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On positive solutions for p-Laplacian systems with sign-changing nonlinearities

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Abstract. We consider the existence and multiplicity of positive solutions to the quasilinear system

$$\begin{cases} -\Delta_{p_i} u_i = \mu_i a_i(x) f_i(u_1, \dots, u_n) \text{ in } \Omega,\ i = 1, \dots, n, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $\Delta_{p_i} u_i = \operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$, $p_i > 1$, μ_i are positive parameters, and f_i are allowed to change sign.

Key words: p-Laplace, systems, sign-changing, positive solutions.

1. Introduction

Consider the system

$$\begin{cases} -\Delta_{p_i} u_i = \mu_i a_i(x) f_i(u_1, \dots, u_n) \text{ in } \Omega,\ i = 1, \dots, n, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases} \tag{I}$$

where $\Delta_{p_i} u_i = div(|\nabla u_i|^{p_i-2}\nabla u_i)$, $p_i > 1$, $f_i : \mathbb{R}_+^n \to \mathbb{R}$, μ_i are positive constants, $i = 1, \dots, n$, and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.

The p-Laplace operator arises in the theory of non-Newtonian fluids, reaction-diffusion problems, flow through porous media, and petroleum extraction (see [9]). The system (I) with f_i nonnegative has been studied extensively in recent years (see e.g., [3], [5], [13] and the references therein). In this paper, we are interested in the case when f_i may take negative values. We shall establish the existence of positive solutions for (I) for large μ_i when f_i are positive near $\mathbf{0}$ or when f_i are eventually positive and p-sublinear at infinity. In particular, Theorems 1.1 and 1.2 below unify and extend results

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in [11], [18] to systems. Our approach is based on the method of sub- and supersolutions.

We make the following assumptions:

(A.1) $a_i \in L^\infty(\Omega)$, $a_i \geq 0$ and $a_i \not\equiv 0$, $i = 1, \ldots, n$.

(A.2) $f_i : [0, \infty] \times \cdots \times [0, \infty] \to \mathbb{R}$ are continuous for all i.

(A.3) There exist $r_1, \ldots, r_n > 0$ such that for all i, $f_i(u_1, \ldots, u_n) = 0$ if $u_i = r_i$, $u_j \in [0, r_j]$ for $j \neq i$ and $f_i(u_1, \ldots, u_n) > 0$ if $u_i \in (0, r_i)$, $u_j \in [0, r_j]$ for $j \neq i$.

(A.4) There exist nonnegative numbers k, A, L, γ with A, $L > 0$, $\gamma < 1/N$, such that for all i,

$$f_i(u_1, \ldots, u_n) \geq -k$$

for all $u = (u_1, \ldots, u_n)$, and

$$f_i(u_1, \ldots, u_n) \geq \frac{L}{u_i^\gamma} \text{ when } u_i > A.$$

(A.5) $\lim_{\|u\| \to \infty} \frac{f_i(u_1, \ldots, u_n)}{\|u\|^{p_i - 1}} = 0$ for all i, where $\|u\| = \max_{1 \leq i \leq n} |u_i|$, $u = (u_1, \ldots, u_n)$.

By a positive solution of (I), we mean a function $u = (u_1, \ldots, u_n) \in W_0^{1,p_1}(\Omega) \times \cdots \times W_0^{1,p_n}(\Omega)$ that satisfies (I) in the weak sense i.e.,

$$\int_\Omega |\nabla u_i|^{p_i - 2} \nabla u_i . \nabla \xi dx = \int_\Omega \mu_i a_i(x) f_i(u_1, \ldots, u_n) \xi dx \ \forall \xi \in W_0^{1,p_i}(\Omega),$$

and $u_i > 0$ in Ω for all i.

Theorem 1.1 *Let* (A.1)–(A.3) *hold. Then there exists* $\mu_0 > 0$ *such that problem* (I) *has a positive solution* $u = (u_1, \ldots, u_n)$ *when* $\min_{1 \leq i \leq n} \mu_i > \mu_0$. *Furthermore*

$$\|u_i\|_\infty \to r_i \text{ as } \min_{1 \leq i \leq n} \mu_i \to \infty$$

for $i = 1, \ldots, n$. *If, in addition,* $\lim_{\|u\| \to 0} \frac{f_i(u_1, \ldots, u_n)}{\|u\|^{p_i - 1}} = 0$ *for all* i, *then* (I) *has a second nontrivial nonnegative solution when* $\min_{1 \leq i \leq n} \mu_i$ *is large.*

Theorem 1.2 *Let* (A.1), (A.2), (A.4) *and* (A.5) *hold. Then there exists*

$\mu_0 > 0$ *such that problem* (I) *has a positive solution* $u = (u_1, \ldots, u_n)$ *when* $\min_{1 \le i \le n} \mu_i > \mu_0$. *Furthermore*

$$\|u_i\|_\infty \to \infty \text{ as } \min_{1 \le i \le n} \mu_i \to \infty$$

for $i = 1, \ldots, n$. *If, in addition,* $f_i \ge 0$ *and* $\lim_{\|u\| \to 0} \frac{f_i(u_1,\ldots,u_n)}{\|u\|^{p_i - 1}} = 0$ *for all* i, *then* (I) *has a second nonnegative nontrivial solution when* $\min_{1 \le i \le n} \mu_i$ *is large.*

Theorem 1.3 *Let* (A.1)–(A.2) *hold and suppose there exists* $C > 0$ *such that*

$$f_i(u_1, \ldots, u_n) \le C\|u\|^{p_i - 1}$$

for all u *and* i. *Then there exists* $\tilde{\mu}_0 > 0$ *such that* (I) *has no nonnegative nontrivial solutions when* $\max_{1 \le i \le n} \mu_i < \tilde{\mu}_0$.

Note that $\tilde{\mu}_0 \le \mu_0$.

Remark 1 Theorems 1.1 and 1.2 generalize and improve results in [11], [18]. In [18], the existence of at least two positive solutions to

$$-\Delta_p u = \lambda f(u) \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega \tag{*}$$

was established for λ large for $f \in C^1(\mathbb{R}^+)$ satisfying

(F.1) $f(0) = f(\beta) = 0$, $f > 0$ on $(0, \beta)$, $f < 0$ on (β, ∞), $\lim_{u \to 0} \frac{f(u)}{u^{p-1}} = 0$, and $(f(s)/s^{p-1})'' < 0$ on $(0, \beta)$ if $p > 2$,

or

(F.2) f is strictly increasing on $\mathbb{R}^+$, $f(0) = 0$, $\lim_{u \to 0} \frac{f(u)}{u^{p-1}} = 0$, and there exist α_1, $\alpha_2 > 0$, $\mu \in (0, p-1)$ so that $f(u) \le \alpha_1 + \alpha_2 u^\mu$ for $u \ge 0$.

In [11], it was proved that (*) has a positive solution for λ large for f satisfying $\lim_{u \to \infty} \frac{f(u)}{u^{p-1}} = 0$ and $f(u) > L > 0$ for u large. A second positive solution was established under the additional assumptions that $f \ge 0$ and $\lim_{u \to 0} \frac{f(u)}{u^{p-1}} = 0$.

Examples 1 Let a_i, $i = 1, 2$, satisfy (A.1) and let p_1, $p_2 > 1$. Consider the problem

$$\begin{cases} -\Delta_{p_1} u = \mu_1 a_1(x)\frac{(u+v)^{r_1}(1-u^2)}{1+uv} & \text{in } \Omega \\ -\Delta_{p_2} v = \mu_2 a_2(x)(u+v)^{r_2}(1-v)e^{-uv} & \text{in } \Omega, \end{cases}, \ u = v = 0 \text{ on } \partial\Omega. \quad (**)$$

Then it follows from Theorems 1.1 and 1.3 that

a. For all r_1, $r_2 \geq 0$, (**) has a positive solution when $\min(\mu_1, \mu_2)$ is large.

b. For $r_i > p_i - 1$, $i = 1, 2$, (**) has at least two positive solutions when $\min(\mu_1, \mu_2)$ is large.

c. For $r_i = p_i - 1$, $i = 1, 2$, (**) has no positive solutions when $\max(\mu_1, \mu_2)$ is small

Examples 2 The problem

$$\begin{cases} -\Delta_{p_1} u = \mu_1 a_1(x)\frac{(u+v)^{s_1}+A}{u^{\gamma_1}+1} & \text{in } \Omega \\ -\Delta_{p_2} v = \mu_2 a_2(x)\frac{(u+v)^{s_2}+B}{v^{\gamma_2}+1} & \text{in } \Omega, \end{cases}, \ u = v = 0 \text{ on } \partial\Omega,$$

where $\gamma_i \in (0, 1/N)$, $s_i \in (0, p_i - 1)$, $i = 1, 2$, and $A, B < 0$ has a positive solution when $\min(\mu_1, \mu_2)$ is large and no positive solutions when $\max(\mu_1, \mu_2)$ is small.

Examples 3 The problem

$$\begin{cases} -\Delta_{p_1} u = \mu_1 a_1(x)(1 - e^{-u^{\gamma_1}})e^{\frac{1}{1+uv}} & \text{in } \Omega \\ -\Delta_{p_2} v = \mu_2 a_2(x)(1 - e^{-v^{\gamma_2}})e^{\frac{1}{2+uv}} & \text{in } \Omega, \end{cases}, \ u = v = 0 \text{ on } \partial\Omega,$$

where $\gamma_i > p_i - 1$, $i = 1, 2$, has at least two positive solutions when $\min(\mu_1, \mu_2)$ is large by Theorem 1.2.

2. Preliminary results

Let $\Phi = (\Phi_1, \ldots, \Phi_n)$, $\Psi = (\Psi_1, \ldots, \Psi_n) \in \prod_{i=1}^n (W_0^{1,p_i}(\Omega) \cap L^\infty(\Omega))$ with $\Phi \leq \Psi$ in Ω i.e., $\Phi_i \leq \Psi_i$ in Ω for each $i \in \{1, \ldots, n\}$. Then we say that $\{\Phi, \Psi\}$ forms a system of sub-supersolutions for (I) if for each $i \in \{1, \ldots, n\}$, $\Phi_i \leq 0$ on $\partial\Omega$, $\Psi_i \geq 0$ on $\partial\Omega$,

$$\int_\Omega |\nabla \Phi_i|^{p_i - 2} \nabla \Phi_i . \nabla \xi dx \leq \int_\Omega \mu_i a_i(x) f_i(\tilde{\Phi}) \xi dx \quad \forall \xi \in W_0^{1,p_i}(\Omega), \ \xi \geq 0, \tag{2.1}$$

where $\tilde{\Phi} = (\tilde{\Phi}_1, \ldots, \tilde{\Phi}_n)$, $\tilde{\Phi}_i = \Phi_i$, $\tilde{\Phi}_k \in [\Phi_k, \Psi_k]$ for $k \neq i$, and

$$\int_\Omega |\nabla \Psi_i|^{p_i-2} \nabla \Psi_i . \nabla \xi dx \geq \int_\Omega \mu_i a_i(x) f_i(\tilde{\Psi}) \xi dx \quad \forall \xi \in W_0^{1,p_i}(\Omega),\ \xi \geq 0, \tag{2.2}$$

where $\tilde{\Psi} = (\tilde{\Psi}_1, \ldots, \tilde{\Psi}_n)$, $\tilde{\Psi}_i = \Psi_i$, $\tilde{\Psi}_k \in [\Phi_k, \Psi_k]$ for $k \neq i$. It is well known (see e.g., [4] or Appendix) that if such a system exists then (I) has a solution u with $\Phi \leq u \leq \Psi$ in Ω.

We shall denote the norms in $C^{1,\alpha}(\bar{\Omega})$ and $L^r(\Omega)$ by $|.|_{1,\alpha}$ and $\|.\|_r$ respectively.

Lemma 2.1 ([12]) *Let $f \in L^q(\Omega)$ for some $q > N$, and let $u \in W_0^{1,p}(\Omega)$ be the solution of*

$$\begin{aligned} -\Delta_p u &= f \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{2.3}$$

Then there exist $\alpha \in (0,1)$ and $C > 0$ independent of u and f such that $u \in C^{1,\alpha}(\bar{\Omega})$ and

$$|u|_{1,\alpha} \leq C \|f\|_q^{\frac{1}{p-1}}. \tag{2.4}$$

Furthermore, the map $K : L^q(\Omega) \to C^1(\bar{\Omega})$ defined by $Kf = u$ is compact.

Proof. Suppose that $p > N$. Then, by the Sobolev Imbedding Theorem, $u \in C(\bar{\Omega})$ and there exists $C_1 > 0$ such that

$$\|u\|_\infty \leq C_1 \|\nabla u\|_p. \tag{2.5}$$

Multiplying the equation in (2.3) by u and integrating gives

$$\|\nabla u\|_p^p = \int_\Omega f u dx \leq \|f\|_q \|u\|_\infty |\Omega|^{\frac{q-1}{q}}. \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$\|u\|_\infty \leq C_2 \|f\|_q^{\frac{1}{p-1}}, \tag{2.7}$$

where C_2 is independent of u, f.

If $p \le N$ then (2.7) was established in [10, Lemma 1.3]. Let v be the solution of

$$-\Delta v = f \quad \text{in } \Omega,\ v = 0 \text{ on } \partial\Omega.$$

Then $v \in W^{2,q}(\Omega)$ and since $q > N$, there exists $\beta \in (0,1)$ independent of v, f such that $v \in C^{1,\beta}(\bar{\Omega})$ and

$$|v|_{C^{1,\beta}} \le C_3 \|f\|_q,$$

where C_3 is a constant independent of v, f.

Let $\tilde{u} = u\|f\|_q^{-\frac{1}{p-1}}$, $\tilde{v} = v\|f\|_q^{-1}$. Then $\tilde{u}$ satisfies

$$\operatorname{div}\big(|\nabla \tilde{u}|^{p-2}\nabla \tilde{u} - \nabla \tilde{v}(x)\big) = 0 \text{ in } \Omega,\ \tilde{u} = 0 \text{ on } \partial\Omega.$$

Since $\|\tilde{u}\|_\infty \le C_2$ and

$$|\nabla \tilde{v}(x) - \nabla \tilde{v}(y)| \le C_3 |x-y|^\beta \text{ for all } x, y \in \Omega,$$

it follows from Lieberman [14] that there exists $\alpha \in (0,1)$ independent of u and f such that $|\tilde{u}|_{1,\alpha} \le C$.

Next, we verify that K is compact. In view of (2.4), we only need to show that K is continuous. Let $f_n \to f$ in $L^q(\Omega)$ and $u_n = Kf_n$. Then, by (2.4), (u_n) is bounded in $C^{1,\alpha}(\bar{\Omega})$. Multiplying the equation

$$-(\Delta_p u_n - \Delta_p u) = f_n - f \text{ in } \Omega$$

by $u_n - u$ and integrating, we obtain

$$\int_\Omega \big(|\nabla u_n|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla u\big).(u_n - u)dx = \int_\Omega (f_n - f)(u_n - u)dx.$$

From this, (2.4), and the inequality (see [16])

$$\big(|x|^{p-2}x - |y|^{p-2}y\big).(x-y) \ge \frac{c|x-y|^{\max(p,2)}}{(|x|+|y|)^{2-\min(p,2)}} \tag{2.8}$$

for all $x, y \in \mathbb{R}^n$, where c is a positive constant depending only on p, we obtain

$$\|\nabla(u_n - u)\|_r^r \le C\|f_n - f\|_q,$$

where $r = \max(p, 2)$ and C depends only on Ω, f, p, q. Thus $u_n \to u$ in $W_0^{1,r}(\Omega)$ and since $C^{1,\alpha}(\bar{\Omega})$ is compactly imbedded in $C^1(\bar{\Omega})$, $u_n \to u$ in $C^1(\bar{\Omega})$. This completes the proof of Lemma 2.1.

Lemma 2.2 *Let $p > 1$, $M > 0$, $q > N$ and f, $g \in L^q(\Omega)$ with $\|f\|_q$, $\|g\|_q < M$. Let u, v satisfy*

$$\begin{aligned} -\Delta_p u &= f \ \text{in } \Omega, \\ u &= 0 \ \text{on } \partial\Omega, \end{aligned} \quad , \quad \begin{aligned} -\Delta_p v &= g \ \text{in } \Omega, \\ v &= 0 \ \text{on } \partial\Omega. \end{aligned}$$

Then $|u - v|_{1,0} \to 0$ as $\|f - g\|_1 \to 0$.

Proof. By Lemma 2.1, there exist constants α, $C_M > 0$ such that u, $v \in C^{1,\alpha}(\bar{\Omega})$ and $|u|_{1,\alpha}$, $|v|_{1,\alpha} < C_M$.

Multiplying the equation

$$-\Delta_p u - (-\Delta_p v) = f - g \ \text{ in } \Omega$$

by $u - v$ and integrating gives

$$\int_\Omega \big(|\nabla u|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla u\big).\nabla(u - v)dx = \int_\Omega (f - g)(u - v)dx.$$

Hence it follows from (2.8) that

$$\int_\Omega |\nabla(u - v)|^r dx \le c_1\|f - g\|_1\|u - v\|_\infty \le c_2\|f - g\|_1,$$

where $r = \max(p, 2)$ and c_1, c_2 are constants depending only on p, M.

Thus $\|\nabla(u - v)\|_r \to 0$ as $\|f - g\|_1 \to 0$. Since $|u - v|_{1,\alpha} < 2C_M$ and $C^{1,\alpha}(\bar{\Omega})$ is compactly imbedded in $C^1(\bar{\Omega})$, Lemma 2.2 follows.

For each i, let $\lambda_{1,i}$ be the first eigenvalue of

$$\begin{aligned} -\Delta_{p_i} u &= \lambda a_i(x)|u|^{p_i - 2}u \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega, \end{aligned}$$

and let $\phi_{1,i}$ be the corresponding positive eigenfunction with $\|\phi_{1,i}\|_\infty = 1$. It is well known that $\lambda_{1,i} > 0$ and $\phi_{1,i} \in C^1(\bar{\Omega})$ (see e.g., [2]). Let ψ_i satisfy

$$-\Delta_{p_i}\psi_i = a_i(x) \text{ in } \Omega, \quad \psi_i = 0 \text{ on } \partial\Omega.$$

Then $\psi_i > 0$ in Ω and $\frac{\partial \psi_i}{\partial n} < 0$ on $\partial\Omega$, where n denotes the outer unit normal vector.

For $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in C(\bar{\Omega})^n$, we say that $u \ll v$ if there exists $\varepsilon > 0$ such that $u_i + \varepsilon\phi_{1,i} \leq v_i$ in Ω for all i.

We are now ready to give the proofs of the main results.

3. Proofs of main results

Proof of Theorem 1.1. Let $x_i \in \Omega$ be such that $\phi_{1,i}(x_i) = \|\phi_{1,i}\|_\infty = 1$ and D be such that $\bar{D} \subset \Omega$ and $x_i \in D$ for all i.

Let $0 < 2\varepsilon < \min_{1\leq i\leq n} r_i$, $d_i \equiv r_i - \varepsilon < c_i < r_i$, and let Φ_i be the solution of

$$-\Delta_{p_i}\Phi_i = \begin{cases} c_i^{p_i-1}\lambda_{1,i}a_i(x)\phi_{1,i}^{p_i-1} & \text{in } D \\ 0 & \text{in } \Omega\backslash\bar{D}, \end{cases}, \ \Phi_i = 0 \text{ on } \partial\Omega.$$

Since

$$-\Delta_{p_i}(c_i\phi_{1,i}) = c_i^{p_i-1}\lambda_{1,i}a_i(x)\phi_{1,i}^{p_i-1} \quad \text{in } \Omega,$$

it follows from the weak comparison principle [15] and the strong maximum principle [17] that $0 < \Phi_i \leq c_i\phi_{1,i}$ in Ω. By Lemma 2.2, $|\Phi_i - c_i\phi_{1,i}|_{C^1} \to 0$ as $|\Omega\backslash\bar{D}| \to 0$, where $|\Omega\backslash\bar{D}|$ denotes the Lebesgue measure of $\Omega\backslash\bar{D}$. Thus we can choose D so that

$$(c_i - \varepsilon)\phi_{1,i} \leq \Phi_i \leq c_i\phi_{1,i} \quad \text{in } \Omega$$

for all i. Let $\Phi \equiv \Phi_D = (\Phi_1, \ldots, \Phi_n)$ and $\Psi = (r_1, \ldots, r_n)$. We shall verify that $\{\Phi, \Psi\}$ forms a system of sub-supersolutions for (I). Let $\tilde{\Phi} = (\tilde{\Phi}_1, \ldots, \tilde{\Phi}_n)$, where $\tilde{\Phi}_i = \Phi_i$ and $\tilde{\Phi}_k \in [\Phi_k, r_k]$ for $k \neq i$. By (A.3), $f_i(\tilde{\Phi}) \geq 0$ in Ω and $f_i(\tilde{\Phi}) \geq m_i$ in D, where

$$m_i = \min\left\{f_i(x) : (c_i - \varepsilon)\min_{\bar{D}}\phi_{1,i} \leq x_i \leq c_i, \, 0 \leq x_j \leq r_j, \, j \neq i\right\} > 0.$$

Let $\mu_0 > 0$ be such that $\mu_0 m_i > \lambda_{1,i} c_i^{p_i-1}$ for all i and suppose $\min_{1\le i\le n}\mu_i > \mu_0$. For $\xi \in W_0^{1,p_i}(\Omega)$, $\xi \ge 0$, we have

$$\int_\Omega |\nabla\Phi_i|^{p_i-2}\nabla\Phi_i.\nabla\xi dx = \int_D c_i^{p_i-1}\lambda_{1,i}a_i(x)\phi_{1,i}^{p_i-1}\xi dx \le \mu_0 m_i \int_D a_i(x)\xi dx$$
$$\le \mu_i \int_D a_i(x) f_i(\tilde\Phi)\xi dx \le \mu_i \int_\Omega a_i(x) f_i(\tilde\Phi)\xi dx,$$

i.e., Φ satisfies (2.1). Also Ψ satisfies (2.2) because of (A.3), which proves the claim. Hence (I) has a solution $u = (u_1, \dots, u_n)$ with

$$(c_i - \varepsilon)\phi_{1,i} \le \Phi_i \le u_i \le r_i \text{ in } \Omega.$$

In particular, $r_i - 2\varepsilon \le \|u_i\|_\infty \le r_i$. Replacing 2ε by $\frac{2\varepsilon}{m}$, $m \in \mathbb{N}$, we obtain an increasing sequence $(\mu_{0,m})$ of positive numbers with $\mu_{0,1} = \mu_0$ such that (I) has a positive solution $u_m = (u_{m,1}, \dots, u_{m,n})$ with

$$r_i - \frac{2\varepsilon}{m} \le \|u_{m,i}\|_\infty \le r_i$$

for all i when $\min_{1\le i\le n}\mu_i > \mu_{0,m}$. Define $u = u_m$ if $\mu_{0,m} < \min_{1\le i\le n}\mu_i \le \mu_{0,m+1}$. Then clearly $\|u_i\|_\infty \to r_i$ as $\min_{1\le i\le n}\mu_i \to \infty$.

Suppose next that $\lim_{\|u\|\to 0}\frac{f_i(u_1,\dots,u_n)}{\|u\|^{p_i-1}} = 0$ for all i. Define $\bar f_i(u_1, \dots, u_n) = f_i(\bar u_1, \dots, \bar u_n)$, where $\bar u_j = \min(u_j^+, r_j)$, $u_j^+ = \max(u_j, 0)$, $j = 1, \dots, n$. Let $\varepsilon > 0$ and $\Phi_0 = (-\varepsilon, \dots, -\varepsilon)$, $\Psi_0 = (\varepsilon\psi_1, \dots, \varepsilon\psi_n)$, $\Psi_1 = (2r_1, \dots, 2r_n)$. Then, if ε is sufficiently small, $\Phi_0 \ll \Psi_0 \ll \Phi \ll \Psi_1$ in Ω. We shall verify that $\{\Phi_0, \Psi_0\}$ forms a system of sub-supersolutions for the system

$$\begin{cases} -\Delta_{p_i} u_i = \mu_i a_i(x)\bar f_i(u_1, \dots, u_n) \text{ in } \Omega,\ i = 1, \dots, n, \\ u_i = 0 \text{ on } \partial\Omega \end{cases} \tag{I*}$$

if ε is sufficiently small. Choose $\delta > 0$ so that $\mu_i\delta(\max_{1\le j\le n}\|\psi_j\|_\infty)^{p_i-1} < (1/2)^{p_i-1}$ for all i. Since $\lim_{\|u\|\to 0}\frac{f_i(u_1,\dots,u_n)}{\|u\|^{p_i-1}} = 0$ for all i, there exists $\varepsilon_0 > 0$ such that

$$f_i(z_1, \dots, z_n) \le \delta\|z\|^{p_i-1} \text{ for all } i \tag{3.1}$$

whenever $\|z\| < \varepsilon_0$, $z = (z_1, \ldots, z_n) \in \mathbb{R}^n_+$. Let $\varepsilon > 0$ be small enough so that $\varepsilon \max_{1\le j\le n} \|\psi_j\|_\infty < \min_{1\le j\le n}(\varepsilon_0, r_j)$. Let $\xi \in W_0^{1,p_i}(\Omega)$, $\xi \ge 0$, $v_i = \varepsilon\psi_i$, $v_k \in [-\varepsilon, \varepsilon\psi_k]$ for $k \ne i$. Then we have

$$\begin{aligned}\mu_i \int_\Omega a_i(x)\bar{f}_i(v_1, \ldots, v_n)\xi dx &= \mu_i \int_\Omega a_i(x) f_i(\bar{v}_1, \ldots, \bar{v}_n)\xi dx \\ &\le \mu_i\delta\|\bar{v}\|^{p_i-1} \int_\Omega a_i(x)\xi dx \le \mu_i\delta\Big(\varepsilon \max_{1\le j\le n} \|\psi_j\|_\infty\Big)^{p_i-1} \int_\Omega a_i(x)\xi dx \\ &\le \varepsilon^{p_i-1} \int_\Omega a_i(x)\xi dx = \int_\Omega |\nabla(\varepsilon\psi_i)|^{p_i-2}\nabla(\varepsilon\psi_i).\nabla\xi dx.\end{aligned}$$

On the other hand, if $w_i = -\varepsilon$, $w_k \in [-\varepsilon, \varepsilon\psi_k]$ for $k \ne i$, we have

$$\mu_i \int_\Omega a_i(x)\bar{f}_i(w_1, \ldots, w_n)\xi dx \ge 0 = \int_\Omega |\nabla w_i|^{p_i-2}\nabla w_i.\nabla\xi dx.$$

Thus $\{\Phi_0, \Psi_0\}$ is a system of sub-supersolutions of (I*). Similarly, it can be verified that $\{\Phi, \Psi_1\}$, $\{\Phi_0, \Psi_1\}$ are systems of sub-supersolutions of (I*).

It follows from the maximum principle that if u is a solution of (I*) then $0 \le u_i \le r_i$ in Ω, and hence u is a nonnegative solution of (I) with $\Phi_0 \ll u \ll \Psi_1$. We shall show next that any solution of (I*) with $\Phi_0 \le u \le \Psi_0$ in Ω satisfies $\Phi_0 \ll u \ll \Psi_0$. Clearly $\Phi_0 \ll u$ since $u \ge 0$. Let u be a solution of (I*) with $\Phi_0 \le u \le \Psi_0$ in Ω. By (3.1),

$$\begin{aligned}-\Delta_{p_i} u_i = \mu_i a_i(x)\bar{f}_i(u_1, \ldots, u_n) &\le \mu_i a_i(x)\delta\|\bar{u}\|^{p_i-1} \\ &\le \mu_i\delta\Big(\varepsilon \max_{1\le j\le n} \|\psi_j\|_\infty\Big)^{p_i-1} a_i(x),\end{aligned}$$

which implies

$$u_i \le (\mu_i\delta)^{\frac{1}{p_i-1}}\Big(\max_{1\le j\le n} \|\psi_j\|_\infty\Big)\varepsilon\psi_i \le (1/2)\varepsilon\psi_i \text{ in } \Omega,$$

i.e., $u \ll \Psi_0$.Using the strong comparison principle [6], [7], we have $\Phi_D \ll \Phi_{D_1}$ if $\bar{D} \subset D_1$, and therefore can assume that there exists a solution u of (I*) with $\Phi \ll u \ll \Psi_1$.

Define $S_i = \{u_i \in C(\bar{\Omega}) : \exists c > 0 \text{ such that } |u_i| \le c\phi_{1,i} \text{ in } \Omega\}$. Then S_i

is a Banach space with norm $\|u_i\|_{\phi_{1,i}} = \inf\{c > 0 : |u_i| \le c\phi_{1,i} \text{ in } \Omega\}$. Let $S = \prod_{i=1}^n S_i$ and define the following open sets in S:

$$\mathcal{O} = \{u \in S : \Phi_0 \ll u \ll \Psi_1\},$$
$$\mathcal{O}_1 = \{u \in S : \Phi_0 \ll u \ll \Psi_0\},$$
$$\mathcal{O}_2 = \{u \in S : \Phi \ll u \ll \Psi_1\}.$$

If every solution v of (I*) with $\Phi \le v \le \Psi_1$ in Ω satisfies $v \in \mathcal{O}_2$ then it follows from Amann's three-solution Theorem (see [1], [8] or Appendix) that (I*) has a solution $u_1 \in \mathcal{O}\backslash(\bar{\mathcal{O}}_1 \cup \bar{\mathcal{O}}_2)$. In particular, $u_1 \ne 0$, $u_1 \ne u$. On the other hand, if there exists a solution v of (I*) with $\Phi \le v \le \Psi_1$ in Ω but $v \notin \mathcal{O}_2$ then v is a second positive solution of (I). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let $k \ge 0$ be given by (A.4). By Lemma 2.1, there exists a solution $v_i \in C^{1,\alpha}(\bar{\Omega})$ of

$$-\Delta_{p_i} v_i = \frac{a_i(x)}{d^\gamma(x)} \text{ in } \Omega, \quad v_i = 0 \text{ on } \partial\Omega,$$

where $d(x)$ denotes the distance from x to $\partial\Omega$. Let c_0, $c_1 > 0$ be such that $c_0 d(x) \le v_i(x) \le c_1 d(x)$ for all i and $x \in \Omega$, and ϕ_i be the solution of

$$-\Delta_{p_i}\phi_i = \begin{cases} \frac{a_i(x)}{d^\gamma(x)} & \text{in } D_i \\ -\frac{m a_i(x)}{d^\gamma(x)} & \text{in } \Omega\backslash\bar{D}_i \end{cases}, \quad \phi_i = 0 \text{ on } \partial\Omega,$$

where $D_i = \left\{x \in \Omega : d(x) > \frac{4A}{c_0\mu_i^{\beta_i/(p_i-1)}}\right\}$, $\beta_i = \left(1 + \frac{\gamma}{p_i-1}\right)^{-1}$, and $m = k\left(\frac{4A}{c_0}\right)^\gamma$. Then $\phi_i \le v_i$ in Ω by the comparison principle. Since

$$\mu_i a_i(x) f_i(u_1, \dots, u_n) = \tilde{\mu}_i a_i(x) F_i(u_1, \dots, u_n),$$

where $\tilde{\mu}_i = \frac{\mu_i L}{2c_1^\gamma}$, $F_i(u_1, \dots, u_n) = \frac{2c_1^\gamma}{L} f_i(u_1, \dots, u_n)$, we can assume that $L > c_1^\gamma$.

By Lemma 2.2, $|\phi_i - v_i|_{C^1} \to 0$ as $\mu_i \to \infty$. Since $v_i > 0$ in Ω with $\frac{\partial v_i}{\partial n} < 0$ on $\partial\Omega$, there exists $\mu_0 > 0$ such that

$$\phi_i \geq \frac{1}{2} v_i \quad \text{in } \Omega$$

for all i provided that $\min_{1\leq i\leq n} \mu_i > \mu_0$, which we shall assume for the rest of the proof.

Define $\tilde{f}_i(z_1, \ldots, z_n) = \sup_{0\leq x_i \leq z_i} f_i(x_1, \ldots, x_n)$. By (A.5),

$$\lim_{\|u\|\to\infty} \frac{\tilde{f}_i(u_1, \ldots, u_n)}{\|u\|^{p_i-1}} = 0$$

for all i, and hence there exists $M > 0$ such that

$$\frac{\tilde{f}_i(M, \ldots, M)}{M^{p_i-1}} < \frac{\|\psi_i\|_\infty^{1-p_i} \mu_i^{-1}}{2^{p_i-1}}$$

for all i. Let $\Phi_i = \mu_i^{\frac{\beta_i}{p_i-1}} \phi_i$, $\Psi_i = M_i \psi_i$, where $M_i = M\|\psi_i\|_\infty^{-1}$ for all i. Then $\Phi \ll \Psi$ if M is large enough. We claim that $\Phi = (\Phi_1, \ldots, \Phi_n)$ and $\Psi = (\Psi_1, \ldots, \Psi_n)$ form a system of sub-supersolutions for (I). Indeed, for $\xi \in W_0^{1,p_i}(\Omega)$, $\xi \geq 0$, we have

$$\begin{aligned} \int_\Omega |\nabla\Phi_i|^{p_i-2}\nabla\Phi_i.\nabla\xi dx &= \mu_i^{\beta_i} \int_\Omega (-\Delta_{p_i}\phi_i)\xi dx \\ &= \mu_i^{\beta_i} \int_{D_i} \frac{a_i(x)}{d^\gamma(x)} \xi dx - \mu_i^{\beta_i} m \int_{\Omega\setminus\bar{D}_i} \frac{a_i(x)}{d^\gamma(x)} \xi dx. \end{aligned} \tag{3.2}$$

In D_i, we have

$$\mu_i^{\frac{\beta_i}{p_i-1}} \phi_i \geq \frac{\mu_i^{\frac{\beta_i}{p_i-1}} v_i}{2} \geq \frac{\mu_i^{\frac{\beta_i}{p_i-1}} c_0 d(x)}{2} > 2A,$$

and hence it follows from (A.4) that

$$f_i(\tilde{\Phi}) \geq \frac{L}{\mu_i^{\frac{\beta_i\gamma}{p_i-1}} \phi_i^\gamma} \geq \frac{L}{\mu_i^{\frac{\beta_i\gamma}{p_i-1}} v_i^\gamma} \geq \frac{L}{\mu_i^{\frac{\beta_i\gamma}{p_i-1}} c_1^\gamma d^\gamma(x)} \text{ in } D_i,$$

where $\tilde{\Phi}_i = \Phi_i$, $\tilde{\Phi}_k \geq \Phi_k$ for $k \neq i$. This implies

$$\mu_i \int_{D_i} a_i(x) f_i(\tilde{\Phi}) \xi dx \geq \frac{\mu_i^{1-\frac{\beta_i \gamma}{p_i - 1}} L}{c_1^\gamma} \int_{D_i} \frac{a_i(x)\xi}{d^\gamma(x)} dx$$

$$= \frac{\mu_i^{\beta_i} L}{c_1^\gamma} \int_{D_i} \frac{a_i(x)\xi}{d^\gamma(x)} dx \geq \mu_i^{\beta_i} \int_{D_i} \frac{a_i(x)}{d^\gamma(x)} \xi dx. \tag{3.3}$$

In $\Omega \backslash \bar{D}_i$, we have

$$\frac{\mu_i^{\beta_i} m}{d^\gamma(x)} \geq \frac{m c_0^\gamma}{(4A)^\gamma} \mu_i^{\beta_i + \frac{\beta_i \gamma}{p_i - 1}} = \frac{m c_0^\gamma \mu_i}{(4A)^\gamma} = k \mu_i,$$

which implies

$$-\mu_i^{\beta_i} m \int_{\Omega \backslash \bar{D}_i} \frac{a_i(x)}{d^\gamma(x)} \xi dx \leq -\mu_i k \int_{\Omega \backslash \bar{D}_i} a_i(x) \xi dx \leq \mu_i \int_{\Omega \backslash \bar{D}_i} a_i(x) f_i(\tilde{\Phi}) \xi dx. \tag{3.4}$$

Combining (3.2)–(3.4), we obtain

$$\int_\Omega |\nabla \Phi_i|^{p_i - 2} \nabla \Phi_i . \nabla \xi dx \leq \mu_i \int_\Omega a_i(x) f_i(\tilde{\Phi}) \xi dx,$$

i.e., Φ satisfies (2.1). Next,

$$\int_\Omega |\nabla \Psi_i|^{p_i - 2} \nabla \Psi_i . \nabla \xi dx = M_i^{p_i - 1} \int_\Omega (-\Delta_{p_i} \psi_i) \xi dx$$

$$= M^{p_i - 1} \|\psi_i\|_\infty^{1 - p_i} \int_\Omega a_i(x) \xi dx \geq \mu_i \tilde{f}_i(M, \ldots, M) \int_\Omega a_i(x) \xi dx$$

$$\geq \mu_i \int_\Omega a_i(x) \tilde{f}_i(\tilde{\Psi}) \xi dx \geq \mu_i \int_\Omega a_i(x) f_i(\tilde{\Psi}) \xi dx,$$

where $\tilde{\Psi}_i = \Psi_i$ and $0 \leq \tilde{\Psi}_k \leq \Psi_k$ for $k \neq i$. Thus $\{\Phi, \Psi\}$ forms a system of sub-supersolutions of (I), as claimed. Hence (I) has a solution u with $\Phi \leq u \leq \Psi$ in Ω. Clearly, $\|u_i\|_\infty \to \infty$ as $\min_{1 \leq i \leq n} \mu_i \to \infty$. We claim that any solution u of (I) with $0 \leq u \leq \Psi$ in Ω satisfies $u \ll \Psi$. Indeed, let u be a solution of (I) with $0 \leq u \leq \Psi$ in Ω. Then we have $0 \leq u_i \leq M_i \|\psi_i\|_\infty = M$ for all i. Hence

$$-\Delta_{p_i} u_i = \mu_i a_i(x) f_i(u_1,\dots,u_n) \le \mu_i a_i(x)\tilde{f}_i(u_1,\dots,u_n)$$
$$\le \mu_i a_i(x)\tilde{f}_i(M,\dots,M) \le \frac{\|\psi_i\|_\infty^{1-p_i}}{2^{p_i-1}} M^{p_i-1} a_i(x) \text{ in } \Omega,$$

which implies

$$u_i \le (1/2) M_i \psi_i \text{ in } \Omega \text{ for all } i.$$

Thus $u \ll \Psi$, as claimed. Next, suppose that $f_i \ge 0$ and $\lim_{\|u\|\to 0} \frac{f_i(u_1,\dots,u_n)}{\|u\|^{p_i-1}} = 0$ for all i. Let $\varepsilon > 0$, $\Phi_0 = (-\varepsilon,\dots,-\varepsilon)$, $\Psi_0 = (\varepsilon\psi_1,\dots,\varepsilon\psi_n)$. Then, if ε is sufficiently small, $\Phi_0 \ll \Psi_0 \ll \Phi \ll \Psi$. As in the proof of Theorem 1.1, we deduce that $\{\Phi_0, \Psi_0\}$ is a system of sub-supersolutions for the system

$$-\Delta_{p_i} u_i = \mu_i a_i(x) f_i\big(u_1^+,\dots,u_n^+\big) \text{ in } \Omega,\ u_i = 0 \text{ on } \partial\Omega,\ i = 1,\dots,n, \quad \text{(I')}$$

and any solution u of (I') with $\Phi_0 \le u \le \Psi_0$ in Ω satisfies $\Phi_0 \ll u \ll \Psi_0$. By modifying the proof for $\{\Phi,\Psi\}$, we see that $\{\Phi_0,\Psi\}$ is also a system of sub-supersolutions of (I') and any solution u of (I') with $\Phi_0 \le u \le \Psi$ in Ω satisfies $\Phi_0 \ll u \ll \Psi$. Also, by replacing A by $A/2$ in the above proof and using the strong comparison principle, we can assume that there exists a solution u of (I') with $\Phi \ll u \ll \Psi$. Hence we obtain, as in the proof of Theorem 1.1, a second nontrivial nonnegative solution u_1 of (I). This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Let u be a nonnegative solution of (I). Then

$$-\Delta_{p_i} u_i \le \mu_i C a_i(x) \|u\|_\infty^{p_i-1}$$

for all i, and the comparison principle implies

$$u_i \le (\mu_i C)^{\frac{1}{p_i-1}} \|u\|_\infty \psi_i \le (\mu_i C)^{\frac{1}{p-1}} \|u\|_\infty \|\psi_i\|_\infty \text{ in } \Omega$$

for all i, where $p = \min_{1\le i\le n} p_i$. Hence,

$$\|u\|_\infty \le (\mu C)^{\frac{1}{p-1}} \max_{1\le i\le n} \|\psi_i\|_\infty \|u\|_\infty,$$

where $\mu = \max_{1\le i\le n} \mu_i$. Thus, if $(\mu C)^{\frac{1}{p-1}} \max_{1\le i\le n} \|\psi_i\|_\infty < 1$ then $u = 0$, which completes the proof.

Appendix

Consider the system

$$\begin{cases} -\Delta_{p_i} u_i = g_i(u_1, \ldots, u_n) \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \quad i = 1, \ldots, n, \end{cases} \tag{II}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $p_i > 1$, and $g_i : \mathbb{R}^n \to \mathbb{R}$ are continuous, $i = 1, \ldots, n$.

For each i, let $\lambda_{1,i}$ be the first eigenvalue of

$$-\Delta_{p_i} u = \lambda a_i(x)|u|^{p_i-2}u \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega,$$

and let $\phi_{1,i}$ be the corresponding positive eigenfunction with $\|\phi_{1,i}\|_\infty = 1$.

For $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in C(\bar{\Omega})^n$, we say that $u \ll v$ if there exists $\varepsilon > 0$ such that $u_i + \varepsilon\phi_{1,i} \le v_i$ in Ω for all i.

Define $S_i = \{u_i \in C(\bar{\Omega}) : \exists c > 0 \text{ such that } |u_i| \le c\phi_{1,i} \text{ in } \Omega\}$. Then S_i is a Banach space with norm $\|u_i\|_{\phi_{1,i}} = \inf\{c > 0 : |u_i| \le c\phi_{1,i} \text{ in } \Omega\}$. Let $S = \prod_{i=1}^n S_i$ with norm $\|u\|_S = \max_{1\le i\le n} \|u_i\|_{\phi_{1,i}}$ and B_R denote the open ball centered at 0 with radius R in S.

For each $v = (v_1, \ldots, v_n) \in C(\bar{\Omega})^n$, let $u = (u_1, \ldots, u_n) = Tv$ be the solution of the system

$$-\Delta_{p_i} u_i = g_i(v_1, \ldots, v_n) \text{ in } \Omega, \quad u_i = 0 \text{ on } \partial\Omega, \ i = 1, \ldots, n.$$

Theorem A i) *Let* $\{\hat{\Phi}, \hat{\Psi}\}$ *be a system of sub-supersolutions for* (II). *Then* (II) *has a solution* u *with* $\hat{\Phi} \le u \le \hat{\Psi}$ *in* Ω. *If, in addition,* $\hat{\Phi} \ll \hat{\Psi}$ *and every solution* v *of* (II) *with* $\hat{\Phi} \le v \le \hat{\Psi}$ *in* Ω *satisfies* $\hat{\Phi} \ll v \ll \hat{\Psi}$ *then there exists* $R > 0$ *such that*

$$\deg(I - T, B_R \cap \mathcal{A}, 0) = 1,$$

where $\mathcal{A} = \{u \in S : \hat{\Phi} \ll u \ll \hat{\Psi}\}$.

ii) *Let* $\{\hat{\Phi}_0, \hat{\Psi}_0\}$, $\{\hat{\Phi}, \hat{\Psi}\}$, $\{\hat{\Phi}_0, \hat{\Psi}\}$ *be systems of sub-supersolutions for* (II). *Suppose* $\hat{\Phi}_0 \ll \hat{\Psi}_0 \ll \hat{\Phi} \ll \hat{\Psi}$ *and every solution* u *of* (II) *with* $\hat{\Phi}_0 \le$

$u \le \hat{\Psi}_0$ *(resp.* $\hat{\Phi} \le u \le \hat{\Psi}$*,* $\hat{\Phi}_0 \le u \le \hat{\Psi}$*) in* Ω *satisfies* $\hat{\Phi}_0 \ll u \ll \hat{\Psi}_0$ *(resp.* $\hat{\Phi} \ll u \ll \hat{\Psi}$*,* $\hat{\Phi}_0 \ll u \ll \hat{\Psi}$*). Then* (I) *has at least three solutions* u_1*,* u_2*,* u_3 *with* $u_1 \in \mathcal{A}_0$*,* $u_2 \in \mathcal{A}_1$*,* $u_3 \in \mathcal{A}_2 \backslash (\bar{\mathcal{A}}_0 \cup \bar{\mathcal{A}}_1)$*, where*

$$\mathcal{A}_0 = \{u \in S : \hat{\Phi}_0 \ll u \ll \hat{\Psi}_0\}, \quad \mathcal{A}_1 = \{u \in S : \hat{\Phi} \ll u \ll \hat{\Psi}\}$$
$$\mathcal{A}_2 = \{u \in S : \hat{\Phi}_0 \ll u \ll \hat{\Psi}\}.$$

Proof. i) Let $\hat{\Phi} = (\phi_1, \ldots, \phi_n)$, $\hat{\Psi} = (\psi_1, \ldots, \psi_n)$. Define $\hat{g}_i(u_1, \ldots, u_n) = g_i(\hat{u}_1, \ldots, \hat{u}_n)$, where $\hat{u}_i = \min(\max(u_i, \phi_i), \psi_i)$, $i = 1, \ldots, n$. Consider the system

$$-\Delta_{p_i} u_i = \hat{g}_i(u_1, \ldots, u_n) \text{ in } \Omega, \quad u_i = 0 \text{ on } \partial\Omega,\ i = 1, \ldots, n. \tag{II*}$$

For each $v = (v_1, \ldots, v_n) \in C(\bar{\Omega})^n$, let $u = (u_1, \ldots, u_n) = \hat{T}v$ be the solution of

$$-\Delta_{p_i} u_i = \hat{g}_i(v_1, \ldots, v_n) \text{ in } \Omega, \quad u_i = 0 \text{ on } \partial\Omega,\ i = 1, \ldots, n.$$

Then $\hat{T} : S \to S$ is a bounded compact operator and therefore there exists $R > 0$ such that $\deg(I - \hat{T}, B_R, 0) = 1$.

Thus $\hat{T}$ has a fixed point u in $B(0, R)$. We shall show that $\hat{\Phi} \le u \le \hat{\Psi}$ in Ω. Let $\xi = (\phi_i - u_i)^+$ and suppose $\xi \not\equiv 0$. Then $\xi \in W_0^{1,p_i}(\Omega)$, $\xi \ge 0$, and

$$\begin{aligned}\int_{\{x : u_i(x) < \phi_i(x)\}} |\nabla u_i|^{p_i - 2} \nabla u_i . \nabla \xi dx &= \int_\Omega |\nabla u_i|^{p_i - 2} \nabla u_i . \nabla \xi dx \\ = \int_\Omega \hat{g}_i(u_1, \ldots, u_n) \xi dx &= \int_{\{x : u_i(x) < \phi_i(x)\}} g_i(\tilde{u}_1, \ldots, \tilde{u}_n) \xi dx \\ &\ge \int_{\{x : u_i(x) < \phi_i(x)\}} |\nabla \phi_i|^{p_i - 2} \nabla \phi_i . \nabla \xi dx,\end{aligned}$$

where $\tilde{u}_i = \phi_i$, $\tilde{u}_k \in [\phi_k, \psi_k]$ for $k \ne i$. Hence

$$\int_{\{x : u_i(x) < \phi_i(x)\}} \big(|\nabla u_i|^{p_i - 2} \nabla u_i - |\nabla \phi_i|^{p_i - 2} \nabla \phi_i\big) . \nabla (u_i - \phi_i) dx \le 0,$$

a contradiction. Thus $\xi \equiv 0$ i.e., $u_i \ge \phi_i$ in Ω. Similarly, we have $u_i \le \psi_i$ in Ω. Thus u is a solution of (II) with $\hat{\Phi} \le u \le \hat{\Psi}$ in Ω. If $\hat{\Phi} \ll \hat{\Psi}$ and every solution v of (II) with $\hat{\Phi} \leq v \leq \hat{\Psi}$ in Ω satisfies $\hat{\Phi} \ll v \ll \hat{\Psi}$ then we deduce from the excision property of the degree that

$$1 = \deg(I - \hat{T}, B_R, 0) = \deg(I - \hat{T}, B_R \cap \mathcal{A}, 0) = \deg(I - T, B_R \cap \mathcal{A}, 0).$$

ii) By (i), there exists $R > 0$ such that

$$\deg(I - T, B_R \cap \mathcal{A}_k, 0) = 1, \ k = 0, 1, 2.$$

Consequently, (II) has solutions $u_1 \in \mathcal{A}_0$, $u_2 \in \mathcal{A}_1$. Since

$$\begin{aligned}\deg(I - T, B_R \cap \mathcal{A}_2, 0) &= \deg(I - T, B_R \cap \mathcal{A}_0, 0) + \deg(I - T, B_R \cap \mathcal{A}_1, 0) \\ &\quad + \deg\big(I - T, B_R \cap (\mathcal{A}_2 \backslash (\bar{\mathcal{A}}_0 \cup \bar{\mathcal{A}}_1)\big),\end{aligned}$$

it follows that

$$\deg\big(I - T, B_R \cap (\mathcal{A}_2 \backslash (\bar{\mathcal{A}}_0 \cup \bar{\mathcal{A}}_1)\big) = -1,$$

and the existence of a third solution $u_3 \in \mathcal{A}_2 \backslash (\bar{\mathcal{A}}_0 \cup \bar{\mathcal{A}}_1)$ follows.

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