2R}\left|\frac{\Omega(y-x)}{|y-x|^{n-\rho}} - \frac{\Omega(y)}{|y|^{n-\rho}}\right|^2 dy\right)^{1/2} \le CR^{n/2-(n-\rho)}\left\{\frac{|x|}{R} + \int_{|x|/2R}^{|x|/R}\frac{\omega_2(\delta)}{\delta}d\delta\right\},$$

where the constant $C > 0$ is independent of R and x.

Using Lemma 3.4, one can obtain the following lemma, which shows that the sharp function of the commutator $\mu_S^{\rho,b}(f)$ can be controlled by the maximal function of $\mu_S^{\rho}(f)$ and the Hardy-Littlewood maximal function $M^2(f)$.

Lemma 3.5 *Suppose that $n/2 < \rho < n$, $\lambda > 2$ and $\Omega \in L^2(S^{n-1})$ satisfying (1.2) with $\sigma > 2$. If $b \in \mathrm{BMO}$ and $0 < \delta < \ell < 1$, then for any smooth function f with compact support set, there exists $0 < C = C_\delta$ such that*

$$M_\delta^\sharp[\mu_{S,b}^\rho(f)](x) \le C\|b\|_*\big[M_\ell[\mu_S^\rho(f)](x) + M^2(f)(x)\big] \tag{3.3}$$

and

$$M_\delta^\sharp[\mu_{\lambda,b}^{*,\rho}(f)](x) \le C\|b\|_*\big[M_\ell[\mu_\lambda^{*,\rho}(f)](x) + M^2(f)(x)\big]. \tag{3.4}$$

Proof. For any $x \in \mathbb{R}^n$, Let $B = B(x_0, r_0)$ be an arbitrary ball containing x with center at x_0 and radius r_0. Since $0 < \delta < 1$, then $\big||a|^\delta - |d|^\delta\big| \le |a-d|^\delta$ for any $a, d \in \mathbb{R}$. Denote $B^* = B(x_0, 8r_0)$ and decompose $f = f_1 + f_2$ with $f_1 = f\chi_{B^*}$. Then we have

$$\begin{aligned} b(x)F(f)(x, y, t) &- F(bf)(x, y, t) \\ &= [b(x) - b_{B^*}]F(f)(x, y, t) - F[(b - b_{B^*})f_1](x, y, t) \\ &\quad - F[(b - b_{B^*})f_2](x, y, t). \end{aligned}$$

where $F(f)(x, y, t)$ is defined in (1.1).

First, we give the proof of (3.3). Take

$$c_B = \frac{1}{|B|}\int_B \mu_S^\rho[(b - b_{B^*})f_2](u)du,$$

then it is easy to check that $c_B < \infty$ by the conclusion of Theorem 3 with

$w \equiv 1$. Hence

$$\begin{aligned}
&\Big(\frac{1}{|B|}\int_B \big|\mu^\rho_{S,b}(f)(u)-c_B\big|^\delta du\Big)^{1/\delta}\\
&\le C\Big(\frac{1}{|B|}\int_B \big|(b(u)-b_{B^*})\mu^\rho_S(f)(u)\big|^\delta du\Big)^{1/\delta}\\
&\quad + C\Big(\frac{1}{|B|}\int_B \big|\mu^\rho_S\big((b-b_{B^*})f_1\big)(u)\big|^\delta du\Big)^{1/\delta}\\
&\quad + C\Big(\frac{1}{|B|}\int_B \big|\mu^\rho_S\big((b-b_{B^*})f_2\big)(u)-c_B\big|^\delta du\Big)^{1/\delta} := \mathrm{I}+\mathrm{II}+\mathrm{III}
\end{aligned}$$

To estimate I, we choose $1<\gamma<\ell/\delta$ and by Hölder's inequality,

$$\begin{aligned}
\mathrm{I} &\le C\Big(\frac{1}{|B|}\int_B |b(u)-b_{B^*}|^{\delta\gamma'}du\Big)^{1/\delta\gamma'}\Big(\frac{1}{|B|}\int_B |\mu^\rho_S(f))(u)|^{\delta\gamma}du\Big)^{1/\delta\gamma}\\
&\le C\|b\|_* M_{\delta\gamma}(\mu^\rho_S(f))(x) \le C\|b\|_* M_\ell(\mu^\rho_S(f))(x). \qquad (3.5)
\end{aligned}$$

Note that μ^ρ_S is of weak type (1,1) (by Theorem C) and $0<\delta<1$. Applying Kolmogorov's inequality (Lemma 3.1), weak (1,1) boundedness of μ^ρ_S and the generalized Hölder's inequality (3.1), we get

$$\begin{aligned}
\mathrm{II} &\le \frac{C}{|B|}\int_B |(b(u)-b_{B^*})f_1(u)|du \le \frac{C}{|B^*|}\int_{B^*}|b(u)-b_{B^*}||f_1(u)|du\\
&\le C\|b(x)-b_{B^*}\|_{\exp L,B^*}\|f\|_{L\log L,B^*} \le C\|b\|_* M_{L\log L}(f)(x).
\end{aligned}$$

In the last step above, we used John-Nirenberg inequality (see [P, p. 169]) and the definition of $M_{L\log L}$. Note that $M^2f(x)\approx M_{L\log L}f(x)$ (by (3.2)), we have

$$\mathrm{II} \le C\|b\|_* M^2 f(x). \qquad (3.6)$$

Finally, let us give the estimate of III. By Theorem B with $w\equiv 1$, we know that $\mu^{*,\rho}_\lambda$ is bounded on L^p under the conditions of Lemma 3.5. Thus

$$\begin{aligned}
\int_B |\mu^\rho_S(f_2)(u)|^p du &\le 2^{\lambda n}\int_B |\mu^{*,\rho}_\lambda(f_2)(u)|^p du\\
&\le C\int_{\mathbb{R}^n}|\mu^{*,\rho}_\lambda(f_2)(u)|^p du\\
&\le C\int_{\mathbb{R}^n}|f_2(u)|^p du \le C\int_{\mathbb{R}^n}|f(u)|^p du.
\end{aligned}$$

Hence

$$\begin{aligned}\int_B |\mu_S^\rho(f_2)(u)|du &\leq 2^{\lambda n}\int_B |\mu_\lambda^{*,\rho}(f_2)(u)|du\\ &\leq C|B|^{1/p'}\bigg(\int_B |\mu_\lambda^{*,\rho}(f_2)(u)|^p du\bigg)^{1/p}\\ &\leq C|B|^{1/p'}\bigg(\int_{R^n} |f(u)|^p du\bigg)^{1/p}.\end{aligned}$$

This shows that both of $\mu_S^\rho(f_2)(x)$ and $\mu_\lambda^{*,\rho}(f_2)(x)$ are bounded a.e. on B. Thus, by Jensen's inequality we have

$$\begin{aligned}\mathrm{III} &\leq \frac{C}{|B|}\int_B \big|\mu_S^\rho((b-b_{B^*})f_2)(u) - (\mu_S^\rho((b-b_{B^*})f_2))_B\big|du\\ &\leq \frac{C}{|B|^2}\int_B\int_B \big|\mu_S^\rho((b-b_{B^*})f_2)(u) - (\mu_S^\rho((b-b_{B^*})f_2))(v)\big|dvdu\\ &= \frac{C}{|B|^2}\int_{B\backslash E}\int_{B\backslash E} \big|\mu_S^\rho((b-b_{B^*})f_2)(u)\\ &\qquad\qquad - (\mu_S^\rho((b-b_{B^*})f_2))(v)\big|dvdu,\end{aligned}$$

where $E \subset B$ with $|E| = 0$ and $\mu_S^\rho(f_2)(u) < \infty$ for any $u \in B\backslash E$. Therefore, we have the following fact for any u, $v \in B \setminus E$, which will be proved in Lemma 3.6:

$$\begin{aligned}&\big|\mu_S^\rho((b-b_{B^*})f_2)(u) - \mu_S^\rho((b-b_{B^*})f_2)(v)\big|\\ &\leq Cr^\varepsilon\int_{(8B^*)^c}\frac{|f(z)||b(z)-b_{B^*}|}{|z-x|^{n+\varepsilon}}dz\\ &\quad + Cr^{\rho-n/2}\int_{(8B^*)^c}\frac{|f(z)||b(z)-b_{B^*}|}{|z-x|^{\rho+n/2}}dz\\ &\quad + C\int_{(8B^*)^c}\frac{|f(z)||b(z)-b_{B^*}|}{|z-x|^n\big(\log(|z-x|/r)\big)^{2+\varepsilon}}dz\\ &:= L_1 + L_2 + L_3,\end{aligned}\tag{3.7}$$

where $\varepsilon < \min\big\{1/2,\ (\lambda-2)n/2,\ \rho - n/2,\ \sigma - 1\big\}$. Next we show

$$L_i \leq C\|b\|_* M^2(f)(x) \quad \text{for} \quad i = 1,\ 2,\ 3.$$

In fact, if we denote $b_{B_j} = \{z\colon |z-x_0| < 2^j r_0\}$, then $|b_{B^{j+1}} - b_{B^*}| \leq 2j\|b\|_*$.

By (3.2) we get

$$\begin{aligned}
L_1 &\le Cr^{\varepsilon}\sum_{j=3}^{\infty}\int_{2^j r\le|z-x|<2^{j+1}r}\frac{|f(z)||b(z)-b_{B^*}|}{|z-x|^{n+\varepsilon}}dz\\
&\le C\sum_{j=3}^{\infty}\frac{1}{2^{j\varepsilon}}\bigg(\int_{B_{j+1}}\frac{|f(z)||b(z)-b_{B_{j+1}}|}{(2^j r)^n}dz+|b_{B_{j+1}}-b_{B^*}|M(f)(x)\bigg)\\
&\le C\sum_{j=3}^{\infty}\frac{1}{2^{j\varepsilon}}\big(\|b-b_{B_{j+1}}\|_{\exp L,B_{j+1}}\|f\|_{L\log L,B_{j+1}}+2j\|b\|_*M(f)(x)\big)\\
&\le C\sum_{j=3}^{\infty}\frac{1}{2^{j\varepsilon}}\big(\|b\|_*M_{L\log L}(f)(x)+2j\|b\|_*M(f)(x)\big)\\
&\le C\|b\|_*\big(M_{L\log L}(f)(x)+M(f)(x)\big)\le C\|b\|_*M^2(f)(x).
\end{aligned}$$

Take $\varepsilon=\rho-n/2$ in the above inequality, we get $L_2\le C\|b\|_*M^2(f)(x)$. Using the same way in estimating L_1 to deal with L_3, we obtain

$$\begin{aligned}
L_3 &\le C\sum_{j=3}^{\infty}\int_{2^j r\le|z-x|<2^{j+1}r}\frac{|f(z)|b(z)-b_{B^*}|}{|z-x|^n\big(\log(|z-x|/r)\big)^{2+\varepsilon}}dz\\
&\le C\sum_{j=3}^{\infty}\frac{1}{j^{2+\varepsilon}}\int_{2^j r\le|z-x|<2^{j+1}r}\frac{|f(z)|b(z)-b_{B^*}|}{|z-x|^n}dz\\
&\le C\sum_{j=3}^{\infty}\frac{1}{j^{2+\varepsilon}}\bigg(\|b\|_*M_{L\log L}(f)(x)+2j\|b\|_*M(f)(x)\bigg)\\
&\le C\|b\|_*M^2(f)(x).
\end{aligned}$$

Thus we get III $\le C\|b\|_*M^2(f)(x)$ from the estimates of L_1, L_2 and L_3, and (3.3) follows. Let us now consider (3.4). For any u, $v\in B\backslash E$, and $\rho>n/2$, $\lambda>2$, the following inequality holds: (Once again, this fact will be proved in Lemma 3.6.)

$$\begin{aligned}
&\big|\mu^{*,\rho}_{\lambda,b}((b-b_{B^*})f_2)(u)-(\mu^{*,\rho}_{\lambda,b}((b-b_{B^*})f_2))(v)\big|\\
&\le Cr^{\varepsilon}\int_{(8B^*)^c}\frac{|f(z)||b(z)-b_{B^*}|}{|z-x|^{n+\varepsilon}}dz\\
&\quad+Cr^{\rho-n/2}\int_{(8B^*)^c}\frac{|f(z)||b(z)-b_{B^*}|}{|z-x|^{\rho+n/2}}dz
\end{aligned}$$

$$+ C\int_{(8B^*)^c}\frac{|f(z)||b(z)-b_{B^*}|}{|z-x|^n\big(\log(|z-x|/r)\big)^{2+\varepsilon}}dz. \tag{3.8}$$

Based on the inequality (3.8), and repeating the same steps as done in estimating μ_S^ρ, we can obtain (3.4). Hence we complete the proof of Lemma 3.5. □

Below we give the proofs of (3.7) and (3.8).

Lemma 3.6 *Let B and E are the same as in the proof of* Lemma 3.5*, $u,\, v\in B\backslash E$, $\rho>n/2$ and $\lambda>2$, then the inequalities* (3.7) *and* (3.8) *hold.*

Proof. The proof is similar to the proof in [DX2, Theorem 1], here we only give the main steps and show the difference from there. First we consider (3.7). Note that

$$\begin{aligned}
&\big|\mu_S^\rho\big((b-b_{B^*})f_2\big)(u)-\mu_S^\rho\big((b-b_{B^*})f_2\big)(v)\big|\\
&=\big|\|F((b-b_{B^*})f_2)(u,\cdot,\cdot)\|_{\mathcal{H}_1}-\|F((b-b_{B^*})f_2)(v,\cdot,\cdot)\|_{\mathcal{H}_1}\big|\\
&\le \|F((b-b_{B^*})f_2)(u,\cdot,\cdot)-F((b-b_{B^*})f_2)(v,\cdot,\cdot)\|_{\mathcal{H}_1}\\
&=\bigg(\int_0^\infty\int_{|y|<1}\bigg|\int t^{-n}\Big(\phi\Big(\frac{u-z}{t}-y\Big)-\phi\Big(\frac{v-z}{t}-y\Big)\Big)\\
&\qquad\qquad\times(b(z)-b_{B^*})f_2(z)dz\bigg|^2\frac{dydt}{t}\bigg)^{1/2}\\
&\le\bigg(\int_0^\infty\int_{|y|<1}\bigg|\int_{\substack{|(u-z)/t-y|<1\\|(v-z)/t-y|\ge1}}t^{-n}\phi\Big(\frac{u-z}{t}-y\Big)\\
&\qquad\qquad\times(b(z)-b_{B^*})f_2(z)dz\bigg|^2\frac{dydt}{t}\bigg)^{1/2}\\
&\quad+\bigg(\int_0^\infty\int_{|y|<1}\bigg|\int_{\substack{|(u-z)/t-y|\ge1\\|(v-z)/t-y|<1}}t^{-n}\phi\Big(\frac{v-z}{t}-y\Big)\\
&\qquad\qquad\times(b(z)-b_{B^*})f_2(z)dz\bigg|^2\frac{dydt}{t}\bigg)^{1/2}\\
&\quad+\bigg(\int_0^\infty\int_{|y|<1}\bigg|\int_{\substack{|(u-z)/t-y|<1\\|(v-z)/t-y|<1}}t^{-n}\Big(\phi\Big(\frac{u-z}{t}-y\Big)-\phi\Big(\frac{v-z}{t}-y\Big)\Big)\\
&\qquad\qquad\times(b(z)-b_{B^*})f_2(z)dz\bigg|^2\frac{dydt}{t}\bigg)^{1/2}.
\end{aligned}\tag{3.9}$$

Let $\widetilde{\Omega}(u,\, v,\, y,\, z)=\Omega(y-z)/|y-z|^{n-\rho}-\Omega(v-u+y-z)/|v-u+y-z|^{n-\rho}$, from (3.9), we have

$$\begin{aligned}
&\left|\mu_S^\rho((b-b_{B^*})f_2)(u)-\mu_S^\rho((b-b_{B^*})f_2)(v)\right| \\
&\le \bigg(\int_0^\infty\int_{|u-y|<t}\bigg|\int_{\substack{|y-z|<t\\ v-u+y-z|\ge t}}\frac{\Omega(y-z)}{|y-z|^{n-\rho}} \\
&\qquad\qquad \times(b(z)-b_{B^*})f_2(z)dz\bigg|^2\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2} \\
&\quad+\bigg(\int_0^\infty\int_{|u-y|<t}\bigg|\int_{\substack{|y-z|\ge t\\ |v-u+y-z|<t}}\frac{\Omega(v-u+y-z)}{|v-u+y-z|^{n-\rho}} \\
&\qquad\qquad \times(b(z)-b_{B^*})f_2(z)dz\bigg|^2\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2} \\
&\quad+\bigg(\int_0^\infty\int_{|u-y|<t}\bigg|\int_{\substack{|y-z|<t\\ |v-u+y-z|<t}}\widetilde{\Omega}(u,\,v,\,y,\,z) \\
&\qquad\qquad \times(b(z)-b_{B^*})f_2(z)dz\bigg|^2\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2} \\
&:=I_1+I_2+I_3.
\end{aligned}\tag{3.10}$$

By the estimates in the proof of Theorem 1 in [DX2], if we replace $f_2(z)$ with $(b(z)-b_{B^*})f_2(z)$ then for $i=1,\,2$, the following inequality holds,

$$\begin{aligned}
I_i &\le Cr^\varepsilon\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^{n+\varepsilon}}dz \\
&\quad + Cr^{\rho-n/2}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^{\rho+n/2}}dz.
\end{aligned}\tag{3.11}$$

To prove (3.7), it remains to estimate I_3. Apply the Minkowski inequality to I_3 and divide the region by $|y-z|\ge 8r$ and $|y-z|<8r$, we get

$$\begin{aligned}
I_3 &\le C\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)| \\
&\qquad \times\bigg(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t\\ |y-u|<t,\ |v-u+y-z|<t\\ |y-z|<8r}}\left|\widetilde{\Omega}(u,\,v,\,y,\,z)\right|^2\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2}dz \\
&\quad + C\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|
\end{aligned}$$

$$\times\Big(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t\\ |y-u|<t,\ |v-u+y-z|<t\\ |y-z|\ge 8r}} \big|\widetilde{\Omega}(u,\,v,\,y,\,z)\big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz$$
$$:= I_{3,1}+I_{3,2}. \tag{3.12}$$

It is easy to see that when $z\in(8B^*)^c$ and $|y-z|<8r$, $|v-u+y-z|\le|v-u|+8r\le 9r$ and $|y-u|\sim|z-u|$. Thus

$$\begin{aligned}I_{3,1}&\le C\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|\\&\quad\times\Big(\int_{\substack{y\in(2B^*)^c,\ |y-z|<8r\\ |v-u+y-z|<9r}}\Big(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}+\frac{|\Omega(v-u+y-z)|^2}{|v-u+y-z|^{2n-2\rho}}\Big)\\&\qquad\times\int_{|y-u|}^{\infty}\frac{dt}{t^{n+2\rho+1}}dy\Big)^{1/2}dz\\&\le C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^{n/2+\rho}}\\&\quad\times\Big(\int_{\substack{y\in(2B^*)^c,\ |y-z|<8r\\ |v-u+y-z|<9r}}\Big(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}+\frac{|\Omega(v-u+y-z)|^2}{|v-u+y-z|^{2n-2\rho}}\Big)dy\Big)^{1/2}dz\\&\le Cr^{\rho-n/2}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^{n/2+\rho}}dz.\end{aligned}\tag{3.13}$$

Let us turn to $I_{3,2}$. Note that $|z-u|<|u-y|+|y-z|<2t$, so $t>|z-u|/2$. Since $|y-z|/r\ge 8$ and $2\rho-n>0$, we get

$$\int_{|y-z|}^{\infty}\frac{\big(\log(t/r)\big)^{4+2\varepsilon}}{t^{2\rho-n+1}}dt\le C\frac{\big[\log(|y-z|/r)\big]^{4+2\varepsilon}}{|y-z|^{2\rho-n}}. \tag{3.14}$$

By (3.14) and using Lemma 3.4, we have

$$\begin{aligned}I_{3,2}&\le C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^n\big(\log(|z-u|/r)\big)^{2+\varepsilon}}\\&\quad\times\Big(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t\\ |y-u|<t,\ |v-u+y-z|<t\\ |y-z|\ge 8r,\ t>|z-u|/2}}\big|\widetilde{\Omega}(u,\,v,\,y,\,z)\big|^2\frac{\big(\log(t/r)\big)^{4+2\varepsilon}dt}{t^{2\rho-n+1}}dy\Big)^{1/2}dz\\&\le C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^n\big(\log(|z-u|/r)\big)^{2+\varepsilon}}\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_{|y-z|\geq 8r} \big|\widetilde{\Omega}(u,\,v,\,y,\,z)\big|^2 \frac{\big(\log(|y-z|/r)\big)^{4+2\varepsilon}}{|y-z|^{2\rho-n}} dy \right)^{1/2} dz \\
&\leq C \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^n \big(\log(|z-u|/r)\big)^{2+\varepsilon}} \\
&\times \left(\sum_{j=3}^{\infty} \int_{2^j r \leq |y-z| < 2^{j+1} r} \big|\widetilde{\Omega}(u,\,v,\,y,\,z)\big|^2 \frac{\big(\log(|y-z|/r)\big)^{4+2\varepsilon}}{|y-z|^{2\rho-n}} dy \right)^{1/2} dz \\
&\leq C \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^n \big(\log(|z-u|/r)\big)^{2+\varepsilon}} \sum_{j=3}^{\infty} \frac{\big(\log(2^{j+1}r/r)\big)^{2+\varepsilon}}{(2^j r)^{\rho-n/2}} \\
&\times \left(\int_{2^j r \leq |y-z| < 2^{j+1} r} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\
&\left. \left. - \frac{\Omega(v-u+y-z)}{|v-u+y-z|^{n-\rho}} \right|^2 dy \right)^{1/2} dz \\
&\leq C \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^n \big(\log(|z-u|/r)\big)^{2+\varepsilon}} \sum_{j=3}^{\infty} \frac{(j+1)^{2+\varepsilon}}{(2^j r)^{\rho-n/2}} \\
&\times (2^j r)^{n/2-(n-\rho)} \left\{ \frac{|v-u|}{2^j r} + \int_{|v-u|/(2^{j+1}r)}^{|v-u|/(2^j r)} \frac{\omega_2(\delta)}{\delta} d\delta \right\} dz \\
&\leq C \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^n \big(\log(|z-u|/r)\big)^{2+\varepsilon}} \sum_{j=3}^{\infty} (j+1)^{2+\varepsilon} \\
&\times \left\{ \frac{1}{2^j} + \frac{1}{(1+j\log 2)^{\sigma}} \int_{|v-u|/(2^{j+1}r)}^{|v-u|/(2^j r)} \frac{\omega_2(\delta)}{\delta} (1+\log\delta)^{\sigma} d\delta \right\} dz \quad (3.15) \\
&\leq C \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^n \big(\log(|z-u|/r)\big)^{2+\varepsilon}} dz
\end{aligned}
$$

Add up from (3.10), (3.11), (3.13) and (3.15), we get the desired estimate (3.7). Now we show the inequality (3.8) holds.

For any $u,\,v \in B\backslash E$, we denote $J := |\mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(u) - \mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(v)|$. Since

$$
\begin{aligned}
J &= \big| \|F((b-b_{B^*})f_2)(u,\,\cdot\,,\,\cdot\,)\|_{\mathcal{H}_2} - \|F((b-b_{B^*})f_2)(v,\,\cdot\,,\,\cdot\,)\|_{\mathcal{H}_2} \big| \\
&\leq \|F((b-b_{B^*})f_2)(u,\,\cdot\,,\,\cdot\,) - F((b-b_{B^*})f_2)(v,\,\cdot\,,\,\cdot\,)\|_{\mathcal{H}_2},
\end{aligned}
$$

we have

$$\begin{aligned} J \leq & \Big(\int_0^\infty \int_{|y|<1} \Big(\frac{1}{1+|y|}\Big)^{\lambda n} \Big| \int t^{-n} \Big[\phi\Big(\frac{u-z}{t} - y\Big) - \phi\Big(\frac{v-z}{t} - y\Big) \Big] \\ & \qquad \times (b(z) - b_{B^*}) f_2(z) dz \Big|^2 \frac{dydt}{t} \Big)^{1/2} \\ & + \Big(\int_0^\infty \int_{|y|\geq 1} \Big(\frac{1}{1+|y|}\Big)^{\lambda n} \Big| \int t^{-n} \Big[\phi\Big(\frac{u-z}{t} - y\Big) - \phi\Big(\frac{v-z}{t} - y\Big) \Big] \\ & \qquad \times (b(z) - b_{B^*}) f_2(z) dz \Big|^2 \frac{dydt}{t} \Big)^{1/2} \\ := & J_1 + J_2. \end{aligned} \tag{3.16}$$

Since $\big(1/(1+|y|)\big)^{\lambda n} \leq 1$, then $J_1 \leq I_1 + I_2 + I_3$, by (3.11), (3.13) and (3.15), we get

$$\begin{aligned} J_1 \leq & C r^{\varepsilon} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}||f(z)}{|z-x|^{n+\varepsilon}} dz \\ & + C r^{\rho - n/2} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}||f(z)|}{|z-x|^{\rho + n/2}} dz \\ & + C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}||f(z)|}{|z-x|^n \big(\log(|z-x|/r)\big)^{2+\varepsilon}} dz. \end{aligned} \tag{3.17}$$

On the other hand, for J_2, we have

$$\begin{aligned} J_2 \leq & \Big(\int_0^\infty \int_{|y|\geq 1} \Big(\frac{1}{1+|y|}\Big)^{\lambda n} \Big| \int_{\substack{|(u-z)/t-y|<1 \\ |(v-z)/t-y|\geq 1}} t^{-n} \phi\Big(\frac{u-z}{t} - y\Big) \\ & \qquad \times (b(z) - b_{B^*}) | f_2(z) dz \Big|^2 \frac{dydt}{t} \Big)^{1/2} \\ & + \Big(\int_0^\infty \int_{|y|\geq 1} \Big(\frac{1}{1+|y|}\Big)^{\lambda n} \Big| \int_{\substack{|(u-z)/t-y|\geq 1 \\ |(v-z)/t-y|<1}} t^{-n} \phi\Big(\frac{v-z}{t} - y\Big) \\ & \qquad \times (b(z) - b_{B^*}) f_2(z) dz \Big|^2 \frac{dydt}{t} \Big)^{1/2} \\ & + \Big(\int_0^\infty \int_{|y|\geq 1} \Big(\frac{1}{1+|y|}\Big)^{\lambda n} \Big| \int_{\substack{|(u-z)/t-y|<1 \\ |(v-z)/t-y|<1}} t^{-n} \\ \times & \Big[\phi\Big(\frac{u-z}{t} - y\Big) - \phi\Big(\frac{v-z}{t} - y\Big) \Big] (b(z) - b_{B^*}) f_2(z) dz \Big|^2 \frac{dydt}{t} \Big)^{1/2}. \end{aligned}$$

Using a transform, we have

$$
\begin{aligned}
J_2 \leq & \Big(\int_0^\infty \int_{|u-y|\geq t}\Big(\frac{t}{t+|u-y|}\Big)^{\lambda n} \\
& \times\Big|\int_{\substack{|y-z|<t\\ |v-u+y-z|\geq t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(z)-b_{B^*})f_2(z)dz\Big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2} \\
& +\Big(\int_0^\infty \int_{|u-y|\geq t}\Big(\frac{t}{t+|u-y|}\Big)^{\lambda n} \\
& \times\Big|\int_{\substack{|y-z|\geq t\\ |v-u+y-z|<t}} \frac{\Omega(v-u+y-z)}{|v-u+y-z|^{n-\rho}}(b(z)-b_{B^*})f_2(z)dz\Big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2} \\
& +\Big(\int_0^\infty \int_{|u-y|\geq t}\Big(\frac{t}{t+|u-y|}\Big)^{\lambda n} \\
& \times\Big|\int_{\substack{|y-z|<t\\ |v-u+y-z|<t}} \widetilde{\Omega}(u,\, v,\, y,\, z)(b(z)-b_{B^*})f_2(z)dz\Big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2} \\
& := K_1+K_2+K_3 \qquad (3.18)
\end{aligned}
$$

where $\widetilde{\Omega}(u,\, v,\, y,\, z) = \Omega(y-z)/|y-z|^{n-\rho} - \Omega(v-u+y-z)/|v-u+y-z|^{n-\rho}$ is the same as before.

By the estimates in Theorem 1 of [DX2], we know that for $i = 1,\, 2,\, K_i$ satisfies the following estimates.

$$
K_i \leq Cr^\varepsilon \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^{n+\varepsilon}}dz. \qquad (3.19)
$$

Finally, we deal with K_3. By the Minskowski inequality

$$
\begin{aligned}
K_3 \leq & C\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|\Big(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t\\ |u-y|\geq t,\ |v-u+y-z|<t}}\Big(\frac{t}{t+|u-y|}\Big)^{\lambda n} \\
& \qquad \times\big|\widetilde{\Omega}(u,\, v,\, y,\, z)\big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2} dz \\
\leq & C\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|\Big(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t\\ |u-y|\geq t,\ |v-u+y-z|<t\\ |y-z|\leq 8r}}\Big(\frac{t}{t+|u-y|}\Big)^{\lambda n} \\
& \qquad \times\big|\widetilde{\Omega}(u,\, v,\, y,\, z)\big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2} dz \\
& + C\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|\Big(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t\\ |u-y|\geq t,\ |v-u+y-z|<t\\ |y-z|>8r}}\Big(\frac{t}{t+|u-y|}\Big)^{\lambda n}
\end{aligned}
$$

$$\times \left|\widetilde{\Omega}(u,\,v,\,y,\,z)\right|^2 \frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2} dz$$

$$:= K_{3.1} + K_{3.2}. \tag{3.20}$$

For $K_{3.1}$, note that when $|y-z|<8r$, then $|v-u+y-z|<9r$, so

$$\begin{aligned} K_{3.1} \le & C\int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \\ & \times \Bigg(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t \\ |u-y|\ge t,\ |v-u+y-z|<t \\ |y-z|\le 8r}} \Big(\frac{t}{t+|u-y|}\Big)^{\lambda n} \\ & \times \Big(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} + \frac{|\Omega(v-u+y-z)|^2}{|v-u+y-z|^{2n-2\rho}}\Big)\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2} dz \\ \le & C\int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \\ & \times \Bigg(\iint_{\substack{y\in(2B^*)^c \\ |u-y|\ge t \\ |y-z|\le 8r}} \Big(\frac{t}{t+|u-y|}\Big)^{\lambda n} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2} dz \\ & + C\int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \\ & \times \Bigg(\iint_{\substack{y\in(2B^*)^c \\ |u-y|\ge t \\ |v-u+y-z|<9r}} \Big(\frac{t}{t+|u-y|}\Big)^{\lambda n} \frac{|\Omega(v-u+y-z)|^2}{|v-u+y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2} dz \\ := & K_{3.1}^1 + K_{3.1}^2. \end{aligned}$$

Also by the proof in Theorem 1 of [DX2], for $i=1,\,2$, we get

$$\begin{aligned} K_{3.1}^i \le & Cr^{\varepsilon}\int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x|^{n+\varepsilon}}dz \\ & + Cr^{\rho-n/2}\int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x|^{n+\rho/2}}dz \end{aligned} \tag{3.21}$$

Let us give the estimate of $K_{3.2}$. As before, we divide the region by $2|y-z|\ge|z-u|$ and $2|y-z|<|z-u|$. Hence

$$\begin{aligned} K_{3.2} \le & C\int_{(8B^*)^c} |f(z)| \\ & \times \Bigg(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t \\ |v-u+y-z|<t \\ |y-z|>8r,\ 2|y-z|\ge|z-u|}} \left|\widetilde{\Omega}(u,\,v,\,y,\,z)\right|^2 \frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2} dz \end{aligned}$$

$$+ C\int_{(8B^*)^c} |f(z)|\bigg(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t\\ |u-y|\ge t,\ |v-u+y-z|<t\\ |y-z|>8r,\ 2|y-z|<|z-u|}} \Big(\frac{t}{t+|u-y|}\Big)^{\lambda n} \times \big|\widetilde{\Omega}(u,\,v,\,y,\,z)\big|^2 \frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2} dz$$

$$:= K_{3.2}^1 + K_{3.2}^2.$$

Since $t > |y-z| > |z-u|/2$, we have

$$\begin{aligned}
K_{3.2}^1 &\le C\int_{(8B^*)^c} |f(z)|\bigg(\iint_{\substack{y\in(2B^*)^c\\ |y-z|>8r\\ t\ge|z-u|/2}} \big|\widetilde{\Omega}(u,\,v,\,y,\,z)\big|^2 \frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2} dz\\
&\le C\int_{(8B^*)^c} |f(z)|\bigg(\int_{\substack{y\in(2B^*)^c\\ |y-z|>8r}} \big|\widetilde{\Omega}(u,\,v,\,y,\,z)\big|^2 \times \Big(\int_{\max\{|y-z|,\,|z-u|/2\}}^{\infty} \frac{dt}{t^{n+2\rho+1}}\Big)dy\bigg)^{1/2} dz\\
&\le C\int_{(8B^*)^c} |f(z)|\bigg(\int_{\substack{y\in(2B^*)^c\\ |y-z|>8r}} \big|\widetilde{\Omega}(u,\,v,\,y,\,z)\big|^2\\
&\quad\times \Big(\int_{\max\{|y-z|,\,|z-u|/2\}}^{\infty} \frac{\big(\log(t/r)\big)^{4+2\varepsilon}dt}{t^{2\rho-n+1}|z-u|^n\big(\log(|z-u|/r)\big)^{4+2\varepsilon}}\Big)dy\bigg)^{1/2} dz\\
&\le C\int_{(8B^*)^c} \frac{|f(z)|}{|z-u|^n\big(\log(|z-u|/r)\big)^{2+\varepsilon}}\\
&\quad\times \bigg(\int_{\substack{y\in(2B^*)^c\\ |y-z|>8r}} \big|\widetilde{\Omega}(u,\,v,\,y,\,z)\big|^2 \Big(\int_{|y-z|}^{\infty} \frac{\big(\log(t/r)\big)^{4+2\varepsilon}dt}{t^{2\rho-n+1}}\Big)dy\bigg)^{1/2} dz.
\end{aligned}$$

By the estimate of $I_{3.2}$ (see (3.15)), we get

$$K_{3.2}^1 \le C\int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^n\big(\log(|z-u|/r)\big)^{2+\varepsilon}} dz. \tag{3.22}$$

Now we consider $K_{3.2}^2$. Denote $C(\varepsilon) = e^{(2+2\varepsilon)/\varepsilon}$. By $2|y-z| < |z-u|$, we get $|u-y| \ge |z-u| - |y-z| \ge |z-u|/2$. Hence

$$K_{3.2}^2 \le C\int_{(8B^*)^c} |f(z)|\bigg(\iint_{\substack{y\in(2B^*)^c,\ |u-y|\ge t\\ |y-z|>8r\\ |u-y|>|z-u|/2,\ |y-z|<t}} \Big(\frac{t}{t+|u-y|}\Big)^{\lambda n} \times\big|\widetilde{\Omega}(u,\,v,\,y,\,z)\big|^2 \frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2} dz$$

$$\leq C\int_{(8B^*)^c}|f(z)|\bigg(\iint_{\substack{y\in(2B^*)^c,\ |u-y|\geq t\\ |y-z|>8r\\ |u-y|>|z-u|/2,\ |y-z|<t}}\frac{t^{\lambda n}\log\big(\frac{t+|u-y|+8C(\varepsilon)}{r}\big)^{4+2\varepsilon}}{(t+|u-y|)^{\lambda n-2n}|z-u|^{2n}}$$

$$\times\frac{1}{\big(\log\frac{t+|u-y|+8C(\varepsilon)}{r}\big)^{4+2\varepsilon}}\times\big|\widetilde{\Omega}(u,v,y,z)\big|^2\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2}dz$$

$$\leq C\int_{(8B^*)^c}\frac{|f(z)|}{(|z-u|)^n\big(\log\frac{|z-u|/2}{r}\big)^{2+\varepsilon}}$$

$$\times\bigg(\iint_{\substack{y\in(2B^*)^c\\ |u-y|\geq t,\ |y-z|>8r\\ |y-z|<t}}\big|\widetilde{\Omega}(u,v,y,z)\big|^2\frac{t^{\lambda n}\big(\log\frac{t+|u-y|+8C(\varepsilon)}{r}\big)^{4+2\varepsilon}}{(t+|u-y|)^{\lambda n-2n}t^{n+2\rho+1}}dydt\bigg)^{1/2}dz$$

Notice that the function $G(s)=(\log s)^{4+2\varepsilon}/s^{\varepsilon}$ is decreasing when $s>e^{(4+2\varepsilon)/\varepsilon}$ and the fact that

$$\int_{|y-z|}^{|u-y|}\frac{t^{\lambda n}\big(\log\frac{t+|u-y|+8C(\varepsilon)r}{r}\big)^{4+2\varepsilon}}{(t+|u-y|)^{\lambda n-2n}}\frac{dt}{t^{n+2\rho+1}}$$

$$\leq C\frac{\big(\log\frac{|y-z|+8C(\varepsilon)}{r}\big)^{4+2\varepsilon}}{|y-z|^{2\rho-2n}}.$$

Using the estimate in (3.15), we get

$$K_{3.2}^2\leq C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-u|^n\big(\log(|z-u|/r)\big)^{2+\varepsilon}}dz. \tag{3.23}$$

From (3.17)–(3.23), we obtain the estimate (3.8) and the proof of this Lemma is finished. □

The following result is also needed in the proof of Theorem 1.

Lemma 3.7 *Suppose $\rho>n/2$, Ω satisfies the condition (1.2) with $\sigma>2$ and $0<\delta<1$. Then for all smooth functions with compact support f, there exists a positive constant $0<C=C_\delta$ such that*

$$M_\delta^\sharp(\mu_S^\rho(f))(x)\leq CM(f)(x)$$

$$\text{and}\quad M_\delta^\sharp(\mu_\lambda^{*,\rho}(f))(x)\leq CM(f)(x)\quad\text{for }\lambda>2.$$

Proof. Because the proofs are similar for $\mu_\lambda^{*,\rho}(f)$ and $\mu_S^\rho(f)$, we only give the proof for $\mu_S^\rho(f)$. Let f_1, f_2 and B, B^* be the same one as in Lemma 3.5. Then the Kolmogorov inequality (Lemma 3.1) and the weak (1,1) boundedness of μ_S^ρ yields

$$\left(\frac{1}{|B|}\int_B [\mu_S^\rho(f_1)(y)]^\delta dy\right)^{1/\delta} \leq \frac{C}{|B|}\int_B |f_1(y)|dy \leq CM(f)(x).$$

By Lemma 3.5, there exists a measurable set $E \subset B$ with $|E| = 0$ such that $\mu_S^\rho(f_2)(x) < \infty$ holds for any $u,\, v \in B\backslash E$. Obviously,

$$|\mu_S^\rho(f_2)^\delta(u) - \mu_S^\rho(f_2)^\delta(v)| \leq |\mu_S^\rho(f_2)(u) - \mu_S^\rho(f_2)(v)|^\delta \qquad \text{for } 0 < \delta < 1.$$

Now we claim the following fact:

$$|\mu_S^\rho(f_2)(u) - \mu_S^\rho(f_2)(v)| \leq CM(f)(x) \qquad \text{holds for any } u,\, v \in B\backslash E. \tag{3.24}$$

In fact, by Lemma 3.5, we know that

$$\begin{aligned}
&|\mu_S^\rho(f_2)(u) - \mu_S^\rho(f_2)(v)| \\
&\leq Cr^\varepsilon \int_{(B^*)^c} \frac{|f(z)|}{|z-x|^{n+\varepsilon}}dz + Cr^{\rho-n/2}\int_{(B^*)^c}\frac{|f(z)|}{|z-x|^{\rho+n/2}}dz \\
&\quad + C\int_{(B^*)^c}\frac{|f(z)|}{|z-x|^n\big(\log(|z-x|/r)\big)^{2+\varepsilon}}dz \\
&\leq CM(f)(x) + C\sum_{j=3}^\infty \int_{2^j r\leq|z-x|<2^{j+1}r}\frac{|f(z)|}{|z-x|^n\big(\log(|z-x|/r)\big)^{2+\varepsilon}}dz \\
&\leq C\bigg(M(f)(x) + \sum_{j=3}^\infty \frac{1}{j^{2+\varepsilon}}\int_{|z-x|<2^{j+1}r}\frac{|f(z)|}{(2^j r)^n}dz\bigg) \\
&\leq CM(f)(x).
\end{aligned}$$

So take $c_B = (\mu_S^\rho(f_2))_B$ then by (3.24)

$$\begin{aligned}
M_\delta^\sharp(\mu_S^\rho(f))(x) &= \sup_{x\in Q}\left(\frac{1}{|Q|}\int_Q |\mu_S^\rho(f_2)^\delta(u) - (\mu_S^\rho(f_2))_B^\delta|du\right)^{1/\delta} \\
&\leq C\sup_{x\in B}\left(\frac{1}{|B|}\int_B |\mu_S^\rho(f_2)(u) - (\mu_S^\rho(f_2))_B|du\right) \\
&\leq \sup_{x\in B}\left(\frac{1}{|B|^2}\int_{B\backslash E}\int_{B\backslash E}|\mu_S^\rho(f_2)(u) - \mu_S^\rho(f_2)(v)|dvdu\right)
\end{aligned}$$

$$\leq CM(f)(x).$$

□

Remark 3.1 From the proof of Lemma 3.5, with the same condition, the following inequalities are also valid.

$$M_\delta^{\triangle,\sharp}(\mu_S^{\rho,b}(f))(x) \leq C\|b\|_*(M_\ell^\triangle(\mu_S^\rho(f))(x) + M^2(f)(x))$$

and

$$M_\delta^{\triangle,\sharp}(\mu_{\lambda,b}^{*,\rho}(f))(x) \leq C\|b\|_*(M_\ell^\triangle(\mu_\lambda^{*,\rho}(f))(x) + M^2(f)(x)).$$

Similarly, by Lemma 3.7 we may obtain the following inequality:

$$M_\delta^{\triangle,\sharp}(\mu_S^\rho(f))(x) \leq CM(f)(x)) \leq CM^2(f)(x)$$
$$\text{and} \quad M_\delta^{\triangle,\sharp}(\mu_\lambda^{*,\rho}(f))(x) \leq CM(f)(x)) \leq CM^2(f)(x).$$

For $b \in \mathrm{BMO}(\mathbb{R}^n)$, let $b_k(x) = b(x)$ if $|b(x)| \leq k$, $b_k(x) = k$ if $b(x) > k$ and $b_k(x) = -k$ if $b(x) < -k$ for $k = 1, 2, 3, \ldots$. Then $b_k \in L^\infty$ and $\|b_k\|_* \leq \|b\|_*$. The following main Lemma shows that $\mu_{S,b_k}^\rho(f)(x)$ and $\mu_{\lambda,b_k}^{*,\rho}(f)(x)$ can be controlled by maximal operators.

Lemma 3.8 *Suppose that* $\Omega \in L^2(S^{n-1})$ *satisfying* (1.2) *with* $\sigma > 1$. *If* $\rho > n/2$, $\lambda > 2$, *and* $\mathrm{supp}(f) \subset B(0, R)$, $|x| > 2R$, *then*

$$\mu_{S,b_k}^\rho(f)(x) \leq CkM(f)(x) \quad \text{and} \quad \mu_{\lambda,b_k}^{*,\rho} \leq CkM(f)(x). \tag{3.25}$$

Proof. First we show that $\mu_{S,b_k}^\rho(f)(x) \leq CkM(f)(x)$.

$$\begin{aligned}\mu_{S,b_k}^\rho(f)(x) \leq & \Bigg(\iint_{\substack{(y,\,t)\in\Gamma(x)\\ y\in B(0,\,3/2R)}} \Bigg|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \\ & \times \big(b_k(x)-b_k(z)\big)f(z)dz\Bigg|^2 \frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2} \\ & + \Bigg(\iint_{\substack{(y,\,t)\in\Gamma(x)\\ y\in B^c(0,\,3/2R)}} \Bigg|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \\ & \times \big(b_k(x)-b_k(z)\big)f(z)dz\Bigg|^2 \frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2} := N + T.\end{aligned}$$

Since $x \in B^c(0, 2R)$, $y \in B(0, 3/2R)$ and $z \in B(0, R)$, so $t > |x-y| >$

$|x|-|y|\geq |x|/4\geq R/2$ and $|y-z|<3R$. By the Minkowski inequality

$$\begin{aligned}N&\le 2Ck\int_{B(0,\,R)}|f(z)|\bigg(\int_{|x|/4}^{\infty}\int_{|y-z|<3R}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2}dz\\ &\le Ck\int_{B(0,\,R)}|f(z)|\bigg(\int_{|x|/4}^{\infty}R^{2\rho-n}\frac{dt}{t^{n+2\rho+1}}\bigg)^{1/2}dz\\ &\le Ck\frac{1}{(2|x|)^n}\int_{B(0,2|x|)}|f(z)|dz\le CkM(f)(x). \qquad (3.26)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}T\le&\bigg(\iint_{\substack{|y-x|<t\\ y\in B^c(0,\,3/2R)\\ t\le |y|+2R}}\bigg|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\\ &\qquad\times\big(b_k(x)-b_k(z)\big)f(z)dz\bigg|^2\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2}\\ &+\bigg(\iint_{\substack{|y-x|<t\\ y\in B^c(0,\,3/2R)\\ t>|y|+2R}}\bigg|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\\ &\qquad\times\big(b_k(x)-b_k(z)\big)f(z)dz\bigg|^2\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2}:=T_1+T_2.\end{aligned}$$

For T_1, note that $\max\{|y-x|,\,|y-z|\}<t\le |y|+2R$ and $x\in B^c(0,\,2R)$, $y\in B^c(0,\,3/2R)$, $z\in B(0,\,R)$ we get

(a) $|y-z|\sim|y|$;

(b) $|y|-R\le |y|-|z|\le|y-z|<t\le|y|+2R$;

(c) $|x|\le|x-y|+|y|\le t+|y|\le 2|y|+2R\le 4|y|$.

Thus

$$\bigg|\frac{1}{(|y|-R)^{n+2\rho}}-\frac{1}{(|y|+2R)^{n+2\rho}}\bigg|\le\frac{CR}{|y|^{n+2\rho+1}}.$$

By the Minkowski inequality

$$\begin{aligned}T_1\le&\, 2Ck\int_{B(0,\,R)}|f(z)|\\ &\times\bigg(\int_{\substack{|x|\le 4|y|\\ y\in B^c(0,\,3/2R)}}\bigg(\int_{|y|-R}^{|y|+2R}\frac{1}{t^{n+2\rho+1}}dt\bigg)\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy\bigg)^{1/2}dz\end{aligned}$$

$$
\begin{aligned}
&\leq Ck\int_{B(0,R)}|f(z)|\Bigg(\int_{\substack{|x|\leq 4|y|\\ y\in B^c(0,3/2R)}}\frac{R}{|y|^{n+2\rho+1}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy\Bigg)^{1/2}dz\\
&\leq Ck\frac{1}{|x|^n}\int_{B(0,R)}|f(z)|\Bigg(\int_{y\in B^c(0,3/2R)}\frac{R|\Omega(y-z)|^2}{|y-z|^{n+1}}dy\Bigg)^{1/2}dz\\
&\leq Ck\frac{1}{|x|^n}\int_{B(0,2|x|)}|f(z)|dz\leq CkM(f)(x). \qquad (3.27)
\end{aligned}
$$

Now we consider T_2, we divide it by the relationship between $|x|$ and $2|y|$.

$$
\begin{aligned}
T_2\leq{}& 2Ck\int_{B(0,R)}|f(z)|\\
&\times\Bigg(\iint_{\substack{\max\{|y-x|,\,|y-z|\}<t\\ y\in B^c(0,3/2R),\ t\geq|y|+2R}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz\\
\leq{}& Ck\int_{B(0,R)}|f(z)|\\
&\times\Bigg(\iint_{\substack{\max\{|y-x|,\,|y-z|\}<t\\ y\in B^c(0,3/2R)\\ t\geq|y|+2R,\ |x|\leq 2|y|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz\\
&+Ck\int_{B(0,R)}|f(z)|\\
&\times\Bigg(\iint_{\substack{\max\{|y-x|,\,|y-z|\}<t\\ y\in B^c(0,3/2R)\\ t\geq|y|+2R,\ |x|>2|y|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz\\
:={}& Ck(T_{2,1}+T_{2,2}).
\end{aligned}
$$

Since $|y-z|\geq|y|-|z|\geq|y|-R$, so $1/|y-z|\leq 1/(|y|-R)\leq C/|x|$, together this and by the fact that $t>|y|+2R>|x|/2$ and $n/2<\rho\leq n$, we get

$$
\begin{aligned}
T_{2,1}\leq{}&\int_{B(0,R)}|f(z)|\\
&\times\Bigg(\int_{|x|/2}^{\infty}\Bigg(\int_{\substack{\max\{|y-x|,\,|y-z|\}<t\\ y\in B^c(0,3/2R)\\ t\geq|y|+2R\\ |x|\leq 2|y|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy\Bigg)\frac{dt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz\\
\leq{}& C\int_{B(0,R)}|f(z)|\Bigg(\int_{|x|/2}^{\infty}\Bigg(\int_{|y-z|<t}\frac{|\Omega(y-z)|^2}{|x|^{2n-2\rho}}dy\Bigg)\frac{dt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz
\end{aligned}
$$

$$\le C\frac{\|\Omega\|_{L^2(S^{n-1})}}{|x|^{n-\rho}}\int_{B(0,\,R)}|f(z)|\Big(\int_{|x|/2}^{\infty}\frac{dt}{t^{2\rho+1}}\Big)^{1/2}dz$$
$$\le C\frac{1}{|x|^n}\int_{B(0,\,|x|)}|f(z)|dz\le CM(f)(x). \tag{3.28}$$

As for $T_{2.2}$, note that $R/2<|y-z|\le|y|+R\le|x|/2+R\le|x|$, $t>|y-x|\ge|x|-|y|>|x|/2$. So by the method of rotation, we have

$$T_{2,2}\le\int_{B(0,\,R)}|f(z)|\Big(\int_{|x|/2}^{\infty}\Big(\int_{|y-z|<|x|}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy\Big)\frac{dt}{t^{n+2\rho+1}}\Big)^{1/2}dz$$
$$\le C\int_{B(0,\,R)}|f(z)|\Big(\int_{|x|/2}^{\infty}|x|^{2\rho-n}\frac{dt}{t^{n+2\rho+1}}\Big)^{1/2}dz$$
$$\le C\frac{1}{|x|^n}\int_{B(0,\,|x|)}|f(z)|dz\le CM(f)(x). \tag{3.29}$$

Thus, by (3.26)–(3.29), we finish the proof of Lemma 3.8 for $\mu^{\rho}_{S,\,b_k}$.

Now let us turn to $\mu^{*\rho}_{\lambda,\,b_k}$. Denote

$$A=\Big(\iint_{|x-y|\ge t}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\times\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b_k(x)-b_k(z))f(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}.$$

By $\big(t/(t+|x-y|)\big)^{\lambda n}\le1$ and $\mu^{\rho}_{S,\,b_k}(f)(x)\le CkM(f)(x)$, we have

$$\mu^{*,\rho}_{\lambda,\,b_k}(f)(x)\le\mu^{\rho}_{S,\,b_k}(f)(x)+A\le CkM(f)(x)+A. \tag{3.30}$$

It remains to show $A\le CkM(f)(x)$. We divide A into two parts.

$$A\le\Big(\iint_{\substack{|x-y|\ge t\\ y\in B(0,\,3/2R)}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\times\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b_k(x)-b_k(z))f(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}$$
$$+\Big(\iint_{\substack{|x-y|\ge t\\ y\in B^c(0,\,3/2R)}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\times\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b_k(x)-b_k(z))f(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}$$

$$:= A_1 + A_2.$$

First we consider A_1. Since $x \in B^c(0,\, 2R)$, $y \in B(0,\, 3/2R)$ and $z \in B(0,\, R)$, we have $|x-y| > |x|-|y| \geq |x|/4 \geq R/2$, $|y-z| \leq |y|+|z| < 3R$ and $1/t \leq 1/|y-z|$. Hence, if we take $0 < \varepsilon < \min\{1,\, (\lambda-2)n/2\}$, then by the Minkowski inequality we get

$$\begin{aligned}
A_1 \leq & 2Ck \int_{B(0,\,R)} |f(z)| \\
& \times \left(\iint_{\substack{|x-y|\geq t \\ y\in B(0,\,3/2R) \\ |y-z|<t}} \left(\frac{t}{t+|x-y|}\right)^{2n+2\varepsilon} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} dz \\
\leq & Ck \int_{B(0,\,R)} |f(z)| \\
& \times \left(\iint_{\substack{|x-y|\geq t \\ y\in B(0,\,3/2R) \\ |y-z|<t}} \frac{1}{|x-y|^{2n+\varepsilon}} \frac{t^{2n+2\varepsilon}}{|x-y|^{\varepsilon}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} dz \\
\leq & Ck \int_{B(0,\,R)} |f(z)| \\
& \times \left(\int_{|y-z|<3R} \frac{1}{|x|^{2n+\varepsilon}} \frac{1}{|x-y|^{\varepsilon}} \frac{|\Omega(y-z)|^2}{|y-z|^{n-\varepsilon}} \left(\int_0^{|x-y|} \frac{dt}{t^{1-\varepsilon}} \right) dy \right)^{1/2} dz \\
\leq & Ck \int_{B(0,\,R)} \frac{|f(z)|}{|x|^{n+\varepsilon/2}} \left(\int_{|y-z|<3R} \frac{|\Omega(y-z)|^2}{|y-z|^{n-\varepsilon}} dy \right)^{1/2} dz \\
\leq & Ck \frac{1}{(2|x|)^n} \int_{B(0,\,2|x|)} |f(z)| dz \leq CkM(f)(x).
\end{aligned} \tag{3.31}$$

As for A_2, we have

$$\begin{aligned}
A_2 \leq & \left(\iint_{\substack{|y-x|\geq t \\ y\in B^c(0,\,3/2R) \\ t\leq |y|+2R}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \right. \\
& \left. \times \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b_k(x)-b_k(z)) f(z) dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} \\
& + \left(\iint_{\substack{|y-x|\geq t \\ y\in B^c(0,\,3/2R) \\ t> |y|+2R}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \right. \\
& \left. \times \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b_k(x)-b_k(z)) f(z) dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2}
\end{aligned}$$

$$:= A_{2,1} + A_{2,2}.$$

We further divide $A_{2,1}$ into two parts as $2|y| > |x|$ and $2|y| \le |x|$.

$$\begin{aligned}
A_{2,1} \le & \Bigg(\iint_{\substack{|y-x|\ge t \\ y\in B^c(0,3/2R) \\ t\le |y|+2R \\ 2|y|>|x|}} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \\
& \times \Big| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b_k(x)-b_k(z)) f(z)dz \Big|^2 \frac{dydt}{t^{n+2\rho+1}} \Bigg)^{1/2} \\
& + \Bigg(\iint_{\substack{|y-x|\ge t \\ y\in B^c(0,3/2R) \\ t\le |y|+2R \\ 2|y|\le|x|}} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \\
& \times \Big| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b_k(x)-b_k(z)) f(z)dz \Big|^2 \frac{dydt}{t^{n+2\rho+1}} \Bigg)^{1/2} \\
:= & A_{2,1}^1 + A_{2,1}^2.
\end{aligned}$$

For $A_{2,1}^1$, since $x \in B^c(0, 2R)$, $y \in B^c(0, 3/2R)$ and $z \in B(0, R)$, we have $|y-z| \sim |y|$, $|y| - R \le |y| - |z| \le |y-z| < t \le |y| + 2R$. Note $\big(t/(t+|x-y|)\big)^{\lambda n} < 1$ and $|x| < 2|y|$, then by the Minkowski inequality and the estimate for T_1 (3.27), we get

$$\begin{aligned}
A_{2,1}^1 \le & 2Ck \int_{B(0,R)} |f(z)| \\
& \times \Bigg(\int_{\substack{|x|<2|y| \\ y\in B^c(0,3/2R)}} \Big(\int_{|y|-R}^{|y|+2R} \frac{1}{t^{n+2\rho+1}} dt \Big) \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} dy \Bigg)^{1/2} dz \\
\le & CkM(f)(x). \qquad (3.32)
\end{aligned}$$

By the Minkowski inequality and the fact that $|x-y| > |x| - |y| \ge |x|/2$

$$\begin{aligned}
A_{2,1}^2 \le & Ck \int_{B(0,R)} |f(z)| \\
\times & \Bigg(\iint_{\substack{|y-x|\ge t \\ y\in B^c(0,3/2R) \\ t\le |y|+2R \\ |y-z|<t \\ 2|y|\le|x|}} \Big(\frac{t}{t+|x-y|}\Big)^{2n} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \Bigg)^{1/2} dz \\
\le & Ck \int_{B(0,R)} |f(z)|
\end{aligned}$$

$$\times\bigg(\int_{y\in B^c(0,\,3/2R)}\Big(\frac{2}{|x|}\Big)^{2n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\Big(\int_{|y|-R}^{|y|+2R}\frac{1}{t^{2\rho-n+1}}dt\Big)dy\bigg)^{1/2}dz$$
$$\leq CkM(f)(x). \tag{3.33}$$

Similarly, we divide $A_{2,2}$ into two parts as following:

$$A_{2,2}\leq Ck\int_{B(0,\,R)}|f(z)|$$
$$\times\bigg(\iint_{\substack{|x-y|\geq t\\ y\in B^c(0,\,3/2R)\\ t>|y|+2R,\ |y-z|<t}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2}dz$$
$$\leq Ck\int_{B(0,\,R)}|f(z)|$$
$$\times\bigg(\iint_{\substack{|x-y|\geq t\\ y\in B^c(0,\,3/2R)\\ t>|y|+2R\\ |y-z|<t,\ |x|\leq 2|y|}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2}dz$$
$$+Ck\int_{B(0,\,R)}|f(z)|$$
$$\times\bigg(\iint_{\substack{|x-y|\geq t\\ y\in B^c(0,\,3/2R)\\ t>|y|+2R\\ |y-z|<t,\ |x|>2|y|}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2}dz$$
$$:=Ck\big(A_{2,2}^1+A_{2,2}^2\big).$$

For $A_{2,2}^1$, it is easy to see that $|y-z|\sim|y|$ and $1/|y-z|\leq C/|x|$. Note $t>|y|+2R>|x|/2$ and using the same steps as deal $T_{2,1}$ (see (3.28)), we may get $A_{2,2}^1\leq CkM(f)(x)$. We now consider $A_{2,2}^2$. Since $R/2<|y-z|\leq |y|+R\leq|x|/2+R\leq|x|$ and $t+|y-x|\geq t+|x|-|y|>t+|x|/2>|x|/2$, if take $0<\varepsilon<\min\{1/2,\,(\lambda-2)n/2,\,\rho-n/2\}$, we have

$$A_{2,2}^2\leq 2Ck\int_{B(0,\,R)}|f(z)|$$
$$\times\bigg(\iint_{\substack{|x-y|\geq t\\ y\in B^c(0,\,3/2R)\\ t>|y|+2R\\ |y-z|<t,\ |x|>2|y|}}\Big(\frac{t}{t+|x-y|}\Big)^{2n+2\varepsilon}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2}dz$$
$$\leq Ck\int_{B(0,\,R)}\frac{|f(z)|}{|x|^n}$$

$$\times \left(\iint_{\substack{|x-y|\geq t\\ y\in B^c(0,\,3/2R)\\ t>|y|+2R\\ |y-z|<t,\ |x|>2|y|}} \frac{1}{|x|^{2\varepsilon}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{2\rho-n+1-2\varepsilon}}\right)^{1/2} dz$$

$$\leq Ck \int_{B(0,\,R)} \frac{|f(z)|}{|x|^n}$$

$$\times \left(\int_{|y-z|<|x|} \frac{1}{|x|^{2\varepsilon}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \left(\int_{|y-z|}^{\infty} \frac{dt}{t^{2\rho-n+1-2\varepsilon}} dt\right) dy\right)^{1/2} dz$$

$$\leq Ck \int_{B(0,\,R)} \frac{|f(z)|}{|x|^n} \left(\int_0^{|x|} \int_{S^{n-1}} \frac{1}{|x|^{2\varepsilon}} \frac{|\Omega(\theta)|^2 s^{n-1}}{s^{n-2\varepsilon}} d\sigma(\theta) ds\right)^{1/2} dz$$

$$\leq Ck \frac{1}{|x|^n} \int_{B(0,\,|x|)} |f(z)| dz \leq CkM(f)(x). \tag{3.34}$$

From (3.30)–(3.34), we finish the proof of Lemma 3.8 for $\mu^{*,\,\rho}_{\lambda,\,b_m}$. □

4. Proof of Theorem 1

As shown in Remark 1.3, we give only the proof of the conclusion (ii) of Theorem 1 here. First let us consider the case where $m = 1$. The proof depends on the following lemma.

Lemma 4.1 *Let $\Phi(t) = t(1+\log^+ t)$, then there exists a positive constant C, such that for any smooth function with compact support f,*

$$\sup_{t>0} \frac{1}{\Phi(1/t)} \left|\{v \in \mathbb{R}^n : \mu^{*,\,\rho}_{\lambda,\,b}(f)(v) > t\}\right|$$
$$\leq C_{\|b\|_*} \sup_{t>0} \frac{1}{\Phi(1/t)} \left|\{v \in \mathbb{R}^n : M^2(f)(v) > t\}\right|.$$

Proof. Obviously, if $\|b\|_* = 0$ (4.1) holds, we therefore may assume $\|b\|_* > 0$. Denote

$$L_{\Phi,\,\delta}(f) = L_\delta(f) = \sup_{t>0} \frac{1}{\Phi(1/t)} \left|\{v \in \mathbb{R}^n : M^\triangle_\delta(\mu^{*,\,\rho}_{\lambda,\,b}(f))(v) > t\}\right|,$$

then it is easy to see that

$$\sup_{t>0} \frac{1}{\Phi(1/t)} \left|\{v \in \mathbb{R}^n : \mu^{*,\,\rho}_{\lambda,\,b}(f)(v) > t\}\right| \leq L_\delta(f). \tag{4.1}$$

We will prove that for arbitrary $0 < \delta < 1$, $\gamma > 0$, the operator $\mu^{*,\,\rho}_{\lambda,\,b}(f)$

satisfies the following inequality

$$L_\delta(f) \le C\gamma L_\delta(f) + C_{\delta,\gamma,\|b\|_*} \sup_{t>0} \frac{1}{\Phi(1/t)} \big|\{v \in \mathbb{R}^n\colon M^2(f)(v) > t\}\big|. \tag{4.2}$$

To do this, by Lemma 3.2 (i) we get for $t > 0$ and $\delta > 0$,

$$\begin{aligned}
&\big|\{v \in \mathbb{R}^n\colon M_\delta^\triangle(\mu_{\lambda,b}^{*,\rho}(f))(v) > t\}\big| \\
&= \big|\{v \in \mathbb{R}^n\colon M^\triangle([\mu_{\lambda,b}^{*,\rho}(f)]^\delta)(v) > t^\delta\}\big| \\
&\le \big|\{v \in \mathbb{R}^n\colon M^\triangle([\mu_{\lambda,b}^{*,\rho}(f)]^\delta)(v) > t^\delta,\ M^{\triangle,\sharp}([\mu_{\lambda,b}^{*,\rho}(f)]^\delta)(v) \le \gamma t^\delta\}\big| \\
&\quad + \big|\{v \in \mathbb{R}^n\colon M^{\triangle,\sharp}([\mu_{\lambda,b}^{*,\rho}(f)]^\delta)(v) > \gamma t^\delta\}\big| \\
&\le C\gamma \left|\left\{v \in \mathbb{R}^n\colon M_\delta^\triangle(\mu_{\lambda,b}^{*,\rho}(f))(v) > \frac{t}{2^{1/\delta}}\right\}\right| \\
&\quad + \big|\{v \in \mathbb{R}^n\colon M_\delta^{\triangle,\sharp}(\mu_{\lambda,b}^{*,\rho}(f))(v) > \gamma^{1/\delta} t\}\big|.
\end{aligned} \tag{4.3}$$

Since $0 < \delta < 1$, we may choose $0 < \delta < \ell < 1$. If denote $\ell = r\delta$, then $1 < r < 1/\delta$. By Remark 3.1, we know that

$$M_\delta^{\triangle,\sharp}(\mu_{\lambda,b}^{*,\rho}(f))(x) \le C\|b\|_* \big(M_\ell^\triangle(\mu_\lambda^{*,\rho}(f))(x) + M^2(f)(x)\big). \tag{4.4}$$

Since $\|b\|_* > 0$, by (4.4) we have

$$\begin{aligned}
&\big|\{v \in \mathbb{R}^n\colon M_\delta^{\triangle,\sharp}(\mu_{\lambda,b}^{*,\rho}(f))(v) > \gamma^{1/\delta} t\}\big| \\
&\le \left|\left\{v \in \mathbb{R}^n\colon M_\ell^\triangle(\mu_\lambda^{*,\rho}(f))(v) > \frac{\gamma^{1/\delta} t}{2C\|b\|_*}\right\}\right| \\
&\quad + \left|\left\{v \in \mathbb{R}^n\colon M^2(f)(v) > \frac{\gamma^{1/\delta} t}{2C\|b\|_*}\right\}\right|.
\end{aligned} \tag{4.5}$$

Note that $\Phi(ab) \le \Phi(a)\Phi(b)$ for $a, b \ge 0$ and Φ is increasing and doubling, the above inequality together with (4.3) and (4.5) yields

$$\begin{aligned}
&\frac{1}{\Phi(1/t)} |\{v \in \mathbb{R}^n\colon M_\delta^\triangle(\mu_{\lambda,b}^{*,\rho}(f))(v) > t\}| \\
&\le \frac{C\gamma}{\Phi(1/t)} \left|\left\{v \in \mathbb{R}^n\colon M_\delta^\triangle(\mu_{\lambda,b}^{*,\rho}(f))(v) > \frac{t}{2^{1/\delta}}\right\}\right| \\
&\quad + \frac{1}{\Phi(1/t)} \left|\left\{v \in \mathbb{R}^n\colon M_\ell^\triangle(\mu_\lambda^{*,\rho}(f))(v) > \frac{\gamma^{1/\delta} t}{2C\|b\|_*}\right\}\right|
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{1}{\Phi(1/t)}\Big|\Big\{v\in\mathbb{R}^n\colon M^2(f)(v) > \frac{\gamma^{1/\delta}t}{2C\|b\|_*}\Big\}\Big| \\
&\le C_\delta\gamma L_\delta(f) \\
&\quad + \Phi\Big(\frac{2C\|b\|_*}{\gamma^{1/\delta}}\Big)\sup_{t>0}\frac{1}{\Phi(1/t)}\big|\{v\in\mathbb{R}^n\colon M_\ell^\triangle(\mu_\lambda^{*,\rho}(f))(v) > t\}\big| \\
&\quad + \Phi\Big(\frac{2C\|b\|_*}{\gamma^{1/\delta}}\Big)\sup_{t>0}\frac{1}{\Phi(1/t)}\big|\{v\in\mathbb{R}^n\colon M^2(f)(v) > t\}\big| \\
&\le C_\delta\gamma L_\delta(f) + C_{\delta,\gamma,\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\big|\{v\in\mathbb{R}^n\colon M_\ell^\triangle(\mu_\lambda^{*,\rho}(f))(v) > t\}\big| \\
&\quad + C_{\delta,\gamma,\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\big|\{v\in\mathbb{R}^n\colon M^2(f)(v) > t\}\big|. \qquad (4.6)
\end{aligned}
$$

By (4.6) and applying Lemma 3.2 (ii) and Remark 3.1, we have

$$
\begin{aligned}
L_\delta(f) &\le C_\delta\gamma L_\delta(f) \\
&\quad + C_{\delta,\gamma,\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\big|\{v\in\mathbb{R}^n\colon M_\ell^{\triangle,\sharp}(\mu_\lambda^{*,\rho}(f))(v) > t\}\big| \\
&\quad + C_{\delta,\gamma,\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\big|\{v\in\mathbb{R}^n\colon M^2(f)(v) > t\}\big| \\
&\le C_\delta\gamma L_\delta(f) + C_{\delta,\gamma,\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\big|\{v\in\mathbb{R}^n\colon M^2(f)(v) > t\}\big|.
\end{aligned}
$$

Thus, for any $\gamma > 0$, we have

$$
\begin{aligned}
L_\delta(f) &\le C_\delta\gamma L_\delta(f) \\
&\quad + C_{\delta,\gamma,\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\big|\{v\in\mathbb{R}^n\colon M^2(f)(v) > t\}\big|. \qquad (4.7)
\end{aligned}
$$

Now we show that

$$
L_\delta(f) \le C_{\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\big|\{v\in\mathbb{R}^n\colon M^2(f)(v) > t\}\big|. \qquad (4.8)
$$

To do this, for any $b \in \mathrm{BMO}$, let b_k ($k = 1, 2, \ldots$) be the same as in the Lemma 3.8. Thus, $\|b_k\|_{L^\infty} \le k$ and $\|b_k\|_* \le \|b\|_*$. We first show that the above inequality (4.8) holds for b_k with constant C is independent of k. Since f is smooth and with compact support, we may assume $\mathrm{supp}(f) \subset B(0, R)$ ($R > 0$). Now, we fix k, by Lemma 3.8, we have $\mu_{\lambda,b_k}^{*,\rho}(f)(x) \le CkM(f)(x)$ for $|x| > 2R$. Since $t\Phi(1/t) \ge 1$ for $t > 0$ and $0 < \delta < 1$, we get

$$
\begin{aligned}
&\frac{1}{\Phi(1/t)}\big|\{v\in\mathbb{R}^n\colon M_\delta^{\triangle}(\mu^{*,\,\rho}_{\lambda,\,b_k}(f))(v)>t\}\big|\\
&\le \frac{1}{\Phi(1/t)}\left|\left\{v\in\mathbb{R}^n\colon M(\chi_{B(0,\,2R)}\mu^{*,\,\rho}_{\lambda,\,b_k}(f))(v)>\frac{t}{2}\right\}\right|\\
&\quad+\frac{1}{\Phi(1/t)}\left|\left\{v\in\mathbb{R}^n\colon M(\chi_{\mathbb{R}^n\setminus B(0,\,2R)}\mu^{*,\,\rho}_{\lambda,\,b_k}(f))(v)>\frac{t}{2}\right\}\right|\\
&\le \frac{2}{t\Phi(1/t)}\int_{B(0,\,2R)}|\mu^{*,\,\rho}_{\lambda,\,b_k}(f)(v)|dv\\
&\quad+\frac{1}{\Phi(1/t)}\left|\left\{v\in\mathbb{R}^n\colon M^2(f)(v)>\frac{t}{Ck}\right\}\right|\\
&\le C|B(0,\,2R)|^{1/2}\bigg(\int_{B(0,\,2R)}|\mu^{*,\,\rho}_{\lambda,\,b_k}(f)(v)|^2dv\bigg)^{1/2}\\
&\quad+\frac{C}{\Phi(1/t)}\int_{R^n}\Phi\bigg(\frac{Ck|f(v)|}{t}\bigg)dv\\
&\le C|B(0,\,2R)|^{1/2}\bigg(\int_{B(0,\,2R)}|f(v)|^2dv\bigg)^{1/2}\\
&\quad+C_k\int_{B(0,\,2R)}\Phi(|f(v)|)dv<\infty.
\end{aligned}
$$

The last inequality we use the L^2-boundedness of $\mu^{*,\,\rho}_{\lambda,\,b_k}(f)$ (see Theorem 3 with $w\equiv 1$) and the submultiplicative property of Φ. Since f is smooth with compact support, the last expression is finite for fixed k. Then we can choose a $\gamma>0$ with $\gamma<1/C_\delta$. Applying (4.7) for b_k, we get

$$
\begin{aligned}
(1-C_\delta\gamma)L_{\delta,b_k}(f)&\le C_{\delta,\,\gamma,\,\|b_k\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\big|\{v\in\mathbb{R}^n\colon M^2(f)(v)>t\}\big|\\
&\le C_{\delta,\,\gamma,\,\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\big|\{v\in\mathbb{R}^n\colon M^2(f)(v)>t\}\big|.
\end{aligned}
$$

That is,

$$
L_{\delta,\,b_k}(f)\le C_{\|b\|_*}\sup_{t>0}\frac{1}{\Phi(t)}\big|\{v\in\mathbb{R}^n\colon M^2(f)(v)>t\}\big|, \tag{4.9}
$$

where C is independent of k. Thus we get (4.8) by letting $k\to\infty$ in (4.9), and Lemma 4.1 follows from (4.1) and (4.8). □

Now let us return to the proof of the conclusion (ii) of Theorem 1. By homogeneity, it suffices to prove the conclusion (ii) holds for $\beta=1$.

Applying Lemma 4.1 and Lemma 3.3, we obtain

$$\begin{aligned}
&\big|\{x\in\mathbb{R}^n\colon |\mu^{*,\rho}_{\lambda,b}(f)(x)|>1\}\big|\\
&\le \sup_{\beta>0}\frac{1}{\Phi(1/\beta)}\big|\{x\in\mathbb{R}^n\colon |\mu^{*,\rho}_{\lambda,b}(f)(x)|>\beta\}\big|\\
&\le C_{\|b\|_*}\sup_{\beta>0}\frac{1}{\Phi(1/\beta)}\big|\{x\in\mathbb{R}^n\colon M^2(f)(x)>\beta\}\big|\\
&\le C_{\|b\|_*}\sup_{\beta>0}\frac{1}{\Phi(1/\beta)}\int_{\mathbb{R}^n}\frac{|f(x)|}{\beta}\Big(1+\log^+\Big(\frac{|f(x)|}{\beta}\Big)\Big)dx.
\end{aligned}$$

Thus we complete the proof of Theorem 1 (ii) for $m=1$.

Finally, we show how to prove the conclusion (ii) of Theorem 1 for $m>1$. In fact, the conclusion can be obtained by using the same idea as we used above and combining with the following lemmas, whose proofs will be omitted here.

Lemma 4.2 ([P, p. 179]) *There exists a positive constant C such that for any function f and for all $\beta>0$,*

$$\big|\{x\in\mathbb{R}^n\colon M^{m+1}(f)(x)>\beta\}\big| \le C\int_{\mathbb{R}^n}\frac{|f(x)|}{\beta}\Big(1+\log^+\Big(\frac{|f(x)|}{\beta}\Big)\Big)^m dx.$$

Lemma 4.3 *Let $b\in \mathrm{BMO}$, $0<\delta<\ell<1$, then there exists a positive constant C such that for all smooth function f with compact support,*

$$\begin{aligned}
&M^\sharp_\delta(\mu^{*,\rho}_{\lambda,b^m}(f))(x)\\
&\qquad\le C\sum_{j=0}^{m-1}\|b\|_* M_\ell(\mu^{*,\rho}_{\lambda,b^j}(f))(x)+C\|b\|_*^m M^{m+1}(f)(x),\\
&M^\sharp_\delta(S^\rho_{\Omega,b^m}(f))(x)\\
&\qquad\le C\sum_{j=0}^{m-1}\|b\|_* M_\ell(S^\rho_{\Omega,b^j}(f))(x)+C\|b\|_*^m M^{m+1}(f)(x),
\end{aligned}$$

where $M_\ell(f)=M(|f|^\ell)^{1/\ell}$.

Lemma 4.4 *Let $\Phi_m(t)=t(1+\log^+t)^m$ Then there exists a positive constant C, such that for any smooth function with compact support f,*

$$\sup_{t>0} \frac{1}{\Phi_m(1/t)} \big|\{x \in \mathbb{R}^n : |(\mu^{*,\rho}_{\lambda, b_m}(f))(x)| > t\}\big|$$
$$\leq C_{\|b\|_*^m} \sup_{t>0} \frac{1}{\Phi_m(1/t)} \big|\{x \in \mathbb{R}^n : M^{m+1}(f)(x) > t\}\big|.$$

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