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Superconvergence of monotone difference schemes for piecewise smooth solutions of convex conservation laws

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Abstract. In this paper we show that the monotone difference methods with smooth numerical fluxes possess superconvergence property when applied to strictly convex conservation laws with piecewise smooth solutions. More precisely, it is shown that not only the approximation solution converges to the entropy solution, its central difference also converges to the derivative of the entropy solution away from the shocks.

Key words: superconvergence, finite difference, conservation laws, monotone scheme.

1. Introduction

In this paper we consider numerical approximations to weak solutions of the initial value problem for strictly convex conservation laws

$$u_t + f(u)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

which is subject to the initial condition prescribed at $t = 0$,

$$u(x, 0) = u_0(x). \tag{1.2}$$

There has been an enormous amount of papers related to the error estimates for the viscosity or more general approximations to scalar conservation laws, see, e.g., the review article of Tadmor [16]. The results on error estimates include

- For BV entropy solutions to (1.1), an $\mathcal{O}(\sqrt{\epsilon})$ convergence rate in L^1 obtained by Kuznetsov [8], Lucier [12], Cockburn-Gremaud-Yang [4] etc (in the BV-solution space, it is shown by Sabac [14] and Tang & Teng [19] that the L^1-convergence rate of order $\mathcal{O}(\sqrt{\epsilon})$ is optimal);
- For BV entropy solutions, an $\mathcal{O}(\epsilon)$ convergence rate in $W^{-1,1}$ obtained by Tadmor [15], Nessyahu & Tadmor [13], Liu-Wang-Warnecke [9] etc;
- For piecewise smooth solutions for (1.1), an $\mathcal{O}(\epsilon)$ convergence rate in

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L^1 obtained by Bakhvalov [1], Harabetian [6], Teng & Zhang [23], Fan [2], Tang & Teng [20], Teng [22], Liu [10], Wang [24], etc;

- For piecewise smooth solutions, an $\mathcal{O}(\epsilon)$ convergence rate in smooth region of the entropy solution obtained by Goodman & Xin [7], Engquist & Sjogreen [5], Tadmor & Tang [17, 18] etc.

For multidimensional conservations laws, there are also error estimates for numerical approximations, see, e.g., [3]. In [21], the authors consider the errors between the solutions of (1.1) and its parabolic regularization. We addressed the questions of the convergence rate in a weighted $W^{1,1}$ space when (1.1) pocesses piecewise smooth solutions. Convergence rate for the *derivative* of the approximate solutions is established under the assumption that a weak *pointwise-error* estimate is given. In other words, we are able to convert weak pointwise-error estimates to error bounds in a weighted $W^{1,1}$ space.

The approximation dealt with in [21] is essentially a viscosity method which can not be used as a practical numerical scheme. When the approximation is truly discrete such as finite difference/element/volume methods, the analysis is much more complicated. In this work, we wish to extend the results in [21] to a fully discretized case. More precisely, we will consider the so-called *monotone* finite-difference schemes for (1.1). To this end, we first introduce the definitions of the monotone schemes. The numerical approximations v_j^n are obtained by $(2k+1)$-point explicit schemes in conservation form

$$\begin{aligned} v_j^{n+1} &= H(v_{j-k}^n, v_{j-k+1}^n, \dots, v_{j+k}^n) \\ &= v_j^n - \lambda(\bar{f}_{j+1/2}^n - \bar{f}_{j-1/2}^n) \end{aligned} \tag{1.3}$$

which is subject to the initial condition:

$$v_j^0 = u_0(x_j),$$

where

$$\bar{f}_{j+1/2}^n = \bar{f}(v_{j-k+1}^n, \dots, v_{j+k}^n). \tag{1.4}$$

Here $v_j^n = v(j\Delta x, n\Delta t)$, $\lambda = \Delta t/\Delta x$, and $\bar{f}$ is a numerical flux function. We require that the numerical flux function to be consistent with the flux $f(u)$ in the following sense:

$$\bar{f}(u, \dots, u) = f(u). \tag{1.5}$$

Definition 1.1 The finite difference scheme (1.2) is a *monotone scheme* if H in (1.3) is a monotone nondecreasing function of each of its $2k+1$ arguments.

The main results of this work are the following: for the monotone finite difference schemes we first show that error bounds in a weighted $W^{1,1}$ space can be obtained provided a weak pointwise-error estimate is given. The weak pointwise-error estimate with monotone schemes is indeed established for the convex conservation lows by Nassyahu and Tadmor [13]. Furthermore, with the $W^{1,1}$-error bounds we can obtain the optimal pointwise-error bounds for the monotone schemes.

The paper is organized as follows. In Section 2, we re-consider the viscosity approximations to the conservation laws (1.1). The convergence rate in the weighted $W^{1,1}$ for the viscosity approximations are recovered, but by proofing techniques different with those used in [21]. The main tool for recovering the viscosity approximation results is the maximum principle, which can be applied to the discrete approximations. In Section 3.1, we will obtain some properties for the monotone schemes. By multiplying the weighted distant function, the error bounds with monotone finite difference schemes for the piecewise smooth solutions in the weighted $W^{1,1}$ are established.

2. Viscosity approximation

In this section, we wish to recover the main results obtained in Tang and Teng [21] by using the maximum principle for differential equations. In [21], the energy-type methods are used to obtain the convergence rates. Since we will in this paper obtain the convergence for the finite difference methods using the (discrete) maximum principle, the brief re-consideration of the viscosity approximation will be useful.

2.1. Weighted error for the viscosity solution

Viscosity approximation for the nonlinear conservation laws (1.1)–(1.2) is to find the solution of the following differential equation:

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_{xx} u^\epsilon, \tag{2.1}$$

which is subject to the initial data

$$u^\epsilon(x, 0) = u_0(x). \tag{2.2}$$

For ease of exposition, we will make the following assumptions:

(A1) the initial data u_0 is piecewise C^3-smooth and compactly supported;

(A2) there exists a smooth curve, $x = X(t)$, such that $u(x, t)$ is smooth at any point away from $x = X(t)$;

(A3) there is a constant $0 < \gamma \le 1$ and a constant $C > 0$ such that

$$|u^\epsilon(x, t) - u(x, t)| \le C\epsilon^\gamma \quad \text{for } |x - X(t)| \ge C\epsilon^\gamma \tag{2.3}$$

and

$$|v_j^n - u(x_j, t_n)| \le C\Delta x^\gamma \quad \text{for } |x_j - X(t_n)| \ge C\Delta x^\gamma, \tag{2.4}$$

where u^ϵ is viscosity solution and v_j^n is difference solution.

Remark Assumption (A3) is satisfied for convex conservation laws with Lip^+-bounded initial data, with $\gamma = 1/3$ (see Tadmor [13]); it can be improved to $\gamma = 1/2$ (see Tadmor and Tang [17]).

Let $u(x, t)$ be the piecewise smooth entropy solution of (1.1). Without loss of generality, we assume that the entropy solution of (1.1)–(1.2) contains one shock discontinuity $x = X(t)$ only with starting point at origin, i.e., $X(0) = 0$. Therefore, u satisfies

$$\partial_t u + \partial_x f(u) = \epsilon\partial_{xx} u + \mathcal{O}(\epsilon)$$

in the region of $x \ne X(t)$, where $\mathcal{O}(\epsilon) = -\epsilon\partial_{xx} u$.

It can be further verified that the error function $e := u^\epsilon - u$ satisfies

$$\partial_t e + \partial_x(\mathcal{A}(u^\epsilon, u)e) = \epsilon\partial_{xx} e + \mathcal{O}(\epsilon), \tag{2.5}$$

where

$$\mathcal{A}(u, v) := \int_0^1 f'(su + (1 - s)v)ds.$$

Following Tadmor and Tang [17], we introduce a non-negative function $\phi(x) \in C^2(\mathbb{R})$ which satisfies

- (i): $\phi(x) \sim |x|^\alpha, \quad |x| \ll 1,$
- (ii): $x\phi'(x) > 0, \quad x \ne 0,$
- (iii): $\phi(x) \to 1, \quad |x| \to \infty,$

where $\alpha \ge 1$ is a finite constant. The second requirement above implies that ϕ is monotonely decreasing for $x < 0$ and increasing for $x > 0$. More precisely, the distance function ϕ is required to satisfy

$$\begin{cases} \phi(0) = 0, \quad 0 < x\phi'(x) \le \alpha\phi(x), & \text{for } x \ne 0, \\ |\phi(x)| \le |x|^{\alpha}, \quad |\phi^{(k)}(x)| \le C, & k = 0, 1, 2, \end{cases} \tag{2.6}$$

where the constants C are independent of x. The functions satisfy the above requirements include $\phi(x) = (1 - e^{-x^2})^{\alpha/2}$.

In this work, we choose $\alpha = \gamma^{-1} + 3$, where γ is the constant given in the assumption (A3). In other words,

$$\phi(x) \sim \begin{cases} x^{\gamma^{-1}+2}, & |x| \ll 1 \\ 1, & |x| \gg 1. \end{cases} \tag{2.7}$$

For convex conservation laws we have $\gamma = 1/2$. Multiplying the error equation by $\phi := \phi(x - X(t))$ and applying the product rule for $\partial_x(\phi w)$ and $\partial_t(\phi w)$ gives

$$\begin{aligned} &\partial_t(\phi e) + \partial_x\Big(\Big(\mathcal{A}(u^\epsilon, u) + 2\epsilon\frac{\phi'}{\phi}\Big)\phi e\Big) \\ &\quad + \Big(\epsilon\beta\frac{\phi''}{\phi} + \big(\dot{X} - \mathcal{A}(u^\epsilon, u)\big)\frac{\phi'}{\phi}\Big)\phi e - \epsilon\partial_{xx}(\phi e) \\ &= \mathcal{O}(\epsilon) + \epsilon(\beta+1)\phi'' e = \mathcal{O}(\epsilon), \end{aligned}$$

where we have used the fact $\epsilon\phi'' e = \mathcal{O}(\epsilon)$ and β is a constant to be determined later. By letting

$$\begin{aligned} L(w) &:= \partial_t(w) + \partial_x\Big(\Big(\mathcal{A}(u^\epsilon, u) + 2\epsilon\frac{\phi'}{\phi}\Big)w\Big) \\ &\quad + \Big(\epsilon\beta\frac{\phi''}{\phi} + \big(\dot{X} - \mathcal{A}(u^\epsilon, u)\big)\frac{\phi'}{\phi}\Big)w - \epsilon\partial_{xx}(w), \end{aligned} \tag{2.8}$$

we have $L(\phi e) = \mathcal{O}(\epsilon)$. Therefore ϕe satisfies

$$\begin{cases} L(\phi e) = \mathcal{O}(\epsilon), \\ \phi e|_{t=0} = 0. \end{cases} \tag{2.9}$$

We can re-write the differential operator $L(w)$ in a non-conservative form:

$$\begin{cases} L(w) = \partial_t w + \Big(\mathcal{A}(u^\epsilon, u) + 2\epsilon\frac{\phi'}{\phi}\Big)\partial_x w + cw - \epsilon\partial_{xx} w, \\ c(x, t) = \mathcal{B}(u^\epsilon, u, u_x^\epsilon, u_x) \\ \qquad\qquad + \Big(\dot{X} - \mathcal{A}(u^\epsilon, u) + \epsilon\Big((\beta+2)\frac{\phi''}{\phi'} - 2\frac{\phi'}{\phi}\Big)\Big)\frac{\phi'}{\phi}, \end{cases} \tag{2.10}$$

where

$$\mathcal{B}(u^\epsilon, u, u_x^\epsilon, u_x) := \int_0^1 f''(su^\epsilon + (1-s)u)(su_x^\epsilon + (1-s)u_x)ds. \quad (2.11)$$

Now we will prove an important maximum theorem for $L(w)$.

Theorem 2.1 *If $w(x, t)$ satisfies*

$$(A) \quad \begin{cases} L(w) = g(x, t) \geq 0, & (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ w|_{t=0} \geq 0, \end{cases} \quad (2.12)$$

then $w(x, t) \geq 0$ for all $(x, t) \in \mathbb{R} \times \mathbb{R}^+$.

Proof. If there is a constant M such that $c(x, t) \geq -M$ for $(x, t) \in \mathbb{R}\times\mathbb{R}^+$, then the theorem is a classical maximum theorem for parabolic equation. But it follows from $f''(u) > 0$ and $u_x^\epsilon(x, t) \to -\infty$ as $(x, t) \to (0, +0)$ that $\mathcal{B}(u^\epsilon, u, u_x^\epsilon, u_x) \to -\infty$ as $(x, t) \to (0, +0)$. Thus $c(x, t) \geq -M$ for $(x, t) \in \mathbb{R}\times\mathbb{R}^+$ can not be satisfied. Since the singularity of $\mathcal{B}$ is only at the origin, we can deduct a small neighborhood at the origin from the upper half-space. More precisely, we consider an auxiliary problem A_ρ, for any $\rho > 0$

$$(A_\rho) \quad \begin{cases} L(w_\rho) = g(x, t) \geq 0, & (x, t) \in \Omega_\rho \\ w_\rho = w|_{t=0} \geq 0, & (x, t) \in \partial\Omega_\rho, \end{cases} \quad (2.13)$$

where

$$\Omega_\rho := \{(x, t) \mid t > 0 \quad \text{for } |x| \geq \rho\,;\, t > \sqrt{\rho^2 - x^2} \quad \text{for } |x| < \rho\}.$$

The next lemma will show that there is a constant M such that $c(x, t) \geq -M$ for $(x, t) \in \Omega_\rho$. Therefore the maximum principle is true for the problem A_ρ, i.e., $w_\rho \geq 0$ on Ω_ρ. Let ρ go to zero and we get $w \geq 0$ on $\mathbb{R} \times \mathbb{R}^+$. This completes the proof of the theorem. □

In the following lemma, we will lower bound the coefficient $c(x, t)$ on Ω_ρ defined in (2.10).

Lemma 2.1 *Let $\gamma > 2/(\alpha-1)$, then the coefficient function $c(x, t)$ defined in (2.10) can be lower bounded on Ω_ρ. More precisely, there exists a positive constant M such that*

$$c(x, t) \geq -M \quad \textit{for all } (x, t) \in \Omega_\rho.$$

Proof. It follows from $\phi(\xi) \sim |\xi|^\alpha$ as $|\xi| \ll 1$, where $\alpha \geq 2$, that

$$\frac{\phi'(\xi)}{\phi(\xi)} \sim \frac{\alpha}{\xi} \quad \text{and} \quad \frac{\phi''(\xi)}{\phi'(\xi)} \sim \frac{\alpha - 1}{\xi} \tag{2.14}$$

and hence

$$(\beta + 2)\frac{\phi''(\xi)}{\phi'(\xi)} - 2\frac{\phi'(\xi)}{\phi(\xi)} \sim \frac{(\beta + 2)(\alpha - 1) - 2\alpha}{\xi}$$

as $|\xi| \ll 1$. The above relationships and the assumption $\beta > 2/(\alpha - 1)$ show that

$$\frac{\phi'(\xi)}{\phi(\xi)} \to \pm\infty \quad \text{and} \quad (\beta + 2)\frac{\phi''(\xi)}{\phi'(\xi)} - 2\frac{\phi'(\xi)}{\phi(\xi)} \to \pm\infty$$

as $\xi \to \pm 0$. Hence there is an $\delta > 0$ such that when $(x, t) \in \{|x - X(t)| \leq \delta\} \bigcap \Omega_\rho$, $c(x, t) > 0$. On the other hand $|c(x, t)|$ is bounded for $|x - X(t)| \geq \delta$. Here we have used the fact that u, u_x, u^ϵ and u_x^ϵ are bounded on Ω_ρ. This completes the proof of this lemma. □

Lemma 2.2 *Let ω be the solution of*

$$\begin{cases} L(\omega) = |\mathcal{O}(\epsilon)|, \\ \omega|_{t=0} = 0, \end{cases} \tag{2.15}$$

where the term $\mathcal{O}(\epsilon)$ is as same as that in (2.9)*, then $\omega \geq 0$ and $|\phi e| \leq \omega$.*

Proof. It is known that $\omega \pm \phi e$ satisfies

$$\begin{cases} L(\omega \pm \phi e) \geq 0, \\ (\omega \pm \phi e)|_{t=0} = 0. \end{cases} \tag{2.16}$$

The maximum principle for the parabolic equation of $L(\omega \pm \phi e) \geq 0$ shows that

$$\omega(x, t) \pm \phi(x - X(t))e(x, t) \geq 0, \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}^+$$

and this concludes the results of the lemma. □

Theorem 2.2 *The weighted error of ϕe is bounded by $\mathcal{O}(\epsilon)$ in L^1 norm. More precisely, we have*

$$\|\phi(\,\cdot\, - X(t))e(\,\cdot\,, t)\|_{L^1(\mathbb{R})} \leq \mathcal{O}(\epsilon). \tag{2.17}$$

Proof. Integrating the equation of $L(\omega) = |\mathcal{O}(\epsilon)|$ in (2.15) with respect to x from $-\infty$ to ∞ gives

$$\frac{d}{dt}\int_{-\infty}^{\infty}\omega dx = -\int_{-\infty}^{\infty}\Big(\epsilon\beta\frac{\phi''}{\phi} + \big(\dot{X} - \mathcal{A}(u^{\epsilon}, u)\big)\frac{\phi'}{\phi}\Big)\omega dx + \mathcal{O}(\epsilon), \quad (2.18)$$

where the differential operator of $L(w)$ is taking the conservative form of (2.8). It is proved in [21] that for any $F \in L^1(\mathbb{R})$ the following inequality holds:

$$-\int_{-\infty}^{\infty}\big(\dot{X} - \mathcal{A}(u^{\epsilon}, u)\big)\phi'|F|dx \le C\int_{-\infty}^{\infty}\phi|F|dx + C\epsilon.$$

In deriving the above inequality, a weak pointwise-error estimate of $|u^{\varepsilon} - u|$ is used.

Since $\phi''(\xi) > 0$ for $0 < |\xi| \ll 1$, there is a $\delta > 0$ such that $-\epsilon\gamma\phi'' \le 0$ for $\xi \in (-\delta, \delta)$. Therefore, on account of $\phi(\xi)^{-1}$ being uniformly bounded on $\mathbb{R} \setminus (-\delta, \delta)$, we have

$$-\int_{-\infty}^{\infty}\epsilon\beta\phi''|F|dx \le -\int_{\mathbf{R}\setminus(-\delta,\,\delta)}\epsilon\beta\frac{\phi''}{\phi}\phi|F|dx \le C\epsilon\int_{-\infty}^{\infty}\phi|F|dx.$$

Applying the above result to the equation (2.18) and using the result $\omega \ge 0$ yield

$$\frac{d}{dt}\int_{-\infty}^{\infty}\omega dx \le C\int_{-\infty}^{\infty}\omega dx + \mathcal{O}(\epsilon).$$

Solving the above inequality with $\omega|_{t=0} = 0$ yields

$$\|w(\,\cdot\,, t)\|_{L^1(\mathbb{R})} \le \mathcal{O}(\epsilon).$$

The inequality of $|\phi e| \le \omega$ concludes the desired result. □

2.2. Weighted error for the derivatives of the viscosity approximations

Now we derive a differential equation for the weighted error of the derivatives, namely $\phi(x - X(t))e_x(x, t)$. First differentiating (2.1) with respect to x gives

$$\partial_t(u_x^{\epsilon}) + \partial_x(a(u^{\epsilon})u_x^{\epsilon}) = \epsilon\partial_{xx}(u_x^{\epsilon}).$$

In the smooth region of the solution of (1.1), i.e. $x \ne X(t)$, differentiating (1.1) with respect to x gives

$$\partial_t(u_x) + \partial_x(a(u)u_x) = \epsilon\partial_{xx}(u_x) + \mathcal{O}(\epsilon).$$

It follows from the above equations that, away from the shock curve $x = X(t)$, the derivative of the error function $e_x = u^\epsilon_x - u_x$ satisfies

$$\begin{aligned}\partial_t(e_x) + \partial_x(a(u^\epsilon)e_x) + a'(u^\epsilon)e_x \\ = \epsilon\partial_{xx}(e_x) + \mathcal{O}(\epsilon) + \mathcal{O}(|u^\epsilon - u|).\end{aligned} \tag{2.19}$$

In deriving the above equation, we have used the following facts:

$$\begin{aligned}&\partial_x(a(u^\epsilon)u^\epsilon_x - a(u)u_x) \\ &= \partial_x(a(u^\epsilon)e_x) + a'(u^\epsilon)e_x \\ &\qquad + \big(a'(u^\epsilon) - a'(u)\big)(u_x)^2 + \big(a(u^\epsilon) - a(u)\big)u_{xx} \\ &= \partial_x(a(u^\epsilon)e_x) + a'(u^\epsilon)e_x + \mathcal{O}(|u^\epsilon - u|).\end{aligned}$$

Multiplying the equation (2.19) by $\phi(x - X(t))$ and applying the product rulers for $\partial_x(\phi w)$ and $\partial_t(\phi w)$ gives

$$\begin{aligned}&\partial_t(\phi e_x) + \partial_x\Big(\Big(a(u^\epsilon) + 2\epsilon\frac{\phi'}{\phi}\Big)\phi e_x\Big) \\ &\qquad + \Big(\epsilon\beta\frac{\phi''}{\phi} + \big(\dot{X} - a(u^\epsilon)\big)\frac{\phi'}{\phi} + a'(u^\epsilon)\Big)\phi e_x - \epsilon\partial_{xx}(\phi e_x) \\ &= \epsilon(\beta+1)\phi'' e_x + \mathcal{O}(|\phi e|).\end{aligned}$$

By introducing the notation $\mathcal{L}$,

$$\begin{aligned}\mathcal{L}(\phi e_x) &:= \partial_t(\phi e_x) + \partial_x\Big(\Big(a(u^\epsilon) + 2\epsilon\frac{\phi'}{\phi}\Big)\phi e_x\Big) \\ &\qquad + \Big(\epsilon\beta\frac{\phi''}{\phi} + \big(\dot{X} - a(u^\epsilon)\big)\frac{\phi'}{\phi} + a'(u^\epsilon)\Big)\phi e_x - \epsilon\partial_{xx}(\phi e_x),\end{aligned}$$

we obtain that $\mathcal{L}(\phi e_x) = \epsilon(\beta+1)\phi'' e_x + \mathcal{O}(|\phi e|)$. Therefore ϕe_x satisfies

$$\begin{cases}\mathcal{L}(\phi e_x) = \epsilon(\beta+1)\phi'' e_x + \mathcal{O}(|\phi e|), \\ \phi e_x|_{t=0} = 0.\end{cases} \tag{2.20}$$

We can write the differential operator of $\mathcal{L}(w)$ in a non-conservative form:

$$\begin{cases}\mathcal{L}(w) = \partial_t w + \Big(a(u^\epsilon) + 2\epsilon\dfrac{\phi'}{\phi}\Big)\partial_x w + dw - \epsilon\partial_{xx}w, \\ d(x,\, t) = a'(u^\epsilon)(1 + u^\epsilon_x) \\ \qquad + \Big(\dot{X} - a(u^\epsilon) + \epsilon\Big((\beta+2)\dfrac{\phi''}{\phi'} - 2\dfrac{\phi'}{\phi}\Big)\Big)\dfrac{\phi'}{\phi}.\end{cases} \tag{2.21}$$

Similar to the proof of Theorem 2.1, we can obtain the following maximum theorem.

Theorem 2.3 *If $w(x, t)$ satisfies*

$$\begin{cases} \mathcal{L}(w) = F(x, t) \geq 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ w|_{t=0} \geq 0, \end{cases} \tag{2.22}$$

then $w(x, t) \geq 0$ for all $(x, t) \in \mathbb{R} \times \mathbb{R}^+$.

The following comparison lemma will be useful in controlling the derivatives of the approximation errors.

Lemma 2.3 *Let ω be the solution of the following initial value problem:*

$$\begin{cases} \mathcal{L}(\omega) = \epsilon(\gamma + 1)|\phi'' e_x| + |\mathcal{O}(\phi e)|, \\ \omega|_{t=0} = 0, \end{cases} \tag{2.23}$$

where the term $\mathcal{O}(\phi e)$ is as same as the one in (2.20)*, then $\omega \geq 0$ and $|\phi e_x| \leq w$.*

Proof. It follows from (2.20) and (2.23) that $\omega \pm \phi e_x$ satisfies

$$\begin{cases} \mathcal{L}(\omega \pm \phi e_x) \geq 0, \\ (\omega \pm \phi e_x)|_{t=0} = 0. \end{cases} \tag{2.24}$$

The maximum Theorem 2.3 shows that

$$\omega(x, t) \pm \phi(x - X(t))e_x(x, t) \geq 0, \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}^+$$

and this concludes the results of the lemma. □

With the above preparations, we are ready to state and prove the following theorem.

Theorem 2.4 *The weighted error for the derivative of the viscosity approximation, ϕe_x, is bounded by $\mathcal{O}(\epsilon)$ in L^1 norm:*

$$\|\phi(\bullet - X(t))e_x(\cdot, t)\|_{L^1(\mathbb{R})} \leq \mathcal{O}(\epsilon). \tag{2.25}$$

Proof. Integrating the differential equation in (2.23) with respect to x from $-\infty$ to ∞, on account of $\|e_x\|_{L^1} \leq M$ and $\|\phi e\|_{L^1} \leq \mathcal{O}(\epsilon)$ (given by (2.17)), gives

$$\frac{d}{dt}\int_{-\infty}^{\infty}\omega dx \le -\int_{-\infty}^{\infty}\Big(\epsilon\beta\frac{\phi''}{\phi} + \big(\dot{X} - a(u^\epsilon)\big)\frac{\phi'}{\phi}\Big)\omega dx + C\int_{-\infty}^{\infty}\omega dx + \mathcal{O}(\epsilon), \quad (2.26)$$

where the differential operator of $\mathcal{L}(\omega)$ is taking the conservative form of (2.20). Using the inequalities (see Lemma 1 of [21])

$$-\int_{-\infty}^{\infty}\big(\dot{X} - a(u^\epsilon)\big)\phi'|F|dx \le C\int_{-\infty}^{\infty}\phi|F|dx$$

and

$$-\int_{-\infty}^{\infty}\epsilon\beta\phi''|F|dx \le C\epsilon\int_{-\infty}^{\infty}\phi|F|dx$$

again to the above differential equality, on account of $w \ge 0$, gives

$$\frac{d}{dt}\int_{-\infty}^{\infty}\omega dx \le C\int_{-\infty}^{\infty}\omega dx + \mathcal{O}(\epsilon).$$

Solving the above differential inequality with the initial condition $\omega|_{t=0} = 0$ yields

$$\|\omega(\bullet, t)\|_{L^1(\mathbb{R})} \le \mathcal{O}(\epsilon).$$

The above result, together with Lemma 2.3, leads to the desired error bound (2.25). □

As proved in [21], the following results are immediate consequences of Theorems 2.2 and 2.4.

Corollary 2.1 *If* (x, t) *is away from the shock discontinuity* $S(t) = \{(x, t) \mid x = X(t)\}$, *then for any* $h > 0$

- Pointwise convergence for u^ϵ away from the shock

$$|(u^\epsilon - u)(x, t)| \le C(h)\epsilon, \quad \mathrm{dist}(x, S(t)) \ge h.$$

- $\partial_x u^\epsilon$ converges to $\partial_x u$ globally in regions away from shock discontinuity

$$\|\partial_x u^\epsilon(\cdot, t) - \partial_x u(\,\cdot\,, t)\|_{L^1(R\setminus(X(t)-h,\, X(t)+h))} \le C(h)\epsilon.$$

3. Difference approximation

The main tool for obtaining error estimates for the monotone difference schemes is a discrete maximum theorem, which is similar to that one used for the viscosity methods.

3.1. Error and difference quotient error equations

For easy of notations and without loss of generality, we consider only three-point finite difference schemes in the rest of this paper. In this case, we consider conservative finite difference scheme in the following form:

$$\begin{aligned} v_j^{n+1} =& H(v_{j+1}^n, v_j^n, v_{j-1}^n) \\ =& v_j^n - \lambda\big(\bar{f}(v_{j+1}^n, v_j^n) - \bar{f}(v_j^n, v_{j-1}^n)\big). \end{aligned} \tag{3.1}$$

Several notations will be used in this section. To begin with, we let

$$\begin{aligned} &\boldsymbol{v}_\alpha = (v_{\alpha+1/2}, v_{\alpha-1/2}), \quad \bar{\boldsymbol{v}}_\alpha := (v_{\alpha+1}, v_\alpha, v_{\alpha-1}), \\ &D_x v_\alpha := \frac{v_{\alpha+1/2} - v_{\alpha-1/2}}{\Delta x}, \quad D_x \boldsymbol{v}_\alpha := \frac{\boldsymbol{v}_{\alpha+1/2} - \boldsymbol{v}_{\alpha-1/2}}{\Delta x}. \end{aligned}$$

In the remaining of this paper, we will concentrate on the estimates of the errors for $D_x\mathbf{v}_j$:

$$D_x\mathbf{v}_j = \Big(\frac{v_{j+1} - v_j}{\Delta x}, \frac{v_j - v_{j-1}}{\Delta x}\Big).$$

We denote $\dot{\bar{\boldsymbol{f}}}(\boldsymbol{\xi};\boldsymbol{\eta})$ the derivative of the numerical flux in the following sense

$$\dot{\bar{\boldsymbol{f}}}(\boldsymbol{\xi};\boldsymbol{\eta}) = \int_0^1 \boldsymbol{D}\bar{\boldsymbol{f}}\big(\boldsymbol{\xi} + s(\boldsymbol{\xi} - \boldsymbol{\eta})\big)ds = (\bar{f}_1(\boldsymbol{\xi};\boldsymbol{\eta}), \bar{f}_{-1}(\boldsymbol{\xi};\boldsymbol{\eta})), \tag{3.2}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2)$, $\boldsymbol{\eta} = (\eta_1, \eta_2)$ and

$$\begin{aligned} &(\bar{f}_1(\boldsymbol{\xi};\boldsymbol{\eta}), \bar{f}_{-1}(\boldsymbol{\xi};\boldsymbol{\eta})) \\ :=& \Big(\int_0^1 \partial_1\bar{f}\big(\xi_1 + s(\eta_1 - \xi_1), \xi_2\big)ds, \int_0^1 \partial_2\bar{f}\big(\eta_1, \xi_2 + s(\eta_2 - \xi_2)\big)ds\Big). \end{aligned} \tag{3.3}$$

Here $\partial_j\bar{f}$ denotes the partial derivative with respect to the j-th argument of the function $\bar{f}$. Therefore by using the above definitions we have

$$\bar{f}(\boldsymbol{\xi}) - \bar{f}(\boldsymbol{\eta}) = \dot{\bar{\boldsymbol{f}}}(\boldsymbol{\xi};\boldsymbol{\eta})(\boldsymbol{\xi} - \boldsymbol{\eta})$$

It follows from the definition for the monotone scheme and from the definitions for $\bar{f}_1$ and $\bar{f}_{-1}$ that.

Lemma 3.1 *If the numerical scheme* (3.1) *is monotone, then for any* $\mathbf{v}_\alpha$ *and* $\mathbf{u}_\beta$ *the following inequalities hold*:

$$-\bar{f}_1(\mathbf{v}_\alpha;\mathbf{u}_\beta) \geq 0, \quad 1-\lambda\big(\bar{f}_{-1}(\mathbf{v}_\alpha;\mathbf{u}_\beta)-\bar{f}_1(\mathbf{v}_\alpha;\mathbf{u}_\beta)\big) \geq 0,$$
$$\bar{f}_{-1}(\mathbf{v}_\alpha,\mathbf{u}_\beta) \geq 0.$$

The three-point finite difference scheme (3.1) can be rewritten into the following form

$$\begin{aligned} v_j^{n+1} &= v_j^n - \lambda\big(\bar{f}(\boldsymbol{v}_{j+1/2}^n) - \bar{f}(\boldsymbol{v}_{j-1/2}^n)\big) \\ &= v_j^n - \Delta t\ \dot{\bar{\boldsymbol{f}}}\ \big(\boldsymbol{v}_{j+1/2}^n;\boldsymbol{v}_{j-1/2}^n\big) D_x \boldsymbol{v}_j^n. \end{aligned} \tag{3.4}$$

The above finite difference equation also has an equivalent form

$$D_t v_j^n + \dot{\bar{\boldsymbol{f}}}\ \big(\boldsymbol{v}_{j+1/2}^n;\boldsymbol{v}_{j-1/2}^n\big) D_x \boldsymbol{v}_j^n = 0, \tag{3.5}$$

where $D_t v_j^n = (v_j^{n+1} - v_j^n)/\Delta t$. We say that the finite difference scheme (3.1) is of *first-order* if for smooth solution u the following expansion holds:

$$\begin{aligned} u_j^{n+1} &= u_j^n - \lambda\big(\bar{f}(\boldsymbol{u}_{j+1/2}^n) - \bar{f}(\boldsymbol{u}_{j-1/2}^n)\big) + \Delta x^2 R_j^n \\ &= u_j^n - \Delta t\ \dot{\bar{\boldsymbol{f}}}\ \big(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{u}_{j-1/2}^n\big) D_x \boldsymbol{u}_j^n + \Delta x^2 R_j^n, \end{aligned} \tag{3.6}$$

where R_j^n is uniformly bounded by a constant independent of Δx. The last term above is the *truncation error*. The above equation can also be written in the following equivalent form:

$$D_t u_j^n + \dot{\bar{\boldsymbol{f}}}\ \big(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{u}_{j-1/2}^n\big) D_x \boldsymbol{u}_j^n = \Delta x R_j^n. \tag{3.7}$$

We have to point out that if $u(x,\, t)$ is a piecewise smooth solution, which is discontinue along $x = X(t)$, then the above equation takes the following form:

$$D_t u_j^n + \dot{\bar{\boldsymbol{f}}}\ \big(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{u}_{j-1/2}^n\big) D_x \boldsymbol{u}_j^n = TE_j^n, \tag{3.8}$$

where the truncation error TE_j^n will have the form

$$TE_j^n = \begin{cases} \mathcal{O}(\Delta x), & j \neq j_s^n, j_s^n + 1 \\ \mathcal{O}(1/\Delta x), & j = j_s^n, j_s^n + 1, \end{cases} \tag{3.9}$$

where j_s^n satisfies

$$j_s^n \Delta x < X(t_n) \leq (j_s^n + 1)\Delta x.$$

In what follows we will only consider u a piece-wise smooth solution of (1.1). Subtracting (3.5) from (3.8) leads to the *error equation* for the error $e_j^n = v_j^n - u_j^n$:

$$D_t e_j^n + D_x\big(\dot{\boldsymbol{f}}\,(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)\boldsymbol{e}_j^n\big) = TE_j^n, \tag{3.10}$$

where the truncation error TE_j^n is defined by (3.9).

Now we derive a difference equation for the difference quotient error $D_x e_j^n$. It follows from (3.5) that

$$\begin{cases} D_t D_x v_j^n + D_x\big(\dot{\boldsymbol{f}}\,(\boldsymbol{v}_{j+1/2}^n;\boldsymbol{v}_{j-1/2}^n)D_x\boldsymbol{v}_j^n\big) = 0, \\ D_t D_x u_j^n + D_x\big(\dot{\boldsymbol{f}}\,(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{u}_{j-1/2}^n)D_x\boldsymbol{u}_j^n\big) = TE_j^n, \end{cases} \tag{3.11}$$

where TE_j^n is the *truncation error* for the piecewise smooth solution $u(x,\,t)$. Since $u(x,\,t)$ is discontinuous along the shock curve $x = X(t)$, some calculation shows that

$$TE_j^n = \begin{cases} \mathcal{O}(\Delta x), & j \neq j_j^n - 1,\, j_j^n,\, j_j^n + 1,\, j_j^n + 2 \\ \mathcal{O}(1/\Delta x^2), & j = j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2, \end{cases} \tag{3.12}$$

where j_s^n satisfies

$$j_s^n \Delta x < X(t_n) \le (j_s^n + 1)\Delta x.$$

Subtracting the second equation from the first one above leads to the finite difference setting for $D_x\boldsymbol{e}_j^n = (\boldsymbol{e}_{j+1/2}^n - \boldsymbol{e}_{j-1/2}^n)/\Delta x$:

$$\begin{aligned} &D_t D_x e_j^n + D_x\big(\dot{\boldsymbol{f}}\,(\boldsymbol{v}_{j+1/2}^n;\boldsymbol{v}_{j-1/2}^n)D_x\boldsymbol{e}_j^n\big) \\ &+ \underbrace{D_x\big(\big(\dot{\boldsymbol{f}}\,(\boldsymbol{v}_{j+1/2}^n;\boldsymbol{v}_{j-1/2}^n) - \dot{\boldsymbol{f}}\,(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{u}_{j-1/2}^n)\big)D_x\boldsymbol{u}_j^n\big)}_{I} = TE_j^n. \end{aligned} \tag{3.13}$$

We re-write the term I in the following form:

$$\begin{aligned} I &= \big(\dot{\boldsymbol{f}}\,(\boldsymbol{v}_{j+1}^n;\boldsymbol{v}_j^n) - \dot{\boldsymbol{f}}\,(\boldsymbol{u}_{j+1}^n;\boldsymbol{u}_j^n)\big)D_x^2\boldsymbol{u}_j^n \\ &\quad + D_x\big(\dot{\boldsymbol{f}}\,(\boldsymbol{v}_{j+1/2}^n;\boldsymbol{v}_{j-1/2}^n) - \dot{\boldsymbol{f}}\,(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{u}_{j-1/2}^n)\big)D_x\boldsymbol{u}_{j-1/2}^n \\ &= \ddot{\boldsymbol{f}}\,(\bar{\boldsymbol{v}}_{j+1/2}^n;\bar{\boldsymbol{u}}_{j+1/2}^n)\bar{\boldsymbol{e}}_{j+1/2}^n D_x^2\boldsymbol{u}_j^n \\ &\quad + \underbrace{D_x\big(\dot{\boldsymbol{f}}\,(\boldsymbol{v}_{j+1/2}^n;\boldsymbol{v}_{j-1/2}^n) - \dot{\boldsymbol{f}}\,(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{u}_{j-1/2}^n)\big)}_{II} D_x\boldsymbol{u}_{j-1/2}^n. \end{aligned}$$

We further re-write the term II in the form:

$$
\begin{aligned}
II &= \big(\dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{v}_{j+1}^n;\boldsymbol{v}_j^n) - \dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{v}_j^n;\boldsymbol{v}_{j-1}^n)\big)\frac{1}{\Delta x}\\
&\quad - \big(\dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{u}_{j+1}^n;\boldsymbol{u}_j^n) - \dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{u}_j^n;\boldsymbol{u}_{j-1}^n)\big)\frac{1}{\Delta x}\\
&= \ddot{\bar{\boldsymbol{f}}}\,(\bar{\boldsymbol{v}}_{j+1/2}^n;\bar{\boldsymbol{v}}_{j-1/2}^n)D_x\bar{\boldsymbol{v}}_j^n - \ddot{\bar{\boldsymbol{f}}}\,(\bar{\boldsymbol{u}}_{j+1/2}^n;\bar{\boldsymbol{u}}_{j-1/2}^n)D_x\bar{\boldsymbol{u}}_j^n\\
&= \ddot{\bar{\boldsymbol{f}}}\,(\bar{\boldsymbol{v}}_{j+1/2}^n;\bar{\boldsymbol{v}}_{j-1/2}^n)D_x\bar{\boldsymbol{e}}_j^n\\
&\quad + \big(\ddot{\bar{\boldsymbol{f}}}\,(\bar{\boldsymbol{v}}_{j+1/2}^n;\bar{\boldsymbol{v}}_{j-1/2}^n) - \ddot{\bar{\boldsymbol{f}}}\,(\bar{\boldsymbol{u}}_{j+1/2}^n;\bar{\boldsymbol{u}}_{j-1/2}^n)\big)D_x\bar{\boldsymbol{u}}_j^n
\end{aligned}
$$

where

$$
\ddot{\bar{\boldsymbol{f}}}\,(\bar{\boldsymbol{v}}_\alpha;\bar{\boldsymbol{u}}_\beta) = \int_0^1 \boldsymbol{D}\,\dot{\bar{\boldsymbol{f}}}\,\big(\bar{\boldsymbol{u}}_\beta + s((\bar{v}_\alpha - \bar{\boldsymbol{u}}_\beta)\big)ds.
$$

The above result can be written in the following equivalent form:

$$
\begin{aligned}
II &= \ddot{\bar{\boldsymbol{f}}}\,(\bar{\boldsymbol{v}}_{j+1/2}^n;\bar{\boldsymbol{v}}_{j-1/2}^n)D_x\bar{\boldsymbol{e}}_j^n\\
&\quad + \dddot{\bar{\boldsymbol{f}}}\,(\bar{\boldsymbol{v}}_{j+1/2}^n,\,\bar{\boldsymbol{v}}_{j-1/2}^n;\bar{\boldsymbol{u}}_{j+1/2}^n,\,\bar{\boldsymbol{u}}_{j-1/2}^n)(\bar{\boldsymbol{e}}_{j+1/2}^n,\,\bar{\boldsymbol{e}}_{j-1/2}^n)D_x\bar{\boldsymbol{u}}_j^n,
\end{aligned}
$$

where the notation $\dddot{\bar{\boldsymbol{f}}}$ is defined by

$$
\begin{aligned}
&\dddot{\bar{\boldsymbol{f}}}\,(\bar{\boldsymbol{v}}_{\alpha+1/2},\,\bar{\boldsymbol{v}}_{\alpha-1/2};\bar{\boldsymbol{u}}_{\beta+1/2},\,\bar{\boldsymbol{u}}_{\beta-1/2})\\
&:= \int_0^1 \boldsymbol{D}\,\ddot{\bar{\boldsymbol{f}}}\,\big(\bar{\boldsymbol{u}}_{\beta+1/2} + s(\bar{\boldsymbol{v}}_{\alpha+1/2} - \bar{\boldsymbol{u}}_{\beta+1/2});\\
&\qquad\qquad \bar{\boldsymbol{u}}_{\beta-1/2} + s(\bar{\boldsymbol{v}}_{\alpha-1/2} - \bar{\boldsymbol{u}}_{\beta-1/2})\big)ds.
\end{aligned}
$$

Substituting I and II into (3.13) gives the difference quotient error equation for $D_x e_j^n$

$$
\begin{aligned}
&D_tD_xe_j^n + D_x\big(\dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{v}_{j+1/2}^n;\boldsymbol{v}_{j-1/2}^n)D_x\boldsymbol{e}_j^n\big)\\
&\qquad + \ddot{\bar{\boldsymbol{f}}}\,(\bar{\boldsymbol{v}}_{j+1/2}^n;\bar{\boldsymbol{v}}_{j-1/2}^n)D_x\bar{\boldsymbol{e}}_j^n D_x\boldsymbol{u}_{j-1/2}^n\\
&\qquad + \dddot{\bar{\boldsymbol{f}}}\,\big(\bar{\boldsymbol{v}}_{j+1/2}^n,\,\bar{\boldsymbol{v}}_{j-1/2}^n;\bar{\boldsymbol{u}}_{j+1/2}^n,\,\bar{\boldsymbol{u}}_{j-1/2}^n\big)\\
&\qquad\qquad \times (\bar{\boldsymbol{e}}_{j+1/2}^n,\,\bar{\boldsymbol{e}}_{j-1/2}^n)D_x\bar{\boldsymbol{u}}_j^n D_x\boldsymbol{u}_{j-1/2}^n\\
&\qquad + \ddot{\bar{\boldsymbol{f}}}\,\big(\bar{\boldsymbol{v}}_{j+1/2}^n;\bar{\boldsymbol{u}}_{j+1/2}^n\big)\bar{\boldsymbol{e}}_{j+1/2}^n D_x^2\boldsymbol{u}_j^n = TE_j^n.
\end{aligned}
\tag{3.14}
$$

3.2. Weighted error for the difference solution

In this section we consider the case that the entropy solution of (1.1) has one shock discontinuity. Let $x = X(t)$ be a shock curve. Denote

$$\phi_j^n = \phi(x_j - X^n),\ \dot{\phi}_j^n = \phi'(x_j - X^n),\ X^n = X(t_n),\ \dot{X}^n = X'(t_n),$$

where ϕ is the weighted distance function introduced in Section 2. It is easy to show that

$$\begin{cases} D_x\phi_j^n = \dot{\phi}_j^n + \mathcal{O}(\Delta x^2), \quad D_t\phi_j^n = \dot{\phi}_j^n + \mathcal{O}(\Delta t) \\ \phi_j^{n+1} = \phi_{j+\alpha}^n + \mathcal{O}(\Delta t + \Delta x). \end{cases} \tag{3.15}$$

Using the product rule $D_x(a_jb_j) = a_{j+1/2}D_xb_j + (D_xa_j)b_{j-1/2}$ to the following equations gives

$$\begin{aligned} & D_x\big(\dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)(\boldsymbol{\phi e})_j^n\big) \\ &= D_x\big(f_1(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)\phi_{j+1/2}^n e_{j+1/2}^n + f_{-1}(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)\phi_{j-1/2}^n e_{j-1/2}^n\big) \\ &= \phi_{j+1}^n D_x(f_1(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)e_{j+1/2}^n) + D_x\phi_{j+1/2}^n f_1(\boldsymbol{u}_{j-1/2}^n;\boldsymbol{v}_{j-1/2}^n)e_j^n \\ &\qquad \phi_{j-1}^n D_x(f_{-1}(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)e_{j-1/2}^n) + D_x\phi_{j-1/2}^n f_{-1}(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{v}_{j+1/2}^n)e_j^n \\ &= \phi_j^{n+1}D_x\big(\dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)\boldsymbol{e}_j^n\big) \\ &\quad + \dot{\phi}_j^n\big(f_1(\boldsymbol{u}_{j-1/2}^n;\boldsymbol{v}_{j-1/2}^n) + f_{-1}(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{v}_{j+1/2}^n)\big)e_j^n + \mathcal{O}(\Delta x). \end{aligned}$$

It follows from the above result that

$$\begin{aligned} &\phi_j^{n+1}D_x\big(\dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)\boldsymbol{e}_j^n\big) = D_x\big(\dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)(\boldsymbol{\phi e})_j^n\big) \\ &-\dot{\phi}_j^n\big(f_1(\boldsymbol{u}_{j-1/2}^n;\boldsymbol{v}_{j-1/2}^n) + f_{-1}(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{v}_{j+1/2}^n)\big)e_j^n + \mathcal{O}(\Delta x). \end{aligned} \tag{3.16}$$

Multiplying the error equation (3.10) by ϕ_j^n gives

$$\begin{aligned} \phi_j^nD_te_j^n + \phi_j^nD_x\big(\dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)\boldsymbol{e}_j^n\big) &= \phi_j^n\begin{cases} \mathcal{O}(\Delta x), & j \neq j_s^n,\ j_s^n+1 \\ \mathcal{O}(1/\Delta x), & j = j_s^n,\ j_s^n+1 \end{cases} \\ &= \begin{cases} \mathcal{O}(\Delta x), & j \neq j_s^n,\ j_s^n+1 \\ \mathcal{O}(1), & j = j_s^n,\ j_s^n+1 \end{cases} \end{aligned} \tag{3.17}$$

This is due to the fact that $\phi_j^n = O(\Delta x)$ for $j = j_s^n,\ j_s^n+1$. Now we will derive a difference equation for $\phi_j^ne_j^n$ from (3.17) and (3.16). To begin with, we observe

$$\begin{aligned}
&D_t(\phi_j^n e_j^n)\\
&=\phi_j^{n+1} D_t e_j^n + (D_t\phi_j^n)e_j^n\\
&=-\phi_j^{n+1} D_x\big(\dot{\boldsymbol{f}}\,(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)\boldsymbol{e}_j^n\big) - \dot{\phi}_j^n \dot{X}^n e_j^n + \begin{cases}\mathcal{O}(\Delta x), & j\neq j_s^n,\, j_s^n+1\\ \mathcal{O}(1), & j=j_s^n,\, j_s^n+1\end{cases}\\
&=-D_x\big(\dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)(\boldsymbol{\phi e})_j^n\big)\\
&\quad+\dot{\phi}_j^n\big(f_1(\boldsymbol{u}_{j-1/2}^n;\boldsymbol{v}_{j-1/2}^n)+f_{-1}(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{v}_{j+1/2}^n)-\dot{X}^n\big)e_j^n\\
&\quad+\begin{cases}\mathcal{O}(\Delta x), & j\neq j_s^n,\, j_s^n+1\\ \mathcal{O}(1), & j=j_s^n,\, j_s^n+1\end{cases}.
\end{aligned}$$

Rearranging the above difference equation gives the equation for $\phi_j^n e_j^n$:

$$\begin{aligned}
&D_t(\phi_j^n e_j^n) + D_x\big(\dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)(\boldsymbol{\phi e})_j^n\big)\\
&\quad-\frac{\dot{\phi}_j^n}{\phi_j^n}\big(\bar{f}_1(\boldsymbol{u}_{j-1/2}^n;\boldsymbol{v}_{j-1/2}^n)+\bar{f}_{-1}(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{v}_{j+1/2}^n)-\dot{X}^n\big)(\phi_j^n e_j^n)\\
&=\begin{cases}\mathcal{O}(\Delta x), & j\neq j_s^n,\, j_s^n+1\\ \mathcal{O}(1), & j=j_s^n,\, j_s^n+1\end{cases}.
\end{aligned}\tag{3.18}$$

Since for $j=j_s^n,\, j_s^n+1$, we have

$$-\frac{\dot{\phi}_j^n}{\phi_j^n}\big(\bar{f}_1(\boldsymbol{u}_{j-1/2}^n;\boldsymbol{v}_{j-1/2}^n)+\bar{f}_{-1}(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{v}_{j+1/2}^n)-\dot{X}^n\big)(\phi_j^n e_j^n) \tag{3.19}$$

$$=-\dot{\phi}_j^n\big(\bar{f}_1(\boldsymbol{u}_{j-1/2}^n;\boldsymbol{v}_{j-1/2}^n)+\bar{f}_{-1}(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{v}_{j+1/2}^n)-\dot{X}^n\big)e_j^n \tag{3.20}$$

$$=\mathcal{O}(\Delta x), \tag{3.21}$$

where we have used the fact $|\phi'(x)|\le C|x|$ for $|x|\ll 1$. Let

$$L(\bar{\boldsymbol{W}}_j^n) := D_t W_j^n + D_x\big(\dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{u}_j^n;\boldsymbol{v}_j^n)\boldsymbol{W}_j^n\big) + c_j^n W_j^n, \tag{3.22}$$

where

$$c_j^n = \begin{cases} 0, & j=j_s^n,\, j_s^n+1\\ \big(\dot{X}^n-\bar{f}_1(\boldsymbol{u}_{j-1/2}^n;\boldsymbol{v}_{j-1/2}^n)-\bar{f}_{-1}(\boldsymbol{u}_{j+1/2}^n;\boldsymbol{v}_{j+1/2}^n)\big)\dfrac{\dot{\phi}_j^n}{\phi_j^n}, & j\neq j_s^n,\, j_s^n+1. \end{cases} \tag{3.23}$$

Thus $\phi_j^n e_j^n$ satisfies

$$\begin{cases} L((\bar{\boldsymbol{\phi} \boldsymbol{e}})_j^n) & = \begin{cases} \mathcal{O}(\Delta x), & j \neq j_s^n,\, j_s^n+1 \\ \mathcal{O}(1), & j = j_s^n,\, j_s^n+1 \end{cases} \\ (\phi e)_j^0 & = 0. \end{cases} \tag{3.24}$$

It follows from (3.23) and (2.14) that

$$c_j^n = \begin{cases} 0, & j = j_s^n,\, j_s^n+1 \\ \mathcal{O}\Big(\dfrac{1}{\Delta x}\Big), & j \neq j_s^n,\, j_s^n+1 \end{cases}$$

$$\text{or} \quad \Delta t c_j^n = \begin{cases} 0, & j = j_s^n,\, j_s^n+1 \\ \mathcal{O}(\lambda), & j \neq j_s^n,\, j_s^n+1, \end{cases} \tag{3.25}$$

where $\lambda = \Delta t/\Delta x$.

Now we prove an useful discrete maximum theorem for $L(\bar{\boldsymbol{W}}_j^n)$.

Theorem 3.1 *If $L(\bar{\boldsymbol{W}}_j^n) \geq 0$ and $W_j^0 \geq 0$, then*

$$W_j^n \geq 0 \quad \forall\ (n, j) \in \mathbb{Z} \times \mathbb{N} \tag{3.26}$$

provided that the Courant Number λ is suitably small.

Proof. It follows from $L(\bar{\boldsymbol{W}}_j^n) \geq 0$ that if $W_j^n \geq 0$ for all j, then

$$\begin{aligned} W_j^{n+1} \geq & -\lambda \bar{f}_1(\boldsymbol{u}_{j+1/2}^n; \boldsymbol{v}_{j+1/2}^n) W_{j+1}^n + \lambda \bar{f}_{-1}(\boldsymbol{u}_{j-1/2}^n; \boldsymbol{v}_{j-1/2}^n) W_{j-1}^n \\ & + \big(1 - \lambda\big(\bar{f}_{-1}(\boldsymbol{u}_{j+1/2}^n; \boldsymbol{v}_{j+1/2}^n) - \bar{f}_1(\boldsymbol{u}_{j-1/2}^n; \boldsymbol{v}_{j-1/2}^n)\big) - \Delta t c_j^n\big) W_j^n \\ \geq & -\lambda \bar{f}_1(\boldsymbol{u}_{j+1/2}^n; \boldsymbol{v}_{j+1/2}^n) W_{j+1}^n + \lambda \bar{f}_{-1}(\boldsymbol{u}_{j-1/2}^n; \boldsymbol{v}_{j-1/2}^n) W_{j-1}^n \\ & + \big(1 - \lambda\big(\bar{f}_{-1}(\boldsymbol{u}_{j+1/2}^n; \boldsymbol{v}_{j+1/2}^n) - \bar{f}_1(\boldsymbol{u}_{j-1/2}^n; \boldsymbol{v}_{j-1/2}^n) + |\mathcal{O}(1)|\big)\big) W_j^n, \end{aligned} \tag{3.27}$$

where in the last inequality we have used the expression (3.25) and the assumption $W_j^n \geq 0$. The monotonicity of H shows that the first and second coefficients of $\bar{\mathbf{W}}_j^n$ given in (3.27) are nonnegative and under little more restriction on λ than the monotone condition the third coefficient also nonnegative. Since $W_j^0 \geq 0$, the inequality (3.27) indicates that the conclusion (3.26) is right for $n = 1$. By induction on n we can conclude that the inequality (3.26) holds for any $n \in \mathbb{N}$. This completes the proof of this theorem. □

Lemma 3.2 *Let W_j^n satisfy*

$$L(\bar{\boldsymbol{W}}_j^n) = \begin{cases} |\mathcal{O}(\Delta x)|, & j \neq j_s^n,\, j_s^n + 1 \\ |\mathcal{O}(1)|, & j = j_s^n,\, j_s^n + 1 \end{cases} \tag{3.28}$$

with $W_j^0 = 0$, where $\mathcal{O}(\Delta x)$ and $\mathcal{O}(1)$ are defined as in (3.24). Then we have

$$W_j^n \geq 0 \quad \textit{and} \quad |\phi_j^n e_j^n| \leq W_j^n \quad \forall\ (n,\, j) \in \mathbb{Z} \times \mathbb{N}.$$

Proof. It is easy to conclude from Theorem 3.1 that $W_j^n \geq 0$ for all $n \geq 0$ and j. It follows from (3.24) and (3.28) that $L(\bar{\boldsymbol{W}}_j^n \pm (\bar{\boldsymbol{\phi e}})_j^n) \geq 0$ and $W_j^0 \pm (\phi e)_j^0 = 0$ and hence the maximum Theorem 3.1 gives $W_j^n \pm (\phi e)_j^n \geq 0$ or equivalent $|\phi_j^n e_j^n| \leq W_j^n$. □

Theorem 3.2 *The weighted error of $(\phi e)_j^n$ is bounded by $\mathcal{O}(\Delta x)$ in L^1 norm. More precisely, we have*

$$\|\phi(\,\cdot\, - X(t_n))e(\,\cdot\,, t_n)\|_{l^1} := \sum_{j=-\infty}^{\infty} |(\phi e)_j^n| \Delta x \leq \mathcal{O}(\Delta x). \tag{3.29}$$

Proof. Multiplying the difference equation (3.28) by Δx and summing up the equations for j from $-\infty$ to ∞ give

$$D_t \sum_{j=-\infty}^{\infty} W_j^n \Delta x = -\sum_{j=-\infty}^{\infty} c_j^n W_j^n \Delta x + \mathcal{O}(\Delta x), \tag{3.30}$$

where we have used the conservative form (3.22) of $L(\bar{\boldsymbol{W}}_j^n)$. In order to get the desired estimate we need an inequality:

$$-\sum_{j=-\infty}^{\infty} c_j^n W_j^n \Delta x \leq C \sum_{j=-\infty}^{\infty} W_j^n \Delta x + \mathcal{O}(\Delta x). \tag{3.31}$$

From the definition (3.23) of c_j^n we see that $|c_j^n|$ is not uniformly bounded by some constant. So we can not get the inequality (3.31) directly by using $|c_j^n| \leq C$ for all $n \geq 0$ and j. But the following important inequality helps to get the requested estimate, namely for any $F \in L^1(\mathbb{R})$ the following inequality holds:

$$
\begin{aligned}
&-\sum_{j=-\infty}^{\infty}\big(\dot{X}^n-\bar{f}_1(\boldsymbol{u}^n_{j-1/2};\boldsymbol{v}^n_{j-1/2})-\bar{f}_{-1}(\boldsymbol{u}^n_{j+1/2};\boldsymbol{v}^n_{j+1/2})\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\dot{\phi}^n_j|F(x_j,\,t_n)|\Delta x \qquad (3.32)\\
&\le C\sum_{j=-\infty}^{\infty}\phi^n_j|F(x_j,\,t_n)|\Delta x+C\Delta x,
\end{aligned}
$$

which will be proved in next lemma. Setting $\phi^n_j|F(x_j,\,t_n)| = W^n_j$ and substituting it into the above inequality give (3.31). Applying (3.31) to (3.30) yields

$$
\begin{aligned}
D_t\sum_{j=-\infty}^{\infty}W^n_j\Delta x &= -\sum_{j=-\infty}^{\infty}c^n_jW^n_j\Delta x+\mathcal{O}(\Delta x)\\
&\le C\sum_{j=-\infty}^{\infty}W^n_j\Delta x+\mathcal{O}(\Delta x),
\end{aligned}
$$

Solving the above inequality gives

$$
\sum_{j=-\infty}^{\infty}W^n_j\Delta x\le\mathcal{O}(\Delta x)\quad\forall\; n\in\mathbb{N}.
$$

This estimate with $|\phi^n_je^n_j|\le W^n_j$ gives the desired result of (3.29). □

Now we prove the important inequality (3.32) by using the assumption (A3).

Lemma 3.3 *For a weighted distance function ϕ, $\phi\sim\min(|x|^{1/\gamma+1},\,1)$, and for any $F\in L^1(\mathbb{R})$, we have*

$$
\begin{aligned}
&-\sum_{j=-\infty}^{\infty}\big(\dot{X}^n-\bar{f}_1(\boldsymbol{u}^n_{j-1/2};\boldsymbol{v}^n_{j-1/2})-\bar{f}_{-1}(\boldsymbol{u}^n_{j+1/2};\boldsymbol{v}^n_{j+1/2})\big)\dot{\phi}^n_j|F^n_j|\Delta x\\
&\le C\sum_{j=-\infty}^{\infty}\phi^n_j|F^n_j|\Delta x+C\Delta x,
\end{aligned}
\tag{3.33}
$$

where $F^n_j=F(x_j,\,t_n)$.

Proof. We split the left-hand of (3.33) into tow parts: I_1+I_2, where

$$
I_1=\sum_{|x_j-X(t_n)|\ge\Delta x^\gamma}\big(\bar{f}_1(\boldsymbol{u}^n_{j-1/2};\boldsymbol{v}^n_{j-1/2})
$$

$$+\bar{f}_{-1}(\boldsymbol{u}^n_{j+1/2};\boldsymbol{v}^n_{j+1/2}) - \dot{X}^n\big)\dot{\phi}^n_j |F^n_j|\Delta x,$$

$$I_{-1} = \sum_{|x_j - X(t_n)| < \Delta x^\gamma} \big(\bar{f}_1(\boldsymbol{u}^n_{j-1/2};\boldsymbol{v}^n_{j-1/2})$$

$$+\bar{f}_{-1}(\boldsymbol{u}^n_{j+1/2};\boldsymbol{v}^n_{j+1/2}) - \dot{X}^n\big)\dot{\phi}^n_j |F^n_j|\Delta x,$$

where γ is the constant given in (A3). It follows from (2.7) that

$$|\phi'(x)| \le C|x|^{1/\gamma}, \quad |\dot{\phi}^n_j| = |\phi'(x_j - X(t_n))| \le C|x_j - X(t_n)|^{1/\gamma}.$$

This result gives

$$I_2 \le C\Delta x. \tag{3.34}$$

Now for $|x_j - X(t_n)| \ge \Delta x^\gamma$, we use the facts that $\dot{\phi}^n_j \ge 0$ and $\dot{X}_n > f'(u^n_+)$, where $u^n_+ = u(X(t_n)+0, t_n)$, to obtain

$$\begin{aligned}
&\big(\bar{f}_1(\boldsymbol{u}^n_{j-1/2};\boldsymbol{v}^n_{j-1/2}) + \bar{f}_{-1}(\boldsymbol{u}^n_{j+1/2};\boldsymbol{v}^n_{j+1/2}) - \dot{X}^n\big)\dot{\phi}^n_j \\
&\le \big(\bar{f}_1(\boldsymbol{u}^n_{j-1/2};\boldsymbol{v}^n_{j-1/2}) + \bar{f}_{-1}(\boldsymbol{u}^n_{j+1/2};\boldsymbol{v}^n_{j+1/2}) - f'(u^n_j)\big)\dot{\phi}^n_j \\
&\quad + ((f'(u^n_j) - f'(u^n_+))\dot{\phi}^n_j \\
&\le Cf''(\bullet)\Delta x^\gamma \dot{\phi}^n_j + f''(\bullet)u_x(\bullet)(x_n - x(t_n))\dot{\phi}^n_j \qquad \text{(using (A3))} \\
&\le Cf''(\bullet)(x_n - x(t_n))\dot{\phi}^n_j + f''(\bullet)u_x(\bullet)(x_n - x(t_n))\dot{\phi}^n_j \\
&\le C(x_n - x(t_n))\dot{\phi}^n_j \le C\phi^n_j.
\end{aligned}$$

Similarly, by noting that $\dot{\phi}^n_j \le 0$ for $x_n \le X(t_n)$ we can also prove that for $x_n - X(t_n) \le \Delta x^\gamma$

$$\big(\bar{f}_1(\boldsymbol{u}^n_{j-1/2};\boldsymbol{v}^n_{j-1/2}) + \bar{f}_{-1}(\boldsymbol{u}^n_{j+1/2};\boldsymbol{v}^n_{j+1/2}) - \dot{X}^n\big)\dot{\phi}^n_j \le C\phi^n_j.$$

The above results lead to

$$I_1 \le C\sum_{j=-\infty}^{\infty} \phi^n_j |F^n_j|\Delta x.$$

This, together with (3.34), yields the inequality (3.33). The proof of the lemma is complete. □

3.3. Weighted error for the difference quotient of the monotone solution

In this subsection we will consider the *strictly* monotone scheme, which satisfies more strictly monotone condition:

$$\partial_u H(u,\, v,\, w) > 0,\ \partial_v H(u,\, v,\, w) > 0 \quad \text{and} \quad \partial_w H(u,\, v,\, w) > 0.$$

This is equivalent to

$$\partial_u \bar{f}(u,\, v) < 0,\ \partial_w \bar{f}(v,\, w) > 0$$
$$\text{and} \quad 1 - \lambda(\partial_v \bar{f}(u,\, v) - \partial_v \bar{f}(v,\, w)) > 0.$$

The well known (generalized) Lax-Friedrichs scheme ([11]) is a strictly monotone scheme, which is defined by

$$v_j^{n+1} = v_j^n - \frac{\lambda}{2}(f(v_{j+1}^n - f(v_{j-1}^n)) + \frac{\mu}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n),$$

where $\lambda = \Delta x/\Delta t$ satisfies a Courant-Friedrichs-Levy condition,

$$\lambda \sup_{|v| \le \|v^0\|_\infty} |f'(v)| \le \mu.$$

For the strictly monotone scheme we have

Lemma 3.4 *If the numerical scheme* (3.1) *is strictly monotone, then for any* $\mathbf{v}_\alpha$ *and* $\mathbf{u}_\beta$ *the following inequalities hold*:

$$-\bar{f}_1(\mathbf{v}_\alpha; \mathbf{u}_\beta) > 0, \quad 1 - \lambda\big(\bar{f}_{-1}(\mathbf{v}_\alpha; \mathbf{u}_\beta) - \bar{f}_1(\mathbf{v}_\alpha; \mathbf{u}_\beta)\big) > 0,$$
$$\bar{f}_{-1}(\mathbf{v}_\alpha,\, \mathbf{u}_\beta) > 0.$$

Now we will derive a difference equation for $\phi_j^n D_x e_j^n$ from (3.14). The derivation is similar to that for $\phi_j^n e_j^n$. Some calculation on (3.14), account of (3.12), gives

$$\begin{aligned}
&D_t(\phi_j^n D_x e_j^n) + D_x\big(\dot{\bar{\boldsymbol{f}}}\ (\boldsymbol{v}_{j+1/2}^n; \boldsymbol{v}_{j-1/2}^n)(\boldsymbol{\phi} D_x \boldsymbol{e})_j^n\big)\\
&\quad - \frac{\dot{\phi}_j^n}{\phi_j^n}\big(f_1(\boldsymbol{v}_j^n; \boldsymbol{v}_{j-1}^n) + f_{-1}(\boldsymbol{v}_{j+1}^n; \boldsymbol{v}_j^n) - \dot{X}^n\big)(\phi D_x e)_j^n\\
&\quad + \ddot{\bar{\boldsymbol{f}}}\ (\bar{\boldsymbol{v}}_{j+1/2}^n; \bar{\boldsymbol{v}}_{j-1/2}^n)(\bar{\boldsymbol{\phi}} D_x \bar{\boldsymbol{e}})_j^n D_x \boldsymbol{u}_{j-1/2}^n\\
&\quad + \dddot{\bar{\boldsymbol{f}}}\ \big(\bar{\boldsymbol{v}}_{j+1/2}^n,\, \bar{\boldsymbol{v}}_{j-1/2}^n; \bar{\boldsymbol{u}}_{j+1/2}^n,\, \bar{\boldsymbol{u}}_{j-1/2}^n\big)\\
&\quad \times \big((\bar{\boldsymbol{\phi}}\bar{\boldsymbol{e}})_{j+1/2}^n,\, (\bar{\boldsymbol{\phi}}\bar{\boldsymbol{e}})_{j-1/2}^n\big) D_x \bar{\boldsymbol{u}}_j^n D_x \boldsymbol{u}_{j-1/2}^n\\
&\quad + \ddot{\bar{\boldsymbol{f}}}\ \big(\bar{\boldsymbol{v}}_{j+1/2}^n; \bar{\boldsymbol{u}}_{j+1/2}^n\big)(\bar{\boldsymbol{\phi}}\bar{\boldsymbol{e}})_{j+1/2}^n D_x^2 \boldsymbol{u}_j^n\\
&= \phi_j^n \begin{cases} \mathcal{O}(\Delta x), & j \ne j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2\\ \mathcal{O}(1/\Delta x^2), & j = j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2, \end{cases}
\end{aligned}$$

$$= \begin{cases} \mathcal{O}(\Delta x), & j \neq j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2 \\ \mathcal{O}(1), & j = j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2, \end{cases}$$

where

$$j_s^n \Delta x < X(t_n) \leq (j_s^n + 1)\Delta x, \quad |\phi_j^n| \leq \mathcal{O}(\Delta x^2),$$

with $\alpha \geq 3$, for $j = j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2$.

It follows from the definition of ϕ that if $\alpha \geq 3$ then

$$|\phi_j^n| \leq \mathcal{O}(\Delta x^3), \quad |\dot{\phi}_j^n| \leq \mathcal{O}(\Delta x^2)$$

for $j = j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2$. Thus we have

$$\begin{aligned} &-\frac{\dot{\phi}_j^n}{\phi_j^n}\big(f_1(\boldsymbol{v}_j^n; \boldsymbol{v}_{j-1}^n) + f_{-1}(\boldsymbol{v}_{j+1}^n; \boldsymbol{v}_j^n) - \dot{X}^n\big)(\phi D_x e)_j^n \\ &= -\dot{\phi}_j^n\big(f_1(\boldsymbol{v}_j^n; \boldsymbol{v}_{j-1}^n) + f_{-1}(\boldsymbol{v}_{j+1}^n; \boldsymbol{v}_j^n) - \dot{X}^n\big)(D_x e)_j^n \\ &= \mathcal{O}(\Delta x) \quad \text{for } j = j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2 \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} &\ddot{\bar{\boldsymbol{f}}}\,\big(\bar{\boldsymbol{v}}_{j+1/2}^n; \bar{\boldsymbol{v}}_{j-1/2}^n\big)(\bar{\boldsymbol{\phi}} D_x \bar{\boldsymbol{e}})_j^n D_x \boldsymbol{u}_{j-1/2}^n \\ &\qquad = \mathcal{O}(\Delta x) \quad \text{for} \quad j = j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2. \end{aligned} \tag{3.36}$$

Therefore we define

$$\begin{aligned} \mathcal{L}(\bar{\boldsymbol{W}}_j^n) = D_t W_j^n + D_x\big(\dot{\bar{\boldsymbol{f}}}\,(\boldsymbol{v}_{j+1/2}^n; \boldsymbol{v}_{j-1/2}^n)\boldsymbol{W}_j^n\big) \\ + \bar{\boldsymbol{c}}_j^n \bar{\boldsymbol{W}}_j^n + d_j^n W_j^n. \end{aligned} \tag{3.37}$$

where

$$d_j^n = \begin{cases} 0, & j = j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2 \\ -\dfrac{\dot{\phi}_j^n}{\phi_j^n}\big(f_1(\boldsymbol{v}_j^n; \boldsymbol{v}_{j-1}^n) + f_{-1}(\boldsymbol{v}_{j+1}^n; \boldsymbol{v}_j^n) - \dot{X}^n\big), & \\ & j \neq j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2 \end{cases} \tag{3.38}$$

and

$$\bar{\boldsymbol{c}}_j^n = \begin{cases} 0, & j = j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2 \\ \ddot{\bar{\boldsymbol{f}}}\,\big(\bar{\boldsymbol{v}}_{j+1/2}^n; \bar{\boldsymbol{v}}_{j-1/2}^n\big) D_x \boldsymbol{u}_{j-1/2}^n, & \\ & j \neq j_s^n - 1,\, j_s^n,\, j_s^n + 1,\, j_s^n + 2, \end{cases} \tag{3.39}$$

Hence c_j^n are uniformly bounded and $|d_j^n| = \mathcal{O}(\Delta x^{-1})$.

Theorem 3.3 *If W_j^n satisfies $\mathcal{L}(\bar{\boldsymbol{W}}_j^n) \geq 0$ with $W_j^0 \geq 0$, then $W_j^n \geq 0$ $\forall\ (n, j) \in \mathbb{N} \times \mathbb{Z}$ provided that Δt and Courant number are sufficient small.*

Proof. We can write $\mathcal{L}(\bar{\boldsymbol{W}}_j^n) \geq 0$ in the following form

$$\begin{aligned} W_j^{n+1} \geq &\big(-\lambda \bar{f}_1(\boldsymbol{v}_{j+1}^n; \boldsymbol{v}_j^n) - \Delta t c_{j+1}^n\big) W_{j+1}^n \\ &+ \big(1 - \lambda\big(\bar{f}_{-1}(\boldsymbol{v}_{j+1}^n; \boldsymbol{v}_j^n) - \bar{f}_1(\boldsymbol{v}_j^n; \boldsymbol{v}_{j-1}^n)\big) \\ &- \Delta t c_j^n + \mathcal{O}(\lambda)\big) W_j^n \\ &+ \big(\lambda \bar{f}_{-1}(\boldsymbol{v}_j^n; \boldsymbol{v}_{j-1}^n) - \Delta t c_{j-1}^n\big) W_{j-1}^n. \end{aligned} \tag{3.40}$$

It follows from the strictly monotonicity of H that coefficients of W_{j+1}^n, W_j^n and W_{j-1}^n in the above inequality are positive provided Δt and λ are sufficiently small. Since $w_j^0 \geq 0$, $W_j^n \geq 0$ is right for $n = 1$. By induction on n we can conclude that the inequality holds for any $n \in \mathbb{N}$. $\square$

Lemma 3.5 *If W_j^n satisfies*

$$\begin{aligned} \mathcal{L}(\bar{\boldsymbol{W}}_j^n) = \Big| &\dddot{\bar{\boldsymbol{f}}}\ \big(\bar{\boldsymbol{v}}_{j+1/2}^n, \bar{\boldsymbol{v}}_{j-1/2}^n; \bar{\boldsymbol{u}}_{j+1/2}^n, \bar{\boldsymbol{u}}_{j-1/2}^n\big) \\ &\big((\bar{\boldsymbol{\phi}}\bar{\boldsymbol{e}})_{j+1/2}^n, (\bar{\boldsymbol{\phi}}\bar{\boldsymbol{e}})_{j-1/2}^n\big) D_x \bar{\boldsymbol{u}}_j^n D_x \boldsymbol{u}_{j-1/2}^n\Big| \\ &+ \Big|\ddot{\bar{\boldsymbol{f}}}\ \big(\bar{\boldsymbol{v}}_{j+1/2}^n; \bar{\boldsymbol{u}}_{j+1/2}^n\big)(\bar{\boldsymbol{\phi}}\bar{\boldsymbol{e}})_{j+1/2}^n D_x^2 \boldsymbol{u}_j^n\Big| + |\mathcal{O}(\Delta x)| \end{aligned} \tag{3.41}$$

with $W_j^0 = |\phi_j^0 D_x e_j^0|$, then

$$W_j^n \geq 0, \quad |\phi_j^n D_x e_j^n| \leq W_j^n,$$

for any $(n, j) \in \mathbb{N} \times \mathbb{Z}$.

Theorem 3.4

$$\|\phi(\bullet - X(t)) D_x e(\bullet, t)\|_{L^1(\boldsymbol{R})} = \mathcal{O}(\Delta x). \tag{3.42}$$

Proof. Integrating the difference inequality of (3.41) with respect to x, on account of the estimate of (3.29) and

$$-\sum_{j=-\infty}^{\infty} d_j^n W_j^n \leq C \sum_{j=-\infty}^{\infty} W_j^n,$$

gives

$$\sum_{j=-\infty}^{\infty} W_j^n \Delta x = \mathcal{O}(\Delta x).$$

The above inequality comes from the assumption of (A_3). This with the estimate of $|\phi_j^n D_x e_j^n| \le W_j^n$ yields the desired result of (3.42). □

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References

[1] Bakhvalov N.S., *Estimation of the error of numerical integration of a first-order quasilinear equation*. Zh. Vychisl. Mat. i Mat. Fiz. **1** (1961), 771–783 (English transl. in USSR Comput. Math. and Math. Phys. **1** (1962), 926–938).

[2] Fan H., *Existence and uniqueness of traveling waves and error estimates for Godunov schemes of conservation laws*. Math. Comp. **67** (1998), 87–109.

[3] Cockburn B., Coquel F. and LeFloch P., *An error estimate for finite volume methods for multidimensional conservation laws*. Math. Comp. **63** (1994), 77–103.

[4] Cockburn B., Gremaud P.-A. and Yang J., *A priori error estimates for numerical methods for scalar conservation laws Part* III: *multidimensional flux-splitting monotone schemes on non-cartesian grids*. SIAM J. Numer. Anal. **35** (1998), 1775–1803.

[5] Engquist B. andSjogreen B., *The convergence rate of finite difference schemes in the presence of shocks*. SIAM J. Numer. Anal. **35** (1998), 2464–2485.

[6] Harabetian E., *Rarefactions and large time behavior for parabolic equations and monotone schemes*. Comm. Math. Phys. **114** (1988), 527–536.

[7] Goodman J. and Xin Z., *Viscous limits for piecewise smooth solutions to systems of conservation laws*. Arch. Rational Mech. Anal. **121** (1992), 235–265.

[8] Kuznetsov N.N., *Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation*. USSR Comput. Math. and Math. Phys. **16** (1976), 105–119.

[9] Liu H. and Warnecke G., *Convergence rates for relaxation schemes approximating conservation laws*. SIAM J. Numer. Anal. **37** (2000), 1316–1337.

[10] Liu H.X. and Pan T., L^1*-convergence rate of viscosity methods for scalar conservation laws with the interaction of elementary waves and the boundary*. Quart. Appl. Math. **62** (2004), 601–621.

[11] Liu J. and Xin Z., L^1*-stability of stationary discrete shocks*. Math. Comp. **60** (1993), 233–244.

[12] Lucier B.J., *Error bounds for the methods of Glimm, Godunov and LeVeque*. SIAM J. Numer. Anal. **22** (1985), 1074–1081.

[13] Nassyahu H. and Tadmor E., *The convergence rate of approximate solutions for nonlinear scalar conservation laws*. SIAM J. Numer. Anal. **29** (1992), 1505–1519.

[14] Sabac F., *The optimal convergence rate of monotone finite difference methods for hyperbolic conservation laws*. SIAM J. Numer. Anal. **34** (1997), 2306–2318.

[15] Tadmor E., *Local error estimates for discontinuous solutions of nonlinear hyperbolic equations*. SIAM J. Numer. Anal. **28** (1991), 891–906.

[16] Tadmor E., *Approximate solutions of nonlinear conservation laws*. in Advanced numerical approximation of nonlinear hyperbolic equations. A. Quarteroni, Ed., Lecture Notes in Mathematics, vol. 1697, Springer, 1998, pp. 1–149.

[17] Tadmor E. and Tang T., *Pointwise convergence rate for scalar conservation laws with piecewise smooth solutions*. SIAM J. Numer. Anal. **36** (1999), 1739–1758.

[18] Tadmor E. and Tang T., *Pointwise error estimates for relaxation approximations to conservation laws*. SIAM J. Math. Anal. **32** (2000), 870–886.

[19] Tang T. and Teng Z.H., *The sharpness of Kuznetsov's* $O(\sqrt{\Delta x})$ L^1*-error estimate for monotone difference schemes*. Math. Comp. **64** (1995), 581–589.

[20] Tang T. and Teng Z.H., *Viscosity methods for piecewise smooth solutions to scalar conservation laws*. Math. Comp. **66** (1997), 495–526.

[21] Tang T. and Teng Z.H., *On the regularity of approximate solutions to conservation laws with piecewise smooth solutions*. SIAM J. Numer. Anal. **38** (2000), 1483–1495.

[22] Teng Z.H., *First-order* L^1*-convergence for relaxation approximations to conservation laws*. Comm. Pure Appl. Math. **51** (1998), 857–895.

[23] Teng Z.H. and Zhang P.W., *Optimal* L^1*-rate of convergence for viscosity method and monotone scheme to piecewise constant solutions with shocks*. SIAM J. Numer. Anal. **34** (1997), 959–978.

[24] Wang H.Y., *On the* L^2 *and* L^1 *convergence rates of viscous solutions of the Keyfitz-Kranzer system with piecewise smooth and large* BV *data*. J. Diff. Eqn **211** (2005), 382–406.

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