Hokkaido Mathematical Journal Vol. 38 (2009) p. 797–803

On negatively curved G-manifolds of low cohomogeneity

R. Mirzaie

(Received September 22, 2008; Revised November 19, 2008)

Abstract. In this paper, we suppose that M is a negatively curved $C_K - G$-manifold, $K \leq 2$, and $M^G \neq \emptyset$. Then we characterize M from topological view point.

Key words: Riemannian manifold, Lie group, sectional curvature.

1. Introduction

Let M be a complete Riemannian manifold and let G be a closed and connected Lie subgroup of isometries of M. We denote by $G(x) = \{gx : x \in G\}$, the orbit containing x. If $m = \max\{\dim G(x) : x \in M\}$, then the number $K = n - m$, $n = \dim M$, is called the G-cohomogeneity of M, and M is called a $C_K - G$-manifold. A $C_0 - G$-manifold is called homogeneous G-manifold. Homogeneous manifolds are studied from various points of view. A theorem by S. Kobayashi ([5]), states that a homogeneous Riemannian manifold of negative curvature is simply connected. Therefore, it is diffeomorphic to R^n, $n = \dim M$. This fact does not hold any longer for arbitrary cohomogeneity. But there are interesting relations between topology of such manifolds an of their orbits, when cohomogeneity is small.

F. Podesta and A. Spiro got interesting results about $C_1 - G$-manifolds of negative curvature in [10]. Among other results, they proved that, if M is a negatively curved Riemannian $C_1 - G$-manifold and $\dim(M) \geq 3$, then either M is diffeomorphic to $R^k \times T^r$, $r + k = \dim(M)$, or $\pi_1(M) = Z$ and the principal orbits are covered by $S^{n-2} \times R$, $n = \dim(M)$. In this paper we study $C_2 - G$-manifolds of negative curvature. We show that, if M is a non-simply connected and negatively curved $C_2 - G$-manifold and $M^G \neq \emptyset$ then M is diffeomorphic to $S^1 \times R^{n-1}$ or $B^2 \times R^{n-2}$ (B^2 is the mobius band).

2000 Mathematics Subject Classification : 53C30, 57S25.

2. Preliminaries

In the following, M is a complete Riemannian manifold and G is a connected and closed Lie subgroup of isometries of M, and $\widetilde{M}$ is the universal Riemannian covering manifold of M, by the covering map:

$$k : \widetilde{M} \to M$$

We will denote the deck transformation group by Δ. Now we mention some facts, which we will use in the sequel.

Facts 2.1 *Let M be a $C_K - G$-manifold. Then*

(a) *There exists a connected covering group $\widetilde{G}$ of G, such that $\widetilde{G}$ acts by isometries on $\widetilde{M}$, and $\widetilde{M}$ is a $C_K - \widetilde{G}$-manifold. If $\widetilde{g} \in \widetilde{G}$ and $\delta \in \Delta$ then we have $\widetilde{g}\delta = \delta\widetilde{g}$.*

(b) *If G has a fixed point in M then $\widetilde{G} = G$, and $\widetilde{M}^{\widetilde{G}}$ is the full inverse image of M^G.*

(c) *Following* (b), *if $\widetilde{G}$ has only one fixed point in $\widetilde{M}$, then we have $\widetilde{M} = M$.*

Proof. For (a) and (b) see [2, p. 63, 64]. For the proof of (c), note that, if x_0 is the fixed point of G in M then by (b), $k^{-1}(x_0)$ must be a one point set. Thus k is one to one and we have $\widetilde{M} = M$. □

For definition of singular and principal orbits, used in the following facts, and details about cohomogeneity one Riemannian manifolds we refer to [1], [6] or [10].

Fact 2.2 *If R^n is of cohomogeneity one under the action of a connected and compact Lie group G of isometries, then each principal orbit is isometric to $S^{n-1}(c)$, for some $c > 0$, and there is a unique singular orbit, which is a one point set.*

Proof. It is a simple consequence of the theorem 3.1 in [8]. □

Fact 2.3 *If M is a simply connected $C_2 - G$-manifold and G is compact and connected, then there is a compact and connected subgroup H of $O(n)$, $n = \dim M$, such that the G-action on M is orbit-equivalent to the H-action on R^n.*

Proof. It is a simple consequence of the theorem 8.5 in [2, p. 208]. □

Fact 2.4 *Let R^{n+2} be a $C_2 - G$-manifold, such that* $\dim(R^{n+2})^G = 1$. *Then the singular orbits are fixed points of G, and each principal orbit is diffeomorphic to S^n.*

Proof. Let $F = (R^{n+2})^G$. Since the fixed point set of G is a totally geodesic submanifold, then F is a straight line in R^{n+2}. Without loss of generality, we assume that $F = \{(x, 0, \ldots, 0) \in R^{n+2}, x \in R\}$. Let $(b, 0, \ldots, 0) \in F$ and let T_b be the hyperplane in R^{n+2}, which is perpendicular to F at the point $(b, 0, \ldots, 0)$ (i.e., $T_b = \{b\} \times R^{n+1}$). It is easy to show that for each b, $G(T_b) = T_b$. Thus we have $G \subseteq \{I\} \times O(n+1)$ (I is the identity map of R). Let $G = \{I\} \times H$, $H \subseteq O(n+1)$. Since the action of G on R^{n+2} is of cohomogeneity two, then the action of H on R^{n+1} is of cohomogeneity one. The origin of R^{n+1} is a fixed point of H, so H is compact. Using the Fact 2.2, we get that the origin of R^{n+1} is the unique fixed point of H and the other H-orbits of R^{n+1} are diffeomorphic to S^n. Since each G-orbit of R^{n+2} is in the form

$$G(b, x_1, \ldots, x_{n+1}) = \{b\} \times H(x_1, \ldots, x_{n+1})$$

We get the result. $\square$

Fact 2.5 *If M^{n+2} is a simply connected Riemannian $C_2 - G$-manifold and* $\dim M^G = 1$ *then the principal orbits are homeomorphic to S^n.*

Proof. Since the fixed point set of G on M is not empty, then G is compact. Thus by using the Facts 2.3 and 2.4 we get the result. $\square$

Lemma 2.6 *If M is a connected and complete $C_K - G$-manifold of non-positive sectional curvature, then we have $K >$* $\dim M^G$.

Proof. This lemma is true in more general case for Riemannian manifolds. We give a simple proof for our special case. By Fact 2.1, without loss of generality, we can assume that M is simply connected. Let $F = M^G$ and $q \in F$. The map $\exp : T_qM \to M$ is a diffeomorphism (see [4, p. 149]). Let $\widetilde{F_q} = \{\exp(V) : V \in (T_qF)^\perp\}$. $\widetilde{F_q}$ is a submanifold of M. For each $g \in G$ the map $g : F \to F$ is the identity map. Thus the map $dg : T_qF \to T_qF$ is the identity map and we have $dg(T_qF)^\perp = (T_qF)^\perp$, so $G\widetilde{F_q} = \widetilde{F_q}$. Thus for each point $x \in M$, the orbit $G(x)$ is contained in $\widetilde{F_q}$, for some $q \in F$. Also it is easy to show that $\dim G(x) \neq \dim \widetilde{F_q}$. Therefore, we have:

$$\dim G(x) < \dim \widetilde{F_q} = \dim M - \dim F \Rightarrow K > \dim F. \qquad \square$$

3. Results

Let M be a Riemannian manifold of negative sectional curvature. A subset C of M is called totally convex, if it contains every geodesic segment of M, whose endpoints are in C. Thus, when C is a submanifold, it is totally geodesic in M. A c^∞ function $f : M \to R$ is said to be strictly convex at a point, if the hessian $\nabla^2 f(V, W) = VWf - (\nabla_V W)f$ is positive definite at that point.

Fact 3.1 (see [3])

(a) *If δ be an isometry of M then the function $d^2_\delta : M \to R$, $d^2_\delta(x) = d^2(x, \delta x)$ is strictly convex on M except at the minimum point set of d^2_δ, which is at most a one point set or the image of a geodesic. If δ preserves a geodesic γ (i.e., $\delta(\gamma) = \gamma$) then the image of γ is the minimum point set of d^2_δ.*

(b) *If S is a closed and totally convex submanifold of M then the map $\exp : \perp S \to M$ is a diffeomorphism ($\perp S$ is the normal bundle of S).*

Lemma 3.2 *Let M be a Riemannian manifold of negative curvature and let $\widetilde{M}$ be its universal covering, by the covering map $k : \widetilde{M} \to M$. If there is a geodesic γ in $\widetilde{M}$ and an element δ in the center of Δ, such that $\delta\gamma = \gamma$. Then M is diffeomorphic to the one of the following spaces:*

$$S^1 \times R^{n-1}, \quad B^2 \times R^{n-2}$$

($n = \dim M$ and B^2 is the mobius band).

Proof. The proof of this lemma is as like as the proof of Theorem 4.9 in [3]. Consider the function $\widetilde{f} = d^2_\delta : \widetilde{M} \to R$. Since δ preserves a geodesic γ, then the image of γ is the minimum point set of $\widetilde{f}$. Now define the function f on M as:

$$f(x) = \widetilde{f}(y), \quad y \in k^{-1}(x)$$

Since δ belongs to the center of Δ, f is well defined. In fact if y_1, y_2 belong to $k^{-1}(x)$, then there is δ_1 in Δ such that $y_1 = \delta_1 y_2$. So we have:

$$\begin{aligned}\widetilde{f}(y_1) &= d^2(y_1, \delta y_1) = d^2(\delta_1 y_2, \delta\delta_1 y_2) \\ &= d^2(\delta_1 y_2, \delta_1 \delta y_2) = d^2(y_2, \delta y_2) = \widetilde{f}(y_2)\end{aligned}$$

The minimum point set of f is the image of the geodesic $ko\gamma$, which we denote it by C. Since C is totally convex, it is simply closed geodesic in M (i.e., it is diffeomorphic to S^1). By Fact 3.1(b), $\exp : \perp C \to M$ is a diffeomorphism. Thus M is a vector bundle over the circle C. If M is orientable then it is diffeomorphic to $S^1 \times R^{n-1}$, otherwise it is diffeomorphic to $B^2 \times R^{n-2}$. □

Theorem 3.3 *Let M^{n+2} be a complete negatively curved and non-simply connected Riemannian $C_2 - G$-manifold, under the action of a closed and connected Lie subgroup G of isometries. If $M^G \neq \emptyset$, then*

(a) *M is diffeomorphic to $S^1 \times R^{n+1}$ or $B^2 \times R^n$ (B^2 is the mobius band).*
(b) *M^G is homeomorphic to S^1.*
(c) *Each principal orbit is homeomorphic to S^n.*

Proof. (a), (b): Consider $\widetilde{M}, \widetilde{G}$ as Fact 2.1. By Fact 2.1(b), we have $\widetilde{M}^{\widetilde{G}} = k^{-1}(M^G)$. Let $\widetilde{F} = \widetilde{M}^{\widetilde{G}}$. If $\dim \widetilde{F} = 0$, then by Fact 2.1(c), M is simply connected, which is a contradiction. Thus we have $\dim \widetilde{F} > 0$. If $\dim \widetilde{F} \geq 2$, then by Lemma 2.6, the cohomogeneity must be > 2. This contradicts the assumptions of the theorem. So we have $\dim \widetilde{F} = 1$ and $\widetilde{F}$ is equal to the image of a geodesic γ. By Fact 2.1(a) each δ in Δ commutes with the elements of $\widetilde{G}$, so for each x in γ and each $\widetilde{g}$ in $\widetilde{G}$ we have

$$\widetilde{g}(\delta x) = \delta \widetilde{g}(x) = \delta x \Rightarrow \delta x \in \gamma$$

Thus we have $\Delta(\gamma) = \gamma$ and by Lemma 3.3 we get (a). Also by theorem 3.4 in [4, p. 261], Δ is isomorphic to $(Z, +)$. Thus we have

$$M^G = k\big(\widetilde{M}^{\widetilde{G}}\big) = k\gamma = \frac{\gamma}{\Delta} \simeq \frac{R}{Z} \simeq S^1$$

This gives (b).

(c): For each $x = \gamma(t) \in \gamma$, let $N_x = \{\exp(V) : V \in (T_x\gamma)^{\perp}\}$. As like as the proof of the lemma 2.6, we can show that each orbit of $\widetilde{M}$ is contained in N_x, for some $x \in \gamma$. Consider $\delta \in \Delta$ and let $x_i = \gamma(t_i)$, $i = 1, 2$, such that

$t_1 \neq t_2$ and $\delta(x_1) = x_2$. δ preserves γ, so we have:

$$d\delta(\gamma'(t_1)) = \gamma'(t_2) \Rightarrow d\delta(T_{x_1}\gamma)^{\perp} = (T_{x_2}\gamma)^{\perp} \Rightarrow \delta(N_{x_1}) = N_{x_2}$$

Thus, for each $x \in \gamma$, the restriction of the map $k : \widetilde{M} \to M$ on N_x, gives a one to one map $k : N_x \to k(N_x)$ (i.e., it is a diffeomorphism). This means that each orbit of M is diffeomorphic to an orbit in $\widetilde{M}$. By the proof of (a) we have $\dim \widetilde{M}^{\widetilde{G}} = 1$. Thus by Fact 2.5, the principal orbits of $\widetilde{M}$ are homeomorphic to S^n. Therefore we get (c). □

Corollary 3.4 *If M is a negatively curved $C_1 - G$-manifold and $M^G \neq \emptyset$, then M is simply connected and M^G is a one point set.*

Proof. Let $F = M^G$. If $\dim F \geq 2$, then by Lemma 2.6, the cohomogeneity can not be one. Thus $\dim F \leq 1$. If $\dim F = 1$ then we would have infinitely many singular orbits (since each fixed point is a singular orbit). This is a contradiction (because by a theorem in [10], each negatively curved $C_1 - G$-manifold has at most one singular orbit). Thus we have $\dim F = 0$. Since F is connected, it is a one point set. Now by Fact 2.1(c), we get that M is simply connected. □

References

[1] Alekseevsky A. V. and Alekseevsky D. V., *Riemannian G-manifolds with one dimensional orbit space*. Ann. Global Anal. Geom. **11** (1993), 197–211.

[2] Bredon G. E., *Introduction to compact transformation groups*. Acad. Press, New york, London, (1972).

[3] Bishop R. and O'Neill B., *Manifolds of negative curvature*. Trans. Am. Math. Soc. **145** (1969), 1–49.

[4] Do Carmo M. P., *Riemannian Geometry*. Brikhhauser, Boston, Basel, Berlin (1992).

[5] Kobayashi S., *Homogeneous Riemannian manifolds of negative curvature*. Tohoku Math. j. **14** (1962), 413-415.

[6] Michor P. W., *Isomrtric actions of Lie groups and invariants*. Lecture course at the university of Vienna, 1996/97, http://www.mat.univie.ac.at /~michor/tgbook.ps.

[7] Mirzaie R., *Cohomogeneity two actions on flat Riemannian manifolds*. Acta Mathematica Sinica, English series. **23**(9) (2007), 1587–1592.

[8] Mirzaie R. and Kashani S. M. B., *On cohomogeneity one flat Riemannain*

manifolds. Glasgow Math. J. **44** (2002), 185–190.

[9] O'Neill B., *Semi Riemannian geometry with applications to Relativity.* Academic press, INC, London., (1983).

[10] Podesta F. and Spiro A., *Some topological properties of cohomogeneity one manifolds with negative curvature*. Ann. Global. Anal. Geom. **14** (1966), 69–79.

Department of Mathematics
Faculty of Sciences
I. Kh. International University
Qazvin, IRAN
E-mail: R_ Mirzaioe@yahoo.com