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A class of Crouzeix-Raviart type nonconforming finite element methods for parabolic variational inequality problem with moving grid on anisotropic meshes

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Abstract. A class of Crouzeix-Raviart type nonconforming finite element methods are proposed for the parabolic variational inequality problem with moving grid on anisotropic meshes. By using some novel approaches and techniques, the same optimal error estimates are obtained as the traditional ones. It is shown that the classical regularity condition or quasi-uniform assumption on meshes is not necessary for the finite element analysis.

Key words: variational inequality, parabolic, anisotropic, moving grid, optimal error estimates.

1. Introduction

Variational inequality(VI) theory was introduced by Hartman and Stampacchina [15] as a tool for the study of partial differential equations(PDEs). Now the VI theory is playing an important role in contact problem, obstacle problem, elasticity problem, traffic problem, and so on. Finite element method(FEM) for solving VI problems has attracted more and more attentions, see [1, 3-5, 9, 14-16, 21, 30, 32, 37-38]. In 1974, Strang [30] suggested that the error between the exact solution and the approximate solution of the obstacle problem using piecewise quadratic finite elements should be $O(h^{3/2})$. Brezzi and Sacchi [6] first obtained the error bound $O(h^{3/2-\varepsilon})$, for any $\varepsilon > 0$, for the above finite element approximation to the obstacle problem, when the obstacle vanished. Then through a detailed analysis, Brezzi, *et al* [5] obtained the same error bound as [6] under the hypothesis that the free boundary has finite length. Later, Wang L.H. [37] obtained the same error bound as [5] for the same element without the hypothesis of finite length of the free boundary. Mark A., *et al* [21] compared several

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numerical methods for solving the discrete contact problem arising from the FEMs by using the above element. Computational tests illustrate the use of these methods for a large collection of elastic bodies, such as a simplified bidimensional wall made of bricks or stone blocks, deformed under volume and surface forces. Hua D.Y. and Wang L.H. [17] used P_2-P_1 finite element to approximate the displacement field and the normal stress component on the contact hypotheses and proposed a new mixed finite element approximation of the VI resulting from the unilateral contact problem in elasticity was obtained. The same optimal convergence rates as [16] were gained with fewer freedom degrees. Numerical test showed that it was better than P_1-P_1 method [4] under the reasonable regularity hypotheses. Recently, Andreas K., *et al* [1] introduced a *p-version* FEM for some quasi-linear elliptic variational inequalities by partitioning the domain Ω into a finite number of curvilinear quadrilateral, and derived a priori error estimate. Dietrich B. [14] derived a posterior estimate for the obstacle problem from the theory for linear equations, but the theory was simpler only if the Lagrange multiplier does not have a nonconforming contribution as it has in actual finite element computations. Chen Z.M. [10] gave discussion on the augmented Lagrangian approach to Signorini elastic contact problem by transforming it into a saddle point problem and derived the optimal error estimates for general smooth domains which were not necessarily convex. The key in [10] is a discrete inf-sup condition which guarantees the existence of the saddle point. Zhang Y.M. [40] discussed the free boundary obstacle problem by using a piecewise linear finite element and analyzed the convergence under a stability condition on the obstacle in a defined distance(the maximal radius of balls contained in the difference set of two regions enclosed by the exact discrete free boundaries). And also in [38], Wang L.H. obtained the error bound $O(h)$ for Crouzeix-Raviart type nonconforming linear triangular element approximation to the obstacle problem. Based on these research, Suttmeimer F.T. [32] employed the finite element Galerkin method to obtain approximate solutions of VIs, and a unified framework for VIs was developed. At the same time, adaptive mesh was designed for the FEM models, and the posteriori error estimates were obtained.

Moving grid methods have been proved powerful for solving time-dependent PDEs, see [12-13, 17, 21, 31]. The general idea [18] is to apply FEMs in space and choose difference methods with respect to the time variable, but the grid can vary with time interval. For instance, when we consider a time-dependent obstacle problem, since the contact part varies with time, finer meshes near the contact part is usually used. [12, 35, 13] applied this technique to the parabolic inegro-differential equation and the nerve conductive equation in 2-space variables with conforming element, and the stokes equation with nonconforming element, respectively. The optimal L_2-norm and energy norm error estimates were derived.

Recently, Baines M.J., *et al* [3] proposed a moving mesh finite element algorithm for the adaptive solution of time-dependent PDEs with moving boundaries, the optimal error estimates were obtained. In order to reduce the computational time and the computer memory, Sutthisak P., *et al* [31] proposed a nodeless variable FEM for two-dimensional steady-state and transient heat transfer problem. The effectiveness of the combined procedure was demonstrated by a heat transfer problem which has exact solution. Chen H.R., *et al* [7] suggested a self-adaptive grid moving scheme to predict the delamination growth process by using a virtual crack closure technique. The contact effect along the delamination front is considered. The numerical results showed that the influences of the distribution and location of the stiffeners, the configuration and the size of the delamination, the boundary condition and the contact upon the failure behavior of the plates were significant. [23] have studied the conforming linear triangular FEM for parabolic VI problems with moving grid. But all of the above studies are based on the regular assumption or quasi-uniform assumption on the meshes [24,25], i.e., $h_K/\rho_K \le C$ or $h/h_{\min} \le C$, where denote by h_K the diameter of K and by ρ_K the largest inscribed circle in K, $h = \max_K h_K$, $h_{\min} = \min_K h_K$ and C is a positive constant which is independent of h and K. However, the domain considered may be narrow and irregular. For example, in modeling a gap between rotor and stator in an electrical machine, or in modeling a cartilage between a joint and hip, if we employ the regular partition, the cost of calculation will be very high. So for simplicity in the application, it is an obvious idea to employ the anisotropic partition which has fewer freedom degrees than the traditional one. But in the anisotropic case, the above ratio $h/h_{\min}$ may be very large, even tends to infinity, which results in some difficulties in the estimates of interpolation error and consistency error for both conforming and nonconforming finite element methods. The Bramble-Hilbert lemma, i.e., the traditional interpolation theory in Sobolev spaces, can not be directly applied to the interpolation error estimates. On the other hand, when we estimate the consistency error on the longer or longest side F of the element K, there will appear a term $|F|/|K|$, which may tend to infinity and makes the estimate in vain. Recently, there appeared a lot of articles focused on the narrow meshes and anisotropic meshes for the second order elliptic problems, and some valuable results on convergence and superconvergence were obtained, for example [7,8,24-27]. [28] investigated an anisotropic Wilson's element and Carey's element approximation to the second order obstacle problem. Moreover they relaxed the restriction of interpolation and simplified the proof in [36] and [37]. Then, [27] applied a class of anisotropic Crouzeix-Raviart type finite element to Signorini VI problem and [29] was devoted to a nonconforming finite element schemes with moving grid for velocity-pressure mixed formulations of the nonstationary Stokes problem in 2-D. However, to our best knowledge, there are few papers devoting to the anisotropic nonconforming FEMs for the parabolic VI problem.

In this paper, we will discuss a class of anisotropic nonconforming Crouzeix-Raviart type FEMs for the parabolic VI problem with moving grid. The Crouzeix-Raviart type element has the least degrees of freedom among the nonconforming ones. By using some novel approaches, the optimal error estimates are obtained. It is shown that the classical regularity assumption or quasi-uniform assumption on meshes mentioned above is not necessary for the finite element analysis, and the use of anisotropic meshes allows to achieve optimal results with fewer degrees of freedom. Moreover, in the previous studies for time-dependent PDEs and VI problems with moving grid, the Ritz projection was indispensable in the error analysis, for example, in [12, 13, 17, 21, 31]. However, with the property of the finite element spaces, we replace the interpolation of Ritz injection directly [20]. Hence, the proof can be simplified greatly. The idea provided in this paper is helpful to design adaptive algorithms for numerical solutions of the corresponding problems.

2. Construction and anisotropy of the Crouzeix-Raviart type finite elements

Let $\hat{K} = [0, 1] \times [0, 1]$ be a rectangular reference element in $\xi - \eta$ plane with vertices $\hat{M}_1(0, 0)$, $\hat{M}_2(1, 0)$, $\hat{M}_3(1, 1)$, $\hat{M}_4(0, 1)$ and barycenter $\hat{M}_5(1/2, 1/2)$. The four edges are denoted by $\hat{l}_1 = \overline{\hat{M}_1\hat{M}_2}$, $\hat{l}_2 = \overline{\hat{M}_2\hat{M}_3}$, $\hat{l}_3 = \overline{\hat{M}_3\hat{M}_4}$ and $\hat{l}_4 = \overline{\hat{M}_4\hat{M}_1}$. If $\hat{K}$ is a triangular element with vertices

$\hat{M}_1(0,\,0)$, $\hat{M}_2(1,\,0)$ and $\hat{M}_4(0,\,1)$, then the three edges are denoted by $\hat{l}_1 = \overline{\hat{M}_1\hat{M}_2}$, $\hat{l}_2 = \overline{\hat{M}_2\hat{M}_4}$ and $\hat{l}_3 = \overline{\hat{M}_4\hat{M}_1}$. We consider two Crouzeix-Raviart type elements $(\hat{K},\,\hat{P},\,\hat{\sum})$ which are defined on $\hat{K}$ as follows:

$$\hat{\Sigma} = \{\hat{v}_1,\,\hat{v}_2,\,\hat{v}_3,\,\hat{v}_4,\,\hat{v}_5\}, \quad \hat{P} = \operatorname{span}\{1,\,\xi,\,\eta,\,\varphi(\xi),\,\varphi(\eta)\}\ [19], \tag{1}$$

$$\hat{\Sigma} = \{\hat{v}_1,\,\hat{v}_2,\,\hat{v}_3\}, \quad \hat{P} = \operatorname{span}\{1,\,\xi,\,\eta\}\ [34], \tag{2}$$

where $\hat{v}_i = (1/|\hat{l}_i|)\int_{\hat{l}_i}\hat{v}d\hat{s},\ i = 1,\,2,\,3,\,4,\ \hat{v}_5 = (1/|\hat{K}|)\int_{\hat{K}}\hat{v}d\xi d\eta,\ \varphi(t) = (\sqrt{5}/2)[3(2t-1)^2 - 1],\ 0 \le t \le 1.$

For convenience, we remark the above two elements in formulas (1) and (2) as FE1 and FE2, respectively. Then it can be checked that the interpolations of them are well-posed and can be expressed as

$$\begin{aligned}\hat{\Pi}\hat{v} = \hat{v}_5 &+ \frac{1}{2}(\hat{v}_1 - \hat{v}_2 + \hat{v}_3 - \hat{v}_4) + (\hat{v}_2 - \hat{v}_4)\xi + (\hat{v}_3 - \hat{v}_1)\eta \\ &+ \frac{1}{2\sqrt{5}}(\hat{v}_1 + \hat{v}_3 - 2\hat{v}_5)\varphi(\xi) + \frac{1}{2\sqrt{5}}(\hat{v}_2 + \hat{v}_4 - 2\hat{v}_5)\varphi(\eta)\end{aligned} \tag{3}$$

and

$$\hat{\Pi}\hat{v} = \hat{v}_3 + \hat{v}_1 - \hat{v}_2 + 2(\hat{v}_2 - \hat{v}_3)\xi + 2(\hat{v}_2 - \hat{v}_1)\eta, \tag{4}$$

respectively.

For the sake of convenience, we assume that $\Omega \subset R^2$ is a convex polygon, J_h is a rectangular(or triangular) partition of Ω, which does not need to satisfy the classical regularity condition or quasi-uniform assumption. But J_h should satisfy the following maximum angle condition proposed in [2], i.e., there exists $0 < \sigma < \pi$ such that each interior angle of the elements is bounded from above by σ. Let $K \in J_h$ be a rectangle with vertices $M_1(0,\,0)$, $M_2(h_x,\,0)$, $M_3(h_x,\,h_y)$ and $M_4(0,\,h_y)$ in $x-y$ plane, $l_1 = \overline{M_1M_2}$, $l_2 = \overline{M_2M_3}$, $l_3 = \overline{M_3M_4}$, $l_4 = \overline{M_4M_1}$. Without loss of generality, let $h_x \gg h_y$. We divide each rectangle, along the diagonal, into two triangles to form triangular subdivision (see Fig. 1). The affine transformation $F_K\colon \hat{K} \to K$ is defined by

$$\begin{cases} x = h_x\xi, \\ y = h_y\eta. \end{cases} \tag{5}$$

The associated finite element is then defined by

$$V_h = \Big\{v_h\colon \hat{v} = v_h|_K \cdot F_K \in \hat{P},\ \int_F [v_h]ds = 0,\ \text{for any } F \subset \partial K\Big\}, \tag{6}$$

Fig. 1. Partitioned meshes

where $[v_h]$ denote the jump of v_h on F, and $[v_h] = v_h$ if $F \subset \partial\Omega$.

[34] and [26] have proven that the above elements have the anisotropic property, i.e., for any $\hat{v} \in H^2(\hat{K})$, $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = 1$, there holds

$$\|\hat{D}^\alpha(\hat{v} - \hat{\Pi}\hat{v})\|_{0,\hat{K}} \leq C|\hat{D}^\alpha \hat{v}|_{1,\hat{K}}. \tag{7}$$

Here and below, the positive constant C will be used as a generic constant, which is independent of h_K/ρ_K and h.
It is easy to prove that $\|\cdot\|_h = (\sum_K |\cdot|_{1,K}^2)^{1/2}$ is a norm on V_h.
Define the interpolation operator Π: $H^1(\Omega) \to V_h$ as follows, for any $v \in H^1(\Omega)$,

$$\Pi|_K = \Pi_K, \quad \Pi_K v = (\hat{\Pi}\hat{v}) \cdot F_K^{-1}, \tag{8}$$

then for any $u \in H^2(\Omega)$, we have

$$\|u - \Pi u\|_h \leq Ch|u|_2, \quad \|u - \Pi u\|_0 \leq Ch^2|u|_2. \tag{9}$$

Consider the following parabolic variational inequality problem [23]:

$$\begin{cases} \text{Find } u(x,\,y,\,t)\colon J \to V, \text{ such that} \\ \left(\dfrac{\partial u}{\partial t}, v - u\right) + a(u, v - u) \geq (f, v - u), & \text{for any } v \in V, t \in J, \\ u(x,\,0) = u_0, & \text{for any } (x,\,y) \in \Omega, \end{cases} \tag{10}$$

where $J = (0, T]$, $V = \{v \in H_0^1(\Omega) \cap H^2(\Omega)\colon v \geq 0 \text{ a.e. } \Omega\}$, $a(u,\,v) = \int_\Omega \nabla u \nabla v dxdy$, $f(v) = \int_\Omega f v dxdy$, $\partial u/\partial t$ denotes the right derivation of u with respect to t.

Suppose $f \in C(J,\,L_\infty(\Omega))$, $\partial f/\partial t \in L_2(J,\,L_\infty(\Omega))$ and $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then (11) has a unique solution u satisfying [23]

$$u \in L_\infty(J,\,H^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L_2(J,\,H_0^1(\Omega)) \cap L_\infty(J,\,L_\infty(\Omega)) \tag{11}$$

with $\|v\|_{L_p(J,X)} = (\int_0^T \|v(t)\|_X^p dt)^{1/p}$, $\|v\|_{L_\infty(J,X)} = \sup_{t\in J} \|v(t)\|_X$.

The following important lemma which is usually used in consistency error estimate of nonconforming finite element methods can be found in [34].

Lemma 2.1 *For any $F \subset \partial K$, $q \in H^1(K)$, $v_h \in V_h$, there holds*

$$\begin{aligned}
&\Big|\int_F (q - M_F q)(v_h - M_F v_h)ds\Big| \\
&\le C\frac{|F|}{|K|}\Big(h_x^2\Big\|\frac{\partial q}{\partial x}\Big\|_{0,K}^2 + h_y^2\Big\|\frac{\partial q}{\partial y}\Big\|_{0,K}^2\Big)^{1/2} \\
&\qquad\qquad\qquad \times \Big(h_x^2\Big\|\frac{\partial v_h}{\partial x}\Big\|_{0,K}^2 + h_y^2\Big\|\frac{\partial v_h}{\partial y}\Big\|_{0,K}^2\Big)^{1/2}, \quad (12)
\end{aligned}$$

where $M_F q = (1/|F|)\int_F q ds$.

It can be seen that the term of order h_x^2/h_y will appear in the estimate of (12) when F is a longer edge of K. Thus under the anisotropic meshes, when h_y is small enough, the estimate of (12) will tend to infinity. This is the essential difference between the anisotropic finite elements and the conventional elements, and also the key to carry out the consistency error estimates for anisotropic nonconforming finite elements. In order to overcome the above difficulty, we introduce an auxiliary finite element $(K, \tilde{P}_K, \tilde{\sum}_K)$ on K as

$$\tilde{P}_K = \mathrm{span}\{1,\, y\}, \quad \tilde{\sum}_K = \Big\{v_i = \frac{1}{|l_i|}\int_{l_i} \tilde{v} ds\Big\}, \quad (13)$$

where $i = 1, 2$ (or $i = 1, 3$) when K is a triangular (or a rectangular) element.

Then we set the space $\tilde{V}_h$ as

$$\tilde{V}_h = \Big\{\tilde{v}_h \in L^2(\Omega) \colon \tilde{v}_h|_K \in \tilde{P}_K,\ \int_{l_i} [\tilde{v}_h] ds = 0\Big\}. \quad (14)$$

Let $\tilde{\prod}_h \colon V_h \to \tilde{V}_h$ be an interpolation operator defined by

$$v_h \mapsto \tilde{\prod}_h v_h = \tilde{v}_h, \quad (15)$$

where

$$\int_{l_i} \tilde{v}_h ds = \int_{l_i} v_h ds, \qquad i = 1, 2 \quad \text{or} \quad i = 1, 3. \quad (16)$$

Then it is easy to see that $\partial v_h/\partial y$ and $\partial\tilde{v}_h/\partial y$ are constants. Even better, by Green's formula and (16) we can obtain $\partial v_h/\partial y = \partial\tilde{v}_h/\partial y$. Since $\partial v_h/\partial y - \partial\tilde{v}_h/\partial y = 1/|K|\int_K(\partial v_h/\partial y - \partial\tilde{v}_h/\partial y) = 1/|K|\int_{\partial K}(v_h - \tilde{v}_h)n_y ds = 0$.

Now we are ready to estimate the consistency error.

Lemma 2.2 *For any $\omega \in H^2(\Omega)$, $v_h \in V_h$, we have*

$$\Big|\sum_K \int_{\partial K} \frac{\partial\omega}{\partial n} v_h ds\Big| \le Ch|\omega|_2\|v_h\|_h. \tag{17}$$

Proof. The desired result has been proven for the FE1 in [8]. However, since the property and construction of the triangular element are quite different from the rectangular ones, the techniques provided in [8] can not be employed directly to estimate (17) for the FE2 and we have to develop some novel approaches to prove it based on the definition of $\tilde{\Pi}_h$ and $\partial v_h/\partial y = \partial\tilde{v}_h/\partial y$.

Since

$$\begin{aligned}
\sum_K \int_{\partial K} \frac{\partial\omega}{\partial n} v_h ds &= \sum_K \int_{\partial K} \frac{\partial\omega}{\partial n} v_h ds - \sum_K \int_K \frac{\partial\omega}{\partial y}\Big(\frac{\partial v_h}{\partial y} - \frac{\partial\tilde{v}_h}{\partial y}\Big) \\
= \sum_K \int_K \frac{\partial^2\omega}{\partial y^2}(v_h - \tilde{v}_h) &+ \sum_K \int_{\partial K}\Big(\frac{\partial\omega}{\partial x} n_x v_h + \frac{\partial\omega}{\partial y} n_y v_h\Big) ds \\
&- \sum_K \int_{\partial K} \frac{\partial\omega}{\partial y}(v_h - \tilde{v}_h) n_y ds \\
= \sum_K \int_K \frac{\partial^2\omega}{\partial y^2}(v_h - \tilde{v}_h) &+ \sum_K \int_{\partial K}\Big(\frac{\partial\omega}{\partial x} n_x v_h\Big) ds \\
&+ \sum_K \int_{\partial K}\Big(\frac{\partial\omega}{\partial y} n_y \tilde{v}_h\Big) ds = E_1 + E_2 + E_3,
\end{aligned}$$

where $E_1 = \sum_K \int_K \partial^2\omega/\partial y^2(v_h - \tilde{v}_h)$, $E_2 = \sum_K \int_{\partial K} \partial\omega/\partial x n_x v_h ds$, $E_3 = \sum_K \int_{\partial K} \partial\omega/\partial y n_y \tilde{v}_h ds$.

Noticing that $\partial v_h/\partial y = \partial\tilde{v}_h/\partial y$ and $\partial\tilde{\hat{v}}_h/\partial\xi = 0$, by *Poincaré* inequality, we have

$$\begin{aligned}
\|v_h - \tilde{v}_h\|_{0,K} &= h_x^{1/2}h_y^{1/2}\|\hat{v}_h - \tilde{\hat{v}}_h\|_{0,\hat{K}} \\
&\le Ch_x^{1/2}h_y^{1/2}|\hat{v}_h - \tilde{\hat{v}}_h|_{1,\hat{K}} \le Ch_x\Big\|\frac{\partial v_h}{\partial x}\Big\|_{0,K},
\end{aligned}$$

thus

$$E_1 = \sum_K \int_K \frac{\partial^2 \omega}{\partial y^2}(v_h - \tilde{v}_h) \le Ch|\omega|_2 \|v_h\|_h.$$

By Lemma 2.1, we have

$$\begin{aligned}
E_2 &= \sum_K \int_{\partial K} \frac{\partial \omega}{\partial x} n_x v_h ds \\
&= \sum_K \int_{\partial K} \Big(\frac{\partial \omega}{\partial x} - M_F \frac{\partial \omega}{\partial x}\Big) n_x (v_h - M_F v_h) ds \\
&\le C \sum_K \sum_{F \subset \partial K} \frac{|F|}{h_x h_y} \frac{h_y}{|F|} \Big(h_x^2 \Big\|\frac{\partial^2 \omega}{\partial x^2}\Big\|_{0,K}^2 + h_y^2 \Big\|\frac{\partial^2 \omega}{\partial x \partial y}\Big\|_{0,K}^2\Big)^{1/2} h \|v_h\|_h \\
&\le Ch|\omega|_2 \|v_h\|_h.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
E_3 &= \sum_K \int_{\partial K} \frac{\partial \omega}{\partial y} n_y \tilde{v}_h ds \\
&= \sum_K \sum_{F \subset \partial K} \int_F \Big(\frac{\partial \omega}{\partial y} - M_F \frac{\partial \omega}{\partial y}\Big)(\tilde{v}_h - M_F \tilde{v}_h) n_y ds \\
&\le C \sum_K \frac{|F|}{|K|} \frac{h_x}{|F|} \Big(h_x^2 \Big\|\frac{\partial^2 \omega}{\partial x \partial y}\Big\|_{0,K}^2 + h_y^2 \Big\|\frac{\partial^2 \omega}{\partial y^2}\Big\|_{0,K}^2\Big)^{1/2} \\
&\quad \times \Big(h_x^2 \Big\|\frac{\partial \tilde{v}_h}{\partial x}\Big\|_{0,K}^2 + h_y^2 \Big\|\frac{\partial \tilde{v}_h}{\partial y}\Big\|_{0,K}^2\Big)^{1/2} \\
&\le C \sum_K \frac{h_x}{h_x h_y} \Big(h_x^2 \Big\|\frac{\partial^2 \omega}{\partial x \partial y}\Big\|_{0,K}^2 + h_y^2 \Big\|\frac{\partial^2 \omega}{\partial y^2}\Big\|_{0,K}^2\Big)^{1/2} h_y \Big\|\frac{\partial \tilde{v}_h}{\partial y}\Big\|_{0,K} \\
&\le Ch|\omega|_2 \|v_h\|_h.
\end{aligned}$$

Collecting the above estimates of E_1, E_2, E_3, there yields the desired result. □

Lemma 2.3 *For any $v_h \in V_h$, we have*

$$\|v_h\|_0 \le C\|v_h\|_h. \tag{18}$$

Proof. Let ω be the solution of the following second order elliptic problem,

$$\begin{cases} -\Delta\omega = g, & \text{for any } x \in \Omega, \\ \omega = 0, & \text{for any } x \in \partial\Omega. \end{cases} \tag{19}$$

By the PDE theory, we have $\omega \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\|\omega\|_2 \le C\|g\|_0$. Hence, by Green's formula and Lemma 2.2,

$$\begin{aligned} \Big|\int_\Omega g v_h\Big| &= \Big|-\int_\Omega \Delta\omega v_h\Big| \\ &\le \Big|\sum_K \int_K \nabla\omega\nabla v_h\Big| + \Big|\sum_K \int_{\partial K} \frac{\partial\omega}{\partial n} v_h ds\Big| \\ &\le C\|\omega\|_2\|v_h\|_h \le C\|g\|_0\|v_h\|_h. \end{aligned}$$

Taking $g = v_h$ in the above inequality, there yields the desired result.

Now we introduce the moving grid for the two elements on anisotropic meshes. Divide the time axis to N parts with points t_n $(0 = t_0 < t_1 < \cdots < t_N = T)$, and let $J_n = (t_n, t_{n+1}]$, $V_n^h = \{v(x, t_n) \in V_h : v \text{ is positive at each nodal of elements}\}$ be the associated space with respect to t_n, which does not need to satisfy the above regularity or quasi-uniform assumption.

We choose the approximation solution space S^h of $u(x, t)$ as follows [23]: the function $u^h(x, t)$ in the space S^h has N interpolation functions, which are determined by the $N+1$ nodal values of $u^h(x, t)$. For each interval $t_n < t \le t_{n+1}$, $u^h(x, t)$ is a linear interpolation of $u^h(x, t_n)$ and $u^h(x, t_{n+1})$.

The value $u_n^h = u^h(x, t_n)$ is determined by the following scheme with moving grid: For any $v \in V_{n+1}^h$,

$$\begin{cases} \|u_0 - u_0^h\|_0 \le Ch, \\ (\hat{u}_n^h - u_n^h, v - \hat{u}_n^h) \ge 0, \\ (u_{n+1}^h - \hat{u}_n^h, v - u_{n+1/2}^h) + a_h(u_{n+1/2}^h, v - u_{n+1/2}^h)\Delta t_n \\ \qquad\qquad\qquad \ge (f_{n+1/2}, v - u_{n+1/2}^h)\Delta t_n, \end{cases} \tag{20}$$

where

$$\begin{aligned} &a_h(u_h, v_h) = \sum_K \int_K \nabla u_h \nabla v_h dxdy, \\ &u_n^h = u^h(x, t_n), \quad u_{n+1/2}^h = \Big(\frac{1}{2}\Big)(\hat{u}_n^h + u_{n+1}^h), \\ &f_{n+1/2} = \Big(\frac{1}{2}\Big)(f(x, t_n) + f(x, t_{n+1})), \quad n = 0, 1, \ldots, N-1, \end{aligned}$$

and $\hat{u}_n^h$ is a correction of u_n^h in V_{n+1}^h.

The minimum theory of the second order functional on convex set shows that u_{n+1}^h determined by (20) uniquely exists.

Now we divide the domain Ω into two parts:

$$\Omega^+(t) = \{x \in \Omega : u(x,t) > 0\}, \quad \Omega^0(t) = \{x \in \Omega : u(x,t) = 0\}.$$

Denote the difference between $\Omega^+(t_n)$ and $\Omega^+(t_{n+1})$ by D_n, the Lebesgue measure of D_n by $m(D_n)$. Like [23], we assume that

$$\sum_{n=0}^{N-1} m(D_n) < C, \tag{21}$$

which means that u does not change quickly from zero to positive. $\square$

3. Error estimates

The error between the exact solution $u(x,\, t)$ and the approximation $u^h(x,\, t)$ contains three parts: approximation error by finite element method in space, finite difference error in time direction and the error due to mesh changing.

First, we prove the following very important fact. For any $u \in H^1(\Omega)$, $\varphi \in V_n^h$, $K \in J_h$,

$$a_h(u - \Pi u,\, \varphi) = 0. \tag{22}$$

In fact, it is easy to check that, for FE1, $\int_K (u - \Pi u) = \int_{\partial K} (u - \Pi u) ds = 0$, $\partial\varphi/\partial n|_{\partial K}$ and $\Delta\varphi|_K$ are two constants, and for FE2, $\Delta\varphi|_K = \int_{\partial K} (u - \Pi u) ds = 0$, $\partial\varphi/\partial n|_{\partial K}$ is a constant. So we have

$$a_h(u - \Pi u,\, \varphi) = \sum_K \int_{\partial K} \frac{\partial \varphi}{\partial n} (u - \Pi u) ds - \sum_K \int_K \Delta\varphi (u - \Pi u) = 0.$$

For convenience, we also introduce the following remarks as [18]:

$$\begin{aligned}
&v_n = u_n^h - \Pi_n u_n, \quad e_n = u_n - \Pi_n u_n, \qquad n = 0,\, 1,\, \ldots,\, N,\\
&\hat{v}_n = \hat{u}_n^h - \Pi_{n+1} u_n,\ \hat{e}_n = u_n - \Pi_{n+1} u_n, \quad n = 0,\, 1,\, \ldots,\, N-1,\\
&v_{n+1/2} = \frac{1}{2}(v_{n+1} + \hat{v}_n),\ e_{n+1/2} = \frac{1}{2}(e_{n+1} + \hat{e}_n),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 0,\, 1,\, \ldots,\, N-1,
\end{aligned} \tag{23}$$

where Π_n is the restriction of Π when $t = t_n$.

Lemma 3.1 *There holds the following inequality*:

$$\begin{aligned}(v_{n+1}-\hat{v}_n,\, v_{n+1/2}) &+ a_h(v_{n+1/2},\, v_{n+1/2})\\ &\le (e_{n+1}-\hat{e}_n,\, v_{n+1/2}) + a_h(e_{n+1/2},\, v_{n+1/2})\Delta t_n + \sum_{j=1}^{8}\rho_j, \quad (24)\end{aligned}$$

where

$$\begin{aligned}
\rho_1 &= a_h(u_{n+1}-u_{n+1/2},\, v_{n+1/2})\Delta t_n,\\
\rho_2 &= -\frac{1}{2}[(u_{n+1}-u_n, u_{n+1}-\Pi_{n+1}u_{n+1})\\
&\qquad + a_h(u_{n+1},\, u_{n+1}-\Pi_{n+1}u_{n+1})\Delta t_n\\
&\quad - (f_{n+1},\, u_{n+1}-\Pi_{n+1}u_{n+1})\Delta t_n],\\
\rho_3 &= -\frac{1}{2}[(u_{n+1}-u_n,\, u_{n+1}-\Pi_{n+1}u_n)\\
&\qquad + a_h(u_{n+1},\, u_{n+1}-\Pi_{n+1}u_n)\Delta t_n\\
&\quad - (f_{n+1},\, u_{n+1}-\Pi_{n+1}u_n)\Delta t_n],\\
\rho_4 &= (f_{n+1/2}-f_{n+1},\, v_{n+1/2})\Delta t_n,\\
\rho_5 &= \frac{1}{2}\Big[(u_{n+1}-u_n,\, u_{n+1}-\Pi_{n+1}u_{n+1})\\
&\qquad - \Big(\frac{\partial u}{\partial t}(t_{n+1}),\, u_{n+1}-\Pi_{n+1}u_{n+1}\Big)\Delta t_n\Big],\\
\rho_6 &= -\Big[(u_{n+1}-u_n,\, v_{n+1/2}) - \Big(\frac{\partial u}{\partial t}(t_{n+1}),\, v_{n+1/2}\Big)\Delta t_n\Big],\\
\rho_7 &= \frac{1}{2}\Big[(u_{n+1}-u_n,\, u_{n+1}-\Pi_{n+1}u_n)\\
&\qquad - \Big(\frac{\partial u}{\partial t}(t_{n+1}),\, u_{n+1}-\Pi_{n+1}u_n\Big)\Delta t_n\Big],\\
\rho_8 &= \frac{1}{2}[\Gamma_h(u_{n+1},\, u^h_{n+1}-u_{n+1}) + \Gamma_h(u_{n+1},\, \hat{u}^h_n-u_{n+1})]\Delta t_n,
\end{aligned}$$

here $\Gamma_h(u,\, v) = \sum_K \int_{\partial K}(\partial u/\partial n)v ds$.

Proof. Substituting $v_h \in V^h_{n+1}$ for $v \in V$ in (10), we have

$$\begin{aligned}\Big(\frac{\partial u}{\partial t}(t_{n+1}),\, v_h-u_{n+1}\Big) &+ a_h(u_{n+1},\, v_h-u_{n+1})\\ &\ge (f_{n+1},\, v_h-u_{n+1}) + \Gamma_h(u_{n+1},\, v_h-u_{n+1}). \quad (25)\end{aligned}$$

So

$$
\begin{aligned}
&(u_{n+1}-u_n,\, v_h-u_{n+1})+a_h(u_{n+1},\, v_h-u_{n+1})\Delta t_n \\
&\geq (f_{n+1},\, v_h-u_{n+1})\Delta t_n+\Gamma_h(u_{n+1},\, v_h-u_{n+1})\Delta t_n+E(v_h),
\end{aligned}
\tag{26}
$$

where

$$
E(v)=(u_{n+1}-u_n,\, v_h-u_{n+1})-\Big(\frac{\partial u}{\partial t}(t_{n+1}),\, v_h-u_{n+1}\Big)\Delta t_n. \tag{27}
$$

Using the definition $v_{n+1/2}=(1/2)(u^h_{n+1}-\Pi_{n+1}u_{n+1}+\hat{u}^h_n-\Pi_{n+1}u_n)$, we have

$$
\begin{aligned}
&(u_{n+1}-u_n,\, v_{n+1/2})+a_h(u_{n+1/2},\, v_{n+1/2})\Delta t_n \\
&=\frac{1}{2}[(u_{n+1}-u_n,\, u^h_{n+1}-\Pi_{n+1}u_{n+1}) \\
&\qquad +a_h(u_{n+1/2},\, u^h_{n+1}-\Pi_{n+1}u_{n+1})\Delta t_n] \\
&+\frac{1}{2}[(u_{n+1}-u_n,\, \hat{u}^h_n-\Pi_{n+1}u_n)+a_h(u_{n+1/2},\, \hat{u}^h_n-\Pi_{n+1}u_n)\Delta t_n].
\end{aligned}
\tag{28}
$$

Taking $v_h=u^h_{n+1}$ and $v_h=\hat{u}^h_n$ in (26), respectively, it results in two inequalities, and then substituting them into (28), there yields

$$
\begin{aligned}
&(u_{n+1}-u_n,\, v_{n+1/2})+a_h(u_{n+1/2},\, v_{n+1/2})\Delta t_n \\
&\geq \frac{1}{2}[(u_{n+1}-u_n,\, u_{n+1}-\Pi_{n+1}u_{n+1})+(u_{n+1}-u_n,\, u_{n+1}-\Pi_{n+1}u_n)] \\
&\quad +\frac{1}{2}[a_h(u_{n+1/2},\, u_{n+1}-\Pi_{n+1}u_{n+1})\Delta t_n \\
&\qquad +a_h(u_{n+1/2},\, u_{n+1}-\Pi_{n+1}u_n)\Delta t_n] \\
&\quad +\frac{1}{2}a_h(u_{n+1/2}-u_{n+1},\, u^h_{n+1}+\hat{u}^h_n-2u_{n+1})\Delta t_n \\
&\quad +\frac{1}{2}(f_{n+1},\, u^h_{n+1}+\hat{u}^h_n-2u_{n+1})\Delta t_n+\frac{1}{2}[E(u^h_{n+1})+E(\hat{u}^h_n)] \\
&\quad +\frac{1}{2}[\Gamma_h(u_{n+1},\, u^h_{n+1}-u_{n+1})+\Gamma_h(u_{n+1},\, \hat{u}^h_n-u_{n+1})] \\
&=a_h(u_{n+1/2}-u_{n+1},\, v_{n+1/2})\Delta t_n+(f_{n+1},\, v_{n+1/2})\Delta t_n \\
&\quad +\frac{1}{2}[(u_{n+1}-u_n,\, u_{n+1}-\Pi_{n+1}u_{n+1}) \\
&\qquad +a_h(u_{n+1},\, u_{n+1}-\Pi_{n+1}u_{n+1})\Delta t_n \\
&\quad -(f_{n+1},\, u_{n+1}-\Pi_{n+1}u_{n+1})\Delta t_n] \\
&\quad +\frac{1}{2}[(u_{n+1}-u_n,\, u_{n+1}-\Pi_{n+1}u_n)+a_h(u_{n+1},\, u_{n+1}-\Pi_{n+1}u_n)\Delta t_n
\end{aligned}
$$

$$
\begin{aligned}
&-(f_{n+1},\, u_{n+1}-\Pi_{n+1}u_n)\Delta t_n] + \frac{1}{2}[E(u_{n+1}^h) + E(\hat{u}_n^h)] \\
&+\frac{1}{2}[\Gamma_h(u_{n+1},\, u_{n+1}^h - u_{n+1}) + \Gamma_h(u_{n+1},\, \hat{u}_n^h - u_{n+1})],
\end{aligned} \tag{29}
$$

where

$$
\begin{aligned}
&\frac{1}{2}[E(u_{n+1}^h) + E(\hat{u}_n^h)] \\
&= \Big[(u_{n+1}-u_n,\, v_{n+1/2}) - \Big(\frac{\partial u}{\partial t}(t_{n+1}),\, v_{n+1/2}\Big)\Delta t_n\Big] \\
&\quad -\frac{1}{2}\Big[(u_{n+1}-u_n,\, u_{n+1}-\Pi_{n+1}u_{n+1}) \\
&\qquad\qquad -\Big(\frac{\partial u}{\partial t}(t_{n+1}),\, u_{n+1}-\Pi_{n+1}u_{n+1})\Delta t_n\Big] \\
&\quad -\frac{1}{2}\Big[(u_{n+1}-u_n,\, u_{n+1}-\Pi_{n+1}u_n) \\
&\qquad\qquad -\Big(\frac{\partial u}{\partial t}(t_{n+1}),\, u_{n+1}-\Pi_{n+1}u_n\Big)\Delta t_n\Big].
\end{aligned} \tag{30}
$$

Similarly, taking $v = \Pi_{n+1}u_{n+1}$ and $v = \Pi_{n+1}u_n$ in (20), respectively, we have

$$
\begin{aligned}
(u_{n+1}^h - \hat{u}_n^h,\, v_{n+1/2}) + a_h(u_{n+1/2}^h,\, v_{n+1/2})\Delta t_n& \\
\le (f_{n+1/2},\, v_{n+1/2})\Delta t_n&.
\end{aligned} \tag{31}
$$

From (29) and (31), we have

$$
\begin{aligned}
&(u_{n+1}^h - \hat{u}_n^h,\, v_{n+1/2}) + a_h(u_{n+1/2}^h,\, v_{n+1/2})\Delta t_n \\
&\le a_h(u_{n+1/2}-u_{n+1},\, v_{n+1/2})\Delta t_n + (f_{n+1/2}-f_{n+1},\, v_{n+1/2})\Delta t_n \\
&+(u_{n+1}-u_n,\, v_{n+1/2}) + a_h(u_{n+1/2},\, v_{n+1/2})\Delta t_n \\
&-\frac{1}{2}[(u_{n+1}-u_n,\, u_{n+1}-\Pi_{n+1}u_{n+1}) \\
&\quad + a_h(u_{n+1},\, u_{n+1}-\Pi_{n+1}u_{n+1})\Delta t_n \\
&\quad -(f_{n+1},\, u_{n+1}-\Pi_{n+1}u_{n+1})\Delta t_n] \\
&-\frac{1}{2}[(u_{n+1}-u_n,\, u_{n+1}-\Pi_{n+1}u_n) + a_h(u_{n+1},\, u_{n+1}-\Pi_{n+1}u_n)\Delta t_n \\
&\quad -(f_{n+1},\, u_{n+1}-\Pi_{n+1}u_n)\Delta t_n] - \frac{1}{2}[E(u_{n+1}^h) + E(\hat{u}_n^h)] \\
&-\frac{1}{2}[\Gamma_h(u_{n+1},\, u_{n+1}^h - u_{n+1}) + \Gamma_h(u_{n+1},\, \hat{u}_n^h - u_{n+1})].
\end{aligned} \tag{32}
$$

Notice that

$$
\begin{aligned}
&(v_{n+1}-\hat{v}_n,\, v_{n+1/2})+a_h(v_{n+1/2},\, v_{n+1/2})\Delta t_n\\
=\;&(u^h_{n+1}-\hat{u}^h_n,\, v_{n+1/2})+a_h(u^h_{n+1/2},\, v_{n+1/2})\Delta t_n\\
&+(e_{n+1}-\hat{e}_n,\, v_{n+1/2})+a_h(e_{n+1/2},\, v_{n+1/2})\\
&-(u_{n+1}-u_n,\, v_{n+1/2})-a_h(u_{n+1/2},\, v_{n+1/2})\Delta t_n
\end{aligned}
\tag{33}
$$

and substitute (32) into (33), there yields the desired result. □

Lemma 3.2 *If $f\in C(J,\, L_\infty(\Omega))$, $\partial f/\partial t\in L_2(J,\, L_\infty(\Omega))$, $u_0\in H^2(\Omega)\cap H^1_0(\Omega)$, and* (21) *satisfied, then we have the following estimate*:

$$
\|v_N\|_0^2+\sum_{n=0}^{N-1}a_h(v_{n+1/2},\, v_{n+1/2})\le C[(M^2h^2+1)h^2+\Delta t], \tag{34}
$$

where M is the times of grid changing, and Mh is bounded,
$\Delta t=\max_{0\le n\le N-1}\Delta t_n$.

Proof. First, we estimate ρ_i $(i=1,\,\ldots,\,8)$ in (24), respectively.

By the coerciveness of $a_h(u,\, v)$ and Cauchy inequality, we can deduce that

$$
\begin{aligned}
\rho_1&=\frac12 a_h(u_{n+1}-u_n,\, v_{n+1/2})\Delta t_n\\
&=\frac12 a_h\Big(\int_{t_n}^{t_{n+1}}\frac{\partial u}{\partial t}dt,\, v_{n+1/2}\Big)\Delta t_n\\
&\le\frac12\int_{t_n}^{t_{n+1}}\Big\|\frac{\partial u}{\partial t}\Big\|_1 dt\|v_{n+1/2}\|_h\Delta t_n\\
&\le C\Big\|\frac{\partial u}{\partial t}\Big\|^2_{L_2(J_n,H^1_0(\Omega))}(\Delta t_n)^2+\frac19 a_h(v_{n+1/2},\, v_{n+1/2})\Delta t_n\\
&\le C(\Delta t_n)^2+\frac19 a_h(v_{n+1/2},\, v_{n+1/2})\Delta t_n,
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
\rho_2&\le\frac12\Big[\int_{t_n}^{t_{n+1}}\Big\|\frac{\partial u}{\partial t}\Big\|_0 dt\|e_{n+1}\|_0\\
&\qquad+\|u_{n+1}\|_1\|e_{n+1}\|_h\Delta t_n+\|f_{n+1}\|_0\|e_{n+1}\|_0\Delta t_n\Big]\\
&\le[Ch^2(\Delta t_n)^{1/2}+Ch\Delta t_n+Ch^2\Delta t_n]\\
&\le C(h^2+\Delta t_n)
\end{aligned}
\tag{36}
$$

and

$$
\begin{aligned}
\rho_3 &= -\frac{1}{2}[(u_{n+1}-u_n,\, u_{n+1}-\Pi_{n+1}u_n)\\
&\qquad + a_h(u_{n+1},\, u_{n+1}-\Pi_{n+1}u_n)\Delta t_n\\
&\qquad - (f_{n+1},\, u_{n+1}-\Pi_{n+1}u_n)\Delta t_n]\\
&= -\frac{1}{2}\Big[(u_{n+1}-u_n,\, u_n-\Pi_{n+1}u_n) + a_h(u_{n+1},\, u_n-\Pi_{n+1}u_n)\Delta t_n\\
&\qquad - (f_{n+1},\, u_n-\Pi_{n+1}u_n)\Delta t_n\\
&\qquad + \Big(\int_{t_n}^{t_{n+1}}\frac{\partial u}{\partial t}dt,\, \int_{t_n}^{t_{n+1}}\frac{\partial u}{\partial t}dt\Big) + a\Big(u_{n+1},\, \int_{t_n}^{t_{n+1}}\frac{\partial u}{\partial t}dt\Big)\Delta t_n\\
&\qquad - \Big(f_{n+1},\, \int_{t_n}^{t_{n+1}}\frac{\partial u}{\partial t}dt\Big)\Delta t_n\Big]\\
&\le C[h^2(\Delta t_n)^{1/2} + h\Delta t_n + h^2\Delta t_n + \Delta t_n + (\Delta t_n)^{3/2}]\\
&\le C[h^2+\Delta t_n].
\end{aligned}
\tag{37}
$$

By $\|v_{n+1/2}\|_0^2 \le \|v_{n+1/2}\|_h^2 \le a_h(v_{n+1/2},\, v_{n+1/2})$ and Cauchy inequality, we have

$$
\begin{aligned}
\rho_4 &= -\frac{1}{2}\Big(\int_{t_n}^{t_{n+1}}\frac{\partial f}{\partial t}dt,\, v_{n+1/2}\Big)\Delta t_n\\
&\le C\int_{t_n}^{t_{n+1}}\Big\|\frac{\partial f}{\partial t}\Big\|_0 dt\|v_{n+1/2}\|_0\Delta t_n\\
&\le C\Big\|\frac{\partial f}{\partial t}\Big\|^2_{L_2(J_n,L_2(\Omega))}(\Delta t_n)^2 + \frac{1}{9}a_h(v_{n+1/2},\, v_{n+1/2})\\
&\le C(\Delta t_n)^2 + \frac{1}{9}a_h(v_{n+1/2},\, v_{n+1/2})\Delta t_n.
\end{aligned}
\tag{38}
$$

Noticing that $\partial u/\partial t \in L_\infty(J,\, L_\infty(\Omega))$, like [23], ρ_5, ρ_6 and ρ_7 can be estimated as

$$\rho_5 \le C(h^2+\Delta t_n), \tag{39}$$

$$\rho_6 \le \frac{1}{9}a_h(v_{n+1/2},\, v_{n+1/2}) + C\Delta t_n \tag{40}$$

and

$$\rho_7 \le C(\Delta t_n + h^2), \tag{41}$$

respectively.

By $(1/2)(u^h_{n+1}+\hat{u}^h_n) - u_{n+1} = v_{n+1/2} + (1/2)(\Pi_{n+1}u_{n+1}+\Pi_n u_n) - u_{n+1}$

and Lemma 2.2, we can immediately obtain

$$\begin{aligned} \rho_8 &= \frac{1}{2}[\Gamma_h(u_{n+1},\, u^h_{n+1} - u_{n+1}) + \Gamma_h(u_{n+1},\, \hat{u}^h_n - u_{n+1})]\Delta t_n \\ &\leq \frac{1}{9} a_h(v_{n+1/2},\, v_{n+1/2})\Delta t_n + C[h^4 + (\Delta t_n)^2]. \end{aligned} \tag{42}$$

So

$$\sum_{i=1}^{8} \rho_i \leq \frac{4}{9} a_h(v_{n+1/2},\, v_{n+1/2})\Delta t_n + q_{n+1} \tag{43}$$

with

$$\sum_{n=0}^{N-1} q_{n+1} \leq C(\Delta t + h^2). \tag{44}$$

On the other hand,

$$\begin{aligned} &(v_{n+1} - \hat{v}_n,\, v_{n+1/2}) + a_h(v_{n+1/2},\, v_{n+1/2})\Delta t_n \\ &= \frac{1}{2}(v_{n+1} - \hat{v}_n,\, v_{n+1} + \hat{v}_n) + a_h(v_{n+1/2},\, v_{n+1/2})\Delta t_n \\ &= \frac{1}{2}\|v_{n+1}\|_0^2 - \frac{1}{2}\|\hat{v}_n\|_0^2 + a_h(v_{n+1/2},\, v_{n+1/2})\Delta t_n, \\ &\quad (\hat{u}^h_n - u^h_n,\, v - \hat{u}^h_n) \geq 0, \quad \text{for any } v \in V^h_{n+1} \end{aligned}$$

and

$$\hat{u}^h_n - u^h_n = (\hat{v}_n - v_n) - (\hat{e}_n - e_n),$$

we can observe that

$$(\hat{v}_n - v_n,\, v - \hat{u}^h_n) \geq (\hat{e}_n - e_n,\, v - \hat{u}^h_n).$$

Taking $v = \Pi_{n+1} u_n$ in the above inequality, there yields

$$(\hat{v}_n - v_n,\, \hat{v}_n) \leq (\hat{e}_n - e_n,\, \hat{v}_n),$$

which implies that

$$\|\hat{v}_n\|_0^2 - (v_n,\, \hat{v}_n) \leq (\hat{e}_n - e_n,\, \hat{v}_n).$$

Thus for $0 < \xi < 1$,

$$\frac{1}{2}(\|\hat{v}_n\|_0^2 - \|v_n\|_0^2) \leq (\hat{e}_n - e_n,\, \hat{v}_n) \leq \frac{\xi}{2}\|\hat{v}_n\|_0^2 + \frac{1}{2\xi}\|\hat{e}_n - e_n\|_0^2,$$

i.e.,

$$(1-\xi)\|\hat{v}_n\|_0^2 - \|v_n\|_0^2 \le \frac{1}{\xi}\|\hat{e}_n - e_n\|_0^2, \tag{45}$$

where we can assume that $S_{n+1} \neq S_n$. In fact, when $S_{n+1} = S_n$, $\hat{e}_n = e_n$, the right hand side of the above inequality vanishes, then the estimate can be simplified by choosing $\xi = 0$.

By Lemma 3.1 and (43)–(45), we have

$$\begin{aligned}
&(1-\xi)\|v_{n+1}\|_0^2 - \|v_n\|_0^2 + 2(1-\xi)a_h(v_{n+1/2}, v_{n+1/2})\Delta t_n \\
&\le \frac{1}{\xi}\|\hat{e}_n - e_n\|_0^2 \\
&\quad + 2(1-\xi)\Big[(e_{n+1} - \hat{e}_n, v_{n+1/2}) + a_h(e_{n+1/2}, v_{n+1/2})\Delta t_n + \sum_{i=1}^{8}\rho_i\Big] \\
&\le \frac{1}{\xi}\|\hat{e}_n - e_n\|_0^2 + C(\|e_{n+1} - \hat{e}_n\|_0^2 + \|e_{n+1/2}\|_h^2\Delta t_n) \\
&\quad + (1-\xi)a_h(v_{n+1/2}, v_{n+1/2})\Delta t_n + 2(1-\xi)q_{n+1}.
\end{aligned}$$

Thus

$$\begin{aligned}
&(1-\xi)\|v_{n+1}\|_0^2 - \|v_n\|_0^2 + (1-\xi)a_h(v_{n+1/2}, v_{n+1/2})\Delta t_n \\
&\le \frac{1}{\xi}\|\hat{e}_n - e_n\|_0^2 + C(\|e_{n+1} - \hat{e}_n\|_0^2 + \|e_{n+1/2}\|_h^2\Delta t_n + q_{n+1}).
\end{aligned} \tag{46}$$

We rewrite (46) as

$$\begin{aligned}
&\eta_{n+1}\|v_{n+1}\|_0^2 - \|v_n\|_0^2 + \eta_{n+1}a_h(v_{n+1/2}, v_{n+1/2})\Delta t_n \\
&\le \frac{1}{\xi}\|\hat{e}_n - e_n\|_0^2 + C(\|e_{n+1} - \hat{e}_n\|_0^2 + \|e_{n+1/2}\|_h^2\Delta t_n + q_{n+1}),
\end{aligned}$$

where, if $S_{n+1} \neq S_n$, $\eta_{n+1} = 1-\xi$, otherwise, $\eta_{n+1} = 1$.

Multiplying $\prod_{i=1}^{n}\eta_i$ (≤ 1) to the left hand side of the above formula, we have

$$\begin{aligned}
&\prod_{i=1}^{n+1}\eta_i\|v_{n+1}\|_0^2 - \prod_{i=1}^{n}\eta_i\|v_n\|_0^2 + \prod_{i=1}^{n+1}\eta_i a_h(v_{n+1/2}, v_{n+1/2})\Delta t_n \\
&\le \frac{1}{\xi}\|\hat{e}_n - e_n\|_0^2 + C(\|e_{n+1} - \hat{e}_n\|_0^2 + \|e_{n+1/2}\|_h^2\Delta t_n + q_{n+1}).
\end{aligned}$$

Summing it for n, there yields

$$\prod_{i=1}^{N}\eta_i\Big[\|v_N\|_0^2+\sum_{n=0}^{N-1}a_h(v_{n+1/2},\,v_{n+1/2})\Delta t_n\Big]$$
$$\le \|v_0\|_0^2+\frac{1}{\xi}\sum_{n=0}^{N-1}\|\hat{e}_n-e_n\|_0^2 \tag{47}$$
$$+\,C\sum_{n=0}^{N-1}(\|e_{n+1}-\hat{e}_n\|_0^2+\|e_{n+1/2}\|_h^2\Delta t_n+q_{n+1}).$$

On the other hand,

$$e_{n+1}-\hat{e}_n=(I-\Pi_{n+1})\int_{t_n}^{t_{n+1}}\frac{\partial u}{\partial t}dt,$$

thus

$$\sum_{n=0}^{N-1}\|e_{n+1}-\hat{e}_n\|_0^2\le\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}h^2\Big\|\frac{\partial u}{\partial t}\Big\|_1^2dt\Delta t_n$$
$$\le C(h^4+(\Delta t)^2)\Big\|\frac{\partial u}{\partial t}\Big\|_{L_2(J,H_0^1(\Omega))}.$$

Since $\prod_{i=1}^{N}\eta_i=(1-\xi)^M$, it follows from (44) and (47) that

$$\|v_0\|_0^2\le Ch^4\|u_0\|_2^2,\quad \sum_{n=0}^{N-1}\|e_{n+1/2}\|_h^2\Delta t_n\le Ch^2\max_{0\le n\le N-1}\|u_n\|_2^2.$$

Substituting the above estimates into (47), there yields

$$\|v_N\|_0^2+\sum_{n=0}^{N-1}a_h(v_{n+1/2},\,v_{n+1/2})\Delta t_n$$
$$\le(1-\xi)^{-M}\xi^{-1}\sum_{n=0}^{N-1}\|\hat{e}_n-e_n\|_0^2+(1-\xi)^{-M}C(h^2+\Delta t). \tag{48}$$

Choosing $\xi=1/(M+1)$ in (48), we have

$$\|v_N\|_0^2+\sum_{n=0}^{N-1}a_h(v_{n+1/2},\,v_{n+1/2})\Delta t_n$$
$$\le C\sum_{n=0}^{N-1}(M+1)\|\hat{e}_n-e_n\|_0^2+C(h^2+\Delta t).$$

Even since

$$\sum_{n=0}^{N-1}\|\hat{e}_n - e_n\|_0^2 \le M \max_{0\le n\le N-1}\|\hat{e}_n - e_n\|_0^2 \le CMh^4 \max_{0\le n\le N}\|u_n\|_2^2,$$

we have

$$\begin{aligned}\|v_N\|_0^2 + \sum_{n=0}^{N-1} a_h(v_{n+1/2},\, v_{n+1/2})\Delta t_n &\le C[M(M+1)h^4 + h^2 + \Delta t]\\ &\le C[(M^2h^2+1)h^2 + \Delta t].\end{aligned}$$

The proof is completed. □

Now we are ready to state our main conclusion of this paper.

Theorem 3.3 *Under the assumption of* Lemma 3.2*, the error between the approximate of* (20) *and the exact solution of* (10) *can be estimated as follows*:

$$\begin{aligned}\max_{0\le n\le N}\|u_n^h - u_n\|_0^2 + \sum_{n=0}^{N-1} a_h(u_{n+1/2}^h - u_{n+1/2},\, u_{n+1/2}^h - u_{n+1/2})\Delta t_n&\\ \le C(M^2h^2+1)h^2 + \Delta t.&\end{aligned}$$

Proof. By the triangle inequality, we have

$$\|u_n^h - u_n\|_0^2 = \|v_n - e_n\|_0^2 \le \|v_n\|_0^2 + \|e_n\|_0^2 \tag{49}$$

and

$$\begin{aligned}a_h(u_{n+1/2}^h - u_{n+1/2},\, u_{n+1/2}^h - u_{n+1/2})&\\ \le \frac{1}{2}a_h(v_{n+1/2},\, v_{n+1/2}) + \frac{1}{2}a_h(e_{n+1/2},\, e_{n+1/2}).&\end{aligned}\tag{50}$$

From (9) we can derive

$$\begin{aligned}\|e_{n+1/2}\|_h^2 &\le C[\|u_{n+1} - \Pi_{n+1}u_{n+1}\|_h^2 + \|u_n - \Pi_{n+1}u_n\|_h^2]\\ &\le Ch^2 \max_{0\le n\le N-1}\|u_n\|_2^2 \le Ch^2.\end{aligned}\tag{51}$$

By Lemma 3.2, combining (50) and (51), there yields

$$\begin{aligned}\sum_{n=0}^{N-1} a_h(u_{n+1/2}^h - u_{n+1/2},\, u_{n+1/2}^h - u_{n+1/2})\Delta t_n&\\ \le C[(M^2h^2+1)h^2 + \Delta t].&\end{aligned}\tag{52}$$

Similarly,

$$\max_{0\le n\le N}\|u_n^h-u_n\|_0^2\le C\max_{0\le n\le N}\{a_h(v_n,\,v_n)+h^4\max_{0\le n\le N-1}\|u_n\|_2^2\}$$
$$\le C[(M^2h^2+1)h^2+\Delta t+h^4]. \quad (53)$$

Thus the desired result follows from (52) and (53). □

Remark We can observe from the above theorem that the grid can not be changed arbitrarily and the orders of the error estimates are optimal only if Mh is a bounded constant.

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