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On closed manifolds which admit codimension one locally free actions of nilpotent Lie groups

Dedicated to Professor Toshiyuki Nishimori on his sixtieth birthday

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Abstract. We show that if a connected closed orientable manifold M admits a codimension one locally free smooth action ϕ of a connected nilpotent Lie group such that any orbit of ϕ is non-compact, then M is homeomorphic to a nilmanifold. And as an example of such an action, we study also a homogeneous action.

Key words: locally free action, foliation, nilpotent Lie group.

1. Introduction

Let $\phi : G \times M \to M$ be a smooth action of a connected Lie group G on a connected manifold M. If ϕ is locally free (i.e., the isotropy subgroup G_x at each point $x \in M$ is discrete in G), then the set $\mathcal{F}_\phi$ of all orbits of ϕ determines a foliated structure on M. We call $\mathcal{F}_\phi$ the *orbit foliation* of ϕ. And ϕ is called a *codimension one* action if any orbit of ϕ has codimension one. Therefore the study of codimension one locally free actions is closely connected with the study of codimension one foliations. And moreover it is connected with the study of discrete subgroups of Lie groups. As well in this paper, we will work in these fields.

In case that ϕ is a codimension one locally free smooth action of a nilpotent Lie group, dynamical properties of $\mathcal{F}_\phi$ and some topological properties of M are obtained in [3] and [5]. For the topological properties of M, we have the following theorem which is a finer version of Theorem (2.7) of [3] and a general version of Theorem (2.5) of [5].

Theorem 1.1 *Suppose that a conncted closed orientable manifold M admits a codimension one locally free smooth action ϕ of a connected nilpotent Lie group such that any orbit of ϕ is non-compact. Then M is homeomorphic to a nilmanifold.*

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To give the proof of this theorem is the purpose of this paper.

Unless otherwise specified, we consider in the smooth (C^∞) category. Therefore all maps and all actions are smooth.

2. Homogeneous actions

In this section we introduce an example of codimension one locally free actions of nilpotent Lie groups, which suggests our result, Theorem 1.1.

Let $\widetilde{G}$ be a connected and simply connected Lie group of dimension $n+1$. Let G be a connected Lie subgroup of $\widetilde{G}$ and let Γ be a discrete subgroup of $\widetilde{G}$. Then the left action ϕ of G on the homogeneous space $M = \widetilde{G}/\Gamma$ is defined by

$$\phi(g, a\Gamma) = ga\Gamma \quad (g \in G, a\Gamma \in \widetilde{G}/\Gamma).$$

This ϕ is called a *homogeneous action.* Since clearly ϕ is locally free, we obtain the orbit foliation $\mathcal{F}_\phi$. In this case, $\mathcal{F}_\phi$ is called a *homogeneous foliation.*

Now we assume that Γ is a uniform subgroup, i.e., $\widetilde{G}/\Gamma$ is compact. Then we have the following theorem.

Theorem 2.1 *Let $\phi : G \times \widetilde{G}/\Gamma \to \widetilde{G}/\Gamma$ be a codimension one homogeneous action of a nilpotent Lie subgroup G. Assume that $\mathcal{F}_\phi$ has no compact leaves. Then $\widetilde{G}/\Gamma$ is a nilmanifold.*

We need a lemma for the proof. First notice that $\dim(G) = n$ and $\widetilde{G}$ is unimodular.

Lemma 2.2 *Let $\mathfrak{g}$ be the subalgebra of a unimodular Lie algebra $\tilde{\mathfrak{g}}$ (i.e., $\mathrm{tr}(\mathrm{ad}(X)) = 0$ for all $X \in \tilde{\mathfrak{g}}$). If $\mathfrak{g}$ is of codimension one and unimodular, then $\mathfrak{g}$ is an ideal of $\tilde{\mathfrak{g}}$.*

Proof. Let $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{l}$ denote the direct sum decomposition as a vector space and let $p : \tilde{\mathfrak{g}} \to \mathfrak{l}$ be the projection to the second term. Choose a basis L of $\mathfrak{l}$.

Suppose that $\mathfrak{g}$ is not an ideal. Then there exist an element $X \in \tilde{\mathfrak{g}}$ and a real number $\alpha (\neq 0)$ such that X is contained in $\mathfrak{g}$ and $p(\mathrm{ad}(X)(L)) = \alpha L$. It follows that

$$\mathrm{tr}(\mathrm{ad}(X)) = \mathrm{tr}(\mathrm{ad}(X)|_{\mathfrak{g}}) + \alpha = 0 + \alpha = \alpha \neq 0.$$

This contradicts the assumption that $\tilde{\mathfrak{g}}$ is unimodular. Therefore $\mathfrak{g}$ is an ideal. $\square$

Proof of Theorem 2.1. Let $\tilde{\mathfrak{g}}$ be the Lie algebra of $\widetilde{G}$ and let $\mathfrak{g}$ be the Lie algebra of G. Since $\mathfrak{g}$ is nilpotent, it is unimodular. From Lemma 2.2, it follows that $\mathfrak{g}$ is an ideal of $\tilde{\mathfrak{g}}$. Therefore G is a normal subgroup of $\widetilde{G}$. Furthermore, since $\widetilde{G}/G$ is one dimensional, it is abelian and therefore $\widetilde{G}$ is solvable.

Now suppose that $\widetilde{G}$ is not nilpotent. Since G is of codimension one in $\widetilde{G}$, G is the maximal connected normal nilpotent subgroup of $\widetilde{G}$. Hence $G/(G \cap \Gamma)$ is compact (cf., [7, Theorem 3.3]). This contradicts the assumption that $\mathcal{F}_\phi$ has no compact leaves. Therefore $\widetilde{G}$ is nilpotent and so M is a nilmanifold. $\square$

Theorem 2.1 is a stronger version of Theorem 1.1 for homogeneous foliations.

3. Fibrations of M by the orbit foliations

We assume that G is a conncted simply connected nilpotent Lie group of dimension n and M is a connected orientable closed manifold of dimension $n+1$.

And assume that $\mathcal{F}_\phi$ contains no compact leaves, i.e., ϕ has no compact orbits. Denote by $\widehat{G}_x$ the Malcev closure of the isotropy subgroup G_x in G (i.e., $\widehat{G}_x$ is the unique connected closed subgroup of G containing G_x so that $\widehat{G}_x/G_x$ is compact). Some dynamical properties of $\mathcal{F}_\phi$ and the results with respect to $\widehat{G}_x$ are described in [3]. We quote some results.

Lemma 3.1 *With the notation and hypotheses above*

(1) *All leaves of $\mathcal{F}_\phi$ are dense in M and have trivial holonomy groups.*
(2) *$\widehat{G}_x$ is a fixed normal subgroup of G which is independent of the choice of x, and contains the commutator $[G, G]$.*

Now let $N = \widehat{G}_x$. By restricting ϕ to $N \times M$, we obtain a locally free action ϕ_N of N

$$\phi_N : N \times M \to M$$

and its orbit foliation $\mathcal{F}_{\phi_N}$. All leaves of $\mathcal{F}_{\phi_N}$ are compact and have trivial holonomy groups. Therefore we can define a fiber bundle $p_1 : M \to M_1$ whose fibers are leaves of $\mathcal{F}_{\phi_N}$. Since N is a normal subgroup of G, ϕ induces a codimension one locally free action ϕ_1 of the abelian Lie group $G_1 = G/N$ on M_1 such that the following diagram is commutative

$$\begin{tikzcd} G \times M \arrow[r, "\phi"] \arrow[d, "\pi_1 \times p_1"'] & M \arrow[d, "p_1"] \\ G_1 \times M_1 \arrow[r, "\phi_1"] & M_1 \end{tikzcd}$$

where $\pi_1 : G \to G_1 (= G/N)$ is the canonical homomorphism (see [3, Section 2.5 and Section 2.6]).

Furthermore, the above fact can be generalized to a closed subgroup K of N which satisfies some appropriate conditions. Denote by ϕ_K the restriction of ϕ to $K \times M$.

Lemma 3.2 *Assume that the connected closed subgroup K of N is normal in G and $G_{x_0} \cap K$ is a uniform subgroup of K for some $x_0 \in M$. Then, for any $x \in M$, $G_x \cap K$ is a uniform subgroup of K and the orbit foliation $\mathcal{F}_{\phi_K}$ of ϕ_K defines a bundle structure on M.*

Proof. We give only a short sketch of the proof.

First, on the tangent bundle of $\mathcal{F}_\phi$, induce the Riemanian metric through ϕ from the right invariant Riemanian metric of G and extend it to the tangent bundle of M.

Next, let L_{x_0} be the leaf of $\mathcal{F}_{\phi_K}$ through x_0. Consider the volume of each leaf of $\mathcal{F}_{\phi_K}$. Since K is normal in the nilpotent Lie Group G, the adjoint representation of G on the Lie algebra of K is unipotent and $\mathcal{F}_{\phi_K}$ is invariant under the action ϕ of G. Therefore the volume of each leaf of $\mathcal{F}_{\phi_K}$ is invariant under the action ϕ of G. It follows that the union of compact leaves which is deffeomorphic to L_{x_0} and have the same volume as L_{x_0} is dense in M. By this fact, we can see the holonomy group of L_{x_0} is trivial. Therefore, from the stability theorem of compact leaves (cf., [2]), we have a regular neighborhood V of L_{x_0} such that $\mathcal{F}_{\phi_K}|_V$ is a product foliation.

Again from invariance of $\mathcal{F}_{\phi_K}$ under ϕ and denseness of the orbit of ϕ, we can conclude that $\mathcal{F}_{\phi_K}$ defines a bundle structure on M. $\square$

By Lemma 3.2, we see the following proposition.

Proposition 3.3 *Assume that there is a connected closed subgroup K of N such that K is normal in G and homogeneous space $F_1 = K/(G_{x_0} \cap K)$ is compact for some $x_0 \in M$. Then, there exist a fiber bundle structure $p_1 : M \to M_1$ whose fibers are diffeomorphic to F_1 such that ϕ induces a codimension one locally free action ϕ_1 of the nilpotent Lie group $G_1 = G/K$ on M_1 and the following diagram is commutative*

$$\begin{array}{ccc} G \times M & \xrightarrow{\phi} & M \\ {\scriptstyle \pi_1 \times p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ G_1 \times M_1 & \xrightarrow{\phi_1} & M_1 \end{array}$$

where $\pi_1 : G \to G_1$ is the canonical homomorphism.

As an example of such K which satisfies the assumption of Proposition 3.3, we can take the center C of N. In fact, it follows from the result for the uniform subgroups of nilpotent Lie groups (see [7, Corollary 1 to Theorem 2.3]) that $C/(G_x \cap C)$ is compact.

4. Actions of G on the spaces of lattices

We retain the notation of the previous section and fix a point $x_0 \in M$.

Assume that K is contained in the center C of N and satisfies the assumption of Proposition 3.3 for x_0, i.e., K is a connected closed normal subgroup of G which is contained in C and $G_{x_0} \cap K$ is uniform in K. Since K is simply connected, we can identify K with its Lie algebra $\mathfrak{k}$ through the exponential map. Furthermore, fixing an appropriate basis of $\mathfrak{k}$, we can identify $\mathfrak{k}$ with $\mathbf{R}^k$ and accordingly identify $G_{x_0} \cap K$ with an unimodular lattice in $\mathbf{R}^k$. Therefore, denoting by $\mathcal{L}(k)$ the space of unimodular lattices in $\mathbf{R}^k$, we have $G_{x_0} \cap K \in \mathcal{L}(k)$.

Now, there exists the natural action of $\mathrm{SL}(k, \mathbf{R})$ on $\mathcal{L}(k)$ and $\mathcal{L}(k) \cong \mathrm{SL}(k, \mathbf{R})/\mathrm{SL}(k, \mathbf{Z})$. We denote the natural action by ψ. On the other hand, by restricting each $\mathrm{Ad}(g)$ to $\mathfrak{k}$ for the adjoint representation Ad of G, we obtain the homomorphism

$$\mathrm{Ad}_{\mathfrak{k}} : G \to \mathrm{SL}(k, \mathbf{R}).$$

Although G can act directly on $\mathcal{L}(k)$ by the adjoint representation, we will consider $\psi(\mathrm{Ad}_{\mathfrak{k}}(g), \Lambda)$ for $\Lambda \in \mathcal{L}(k)$ by way of $\mathrm{Ad}_{\mathfrak{k}}$ and ψ.

Next we define a map $\chi : M \to \mathcal{L}(k)$ by $\chi(x) = G_x \cap K$. By Proposition 3.3, it is clear that χ is well-defined and smooth. Then we obtain the following lemma.

Lemma 4.1 *With the notation and hypotheses above*

(1) *The diagram*

$$\begin{array}{ccc} G \times M & \xrightarrow{\phi} & M \\ {\scriptstyle \mathrm{Ad}_{\mathfrak{k}} \times \chi}\downarrow & & \downarrow{\scriptstyle \chi} \\ \mathrm{SL}(k, \mathbf{R}) \times \mathcal{L}(k) & \xrightarrow{\psi} & \mathcal{L}(k) \end{array}$$

is commutative.

(2) *The image U of* $\mathrm{Ad}_{\mathfrak{k}}$ *is an abelian and unipotent subgroup of* $\mathrm{SL}(k, \mathbf{R})$.

Proof. (1) is clear from the definition of χ.

(2) Let H be the kernel of $\mathrm{Ad}_{\mathfrak{k}}$. Then H is the centralizer of K in G. Hence H contains N and so $[G, G]$. Therefore $U \cong G/H$ is abelian.

Since G is nilpotent, it is clear that U is unipotent. $\square$

5. The proof of the main theorem

We retain the notation of the preceding two sections.

Recall that K is a connected closed normal subgroup of G which is contained in C and $G_{x_0} \cap K$ is uniform in K. Clearly there is a minimal one in the set of connected closed normal subgroups having same properties as K. Therefore we may assume that K is minimal. First we will show that K is contained in the center of G.

Let $\psi_U : U \times \chi(M) \to \chi(M)$ denote the restriction of ψ to $U \times \chi(M)$ and let $\Gamma_o = \chi(x_0)(= G_{x_0} \cap K)$. Then, since by Lemma 4.1 the diagram

$$\begin{array}{ccc} G \times M & \xrightarrow{\phi} & M \\ {\scriptstyle \mathrm{Ad}_{\mathfrak{k}} \times \chi}\downarrow & & \downarrow{\scriptstyle \chi} \\ U \times \chi(M) & \xrightarrow{\psi_U} & \chi(M) \end{array}$$

is commutative, it follows that each orbit of ψ_U is dense in $\chi(M)$. And moreover the following is satisfied.

Claim 1 *ψ_U is a free action.*

Proof. Suppose that ψ_U is not free. Then, by denseness of the orbit and abelianity of U, we may assume that $u \cdot \mathit{\Gamma}_o (= \psi_U(u, \Gamma_0)) = \mathit{\Gamma}_o$ for some $u (\neq E) \in U$ (where E is the identity matrix).

Since the linear transformation u of $\mathfrak{k}$ maps $\mathit{\Gamma}_o$ onto $\mathit{\Gamma}_o$, the nilpotent transformation $u - E$ maps $\mathit{\Gamma}_o$ into $\mathit{\Gamma}_o$. Letting $\mathfrak{h}$ be the real span of $(u - E)(\mathit{\Gamma}_o)$, it follows that $\dim(\mathfrak{h}) < \dim(\mathfrak{k})$. Moreover, since $(u - E)(\mathit{\Gamma}_o)$ is a lattice in $\mathfrak{h}$, $G_{x_o} \cap \mathfrak{h} = \mathit{\Gamma}_o \cap \mathfrak{h}$ is a lattice in $\mathfrak{h}$.

Clearly $\mathfrak{h}$ is a subalgebra of the Lie algebra $\mathfrak{c}$ of C, but we claim that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. For the proof of this, it suffices to show that $a(\mathfrak{h}) = \mathfrak{h}$ for any $a \in U$.

Since U is abelian,

$$a(\mathfrak{h}) = a((u - E)(\mathfrak{k})) = (u - E)(a(\mathfrak{k})) = (u - E)(\mathfrak{k}) = \mathfrak{h}.$$

This shows that $\mathfrak{h}$ is an ideal. Let H be the Lie subgroup corresponding to $\mathfrak{h}$. Then H is strictly contained in K and satisfies the same conditions as K. This contradicts minimality of K. □

Notice that $\dim(U) = 0$ if and only if K is contained in the center of G, and equivalently $\chi(M)$ consists of a single point.

Now we assume that $\dim(U) \neq 0$. Then, Ratner's orbit closure theorem ([8, Theorem A]) says that there exists a closed connected subgroup $W \subset \mathrm{SL}(k, \mathbf{R})$ such that $W \supset U$ and $W \cdot \mathit{\Gamma}_o = \overline{U \cdot \mathit{\Gamma}_o}$. Letting W_0 denote the isotropy subgroup of W at $\mathit{\Gamma}_o$, it follows that $\chi(M) \cong W/W_0 = W/(W \cap \gamma_0 \mathrm{SL}(k, \mathbf{Z}) \gamma_0^{-1})$, where $\gamma_0 \in \mathrm{SL}(k, \mathbf{R})$ is a representative of $\mathit{\Gamma}_o \in \mathrm{SL}(k, \mathbf{R})/\mathrm{SL}(k, \mathbf{Z}) = \mathcal{L}(k)$. Therefore $\chi(M)$ is a connected closed manifold and the action $\psi_U : U \times \chi(M) \to \chi(M)$ is a homogeneous action. By Claim 1, we see that ψ_U is also a codimension one action as ϕ. Thus, from Theorem 2.1, it follows that $\chi(M)$ is a nilmanifold and W is a nilpotent Lie group. Moreover we obtain the following strict result.

Claim 2 *W is an abelian Lie group.*

Proof. Since U is a simply connected abelian Lie group, by [9], the fundamental group of $\chi(M)$ is a free abelian group. Therefore the universal cover

$\widetilde{W}$ of W contains a uniform discrete subgroup which is abelian. From [7, Theorem 2.3], it follows that $\widetilde{W}$ is abelian and so W is abelian. $\square$

By the fact that W is abelian and contains unipotent subgroup and $\dim(W) \geq 2$, we see that there is an element $h \in W$ such that $h \cdot \Gamma_{\mathrm{o}} = \Gamma_{\mathrm{o}}$ and the nilpotent part h_n of the additive Jordan decomposition of h is nontrivial. Selecting another basis of $\mathfrak{k}$ if necessary, we may consider h to be a integer matrix. Then h_n is a matrix whose entries are all rational number (cf., [1, Proposition 4.2]). Therefore we can choose an integer $m(\neq 0)$ such that all entries of mh_n are integers. It follows that $\{\mathbf{0}\} \neq mh_n(\Gamma_{\mathrm{o}}) \subsetneq \Gamma_{\mathrm{o}}$. Let $\Gamma' = mh_n(\Gamma_{\mathrm{o}})$ and $\mathfrak{k}' = mh_n(\mathfrak{k})$. Clearly Γ' is a lattice in $\mathfrak{k}'$ and $0 < \dim(\mathfrak{k}') < \dim(\mathfrak{k})$. Moreover, we see the following.

Claim 3 *$\mathfrak{k}'$ is an ideal of $\mathfrak{g}$*

Proof. For the proof, it suffices to show that $a(\mathfrak{k}') = \mathfrak{k}'$ for any $a \in U$. Since W is abelian, it follows that h_n is commutative with any $a \in U$. Therefore

$$a(\mathfrak{k}') = a(mh_n(\mathfrak{k})) = mh_n(a(\mathfrak{k})) = mh_n(\mathfrak{k}) = \mathfrak{k}' \qquad \square$$

By the same argument as the proof of Claim 1, nontriviality of $\mathfrak{k}'$ induce a contradiction. Hence we conclude that $\dim(U) = 0$. This proves the following lemma.

Lemma 5.1 *K is contained in the center of G.*

Notice that $\dim(K) = 1$. Applying Proposition 3.3 to K, we obtain a fiber bundle $p_1 : M \to M_1$ whose fibers are diffeomorphic to $\mathbf{S}^1$ and a codimension one locally free action ϕ_1 of $G_1 = G/K$ on M_1 such that the diagram

$$\begin{array}{ccc} G \times M & \xrightarrow{\phi} & M \\ {\scriptstyle \pi_1 \times p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ G_1 \times M_1 & \xrightarrow{\phi_1} & M_1 \end{array}$$

is commutative, where $\pi_1 : G \to G_1$ is the canonical homomorphism. Then, since K is contained in the center G, the restricted action ϕ_K induce a $\mathbf{S}^1$ action on M. Therefore $p_1 : M \to M_1$ is a principal $\mathbf{S}^1$-bundle. Continuing the same arguments as above until the reduced action is free, we obtain the following lemma.

Lemma 5.2 *There exisits a sequence of principal* $\mathbf{S}^1$*-bundles*

$$M = M_0 \xrightarrow{p_1} M_1 \xrightarrow{p_2} M_2 \xrightarrow{p_3} \cdots \xrightarrow{p_r} M_r$$

such that each M_i *admits a codimension one locally free action* ϕ_i *of a connected nilpotent Lie group and* ϕ_r *is a free action.*

For the free action ϕ_r on M_r, from the result of [4] and [9] it follows that M_r is homeomorphic to a torus, therefore homeomorphic to a nilmanifold.

On the other hand, for M_i $(i < r)$, we apply the following lemma which is deduced from a result for the cohomology of a nilmanifold (cf., [6] and [7]).

Lemma 5.3 *Let* $p : E \to B$ *be a principal* $\mathbf{S}^1$*-bundle such that the base space* B *is deffeomorphic* (*resp. homeomorphic*) *to a nilmanifold. Then the total space* E *is deffeomorphic* (*resp. homeomorphic*) *to a nilmanifold.*

By Lemma 5.2 and Lemma 5.3, we see that M is homeomorphic to a nilmanifold. $\square$

Remark 5.4 (1) By simple observation, we notice that a non-orientable closed manifold does not admit a codimension one locally free action of a nilpotent Lie group whose orbits are non-compact. Therefore the assumption of orientability for M is unnecessary.

(2) For a codimension one action ϕ, it is easily shown that the set $\mathcal{F}_\phi$ of all orbits of ϕ is a foliation on M as well as for a locally free action. By this fact and the similar argument as the proof of Lemma 3.1, ϕ can be reduced to the case of a locally free action. Therefore we have the following.

Corollary 5.5 *If a connected closed manifold* M *admits a codimension one smooth action* ϕ *of a connected nilpotent Lie group such that any orbit of* ϕ *is non-compact, then* M *is homeomorphic to a nilmanifold.*

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