Nonlinear stability of stationary solutions
for curvature flow with triple junction

Harald GARCKE, Yoshihito KOHSAKA and Daniel ŠEVČOVIČ

Abstract. In this paper we analyze the motion of a network of three planar curves with a speed proportional to the curvature of the arcs, having perpendicular intersections with the outer boundary and a common intersection at a triple junction. As a main result we show that a linear stability criterion due to Ikota and Yanagida [13] is also sufficient for nonlinear stability. We also prove local and global existence of classical smooth solutions as well as various energy estimates. Finally, we prove exponential stabilization of an evolving network starting from the vicinity of a linearly stable stationary network.

Key words: curvature flow, triple junction, higher order estimates for the curvature, nonlinear stability of stationary solutions.

1. Introduction

The motion of curves under the curvature flow has been widely studied in the past [6], [11], [3]. Less is known about the evolution of networks under the curvature flow [4], [13], [20]. In this case the arcs in the network evolve in the normal direction with a speed proportional to the curvature of the arcs. At intersections with an outer boundary and at triple junctions boundary conditions have to hold. At the outer boundary one can prescribe the position (see [16], [18]), or the angle with the outer boundary [4], [13]. At the triple junction Young's law, a force balance, leads to angle conditions. In this paper we are interested in the stability of stationary solutions to the curvature flow with a triple junction when we prescribe the natural angle condition of 90° at the outer boundary. For this case a linear stability criterion has been derived by Ikota and Yanagida [13] (see also [14]). We will demonstrate here that this criterion also leads to nonlinear stability.

We now specify the problem in detail. Let Ω be a bounded domain in $\mathbb{R}^2$ with C^3-boundary $\partial\Omega$. We introduce a C^3-function $\psi : \mathbb{R}^2 \to \mathbb{R}$ with $\nabla\psi(x) \neq 0$ if $\psi(x) = 0$ such that

$$\Omega = \{x \in \mathbb{R}^2 \mid \psi(x) < 0\}, \qquad \partial\Omega = \{x \in \mathbb{R}^2 \mid \psi(x) = 0\}.$$

We search families of curves Γ_t^1, Γ_t^2, and Γ_t^3 which are parameterized by time t and which are contained in Ω. The three curves are supposed to meet at a triple junction $p(t) \in \Omega$ at their one end point and at the other end point they are required to intersect with $\partial\Omega$, see Figure 1. We require for $i = 1, 2, 3$

$$\beta^i V^i = \gamma^i \kappa^i \qquad \text{on} \quad \Gamma_t^i, \tag{1.1}$$

$$\sum_{i=1}^{3} \gamma^i T^i = 0 \qquad \text{at} \quad p(t), \tag{1.2}$$

$$\Gamma_t^i \perp \partial\Omega \qquad \text{at} \quad \Gamma_t^i \cap \partial\Omega. \tag{1.3}$$

Here V^i and κ^i are the normal velocity and curvature of Γ_t^i, respectively. The constants β^i and γ^i are given physical parameters and T^i are unit tangents to the curve which are chosen such that they point away from the triple junction.

Equation (1.2) is a force balance and one can solve for the T^i's if the condition

$$\gamma^i + \gamma^j \geq \gamma^k \quad \text{for all } \{i, j, k\} \text{ mutually different,}$$

is fulfilled. In the following we assume strict inequalities and an argument as in Bronsard and Reitich [4] gives that the angles θ^i between the tangents T_j and T_k fulfill

$$\frac{\sin\theta^1}{\gamma^1} = \frac{\sin\theta^2}{\gamma^2} = \frac{\sin\theta^3}{\gamma^3}$$

with $0 < \theta^i < \pi$ ($i = 1, 2, 3$) and $\theta^1 + \theta^2 + \theta^3 = 2\pi$. Existence of solutions to the evolution problem (1.1)–(1.3) has been shown by Bronsard and Reitich [4]. We will show later that the energy functional

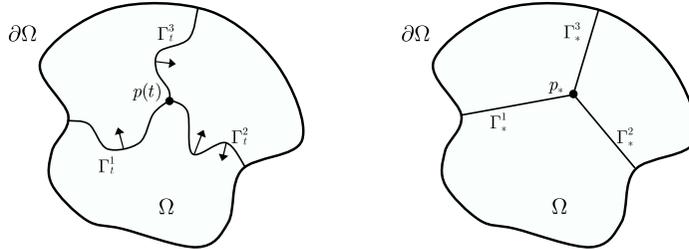$$E[\Gamma_t] = \sum_{i=1}^{3} \gamma^i L[\Gamma_t^i]$$

Figure 1. The curvature driven flow Γ_t of a network with a triple junction at $p(t)$ (left) and a steady state Γ_* (right).

where $\Gamma_t = \bigcup_{i=1}^{3} \Gamma_t^i$ and $L[\Gamma_t^i]$ is the length of Γ_t^i, is a Ljapunov functional. The constants γ^i can be interpreted as surface free energy densities (surface tensions) and the functional E is the total free energy of the systems. Sternberg and Ziemer [21] showed the existence of isolated local minimizers to E, which can be interpreted as solutions to a partitioning problem of two dimensional domains into three subdomains having (locally) least interfacial area.

The paper is organized as follows. In the next section we present a way how to parameterize the problem. We derive a nonlinear nonlocal system of parabolic equations governing the evolution of curves driven by curvature. By means of the semi-group theory due to Lunardi [17] we prove local existence of a classical solution. Section 3 is devoted to the rigorous derivation of the linearized system of equations. We recall the result of Yanagida and Ikota stating an explicit condition for linearized stability of the governing system of equations. In Section 4 we provide a usefull result guaranteeing local uniqueness of a stationary solution proved by the inverse function theorem and the result is to our knowledge the first result in this direction for networks. As a byproduct we also obtain an important bound for the displacement of the network in terms of the curvature. We proceed by deriving useful geometric equations for the curvature and other geometric quantities in Section 5. Using the linearized stability criterion we show how to derive a priori estimates for Sobolev norms of the solution. These geometric equations are then used in order to prove usefull bounds for a solution. With the help of these energy type estimates we prove global existence of a classical solution. In the final Section 7 we prove exponential stability of the stationary solution.
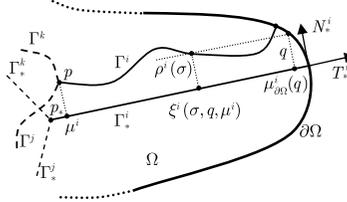
Figure 2.   Description of the local parametrization of the curve Γ^i.

2.   Parameterization and local existence

We consider line segments Γ_*^1, Γ_*^2 and Γ_*^3 meeting the outer boundary with an angle of $90°$ at their one end point and having without loss of generality $p_* = (0,0)^T$ as their common other end point where we assume that (1.2) holds. Then we define an arc-length parameterization of Γ_*^i ($i = 1, 2, 3$) as

$$\Gamma_*^i = \left\{ \Phi_*^i(\sigma) \mid \sigma \in [0, l^i] \right\}$$

with $\Phi_*^i(0) = p_* = (0,0)^T$, $\Phi_*^i(l^i) \in \partial\Omega$. In particular, we obtain that l^i is the length of Γ_*^i. Then we will extend Φ_*^i as an arc-length parameterization of the full line which contains Γ_*^i. We will now introduce a certain stretched coordinate system in order to allow for parameterizations of curves close to Γ_*^i ($i = 1, 2, 3$) over fixed intervals $[0, l^i]$.

Let T_*^i be the unit tangent to Γ_*^i pointing from the triple junction p_* to the outer boundary and let $N_*^i = RT_*^i$ be a unit normal where R is the anticlockwise rotation by $\pi/2$. We then define

$$\mu^i = \left(p, T_*^i\right)_{\mathbb{R}^2}, \quad \mu_{\partial\Omega}^i(q) = \max\left\{\sigma \mid \Phi_*^i(\sigma) + qN_*^i \in \overline{\Omega}\right\}.$$

We remark that the parameter μ^i allows for a tangential movement of the triple junction along Γ_*^i. We now set

$$\Psi^i(\sigma, q, \mu^i) = \Phi_*^i(\xi^i(\sigma, q, \mu^i)) + qN_*^i,$$

where

$$\xi^i(\sigma, q, \mu^i) = \mu^i + \frac{\sigma}{l^i}\left(\mu_{\partial\Omega}^i(q) - \mu^i\right).$$

Note that $\xi^i(\sigma, 0, 0) = \sigma$ and $\xi^i(0, q, \mu^i) = \mu^i$.

We now define the parameterization of curves $\Gamma = (\Gamma^1, \Gamma^2, \Gamma^3)$ close to $\Gamma_* = (\Gamma_*^1, \Gamma_*^2, \Gamma_*^3)$ having their triple junction at the point p with the help of functions

$$\rho^i : [0, l^i] \to \mathbb{R}$$

which fulfill the conditions

$$\rho^i(0) = \left(p, N_*^i\right)_{\mathbb{R}^2} \quad (i = 1, 2, 3) \tag{2.1}$$

(see Fig. 2). Set

$$\Phi^i(\sigma) = \Psi^i(\sigma, \rho^i(\sigma), \mu^i), \quad \sigma \in [0, l^i]. \tag{2.2}$$

Then the functions Φ^i parameterize the curves Γ^i in the neighborhood of Γ_* as $\Gamma^i = \{\Phi^i(\sigma) \mid \sigma \in [0, l^i]\}$. Since $\Phi_*^i(\mu^i) = \mu^i T_*^i$, we have $\Phi^i(0) = \mu^i T_*^i + \rho^i(0) N_*^i = p$, which implies that

$$\mu^1 T_*^1 + \rho^1(0) N_*^1 = \mu^2 T_*^2 + \rho^2(0) N_*^2 = \mu^3 T_*^3 + \rho^3(0) N_*^3. \tag{2.3}$$

By virtue of the definition of μ^i, equation (2.1), and Young's law

$$\sum_{i=1}^3 \gamma^i T_*^i = 0, \quad \sum_{i=1}^3 \gamma^i N_*^i = 0,$$

we are led to

$$\sum_{i=1}^3 \gamma^i \mu^i = 0, \quad \sum_{i=1}^3 \gamma^i \rho^i(0) = 0.$$

Furthermore, identities (2.3) and the angle conditions give the following lemma.

Lemma 2.1 *Let us define the matrix*

$$Q = -\frac{1}{1 - c^1 c^2 c^3} \begin{pmatrix} c^3 c^1 s^2 & s^3 & c^3 s^1 \\ c^1 s^2 & c^1 c^2 s^3 & s^1 \\ s^2 & c^2 s^3 & c^2 c^3 s^1 \end{pmatrix},$$

where $c^i = \cos \theta^i$, $s^i = \sin \theta^i$. *Then, for* $\boldsymbol{\rho} = (\rho^1, \rho^2, \rho^3)$, *and* $\boldsymbol{\mu} = (\mu^1, \mu^2, \mu^3)$, *it holds* $\boldsymbol{\mu}^T = Q \boldsymbol{\rho}^T(0)$.

Proof. It follows from (2.3) that

$$\mu^i = \mu^j \left(T^j_*, T^i_*\right)_{\mathbb{R}^2} + \rho^j(0) \left(N^j_*, T^i_*\right)_{\mathbb{R}^2}$$

for $(i,j) = (1,2)$, $(2,3)$, $(3,1)$. By the angle condition, we have

$$\left(T^i_*, T^j_*\right)_{\mathbb{R}^2} = \cos \theta^k, \quad \left(T^i_*, N^j_*\right)_{\mathbb{R}^2} = \cos(\theta^k + \pi/2) = -\sin \theta^k$$

for $(i,j,k) = (1,2,3)$, $(2,3,1)$, $(3,1,2)$. This implies $\mu^i - c^k \mu^j = -s^k \rho^j(0)$, so that we are led to

$$\begin{pmatrix} -c^2 & 0 & 1 \\ 1 & -c^3 & 0 \\ 0 & 1 & -c^1 \end{pmatrix} \begin{pmatrix} \mu^1 \\ \mu^2 \\ \mu^3 \end{pmatrix} = \begin{pmatrix} -s^2 & 0 & 0 \\ 0 & -s^3 & 0 \\ 0 & 0 & -s^1 \end{pmatrix} \begin{pmatrix} \rho^1(0) \\ \rho^2(0) \\ \rho^3(0) \end{pmatrix}.$$

Then we obtain

$$\begin{pmatrix} \mu^1 \\ \mu^2 \\ \mu^3 \end{pmatrix} = \frac{-1}{1 - c^1 c^2 c^3} \begin{pmatrix} c^3 c^1 & 1 & c^3 \\ c^1 & c^1 c^2 & 1 \\ 1 & c^2 & c^2 c^3 \end{pmatrix} \begin{pmatrix} s^2 & 0 & 0 \\ 0 & s^3 & 0 \\ 0 & 0 & s^1 \end{pmatrix} \begin{pmatrix} \rho^1(0) \\ \rho^2(0) \\ \rho^3(0) \end{pmatrix}$$

$$= Q \begin{pmatrix} \rho^1(0) \\ \rho^2(0) \\ \rho^3(0) \end{pmatrix}$$

which completes the proof. $\square$

We now consider evolving curves

$$\Gamma^i(t) = \{\Phi^i(\sigma, t) \mid \sigma \in [0, l^i]\},$$

where $\Phi^i(\cdot, t)$ are defined as in (2.2) such that $\rho^i(\cdot, t)$ ($i = 1, 2, 3$) satisfy (2.1). We formulate the curvature flow for a network with the help of these parameterizations. For that purpose, the following quantities are needed

$$T^i = \frac{1}{|\Phi^i_\sigma|} \Phi^i_\sigma, \qquad N^i = \frac{1}{|\Phi^i_\sigma|} R\Phi^i_\sigma,$$

$$V^i = \left(\Phi^i_t, N^i\right)_{\mathbb{R}^2}, \qquad \kappa^i = \left(\frac{1}{|\Phi^i_\sigma|} T^i_\sigma, N^i\right)_{\mathbb{R}^2}, \quad i = 1, 2, 3. \tag{2.4}$$

Then we obtain the following formulation for the curvature flow of a network:

$$\beta^i \left(\Phi^i_t, N^i\right)_{\mathbb{R}^2} = \gamma^i \left(\frac{1}{|\Phi^i_\sigma|} T^i_\sigma, N^i\right)_{\mathbb{R}^2}, \quad i = 1, 2, 3, \tag{2.5}$$

$$\sum_{i=1}^{3} \gamma^i \rho^i = 0 \quad \text{at} \quad \sigma = 0, \tag{2.6}$$

$$(T^1, T^2)_{\mathbb{R}^2} = \cos\theta^3, \quad (T^3, T^1)_{\mathbb{R}^2} = \cos\theta^2 \quad \text{at} \quad \sigma = 0, \tag{2.7}$$

$$(N^i, \nabla\psi(\Phi^i))_{\mathbb{R}^2} = 0 \quad \text{at} \quad \sigma = l^i, \quad i = 1, 2, 3. \tag{2.8}$$

Note that the conditions $\psi(\Phi^i(l^i, t)) = 0$ $(i = 1, 2, 3)$ and $\Phi^1(0, t) = \Phi^2(0, t) = \Phi^3(0, t)$ are always fulfilled with our choice of the parameterizations. We can formulate the problem in terms of (ρ^1, ρ^2, ρ^3) and obtain a system of three second order parabolic equations where each equation is defined on a different spatial interval $[0, l^i]$. We obtain one boundary condition at $\sigma = l^i$ for the i-th equation and the three equations are coupled through the three boundary conditions at the triple junction.

Let us derive the form of the nonlinear system for $\rho^i(\sigma, t)$ $(i = 1, 2, 3)$. Set $\mathcal{I}^i = [0, l^i]$ and $\mathcal{Q}^i_{t_0, t_1} = \mathcal{I}^i \times (t_0, t_1)$. Equations (2.5) give

$$\rho^i_t = L^i\left(\rho^i, \rho^i_\sigma, \mu^i\right)\kappa^i\left(\rho^i, \rho^i_\sigma, \rho^i_{\sigma\sigma}, \mu^i\right) + \Lambda^i\left(\rho^i, \rho^i_\sigma, \mu^i\right)\mu^i_t$$

$$\text{for } (\sigma, t) \in \mathcal{Q}^i_{0,T}, \tag{2.9}$$

where $L^i(\rho^i, \rho^i_\sigma, \mu^i)$ and $\Lambda^i(\rho^i, \rho^i_\sigma, \mu^i)$ are

$$L^i\left(\rho^i, \rho^i_\sigma, \mu^i\right) = \frac{\gamma^i}{\beta^i\left(\Psi^i_q, R\Psi^i_\sigma\right)_{\mathbb{R}^2}} J^i\left(\rho^i, \rho^i_\sigma, \mu^i\right),$$

$$\Lambda^i\left(\rho^i, \rho^i_\sigma, \mu^i\right) = -\frac{1}{\left(\Psi^i_q, R\Psi^i_\sigma\right)_{\mathbb{R}^2}}\left\{\left(\Psi^i_\mu, R\Psi^i_\sigma\right)_{\mathbb{R}^2} + \left(\Psi^i_\mu, R\Psi^i_q\right)_{\mathbb{R}^2}\rho^i_\sigma\right\}$$

with the notation $J^i = |\Phi^i_\sigma|$, and the curvature $\kappa^i = \kappa^i(\rho^i, \rho^i_\sigma, \rho^i_{\sigma\sigma}, \mu^i)$ is represented as

$$\kappa^i(\rho^i, \rho^i_\sigma, \rho^i_{\sigma\sigma}, \mu^i)$$

$$= \frac{1}{\{J^i(\rho^i, \rho^i_\sigma, \mu^i)\}^3}\Big[\big(\Psi^i_q, R\Psi^i_\sigma\big)_{\mathbb{R}^2}\rho^i_{\sigma\sigma} + \big\{2\big(\Psi^i_{\sigma q}, R\Psi^i_\sigma\big)_{\mathbb{R}^2} + \big(\Psi^i_{\sigma\sigma}, R\Psi^i_q\big)_{\mathbb{R}^2}\big\}\rho^i_\sigma$$

$$+ \big\{\big(\Psi^i_{qq}, R\Psi^i_\sigma\big)_{\mathbb{R}^2} + 2\big(\Psi^i_{\sigma q}, R\Psi^i_q\big)_{\mathbb{R}^2} + \big(\Psi^i_{qq}, R\Psi^i_q\big)_{\mathbb{R}^2}\rho^i_\sigma\big\}\big(\rho^i_\sigma\big)^2$$

$$+ \big(\Psi^i_{\sigma\sigma}, R\Psi^i_\sigma\big)_{\mathbb{R}^2}\Big].$$

By virtue of Lemma 2.1 and (2.9), we have

$$\begin{pmatrix} \mu^1_t \\ \mu^2_t \\ \mu^3_t \end{pmatrix} = Q\,\mathcal{T}^0 \begin{pmatrix} \rho^1_t \\ \rho^2_t \\ \rho^3_t \end{pmatrix} = Q\big[\mathcal{T}^0 M(\boldsymbol{\rho}, \boldsymbol{\rho}_\sigma, \boldsymbol{\mu})\big]^{-1}\mathcal{T}^0 \begin{pmatrix} L^1(\boldsymbol{u}^1)\kappa^1(\boldsymbol{u}^1) \\ L^2(\boldsymbol{u}^2)\kappa^2(\boldsymbol{u}^2) \\ L^3(\boldsymbol{u}^3)\kappa^3(\boldsymbol{u}^3) \end{pmatrix},$$

where Q is the matrix as in Lemma 2.1, $\mathcal{T}^0$ is the trace operator onto $\sigma = 0$, i.e. $\mathcal{T}^0 f = f\big|_{\sigma=0}$, and $M(\boldsymbol{\rho}, \boldsymbol{\rho}_\sigma, \boldsymbol{\mu})$ is the matrix

$$M(\boldsymbol{\rho}, \boldsymbol{\rho}_\sigma, \boldsymbol{\mu}) = \mathrm{Id} - \mathrm{diag}(\Lambda^1(\boldsymbol{u}^1), \Lambda^2(\boldsymbol{u}^2), \Lambda^3(\boldsymbol{u}^3))Q$$

with the notation $\boldsymbol{u}^i = (\rho^i, \rho^i_\sigma, \mu^i)$.

Remark 2.2   The matrix $M(\boldsymbol{\rho}, \boldsymbol{\rho}_\sigma, \boldsymbol{\mu})$ is invertible provided that

$$\sum_{i=1}^3 \Big(\sup_{t\in[0,T]} \|\rho^i(\cdot, t)\|_{C^1(\mathcal{I}^i)} + \sup_{t\in[0,T]} |\mu^i(t)| \Big) < \delta_0 \qquad (2.10)$$

for some $\delta_0 > 0$. Indeed, we have $\det M(\boldsymbol{\rho}, \boldsymbol{\rho}_\sigma, \boldsymbol{\mu}) = d\{-1 + (c^2 - \Lambda^1(\boldsymbol{u}^1)s^2)(c^3 - \Lambda^2(\boldsymbol{u}^2)s^3)(c^1 - \Lambda^3(\boldsymbol{u}^3)s^1)\}$, where $d = -1/(1 - c^1 c^2 c^3)$. Then, $\Lambda^i(\boldsymbol{0}) = 0$ ($i = 1, 2, 3$) imply that $\det M(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}) = d(-1 + c^1 c^2 c^3) = 1 \neq 0$. Since $\det M(\boldsymbol{\rho}, \boldsymbol{\rho}_\sigma, \boldsymbol{\mu})$ is continuous with respect to $\boldsymbol{\rho}$, $\boldsymbol{\rho}_\sigma$, and $\boldsymbol{\mu}$, we can conclude that for $\varepsilon < 1$ there exists a $\delta_0 > 0$ such that $\det M(\boldsymbol{\rho}, \boldsymbol{\rho}_\sigma, \boldsymbol{\mu}) > 1 - \varepsilon > 0$ provided that (2.10) holds.

As a consequence, we are led to the following nonlinear nonlocal partial differential equations for $\rho^i(\sigma, t)$ ($i = 1, 2, 3$):

$$\rho_t^i = a^i\big(\rho^i, \rho_\sigma^i, \mu^i\big)\rho_{\sigma\sigma}^i + \Lambda^i\big(\rho^i, \rho_\sigma^i, \mu^i\big)\sum_{j=1}^{3} a_1^{ij}\big(\mathcal{T}^0\boldsymbol{\rho}, \mathcal{T}^0\boldsymbol{\rho}_\sigma, \boldsymbol{\mu}\big)\mathcal{T}^0\rho_{\sigma\sigma}^j$$

$$+ f^i\big(\rho^i, \partial_\sigma\rho^i, \mathcal{T}^0\boldsymbol{\rho}, \mathcal{T}^0\boldsymbol{\rho}_\sigma, \boldsymbol{\mu}\big) \quad \text{for } (\sigma, t) \in \mathcal{Q}_{0,T}^i$$

where $a^i(\rho^i, \rho_\sigma^i, \mu^i) = \gamma^i/[\beta^i\{J^i(\rho^i, \rho_\sigma^i, \mu^i)\}^2]$ and $a_1^{ij}(\mathcal{T}^0\boldsymbol{\rho}, \mathcal{T}^0\boldsymbol{\rho}_\sigma, \boldsymbol{\mu})$ is the (i,j)-component of the matrix

$$\boldsymbol{a}_1\big(\mathcal{T}^0\boldsymbol{\rho}, \mathcal{T}^0\boldsymbol{\rho}_\sigma, \boldsymbol{\mu}\big)$$

$$= Q\big[\mathcal{T}^0 M(\boldsymbol{\rho}, \boldsymbol{\rho}_\sigma, \boldsymbol{\mu})\big]^{-1}\mathrm{diag}\big(\mathcal{T}^0 a^1(\boldsymbol{u}^1), \mathcal{T}^0 a^2(\boldsymbol{u}^2), \mathcal{T}^0 a^3(\boldsymbol{u}^3)\big).$$

Furthermore, f^i is a smooth function in $\mathbb{R}$ which is evaluated at lower order terms. Then, recalling the boundary conditions (2.6)–(2.8) and Lemma 2.1, we have the following nonlinear system:

$$(2.11) \quad \begin{cases} \rho_t^i = a^i\big(\rho^i, \rho_\sigma^i, \mu^i\big)\rho_{\sigma\sigma}^i + \Lambda^i\big(\rho^i, \rho_\sigma^i, \mu^i\big)\sum_{j=1}^{3} a_1^{ij}\big(\mathcal{T}^0\boldsymbol{\rho}, \mathcal{T}^0\boldsymbol{\rho}_\sigma, \boldsymbol{\mu}\big)\mathcal{T}^0\rho_{\sigma\sigma}^j \\ \qquad + f^i\big(\rho^i, \rho_\sigma^i, \mathcal{T}^0\boldsymbol{\rho}, \mathcal{T}^0\boldsymbol{\rho}_\sigma, \boldsymbol{\mu}\big) \quad \text{for } (\sigma, t) \in \mathcal{Q}_{0,T}^i, \\ \sum_{i=1}^{3}\gamma^i\rho^i = 0, \quad g^{12}(\boldsymbol{u}^{12}) = 0, \quad g^{13}(\boldsymbol{u}^{13}) = 0 \quad \text{at } \sigma = 0, \\ b_{\partial\Omega}^i(\rho^i, \mu^i)\rho_\sigma^i + g_{\partial\Omega}^i(\rho^i, \mu^i) = 0 \quad \text{at } \sigma = l^i \quad (i = 1, 2, 3), \\ \boldsymbol{\mu}^T = Q\mathcal{T}^0\boldsymbol{\rho}^T \quad \text{for } t \in (0, T] \end{cases}$$

where $\boldsymbol{u}^{1j} = (\rho^1, \rho^j, \rho_\sigma^1, \rho_\sigma^j, \mu^1, \mu^j)$ $(j = 2, 3)$ and

$$g^{12}(\boldsymbol{u}^{12}) = \big(\Psi_\sigma^1, \Psi_\sigma^2\big)_{\mathbb{R}^2} + \big(\Psi_\sigma^1, \Psi_q^2\big)_{\mathbb{R}^2}\rho_\sigma^2 + \big(\Psi_q^1, \Psi_\sigma^2\big)_{\mathbb{R}^2}\rho_\sigma^1 + \big(\Psi_q^1, \Psi_q^2\big)_{\mathbb{R}^2}\rho_\sigma^1\rho_\sigma^2$$

$$- J^1\big(\rho^1, \rho_\sigma^1, \mu^1\big)J^2\big(\rho^2, \rho_\sigma^2, \mu^2\big)\cos\theta^3,$$

$$g^{13}(\boldsymbol{u}^{13}) = \big(\Psi_\sigma^3, \Psi_\sigma^1\big)_{\mathbb{R}^2} + \big(\Psi_\sigma^3, \Psi_q^1\big)_{\mathbb{R}^2}\rho_\sigma^1 + \big(\Psi_q^3, \Psi_\sigma^1\big)_{\mathbb{R}^2}\rho_\sigma^3 + \big(\Psi_q^3, \Psi_q^1\big)_{\mathbb{R}^2}\rho_\sigma^3\rho_\sigma^1$$

$$- J^3\big(\rho^3, \rho_\sigma^3, \mu^3\big)J^1\big(\rho^1, \rho_\sigma^1, \mu^1\big)\cos\theta^2,$$

$$b_{\partial\Omega}^i(\rho^i, \mu^i) = \big(R\Psi_q^i, \nabla\psi(\Psi^i)\big)_{\mathbb{R}^2}, \quad g_{\partial\Omega}^i(\rho^i, \mu^i) = -\big(R\Psi_\sigma^i, \nabla\psi(\Psi^i)\big)_{\mathbb{R}^2}.$$

Now we are ready to state a local existence result.

Theorem 2.3 (Local existence)   *Let $\alpha \in (0,1)$ and let us assume that $\rho_0^i \in C^{2+\alpha}(\mathcal{I}^i)$ and μ_0^i $(i = 1, 2, 3)$ with sufficiently small norms $\|\rho_0^i\|_{C^{1+\alpha}(\mathcal{I}^i)}$ and $|\mu_0^i|$ fulfill the compatibility conditions*

$$\begin{cases} \sum_{i=1}^3 \gamma^i \rho_0^i = 0, \quad g^{12}(\boldsymbol{u}_0^{12}) = 0, \quad g^{13}(\boldsymbol{u}_0^{13}) = 0 \quad \text{at } \sigma = 0, \\ b_{\partial\Omega}^i(\rho_0^i, \mu_0^i)\rho_{0,\sigma}^i + g_{\partial\Omega}^i(\rho_0^i, \mu_0^i) = 0 \quad \text{at } \sigma = l^i \quad (i = 1, 2, 3), \end{cases}$$

where $\boldsymbol{u}_0^{1j} = (\rho_0^1, \rho_0^j, \rho_{0,\sigma}^1, \rho_{0,\sigma}^j, \mu_0^1, \mu_0^j)$ $(j = 2, 3)$. Then there exists a

$$T_0 = T_0\left(1/\sum_{i=1}^3 \|\rho_0^i\|_{C^{2+\alpha}(\mathcal{I}^i)}\right) > 0,$$

T_0 being an increasing function of its argument and such that the problem (2.11) with $(\rho^i(\cdot, 0), \mu^i(0)) = (\rho_0^i, \mu_0^i)$ $(i = 1, 2, 3)$ has a unique solution

$$(\rho^1, \rho^2, \rho^3, \mu^1, \mu^2, \mu^3)$$
$$\in C^{2+\alpha,1}\left(\overline{\mathcal{Q}_{0,T_0}^1}\right) \times C^{2+\alpha,1}\left(\overline{\mathcal{Q}_{0,T_0}^2}\right) \times C^{2+\alpha,1}\left(\overline{\mathcal{Q}_{0,T_0}^3}\right) \times \left[C^1[0, T_0]\right]^3$$

satisfying (2.10).

In order to prove Theorem 2.3 by using a contraction principle, we need some preparations which consist of three steps: 1) the linearization of (2.11) around the initial data; 2) the verification of the complementary conditions for the linearized system; 3) the derivation of suitable a priori estimate for solutions of the linearized system.

Step 1   Let us derive the linearization of (2.11) around the initial data $\rho_0^i \in C^{2+\alpha}(\mathcal{I}^i)$ and μ_0^i $(i = 1, 2, 3)$. First we define differential operators as

$$\mathcal{A}_0 = \text{diag}\left(a^1(\boldsymbol{u}_0^1), a^2(\boldsymbol{u}_0^2), a^3(\boldsymbol{u}_0^3)\right)\partial_\sigma^2,$$
$$\mathcal{A}_1 = \text{diag}\left(\Lambda^1(\boldsymbol{u}_0^1), \Lambda^2(\boldsymbol{u}_0^2), \Lambda^3(\boldsymbol{u}_0^3)\right)\boldsymbol{a}_1\left(\mathcal{T}^0\boldsymbol{\rho}_0, \mathcal{T}^0\boldsymbol{\rho}_{0,\sigma}, \boldsymbol{\mu}_0\right)\mathcal{T}^0\partial_\sigma^2,$$

and also define, for given functions $(\overline{\rho}^i, \overline{\mu}^i) \in C^{2+\alpha,1}(\overline{\mathcal{Q}_{0,T}^i}) \times C^1[0, T]$ $(i = 1, 2, 3)$,

$$F^i(\sigma, t) = \big\{ a^i(\overline{\boldsymbol{u}}^i) - a^i(\boldsymbol{u}_0^i) \big\} \overline{\rho}_{\sigma\sigma}^i$$

$$+ \sum_{j=1}^{3} \big\{ \Lambda^i(\overline{\boldsymbol{u}}^i) a_1^{ij}(\mathcal{T}^0\overline{\boldsymbol{\rho}}, \mathcal{T}^0\overline{\boldsymbol{\rho}}_\sigma, \overline{\boldsymbol{\mu}})$$

$$- \Lambda^i(\boldsymbol{u}_0^i) a_1^{ij}(\mathcal{T}^0\boldsymbol{\rho}_0, \mathcal{T}^0\boldsymbol{\rho}_{0,\sigma}, \boldsymbol{\mu}_0) \big\} \mathcal{T}^0\overline{\rho}_{\sigma\sigma}^j$$

$$+ f^i\big(\overline{\rho}^i, \overline{\rho}_\sigma^i, \mathcal{T}^0\overline{\boldsymbol{\rho}}, \mathcal{T}^0\overline{\boldsymbol{\rho}}_\sigma, \overline{\boldsymbol{\mu}}\big),$$

where $\boldsymbol{u}_0^i = (\rho_0^i, \rho_{0,\sigma}^i, \mu_0^i)$, $\boldsymbol{\rho}_0 = (\rho_0^1, \rho_0^2, \rho_0^3)$, $\boldsymbol{\mu}_0 = (\mu_0^1, \mu_0^2, \mu_0^3)$, $\overline{\boldsymbol{u}}^i = (\overline{\rho}^i, \overline{\rho}_\sigma^i, \overline{\mu}^i)$, $\overline{\boldsymbol{\rho}} = (\overline{\rho}^1, \overline{\rho}^2, \overline{\rho}^3)$, and $\overline{\boldsymbol{\mu}} = (\overline{\mu}^1, \overline{\mu}^2, \overline{\mu}^3)$. Then, setting $\boldsymbol{F} = (F^1, F^2, F^3)$, we have the linearization of the differential equation given as

$$\partial_t \boldsymbol{\rho}^T = \mathcal{A}_0 \boldsymbol{\rho}^T + \mathcal{A}_1 \boldsymbol{\rho}^T + \boldsymbol{F}^T(\sigma, t).$$

Let us derive the linearization of the boundary conditions. For $\rho_0^i \in C^{2+\alpha}(\mathcal{I}^i)$ and μ_0^i $(i = 1, 2, 3)$, we define differential operators as

$$\big(B^{ki}(\mathcal{T}^0\boldsymbol{\rho}_0, \mathcal{T}^0\boldsymbol{\rho}_{0,\sigma}, \boldsymbol{\mu}_0)\partial_\sigma\big)_{i=1,2,3}$$

$$= \begin{cases} (\gamma^1, \gamma^2, \gamma^3) & \text{for} \quad k = 1, \\ \big(b^{21}(\mathcal{T}^0\boldsymbol{u}_0^{12})\partial_\sigma, b^{22}(\mathcal{T}^0\boldsymbol{u}_0^{12})\partial_\sigma, 0\big) & \text{for} \quad k = 2, \\ \big(b^{31}(\mathcal{T}^0\boldsymbol{u}_0^{13})\partial_\sigma, 0, b^{33}(\mathcal{T}^0\boldsymbol{u}_0^{13})\partial_\sigma\big) & \text{for} \quad k = 3. \end{cases}$$

Here the components are represented as follows:

$$b^{21}(\boldsymbol{u}^{12}) = \big(\Psi_q^1, \Psi_\sigma^2\big)_{\mathbb{R}^2} + \big(\Psi_q^1, \Psi_q^2\big)_{\mathbb{R}^2} \rho_\sigma^2 - \big\{(\Psi_\sigma^1, \Psi_q^1)_{\mathbb{R}^2} + |\Psi_q^1|^2 \rho_\sigma^1\big\} \frac{J^2(\boldsymbol{u}^2)}{J^1(\boldsymbol{u}^1)} \cos\theta^3,$$

$$b^{22}(\boldsymbol{u}^{12}) = \big(\Psi_\sigma^1, \Psi_q^2\big)_{\mathbb{R}^2} + \big(\Psi_q^1, \Psi_q^2\big)_{\mathbb{R}^2} \rho_\sigma^1 - \big\{(\Psi_\sigma^2, \Psi_q^2)_{\mathbb{R}^2} + |\Psi_q^2|^2 \rho_\sigma^2\big\} \frac{J^1(\boldsymbol{u}^1)}{J^2(\boldsymbol{u}^2)} \cos\theta^3,$$

$$b^{31}(\boldsymbol{u}^{13}) = \big(\Psi_\sigma^3, \Psi_q^1\big)_{\mathbb{R}^2} + \big(\Psi_q^3, \Psi_q^1\big)_{\mathbb{R}^2} \rho_\sigma^3 - \big\{(\Psi_\sigma^1, \Psi_q^1)_{\mathbb{R}^2} + |\Psi_q^1|^2 \rho_\sigma^1\big\} \frac{J^3(\boldsymbol{u}^3)}{J^1(\boldsymbol{u}^1)} \cos\theta^2,$$

$$b^{33}(\boldsymbol{u}^{13}) = \big(\Psi_q^3, \Psi_\sigma^1\big)_{\mathbb{R}^2} + \big(\Psi_q^3, \Psi_q^1\big)_{\mathbb{R}^2} \rho_\sigma^1 - \big\{(\Psi_\sigma^3, \Psi_q^3)_{\mathbb{R}^2} + |\Psi_q^3|^2 \rho_\sigma^3\big\} \frac{J^1(\boldsymbol{u}^1)}{J^3(\boldsymbol{u}^3)} \cos\theta^2.$$

Also, we define differential operators as

$$\left(B_{\partial\Omega}^{ki}(\mathcal{T}^{l^i}\rho_0^i,\mu_0^i)\partial_\sigma\right)_{i=1,2,3} = \begin{cases} \left(b_{\partial\Omega}^1(\mathcal{T}^{l^1}\rho_0^1,\mu_0^1)\partial_\sigma,0,0\right) & \text{for } k=1, \\ \left(0,b_{\partial\Omega}^2(\mathcal{T}^{l^2}\rho_0^2,\mu_0^2)\partial_\sigma,0\right) & \text{for } k=2, \\ \left(0,0,b_{\partial\Omega}^3(\mathcal{T}^{l^3}\rho_0^3,\mu_0^3)\partial_\sigma\right) & \text{for } k=3, \end{cases}$$

where $\mathcal{T}^{l^i}$ $(i=1,2,3)$ is the trace operator onto $\sigma=l^i$, i.e. $\mathcal{T}^{l^i}f = f\mid_{\sigma=l^i}$. Then we set

$$\mathcal{B}_0(0\,;\partial_\sigma) = \left(B^{ki}(\mathcal{T}^0\boldsymbol{\rho}_0,\mathcal{T}^0\boldsymbol{\rho}_{0,\sigma},\boldsymbol{\mu}_0)\partial_\sigma\right)_{k,i=1,2,3},$$

$$\mathcal{B}_0(l^i\,;\partial_\sigma) = \left(B_{\partial\Omega}^{ki}(\mathcal{T}^{l^i}\rho_0^i,\mu_0^i)\partial_\sigma\right)_{k,i=1,2,3},$$

and also set, for $(\overline{\rho}^i,\overline{\mu}^i) \in C^{2+\alpha,1}(\overline{\mathcal{Q}_{0,T}^i}) \times C^1[0,T]$ $(i=1,2,3)$,

$$G^1(t) = 0,$$

$$G^j(t) = \mathcal{T}^0\Bigg[b^{j1}(\boldsymbol{u}_0^{1j})\rho_{0,\sigma}^1 + b^{jj}(\boldsymbol{u}_0^{1j})\rho_{0,\sigma}^j - \frac{\partial g^{1j}}{\partial\rho^1}(\boldsymbol{u}_0^{1j})(\overline{\rho}^1-\rho_0^1)$$

$$-\frac{\partial g^{1j}}{\partial\rho^j}(\boldsymbol{u}_0^{1j})(\overline{\rho}^j-\rho_0^j) - \frac{\partial g^{1j}}{\partial\mu^1}(\boldsymbol{u}_0^{1j})(\overline{\mu}^1-\mu_0^1) - \frac{\partial g^{1j}}{\partial\mu^j}(\boldsymbol{u}_0^{1j})(\overline{\mu}^j-\mu_0^j)$$

$$-\frac{1}{2}\int_0^1\left\langle D^2g^{1j}(\eta\overline{\boldsymbol{u}}^{1j}+(1-\eta)\boldsymbol{u}_0^{1j})(\overline{\boldsymbol{u}}^{1j}-\boldsymbol{u}_0^{1j}),\overline{\boldsymbol{u}}^{1j}-\boldsymbol{u}_0^{1j}\right\rangle d\eta\Bigg]$$

$$(j=2,3),$$

$$G_{\partial\Omega}^i(t) = \mathcal{T}^{l^i}\Big[-\big\{b_{\partial\Omega}^i(\overline{\rho}^i,\overline{\mu}^i)-b_{\partial\Omega}^i(\rho_0^i,\mu_0^i)\big\}\overline{\rho}_\sigma^i + g_{\partial\Omega}^i(\overline{\rho}^i,\overline{\mu}^i)\Big] \quad (i=1,2,3),$$

where Dg^{1j} is the Fréchet derivative of g^{1j} and the bracket $\langle\cdot,\cdot\rangle$ is the respective inner product. Then we have the linearization of the boundary conditions:

$$\mathcal{B}_0(0\,;\partial_\sigma)\boldsymbol{\rho}^T = \boldsymbol{G}^T(t), \quad \mathcal{B}_0(l^i\,;\partial_\sigma)\boldsymbol{\rho}^T = \boldsymbol{G}_{\partial\Omega}^T(t)$$

for $\boldsymbol{G}(t) = (G^1(t),G^2(t),G^3(t))$ and $\boldsymbol{G}_{\partial\Omega}(t) = (G_{\partial\Omega}^1(t),G_{\partial\Omega}^2(t),G_{\partial\Omega}^3(t))$.

Step 2   Let us verify that the complementary conditions hold for the linearized system. We refer to Lunardi [17] for more information on the role of the complementary conditions. For that purpose, we make some

preparations. Let $\mathcal{L}_0(r, \mathbf{i}\,\zeta) = (a_0^{ij})_{i,j=1,2,3}$ where

$$a_0^{ii} = \frac{1}{\{J^i(\boldsymbol{u}_0^i)\}^2}\zeta^2 + r, \quad a_0^{ij} = 0 \text{ for } i \neq j.$$

Then we have

$$\det \mathcal{L}_0 = \prod_{i=1}^{3}\left[\frac{1}{\{J^i(\boldsymbol{u}_0^i)\}^2}\zeta^2 + r\right].$$

Setting $\hat{\mathcal{L}}_0 = (\hat{a}_0^{ij}) = (\det \mathcal{L}_0)(\mathcal{L}_0)^{-1}$, we are led to

$$\hat{a}_0^{ii} = \prod_{k=1, i \neq k}^{3}\left[\frac{1}{\{J^i(\boldsymbol{u}_0^i)\}^2}\zeta^2 + r\right], \quad \hat{a}_0^{ij} = 0 \text{ for } i \neq j.$$

The matrix of the boundary conditions at $\sigma = 0$ is denoted by

$$\left(B_0^{ki}(0\,;\mathbf{i}\,\zeta)\right)_{i=1,2,3} = \begin{cases} (\gamma^1, \gamma^2, \gamma^3) & \text{for } k = 1, \\ \left(\mathbf{i}\,b^{21}(\mathcal{T}^0\boldsymbol{u}_0^{12})\zeta, \mathbf{i}\,b^{22}(\mathcal{T}^0\boldsymbol{u}_0^{12})\zeta, 0\right) & \text{for } k = 2, \\ \left(\mathbf{i}\,b^{31}(\mathcal{T}^0\boldsymbol{u}_0^{13})\zeta, 0, \mathbf{i}\,b^{33}(\mathcal{T}^0\boldsymbol{u}_0^{13})\zeta\right) & \text{for } k = 3, \end{cases}$$

and at $\sigma = l^i$ by

$$\left(B_0^{ki}(l^i\,;\mathbf{i}\,\zeta)\right)_{i=1,2,3} = \begin{cases} \left(\mathbf{i}\,b_{\partial\Omega}^1(\mathcal{T}^{l^1}\rho_0^1, \mu_0^1)\zeta, 0, 0\right) & \text{for } k = 1, \\ \left(0, \mathbf{i}\,b_{\partial\Omega}^2(\mathcal{T}^{l^2}\rho_0^2, \mu_0^2)\zeta, 0\right) & \text{for } k = 2, \\ \left(0, 0, \mathbf{i}\,b_{\partial\Omega}^3(\mathcal{T}^{l^3}\rho_0^3, \mu_0^3)\zeta\right) & \text{for } k = 3. \end{cases}$$

To verify the complementary condition we will show that the rows of the matrix $\mathcal{B}_0\hat{\mathcal{L}}_0$ are linearly independent for all $r \in \mathbb{C}$, $\operatorname{Re} r > 0$, modulo the polynomial

$$P(r, \zeta) = \prod_{i=1}^{3}\left\{\zeta - \zeta_0^i(r)\right\}$$

where $\zeta_0^i(r) = J^i(\boldsymbol{u}_0^i)|r|^{\frac{1}{2}}e^{\mathbf{i}\left(\frac{\Theta}{2} + \frac{\pi}{2}\right)}$ $(i = 1, 2, 3)$. Here we note that $\zeta_0^i(r)$ are

the roots of the polynomial $\det \mathcal{L}_0(r, \mathbf{i}\,\zeta)$ which have a positive imaginary part.

First let us verify the complementary condition at $\sigma = 0$. To determine whether or not the complementary condition is satisfied, we have to verify that the system

$$\sum_{k=1}^{3} \omega_k \mathcal{B}_0^{ki}(0\,;\mathbf{i}\,\zeta)\hat{a}_0^{ii}(\zeta) \equiv 0 \mod P(r,\zeta) = \prod_{i=1}^{3}(\zeta - \zeta_0^i) \quad (i = 1,2,3)$$

has the unique solution $(\omega_1, \omega_2, \omega_3)^T = \mathbf{0}$ or equivalently that $(\omega_1, \omega_2, \omega_3)^T = \mathbf{0}$ is the only vector satisfying $\sum_{k=1}^{3} \omega_k \mathcal{B}_0^{ki}(0\,;\mathbf{i}\,\zeta) \equiv 0$ mod $\zeta - \zeta_0^i$ $(i = 1,2,3)$. That is, we may investigate that $(\omega_1, \omega_2, \omega_3)^T = \mathbf{0}$ is the only vector satisfying $\sum_{k=1}^{3} \omega_k \mathcal{B}_0^{ki}(0\,;\mathbf{i}\,\zeta_0^i) = 0$ $(i = 1,2,3)$. Thus it suffices to show that

$$\det \begin{pmatrix} \gamma^1 & \gamma^2 & \gamma^3 \\ b^{21}(\mathcal{T}^0 \boldsymbol{u}_0^{12}) & b^{22}(\mathcal{T}^0 \boldsymbol{u}_0^{12}) & 0 \\ b^{31}(\mathcal{T}^0 \boldsymbol{u}_0^{13}) & 0 & b^{33}(\mathcal{T}^0 \boldsymbol{u}_0^{13}) \end{pmatrix} \neq 0. \tag{2.12}$$

Indeed, in the case $(\rho_0^i, \mu_0^i) \equiv (0,0)$ $(i = 1,2,3)$, we have

$$\det \begin{pmatrix} \gamma^1 & \gamma^2 & \gamma^3 \\ b^{21}(\mathbf{0}) & b^{22}(\mathbf{0}) & 0 \\ b^{31}(\mathbf{0}) & 0 & b^{33}(\mathbf{0}) \end{pmatrix} = \det \begin{pmatrix} \gamma^1 & \gamma^2 & \gamma^3 \\ \sin\theta^3 & -\sin\theta^3 & 0 \\ -\sin\theta^2 & 0 & \sin\theta^2 \end{pmatrix}$$

$$= -(\gamma^1 + \gamma^2 + \gamma^3)\sin\theta^2 \sin\theta^3 \neq 0.$$

Since the determinant in (2.12) is continuous with respect to ρ_0^i and μ_0^i $(i = 1,2,3)$, we are led to (2.12) provided that $\|\rho_0^i\|_{C^1(\mathcal{I}^i)}$ and $|\mu_0^i|$ are small enough.

Next let us verify the complementary conditions at $\sigma = l^i$. Similarly as in the case $\sigma = 0$, it suffices to show that

$$\mathrm{diag}\big(b_{\partial\Omega}^1(\mathcal{T}^{l^1}\rho_0^1, \mu_0^1), b_{\partial\Omega}^2(\mathcal{T}^{l^2}\rho_0^1, \mu_0^2), b_{\partial\Omega}^3(\mathcal{T}^{l^3}\rho_0^1, \mu_0^3)\big) \neq 0.$$

Indeed, in the case $(\rho_0^i, \mu_0^i) \equiv (0,0)$ $(i = 1,2,3)$, we have

$$\det\left\{\mathrm{diag}(b_{\partial\Omega}^1(0,0), b_{\partial\Omega}^2(0,0), b_{\partial\Omega}^3(0,0))\right\}$$

$$= \det\left\{\mathrm{diag}(|\nabla\psi(\Phi_*^1)|, |\nabla\psi(\Phi_*^2)|, |\nabla\psi(\Phi_*^3)|)\right\}$$

$$= -\prod_{i=1}^{3}\left|\nabla\psi(\Phi_*^i)\right| \neq 0.$$

Since the determinant is also continuous with respect to ρ_0^i and μ_0^i ($i = 1, 2, 3$), we conclude $\det\left\{\mathrm{diag}(b_{\partial\Omega}^1(\mathcal{T}^{l^1}\rho_0^1, \mu_0^1), b_{\partial\Omega}^2(\mathcal{T}^{l^2}\rho_0^1, \mu_0^2), b_{\partial\Omega}^3(\mathcal{T}^{l^3}\rho_0^1, \mu_0^3))\right\} \neq 0$ provided that $\|\rho_0^i\|_{C^1(\mathcal{I}^i)}$ and $|\mu_0^i|$ are small enough.

Step 3 Let us analyze the linearized system. Set $X = C(\mathcal{I}^1) \times C(\mathcal{I}^2) \times C(\mathcal{I}^3)$ and $Y = C^2(\mathcal{I}^1) \times C^2(\mathcal{I}^2) \times C^2(\mathcal{I}^3)$. Define the realization of $\mathcal{A}_0$ in X with homogeneous boundary conditions as follows

$$D(A_0) = \left\{\boldsymbol{\varphi} \in Y \mid \boldsymbol{\varphi}, \mathcal{A}_0\boldsymbol{\varphi} \in X, \mathcal{B}_0(0\,;\partial_\sigma)\boldsymbol{\varphi} = \mathbf{0}, \mathcal{B}_0(l^i;\partial_\sigma)\boldsymbol{\varphi} = \mathbf{0}\right\},$$

$$A_0\boldsymbol{\varphi} = \mathcal{A}_0\boldsymbol{\varphi}.$$

Then we have the following lemma which, in particular, characterizes the interpolation spaces $D_{A_0}(\beta, \infty)$. For a definition of $D_{A_0}(\beta, \infty)$ we refer to Lunardi [17].

Lemma 2.4
(i) *The linear operator $A_0 : D(A_0) \to X$ is sectorial.*
(ii) *The characterization of the interpolation spaces $D_{A_0}(\alpha, \infty)$ is given as*

$$D_{A_0}(\beta, \infty) = \begin{cases} \left\{\boldsymbol{\varphi} \in X_\beta \mid \sum_{i=1}^{3} \gamma^i \varphi^i(0) = 0\right\} & \text{if } \beta \in (0, 1/2), \\ \left\{\boldsymbol{\varphi} \in X_\beta \mid \mathcal{B}_0(0\,;\partial_\sigma)\boldsymbol{\varphi} = \mathbf{0}, \mathcal{B}_0(l^i;\partial_\sigma)\boldsymbol{\varphi} = \mathbf{0}\right\} \\ & \text{if } \beta \in (1/2, 1), \end{cases}$$

$$(2.13)$$

where $X_\beta := C^{2\beta}(\mathcal{I}^1) \times C^{2\beta}(\mathcal{I}^2) \times C^{2\beta}(\mathcal{I}^3)$.

Proof. See [22, Section 2] or adapt the argument in [17, Section 3.1.5] for the case of systems with the estimates in [2, Theorem 12.2]. □

Set $A_1\boldsymbol{\varphi} = \mathcal{A}_1\boldsymbol{\varphi}$ for $\boldsymbol{\varphi} \in D(A_0)$. Then we obtain the following lemma.

Lemma 2.5   *Let $A = A_0 + A_1$. Then $A : D(A_0) \to X$ is a sectorial operator.*

Proof.   According to [17, Proposition 2.4.1(ii)], A is a sectorial operator in X if A_1 is a bounded linear operator from $D(A_0)$ to $D_{A_0}(\alpha/2, \infty)$. Indeed, by means of (2.13) and the definition of $\boldsymbol{a}_1$, we have, for $\boldsymbol{\varphi} \in D(A_0)$,

$$\|A_1\boldsymbol{\varphi}\|_{D_{A_0}(\alpha/2,\infty)}$$

$$\leq C_0\bigg(\sum_{i=1}^3 \big\|\Lambda^i(\boldsymbol{u}_0^i)\big\|_{C^\alpha(\mathcal{I}^i)}\bigg)\big\|\boldsymbol{a}_1(\mathcal{T}^0\boldsymbol{\rho}_0, \mathcal{T}^0\boldsymbol{\rho}_{0,\sigma}, \boldsymbol{\mu}_0)\big\|_{X_{\alpha/2}}\big\|\mathcal{T}^0\partial_\sigma^2\boldsymbol{\varphi}\big\|_X$$

$$\leq \hat{C}_0\|\boldsymbol{\varphi}\|_{D(A_0)}$$

where $\alpha \in (0,1)$ and C_0, $\hat{C}_0$ are constants which depend on $\|\rho_0^i\|_{C^{1+\alpha}(\mathcal{I}^i)}$ and $|\mu_0^i|$. This completes the proof.   □

Using an estimate as in the proof of [17, Proposition 2.4.1(ii)], we see that $A : D(A) = D(A_0) \to X$ is a sectorial operator such that $c_1\|\boldsymbol{\varphi}\|_{D(A_0)} \leq \|\boldsymbol{\varphi}\|_{D(A)} \leq c_2\|\boldsymbol{\varphi}\|_{D(A_0)}$ for $\boldsymbol{\varphi} \in D(A_0)$ and some constants $c_1, c_2 > 0$. Hence $D_A(\alpha, \infty) = D_{A_0}(\alpha, \infty)$ with equivalence of the respective norms.

By virtue of Lemma 2.5, we find that A generates the analytic semigroup e^{tA}. Then we are led to the following proposition guaranteeing the existence of a unique solution for our linearized system.

Proposition 2.6   *Let us assume that $\rho_0^i \in C^{2+\alpha}(\mathcal{I}^i)$ $(i = 1, 2, 3)$ satisfy the compatibility conditions*

$$\mathcal{B}_0(0\,;\partial_\sigma)\boldsymbol{\rho}_0^T = \boldsymbol{G}^T(0), \quad \mathcal{B}_0\big(l^i;\partial_\sigma\big)\boldsymbol{\rho}_0^T = \boldsymbol{G}_{\partial\Omega}^T(0).$$

For $(\overline{\rho}^i, \overline{\mu}^i) \in C^{2+\alpha,1}(\overline{\mathcal{Q}_{0,T}^i}) \times C^1[0,T]$ $(i = 1, 2, 3)$, the linearized system

$$\begin{cases} \partial_t\boldsymbol{\rho}^T = \mathcal{A}\boldsymbol{\rho}^T + \boldsymbol{F}^T(\sigma, t), \\ \mathcal{B}_0(0\,;\partial_\sigma)\boldsymbol{\rho}^T = \boldsymbol{G}^T(t), \quad \mathcal{B}_0(l^i;\partial_\sigma)\boldsymbol{\rho}^T = \boldsymbol{G}_{\partial\Omega}^T(t), \\ \rho_0^i(\cdot, 0) = \rho_0^i \quad (i = 1, 2, 3) \end{cases} \tag{2.14}$$

with the notation $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ has a unique solution such that

$$\sum_{i=1}^{3} \|\rho^i\|_{C^{2+\alpha,1}(\overline{\mathcal{Q}_{0,T}^i})} \leq C \sum_{i=1}^{3} \left(\|\rho_0^i\|_{C^{2+\alpha}(\mathcal{I}^i)} + \|F^i\|_{C^{\alpha,0}(\overline{\mathcal{Q}_{0,T}^i})} \right.$$
$$\left. + \|G^i\|_{C^{(1+\alpha)/2}[0,T]} + \|G_{\partial\Omega}^i\|_{C^{(1+\alpha)/2}[0,T]} \right).$$
$$(2.15)$$

Proof. Adapt the argument in the proof of [17, Theorem 5.1.19] to our linearized system (2.14). □

Now we are ready to prove Theorem 2.3 by using the contraction principle.

Proof of Theorem 2.3. Set

$$\mathcal{D} = \left\{ (\boldsymbol{\rho}, \boldsymbol{\mu}) \in C^{2+\alpha,1}(\overline{\mathcal{Q}_{0,T}^1}) \times C^{2+\alpha,1}(\overline{\mathcal{Q}_{0,T}^2}) \times C^{2+\alpha,1}(\overline{\mathcal{Q}_{0,T}^3}) \times \left[C^1[0,T] \right]^3 \,\big| \right.$$

$$\left. (\rho^i(\cdot,0), \mu^i(0)) = (\rho_0^i, \mu_0^i) \ (i=1,2,3), \ \ \|\boldsymbol{\rho}\|_{C_T^{2+\alpha,1}} + \|\boldsymbol{\mu}\|_{C_T^1} \leq K \right\}$$

for some bounded positive parameters K and T where

$$\|\boldsymbol{\rho}\|_{C_T^{2+\alpha,1}} = \sum_{i=1}^{3} \|\rho^i\|_{C^{2+\alpha,1}(\overline{\mathcal{Q}_{0,T}^i})}, \quad \|\boldsymbol{\mu}\|_{C_T^1} = \sum_{i=1}^{3} \|\mu^i\|_{C^1[0,T]}.$$

Then, for $(\overline{\boldsymbol{\rho}}, \overline{\boldsymbol{\mu}}) \in \mathcal{D}$, we define the mapping

$$\mathcal{F} : \mathcal{D} \ni (\overline{\boldsymbol{\rho}}, \overline{\boldsymbol{\mu}}) \mapsto (\boldsymbol{\rho}, \boldsymbol{\mu})$$

where $\boldsymbol{\rho}$ is the solution of (2.14) and $\boldsymbol{\mu}$ is given by $\boldsymbol{\mu}^T = Q\,T^0\boldsymbol{\rho}^T$ for such solution $\boldsymbol{\rho}$. Once we prove that the mapping $\mathcal{F}$ is a contraction on $\mathcal{D}$ for suitable K and T, the mapping $\mathcal{F}$ has a unique fixed point in $\mathcal{D}$ which implies that the nonlinear problem (2.11) admits a unique solution in $[0,T]$.

Let us first prove that $\mathcal{F}$ maps $\mathcal{D}$ into itself. Note that the lower order terms in F^i and $G_{\partial\Omega}^i$ can be rewritten as

$$f^i(\overline{\boldsymbol{u}}) = f^i(\boldsymbol{u}_0) + \int_0^1 \left\langle Df^i(\eta\overline{\boldsymbol{u}} + (1-\eta)\boldsymbol{u}_0), \overline{\boldsymbol{u}} - \boldsymbol{u}_0 \right\rangle d\eta,$$

$$g_{\partial\Omega}^i(\overline{\boldsymbol{u}}_1^i) = g_{\partial\Omega}^i(\boldsymbol{u}_{1,0}^i) + \int_0^1 \left\langle Dg_{\partial\Omega}^i(\eta\overline{\boldsymbol{u}}_1^i + (1-\eta)\boldsymbol{u}_{1,0}^i), \overline{\boldsymbol{u}}_1^i - \boldsymbol{u}_{1,0}^i \right\rangle d\eta.$$

Here $\overline{\boldsymbol{u}} = (\overline{\rho}^i, \overline{\rho}^i_\sigma, \mathcal{T}^0\overline{\boldsymbol{\rho}}, \mathcal{T}^0\overline{\boldsymbol{\rho}}_\sigma, \overline{\boldsymbol{\mu}})$, $\boldsymbol{u}_0 = (\rho^i_0, \rho^i_{0,\sigma}, \mathcal{T}^0\boldsymbol{\rho}_0, \mathcal{T}^0\boldsymbol{\rho}_{0,\sigma}, \boldsymbol{\mu}_0)$, $\overline{\boldsymbol{u}}^i_1 = (\overline{\rho}^i, \overline{\mu}^i)$, and $\boldsymbol{u}^i_{1,0} = (\rho^i_0, \mu^i_0)$. Moreover, Df^i and $Dg^i_{\partial\Omega}$ are the Fréchet derivative of f^i and $g^i_{\partial\Omega}$, respectively, and the bracket $\langle\cdot,\cdot\rangle$ is the respective inner product. Then, by means of (2.15) and $\boldsymbol{\mu}^T = Q\,\mathcal{T}^0\boldsymbol{\rho}^T$, we have

$$\|\boldsymbol{\rho}\|_{C_T^{2+\alpha,1}} + \|\boldsymbol{\mu}\|_{C_T^1}$$

$$\leq \hat{C}\Bigg\{ \sum_{i=1}^{3} \big(\|\rho^i_0\|_{C^{2+\alpha}(\mathcal{I}^i)} + \|f^i(\boldsymbol{u}_0)\|_{C^\alpha(\mathcal{I}^i)} + |g^i_{\partial\Omega}(\mathcal{T}^{l^i}\boldsymbol{u}^i_0)|\big)$$

$$+ \sum_{j=2}^{3} \big(|b^{j1}(\mathcal{T}^0\boldsymbol{u}^{1j}_0)||\mathcal{T}^0\rho^1_{0,\sigma}| + |b^{jj}(\mathcal{T}^0\boldsymbol{u}^{1j}_0)||\mathcal{T}^0\rho^j_{0,\sigma}|\big) \Bigg\} + C_K T^\nu$$

for $\nu = \min\{\alpha/2, (1-\alpha)/2\}$. Thus, choosing

$$K = 2\hat{C}\Bigg\{ \sum_{i=1}^{3} \big(\|\rho^i_0\|_{C^{2+\alpha}(\mathcal{I}^i)} + \|f^i(\boldsymbol{u}_0)\|_{C^\alpha(\mathcal{I}^i)} + |g^i_{\partial\Omega}(\mathcal{T}^{l^i}\boldsymbol{u}^i_0)|\big)$$

$$+ \sum_{j=2}^{3} \big(|b^{j1}(\mathcal{T}^0\boldsymbol{u}^{1j}_0)||\mathcal{T}^0\rho^1_{0,\sigma}| + |b^{jj}(\mathcal{T}^0\boldsymbol{u}^{1j}_0)||\mathcal{T}^0\rho^j_{0,\sigma}|\big) \Bigg\}, \quad (2.16)$$

we conclude that there exists a time $T_1 > 0$ such that

$$\|\boldsymbol{\rho}\|_{C_T^{2+\alpha,1}} + \|\boldsymbol{\mu}\|_{C_T^1} \leq K \quad \text{for } T \leq T_1. \qquad (2.17)$$

That is, $\mathcal{F}$ maps $\mathcal{D}$ into itself.

Let us prove that the mapping $\mathcal{F}$ is a contraction. For $(\overline{\boldsymbol{\rho}}_1, \overline{\boldsymbol{\mu}}_1)$, $(\overline{\boldsymbol{\rho}}_2, \overline{\boldsymbol{\mu}}_2) \in \mathcal{D}$ with $T \leq T_1$, let

$$(\boldsymbol{\rho}_1, \boldsymbol{\mu}_1) = \mathcal{F}(\overline{\boldsymbol{\rho}}_1, \overline{\boldsymbol{\mu}}_1), \quad (\boldsymbol{\rho}_2, \boldsymbol{\mu}_2) = \mathcal{F}(\overline{\boldsymbol{\rho}}_2, \overline{\boldsymbol{\mu}}_2)$$

be the solutions associated with the linearized problem (2.14). Then, applying a similar argument to [4, pp. 373–375] with $\boldsymbol{\mu}^T = Q\,\mathcal{T}^0\boldsymbol{\rho}^T$, we are led to

$$\|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\|_{C_T^{2+\alpha,1}} + \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{C_T^1} \leq \hat{C}_K T^\nu \big(\|\overline{\boldsymbol{\rho}}_1 - \overline{\boldsymbol{\rho}}_2\|_{C_T^{2+\alpha,1}} + \|\overline{\boldsymbol{\mu}}_1 - \overline{\boldsymbol{\mu}}_2\|_{C_T^1}\big)$$

for $\nu = \min\{\alpha/2, (1-\alpha)/2\}$. Thus, $\mathcal{F}$ is a contraction for $T \leq T_2$, which satisfies $\hat{C}_K T_2^\nu \leq 1/2$. Consequently, choosing $T_* = \min\{T_1, T_2\}$, we find that $\mathcal{F}$ has a unique fixed point in $\mathcal{D}$ for $T \leq T_*$, so that the nonlinear problem (2.11) has a unique solution in $[0, T]$ with (2.17) for $T \leq T_*$. Further, checking the details of the estimate for the linear system, we obtain for $t \in [0, T]$

$$\sum_{i=1}^{3} \|\rho^i(\cdot, t)\|_{C^{1+\alpha}(\mathcal{I}^i)} \leq m_0 + C_K T^\nu,$$

where m_0 depends on $\|\rho_0^i\|_{C^{1+\alpha}(\mathcal{I}^i)}$ and $|\mu_0^i|$. Then, there exists a time $T_3 > 0$ such that $m_0 + C_K T^\nu \leq 2m_0$ for $T \leq T_3$. Thus, choosing $T_0 = \min\{T_*, T_3\}$, we have $\sum_{i=1}^{3} \|\rho^i(\cdot, t)\|_{C^{1+\alpha}(\mathcal{I}^i)} \leq 2m_0$ for $t \in [0, T]$ with $T \leq T_0$. It is possible to guarantee $2m_0 \leq \delta_0$ for sufficiently small $\|\rho_0^i\|_{C^{1+\alpha}(\mathcal{I}^i)}$ and $|\mu_0^i|$, where δ_0 is as in (2.10). By Lemma 2.1, $|\mu^i(t)|$ is estimated by $\sum_{i=1}^{3} \|\rho^i(\cdot, t)\|_{C^0(\mathcal{I}^i)}$, so that $|\mu^i(t)|$ can be smaller than δ_0 if $\|\rho_0^i\|_{C^{1+\alpha}(\mathcal{I}^i)}$ and $|\mu_0^i|$ are small enough. This completes the proof of Theorem 2.3. $\square$

3. Linearization

In order to linearize the nonlinear system (2.11) around the stationary solution $\Gamma_* = \bigcup_{i=1}^{3} \Gamma_*^i$, we need to establish the following properties of Ψ at $(q, \mu^i) = (0, 0)$.

Lemma 3.1 *For the parameterizations Ψ^i, $i = 1, 2, 3$, in Section 2, the following properties hold*:

(i) $\Psi^i(\sigma, 0, 0) = \Phi_*^i(\sigma)$ *and* $\Psi^i(\sigma, q, 0) = \Phi_*^i(\sigma \mu_{\partial\Omega}^i(q)/l^i) + q N_*^i(\sigma \mu_{\partial\Omega}^i(q)/l^i)$.

(ii) $\Psi_\sigma^i(\sigma, 0, 0) = T_*^i$, $\Psi_q^i(\sigma, 0, 0) = N_*^i$, *and* $\Psi_\mu^i(\sigma, 0, 0) = (1 - \sigma/l^i)T_*^i$.

(iii) $\Psi_{\sigma\sigma}^i(\sigma, 0, 0) = (0, 0)^T$, $\Psi_{\sigma q}^i(\sigma, 0, 0) = (0, 0)^T$, *and* $\Psi_{\sigma\mu}^i(\sigma, 0, 0) = (-1/l^i)T_*^i$.

(iv) $\Psi_{\sigma\sigma q}^i(\sigma, 0, 0) = (0, 0)^T$ *and* $\Psi_{\sigma\sigma\mu}^i(\sigma, 0, 0) = (0, 0)^T$.

Proof. By the definition of Ψ^i, (i) is obvious. Let us prove (ii). Differentiating $\Psi^i(\sigma, 0, 0) = \Phi_*^i(\sigma)$ with respect to σ, we readily derive $\Psi_\sigma^i(\sigma, 0, 0) = T_*^i(\sigma)$. Applying a similar argument to [7], we obtain $\{\mu_{\partial\Omega}^i(q)\}'|_{q=0} = 0$. Thus (i) gives $\Psi_q^i(\sigma, 0, 0) = N_*^i(\sigma)$. Further, by the definition of ξ^i, we have

$$\xi_\mu^i(\sigma, 0, 0) = 1 - \sigma/l^i.$$

It follows from the definition of Ψ^i and the Frenet-Serret formulae that

$$\Psi_\mu^i(\sigma, q, \mu^i) = \xi_\mu(\sigma, q, \mu^i)(1 - q\kappa_*^i)T_*^i(\xi(\sigma, q, \mu^i)) = \xi_\mu(\sigma, q, \mu^i)T_*^i(\xi(\sigma, q, \mu^i)).$$

Putting $(q, \mu^i) = (0,0)$, the third property of (ii) is derived. Finally, by using (ii), we have (iii)–(iv). □

By virtue of Lemma 3.1, we are led to the linearization of (2.11) around the stationary solution $\Gamma_* = \bigcup_{i=1}^3 \Gamma_*^i$.

Proposition 3.2 *The linearization of* (2.11) *is given by*

$$
\begin{cases}
\beta^i \rho_t^i = \gamma^i \rho_{\sigma\sigma}^i & for \ \ \sigma \in (0, l^i), \\[2mm]
\displaystyle\sum_{i=1}^3 \gamma^i \rho^i = 0 & at \ \ \sigma = 0, \\[2mm]
\rho_\sigma^1 = \rho_\sigma^2 = \rho_\sigma^3 & at \ \ \sigma = 0, \\[2mm]
\rho_\sigma + h_*^i \rho^i = 0 & at \ \ \sigma = l^i,
\end{cases}
\tag{3.1}
$$

for $i = 1, 2, 3$, *where* h_*^i *is the curvature of* $\partial\Omega$ *at the point* $\Phi_*^i(l^i) \in \Gamma_*^i \cap \partial\Omega$.

We remark that (3.1) corresponds to the linearized problem which was derived in a formal way by Ikota and Yanagida [13].

Proof of Proposition 3.2. Applying the same argument as in [7, Section 3], using Lemma 3.1 and $\kappa_*^i = 0$ $(i = 1, 2, 3)$ we obtain from equations (2.9) and the boundary conditions at $\sigma = l^i$, the first and fourth equations in (3.1). Thus we only derive the third equation of (3.1). To simplify the notation, we set

$$\widetilde{J}^i(\rho^i, \mu^i) = J(\rho^i, \rho_\sigma^i, \mu^i),$$

$$\widetilde{g}^{1j}(\rho^1, \rho^j, \mu^1, \mu^j) = g^{1j}(\rho^1, \rho^j, \rho_\sigma^1, \rho_\sigma^j, \mu^1, \mu^j) \quad (j = 2, 3).$$

Then it is easy to obtain

$$\widetilde{J}^i(0,0) = 1, \quad \partial \widetilde{J}^i(0,0)[\rho^i, \mu^i] = -\frac{1}{l^i}\mu^i,$$

where $\partial \widetilde{J}^i(0,0)$ is the Fréchet derivative of $\widetilde{J}^i$ at $(0,0)$. Recalling the definition of $\widetilde{g}^{1j}$ $(j = 2, 3)$ and using Lemma 3.1, we have

$$\partial \widetilde{g}^{1j}(0,0,0,0)[\rho^1, \rho^j, \mu^1, \mu^j]$$

$$= -\frac{1}{l^1}\left(T^1_*, T^j_*\right)_{\mathbb{R}^2}\mu^1 - \frac{1}{l^j}\left(T^1_*, T^j_*\right)_{\mathbb{R}^2}\mu^j + \left(T^1_*, N^j_*\right)_{\mathbb{R}^2}\rho^j_\sigma$$

$$+ \left(N^1_*, T^j_*\right)_{\mathbb{R}^2}\rho^1_\sigma - \left(-\frac{1}{l^1}\mu^1 - \frac{1}{l^j}\mu^j\right)\cos\theta^k$$

for $(j, k) = (2, 3)$ or $(3, 2)$, where $\partial \widetilde{g}^{1j}(0,0,0,0)$ is the Fréchet derivative of $\widetilde{g}^{1j}$ at $(0,0,0,0)$. Since the angle conditions at $\sigma = 0$ give

$$\left(T^1_*, T^2_*\right)_{\mathbb{R}^2} = \cos\theta^3, \quad \left(T^1_*, N^2_*\right)_{\mathbb{R}^2} = -\sin\theta^3, \quad \left(N^1_*, T^2_*\right)_{\mathbb{R}^2} = \sin\theta^3,$$

$$\left(T^1_*, T^3_*\right)_{\mathbb{R}^2} = \cos\theta^2, \quad \left(T^1_*, N^3_*\right)_{\mathbb{R}^2} = \sin\theta^2, \quad \left(N^1_*, T^3_*\right)_{\mathbb{R}^2} = -\sin\theta^2,$$

it follows that

$$\partial \widetilde{g}^{12}(0,0,0,0)[\rho^1, \rho^2, \mu^1, \mu^2] = \left(-\rho^2_\sigma + \rho^1_\sigma\right)\sin\theta^3,$$

$$\partial \widetilde{g}^{13}(0,0,0,0)[\rho^1, \rho^3, \mu^1, \mu^3] = \left(\rho^3_\sigma - \rho^1_\sigma\right)\sin\theta^2$$

at $\sigma = 0$. Hence the linearization of $\widetilde{g}^{1j}(\rho^1, \rho^j, \mu^1, \mu^j) = 0$ $(j = 2, 3)$ is

$$\left(-\rho^2_\sigma + \rho^1_\sigma\right)\sin\theta^3 = 0, \quad \left(\rho^3_\sigma - \rho^1_\sigma\right)\sin\theta^2 = 0 \quad \text{at } \sigma = 0,$$

so that, by virtue of $\theta^j \in (0, \pi)$ $(j = 2, 3)$, we have

$$\rho^1_\sigma = \rho^2_\sigma = \rho^3_\sigma \quad \text{at } \sigma = 0.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

In [13] Ikota and Yanagida investigated linearized stability for the curvature flow with a triple junction (3.1). They derived a criterion according to which one can determine whether the stationary solution is linearly stable or unstable. In what follows, we recall their linearized stability criterion. The

main result of [13, Theorem 1.1] is concerned with the analysis of the self-adjoint eigenvalue problem associated to the linearized system of equations (3.1). We now recall their linearized stability result.

Theorem 3.3 *The maximal eigenvalue of the eigenvalue problem corresponding to the linearized problem* (3.1), *i.e. we set* $\lambda\rho^i$ *instead of* $\beta^i\rho_t^i$ *for a constant* λ, *is negative and the stationary solution is linearly asymptotically stable if and only if one of the following conditions is fulfilled*:

(a) *either all* h_*^1, h_*^2, *and* h_*^3 *are positive*,
(b) *or, at most one of them is non-positive, and they satisfy*

$$\gamma^1\big(1+l^1h_*^1\big)h_*^2h_*^3 + \gamma^2\big(1+l^2h_*^2\big)h_*^1h_*^3 + \gamma^3\big(1+l^3h_*^3\big)h_*^1h_*^2 > 0.$$

We will also need a variational characterization of the linearized stability property. To this end, let us introduce the bilinear form

$$I_*[\boldsymbol{\varphi},\boldsymbol{\varphi}] = \sum_{i=1}^{3}\gamma^i\bigg\{\int_0^{l^i}(\varphi_s^i)^2\,ds + h_*^i(\varphi^i)^2|_{s=l^i}\bigg\} \qquad (3.2)$$

for all $\boldsymbol{\varphi}\in\mathcal{E}(\Gamma_*)$, where

$$\mathcal{E}(\Gamma_*) = \bigg\{(\varphi^1,\varphi^2,\varphi^3)\in H^1(0,l^1)\times H^1(0,l^2)\times H^1(0,l^3)\,\bigg|\,\sum_{i=1}^{3}\gamma^i\varphi^i(0)=0\bigg\}.$$

This bilinear form was also considered in [13]. The following lemma is a simple consequence of the variational characterizations of the largest eigenvalue.

Lemma 3.4 *Let* λ *be the maximal eigenvalue of the eigenvalue problem corresponding to the linearized system* (3.1). *Then*

$$I_*[\boldsymbol{\varphi},\boldsymbol{\varphi}] \geq (-\lambda)\sum_{i=1}^{3}\gamma^i\|\varphi^i\|_{L^2(\Gamma_*^i)}^2 \quad \text{for } \boldsymbol{\varphi}\in\mathcal{E}(\Gamma_*).$$

Remark 3.5 In order to simplify the presentation, we will henceforth consider only the case $\beta^i = \gamma^i$, for $i = 1, 2, 3$. It is worth to note that the linearized stability criterion is invariant with respect to the positive constants $\beta^i > 0$ for $i = 1, 2, 3$. As it should be obvious from all the energy

type estimates to follow, the full nonlinear stability of the stationary solution will not be affected by a different choice of positive mobility constants $\beta^i > 0$ for $i = 1, 2, 3$.

4. Uniqueness of the stationary solution

In this section we prove the uniqueness of the stationary solution in a small H^2-neighborhood. The inverse mapping theorem also gives a bound on the H^2-norm of $\boldsymbol{\rho} = (\rho^1, \rho^2, \rho^3)$ in terms of the L^2-norm of the curvature $\boldsymbol{\kappa} = (\kappa^1, \kappa^2, \kappa^3)$.

To this end, let us introduce the function space

$$\mathcal{M} = \left\{ (\rho^1, \rho^2, \rho^3) \in H^2(0, l^1) \times H^2(0, l^2) \times H^2(0, l^3) \,\middle|\, \sum_{i=1}^{3} \gamma^i \rho^i(0) = 0 \right\}.$$

Then $\boldsymbol{\rho}$ via parametrization (2.2) defines a neighboring triple junction configuration such that the end points lie on $\partial \Omega$.

Theorem 4.1 *Let I_* be positive. Then there exists a H^2-neighborhood of $\boldsymbol{\rho} \equiv 0$ in $\mathcal{M}$, such that $\boldsymbol{\rho} \equiv 0$ is the only solution of the problem*

$$\kappa^i = 0, \quad \sphericalangle\left(\partial\Omega, \Gamma_t^i\right) = \pi/2, \tag{4.1}$$

$$\sphericalangle(\Gamma^i(t), \Gamma^j(t)) = \cos\theta^k \quad \text{for } i, j, k \in \{1, 2, 3\} \text{ mutually different.} \tag{4.2}$$

Proof. The idea of the proof is to use the local inverse mapping theorem for the curvature operator with appropriate boundary conditions. The positivity of I_* will ensure invertibility of the linearization.

Using the notation (2.4) of Section 2 we obtain

$$\kappa^i(\boldsymbol{\rho}) = a^i\left(\rho^i, \rho_\sigma^i, \boldsymbol{\mu}(\boldsymbol{\rho}(0))\right)\rho_{\sigma\sigma}^i + f^i\left(\rho^i, \rho_\sigma^i, \boldsymbol{\mu}(\boldsymbol{\rho}(0))\right)$$

with $a^i(\rho^i, \rho_\sigma^i, \boldsymbol{\mu}(\boldsymbol{\rho}(0))) = 1/\{J^i(\rho^i, \rho_\sigma^i, \boldsymbol{\mu}(\boldsymbol{\rho}(0)))\}^2$, a smooth function f^i, and a linear mapping $\boldsymbol{\mu}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$. The boundary conditions in (4.1) and (4.2) can be written as

$$g^i\left(\rho^i(l^i), \rho_\sigma^i(l^i), \boldsymbol{\mu}(\boldsymbol{\rho}(0))\right) = 0 \quad \text{for } i = 1, 2, 3,$$

$$g^i\left(\boldsymbol{\rho}(0), \boldsymbol{\rho}_\sigma(0), \boldsymbol{\mu}(\boldsymbol{\rho}(0))\right) = 0 \quad \text{for } i = 4, 5$$

where $g^1, \ldots, g^5$ are smooth functions. We define $\mathcal{N} = L^2(0, l^1) \times L^2(0, l^2) \times L^2(0, l^3)$ and observe that solving (4.1) and (4.2) is equivalent to finding a zero of the mapping

$$F : B_\varepsilon(0) \to \mathcal{N} \times \mathbb{R}^5, \quad F(\boldsymbol{\rho}) = (\kappa^1(\boldsymbol{\rho}), \, \kappa^2(\boldsymbol{\rho}), \, \kappa^3(\boldsymbol{\rho}), \, g^1, \ldots, g^5)$$

where $B_\varepsilon(0)$ is a ball of a radius $\varepsilon > 0$ around zero in the space $\mathcal{M}$. Since H^2 is embedded in C^1 the mapping is well defined. Arguing similarly as in Section 3 we obtain

$DF(0) :$

$$\mathcal{M} \to \mathcal{N} \times \mathbb{R}^5$$

$$\boldsymbol{\rho} \mapsto \left(\rho^1_{\sigma\sigma}, \rho^2_{\sigma\sigma}, \rho^3_{\sigma\sigma}, \rho^1_\sigma + h^1\rho^1, \rho^2_\sigma + h^2\rho^2, \rho^3_\sigma + h^3\rho^3, \rho^1_\sigma - \rho^2_\sigma, \rho^1_\sigma - \rho^3_\sigma\right).$$

Similarly as in [13], since I_* is positive we can conclude that $DF(0)$ is injective and hence the Fredholm alternative gives that $DF(0)$ is invertible. Now the local inverse mapping theorem (see e.g. [24]) gives that there is a neighborhood around $\mathbf{0}$ such that only $\boldsymbol{\rho} \equiv \mathbf{0}$ solves (4.1) and (4.2). $\qquad \square$

It is worthwhile noting that the mapping F analyzed in the proof of the above theorem is in fact a local diffeomorphism. Therefore its inverse mapping is locally Lipschitz continuous. Hence we have the following corollary.

Corollary 4.2 *There exist constants $C, \delta_1 > 0$ such that*

$$\|\boldsymbol{\rho}\|_{H^2} \leq C\|\boldsymbol{\kappa}\|_{L^2}$$

provided that $\|\boldsymbol{\kappa}\|_{L^2}^2 < \delta_1$ and $\boldsymbol{\rho} \in \mathcal{M}$ fulfills $F(\boldsymbol{\rho}) = (\kappa^1, \kappa^2, \kappa^3, 0, 0, 0, 0, 0)$.

In other words, by means of the above theorem and its corollary, we obtained a bound on the H^2-norm of the solution $\boldsymbol{\rho}$ in terms of the L^2-norm of the curvature $\boldsymbol{\kappa} = (\kappa^1, \kappa^2, \kappa^3)$ in the vicinity of the stationary solution $\boldsymbol{\rho} \equiv \mathbf{0}$ provided that $\boldsymbol{\rho}$ fulfills the boundary conditions. This useful observation will be used several times throughout the rest of the paper. Although we could take the standard L^2-norm in the corollary above we choose the suitable weighted L^2-norm defined in (6.2) in the corollary as this will simplify the further presentation.

5. Governing equations for the curvature and other geometric quantities

In order to show the global existence and the nonlinear stability of solutions for which the bilinear form of [13] is positive, we apply an energy method similar to the one used in [5] and [15]. For such a method it is important to derive evolution equations for the curvature.

Let s be the arc-length parameter along the evolving curve Γ_t and let X be a smooth map such that $X(\cdot, t)$ is an arc-length parameterization of Γ_t with

$$\Gamma_t^i = \{X^i(s,t) \mid s \in [0, r^i(t)]\}$$

where r^i is smooth such that $L[\Gamma_t^i] = r^i(t)$ which is the length of Γ_t^i. Let N^i $(= N^i(s,t))$ be the unit normal vector of Γ_t^i. It can be written as

$$N^i(s,t) = \begin{pmatrix} \cos \omega^i(s,t) \\ \sin \omega^i(s,t) \end{pmatrix}.$$

Then we have

$$\begin{cases} N_s^i = -\kappa^i T^i, & T_s^i = \kappa^i N^i, \\ N_t^i = -\omega_t^i T^i, & T_t^i = \omega_t^i N^i \end{cases} \tag{5.1}$$

where T^i is the unit tangent vector of Γ_t^i and κ^i is the curvature of Γ_t^i. In addition, we define

$$V^i = \left(X_t^i, N^i\right)_{\mathbb{R}^2}, \quad v^i = \left(X_t^i, T^i\right)_{\mathbb{R}^2}$$

and hence

$$X_t^i = V^i N^i + v^i T^i. \tag{5.2}$$

Differentiating (5.2) with respect to s and using (5.1), we have

$$X_{ts}^i = V_s^i N^i + V^i N_s^i + v_s^i T^i + v^i T_s^i \tag{5.3}$$

$$= \left(V_s^i + \kappa^i v^i\right) N^i + \left(-\kappa^i V^i + v_s^i\right) T^i. \tag{5.4}$$

Lemma 5.1 *Let X^i be a smooth arc-length parameterization as above. Then*

$$\omega^i_t = V^i_s + \kappa^i v^i, \quad v^i_s = \kappa^i V^i.$$

Proof. Since $X^i_{ts} = X^i_{st}$ and $X^i_s = T^i$, it follows from (5.1) and (5.2) that

$$\omega^i_t N^i = \left(V^i_s + \kappa^i v^i\right)N^i + \left(-\kappa^i V^i + v^i_s\right)T^i.$$

Thus we obtain the desired results. $\qquad\square$

As a consequence of Lemma 5.1 we have the following lemma.

Lemma 5.2 *Let X^i be a smooth arc-length parameterization of Γ^i_t as above. Then the curvature κ^i $(i = 1, 2, 3)$ satisfies the evolution equation:*

$$\kappa^i_t = V^i_{ss} + (\kappa^i)^2 V^i + \kappa^i_s v^i$$

Proof. By $\omega^i_s = \kappa^i$ and Lemma 5.1, we obtain

$$\kappa^i_t = \omega^i_{st} = \omega^i_{ts} = \left(V^i_s + \kappa^i v^i\right)_s = V^i_{ss} + \kappa^i v^i_s + \kappa^i_s v^i = V^i_{ss} + (\kappa^i)^2 V^i + \kappa^i_s v^i.$$

This completes the proof. $\qquad\square$

By the assumption that Γ^i_t meets $\partial\Omega$ at the one end point with the angle $\pi/2$, we have

$$\psi(X^i(r^i(t), t)) = 0, \quad (\nabla\psi(X^i), N^i)_{\mathbb{R}^2} = 0 \quad \text{at } s = r^i(t).$$

Differentiating the identity $\psi(X^i(r^i(t), t)) = 0$ with respect to t and taking into account the transversality condition $(\nabla\psi(X^i), N^i)_{\mathbb{R}^2} = 0$ and the governing equation $X^i_t = V^i N^i + v^i T^i$ we can derive the following lemma.

Lemma 5.3 *At the point where Γ^i_t $(i = 1, 2, 3)$ meets the outer boundary $\partial\Omega$ we have*

$$v^i + (r^i)' = 0 \quad for \quad s = r^i(t).$$

Next we derive corresponding boundary conditions at the triple junction point $p(t)$. It is assumed that curves Γ^1, Γ^2, and Γ^3 meet at the triple junction. Let $i, j, k \in \{1, 2, 3\}$ be mutually different. Let Γ^k and γ^k be the

interface and the surface energy density between phases i, j. Following the arguments in Bronsard and Reitich [4] the angles θ^i $(i = 1, 2, 3)$ of the curves at the triple junction point $p(t)$ fulfill Young's law

$$\frac{\sin \theta^1}{\gamma^1} = \frac{\sin \theta^2}{\gamma^2} = \frac{\sin \theta^3}{\gamma^3} \tag{5.5}$$

(see [23]). Young's law can be expressed as a force balance in the following form

$$\sum_{i=1}^{3} \gamma^i T^i = \sum_{i=1}^{3} \gamma^i N^i = 0. \tag{5.6}$$

Let $p(t) \in \mathbb{R}^2$ denote a triple junction. At the triple junction the following boundary conditions hold:

$$X^1(0, t) = X^2(0, t) = X^3(0, t) \ (= p(t)), \tag{5.7}$$

$$\left(X_s^i, X_s^j\right)_{\mathbb{R}^2} = \cos \theta^k \quad \text{at } p(t) \tag{5.8}$$

for $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$. Then we obtain the following lemma.

Lemma 5.4 *At the triple junction $p(t)$ we have the following equality*:

$$\sum_{i=1}^{3} \gamma^i v^i = 0.$$

Proof. Differentiating (5.7) with respect to t, we obtain

$$\frac{dp}{dt} = X_t^1 = X_t^2 = X_t^3. \tag{5.9}$$

For $i = 1, 2, 3$, it holds

$$\left(\frac{dp}{dt}, T^i\right)_{\mathbb{R}^2} = \left(X_t^i, T^i\right)_{\mathbb{R}^2} = v^i.$$

This fact and Young's law imply that

$$\sum_{i=1}^{3} \gamma^i v^i = \sum_{i=1}^{3} \gamma^i \left(\frac{dp}{dt}, T^i \right)_{\mathbb{R}^2} = \left(\frac{dp}{dt}, \sum_{i=1}^{3} \gamma^i T^i \right)_{\mathbb{R}^2} = 0.$$

Hence the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the next lemma we derive evolution equations and boundary conditions for the curvature.

Lemma 5.5   *A smooth solution of the curvature flow equations*

$$V^i = \kappa^i, \quad i = 1, 2, 3, \tag{5.10}$$

with the boundary conditions

$$\begin{cases} \sphericalangle(\Gamma^i(t), \Gamma^j(t)) = \cos\theta^k & \text{for} \quad i, j, k \in \{1, 2, 3\} \text{ mutually different,} \\ \sphericalangle(\partial\Omega, \Gamma_t^i) = \pi/2 & \text{at} \quad \Gamma_t^i \cap \partial\Omega, \\ \partial\Gamma_t^i \subset \partial\Omega \end{cases} \tag{5.11}$$

fulfills when expressed in the above arc-length parameterization the evolution equations

$$\kappa_t^i = \kappa_{ss}^i + (\kappa^i)^3 + \kappa_s^i v^i, \quad i = 1, 2, 3.$$

Furthermore, at the triple junction $p(t)$ we have

$$\sum_{i=1}^{3} \gamma^i \kappa^i = 0, \tag{5.12}$$

$$\kappa_s^1 + \kappa^1 v^1 = \kappa_s^2 + \kappa^2 v^2 = \kappa_s^3 + \kappa^3 v^3 \tag{5.13}$$

and at $\Gamma_t^i \cap \partial\Omega$ the identity

$$\kappa_s^i + h^i \kappa^i = 0 \tag{5.14}$$

holds. Here h^i is the curvature of $\partial\Omega$ at the point $X^i(r^i(t), t) \in \Gamma_t^i \cap \partial\Omega$.

Proof.   From (5.9) we deduce

$$\left(\frac{dp}{dt}, N^i\right)_{\mathbb{R}^2} = \left(X_t^i, N^i\right)_{\mathbb{R}^2} = V^i.$$

This fact and Young's law imply that

$$\sum_{i=1}^{3} \gamma^i V^i = \sum_{i=1}^{3} \gamma^i \left(\frac{dp}{dt}, N^i\right)_{\mathbb{R}^2} = \left(\frac{dp}{dt}, \sum_{i=1}^{3} \gamma^i N^i\right)_{\mathbb{R}^2} = 0.$$

Since $V^i = \kappa^i$, we are led to (5.12).

Differentiating (5.8) with respect to t, we obtain

$$\left(X_{st}^i, X_s^j\right)_{\mathbb{R}^2} + \left(X_s^i, X_{st}^j\right)_{\mathbb{R}^2} = 0.$$

It follows from (5.3) and (5.11) that

$$\left(V_s^i + \kappa^i v^i\right) \sin \theta^k + \left(V_s^j + \kappa^j v^j\right)(-\sin \theta^k) = 0.$$

By $0 < \theta^k < \pi$ we have $\sin \theta^k > 0$, so that

$$V_s^i + \kappa^i v^i = V_s^j + \kappa^j v^j.$$

Hence, Lemma 5.2 and the fact that $V^i = \kappa^i$ imply (5.13).

Finally, (5.14) follows as in [7, Section 3]. □

6. A priori estimates and global existence of a smooth solution

The purpose of this section is to derive a priori estimates guaranteeing global existence of a smooth solution and its convergence to a steady state. First, we derive a priori estimates for the L^2-norm of the curvature. Next we proceed with higher order energy estimates yielding a priori estimates for the H^2-norm of the curvature. As a consequence of these estimates we will be able to prove exponential decay of the H^2-norm of the curvature. We remark that due to the parabolic regularization property the solution of Theorem 2.3 will become smoother for positive time such that all derivatives in the following computations exist.

6.1. First order a priori estimates

Let us define the energy functional

$$E[\Gamma] = \sum_{i=1}^{3} \gamma^i L[\Gamma^i]$$

where $\Gamma = \bigcup_{i=1}^{3} \Gamma^i$ and $L[\Gamma^i]$ is the length of Γ^i. Further, throughout Section 6, we use the following notations:

$$\|\varphi^i\|_{L^p} = \left(\int_{\Gamma_t^i} |\varphi^i|^p \, ds \right)^{\frac{1}{p}} (1 \le p < \infty), \quad \|\varphi^i\|_{L^\infty} = \operatorname*{ess\,sup}_{\Gamma_t^i} |\varphi^i|, \quad (6.1)$$

$$\|\boldsymbol{\varphi}\|_{L^p} = \left(\sum_{i=1}^{3} \gamma^i \|\varphi^i\|_{L^p}^p \right)^{\frac{1}{p}}, \quad \|\boldsymbol{\varphi}\|_{L^\infty} = \max_i \|\varphi^i\|_{L^\infty} \qquad (6.2)$$

for a vector function $\boldsymbol{\varphi} = (\varphi^1, \varphi^2, \varphi^3)$.

Lemma 6.1   *A smooth solution of* (5.10)–(5.11) *fulfills the following energy type identities*

(i) $\frac{d}{dt} E[\Gamma_t] + \|\boldsymbol{\kappa}\|_{L^2}^2 = 0$,

(ii) $\frac{d}{dt} \|\boldsymbol{\kappa}\|_{L^2}^2 = -2 \sum_{i=1}^{3} \gamma^i \left\{ \int_{\Gamma_t^i} (V_s^i)^2 \, ds + h^i (V^i)^2 \big|_{s=r^i} \right\} + \|\boldsymbol{\kappa}\|_{L^4}^4 + \sum_{i=1}^{3} \gamma^i (\kappa^i)^2 v^i \big|_{s=0}$ *where* h^i *is evaluated at* $X^i(r^i(t), t)$.

Proof.   By means of the identity $L[\Gamma_t^i] = r^i(t)$, Lemmata 5.1, 5.3 and 5.4, we have

$$\frac{d}{dt} E[\Gamma_t] = \sum_{i=1}^{3} \gamma^i (r^i)' = -\sum_{i=1}^{3} \gamma^i \{ v^i(r^i, t) - v^i(0, t) \}$$

$$= -\sum_{i=1}^{3} \gamma^i \int_{\Gamma_t^i} v_s^i \, ds = -\sum_{i=1}^{3} \gamma^i \int_{\Gamma_t^i} \kappa^i V^i \, ds = -\sum_{i=1}^{3} \gamma^i \int_{\Gamma_t^i} (\kappa^i)^2 \, ds.$$

In order to prove (ii), we compute

$$2 \int_{\Gamma_t^i} \kappa^i \kappa_t^i \, ds = 2 \int_{\Gamma_t^i} \kappa^i \{ \kappa_{ss}^i + (\kappa^i)^3 + \kappa_s^i v^i \} \, ds. \qquad (6.3)$$

The identity

$$\frac{d}{dt} \int_{\Gamma_t^i} (\kappa^i)^2 \, ds = 2 \int_{\Gamma_t^i} \kappa^i \kappa_t^i \, ds + (\kappa^i)^2 \big|_{s=r^i} (r^i)',$$

and Lemma 5.3 imply

$$2 \int_{\Gamma_t^i} \kappa^i \kappa_t^i \, ds = \frac{d}{dt} \int_{\Gamma_t^i} (\kappa^i)^2 \, ds + (\kappa^i)^2 v^i \big|_{s=r^i}. \tag{6.4}$$

For the right hand side of equation (6.3) we can use the boundary condition (5.14). Integration by parts yields

$$2 \int_{\Gamma_t^i} \kappa^i \big\{ \kappa_{ss}^i + (\kappa^i)^3 + \kappa_s^i v^i \big\} \, ds$$

$$= 2 \bigg\{ \big[\kappa^i \kappa_s \big]_{s=0}^{s=r^i} - \int_{\Gamma_t^i} (\kappa_s^i)^2 \bigg\} \, ds + 2 \int_{\Gamma_t^i} (\kappa^i)^4 \, ds + 2 \int_{\Gamma_t^i} \kappa^i \kappa_s^i v^i \, ds$$

$$= -2 \bigg\{ \int_{\Gamma_t^i} \left(\kappa_s^i \right)^2 \, ds + h^i (\kappa^i)^2 \big|_{s=r^i} \bigg\} - 2 \kappa^i \kappa_s^i \big|_{s=0}$$

$$+ 2 \int_{\Gamma_t^i} (\kappa^i)^4 \, ds + 2 \int_{\Gamma_t^i} \kappa^i \kappa_s^i v^i \, ds. \tag{6.5}$$

Using the identities $v_s^i = \kappa^i V^i = (\kappa^i)^2$ and integrating by parts we obtain

$$\int_{\Gamma_t^i} \kappa^i \kappa_s^i v^i \, ds = [(\kappa^i)^2 v^i]_{s=0}^{s=r^i} - \int_{\Gamma_t^i} \kappa^i \left(\kappa_s^i v^i + \kappa^i v_s^i \right) ds$$

$$= [(\kappa^i)^2 v^i]_{s=0}^{s=r^i} - \int_{\Gamma_t^i} \kappa^i \kappa_s^i v^i \, ds - \int_{\Gamma_t^i} (\kappa^i)^4 \, ds.$$

Thus we have

$$\int_{\Gamma_t^i} \kappa^i \kappa_s^i v^i \, ds = \frac{1}{2} \bigg\{ [(\kappa^i)^2 v^i]_{s=0}^{s=r^i} - \int_{\Gamma_t^i} (\kappa^i)^4 \, ds \bigg\}. \tag{6.6}$$

It follows from (6.4), (6.5), and (6.6) that

$$\frac{d}{dt} \int_{\Gamma_t^i} (\kappa^i)^2 \, ds = -2 \left\{ \int_{\Gamma_t^i} (\kappa_s^i)^2 \, ds + h^i (\kappa^i)^2 \big|_{s=r^i} \right\} + \int_{\Gamma_t^i} (\kappa^i)^4 \, ds$$

$$- 2\kappa^i \kappa_s^i \big|_{s=0} - (\kappa^i)^2 v^i \big|_{s=0}.$$

By (5.12) and (5.13) we have

$$\sum_{i=1}^3 \gamma^i \left\{ 2\kappa^i \kappa_s^i \big|_{s=0} + (\kappa^i)^2 v^i \big|_{s=0} \right\}$$

$$= 2 \sum_{i=1}^3 \gamma^i \kappa^i \left(\kappa_s^i + \kappa^i v^i \right) \big|_{s=0} - \sum_{i=1}^3 \gamma^i (\kappa^i)^2 v^i \big|_{s=0}$$

$$= - \sum_{i=1}^3 \gamma^i (\kappa^i)^2 v^i \big|_{s=0}.$$

Thus we are led to the identity

$$\frac{d}{dt} \sum_{i=1}^3 \gamma^i \int_{\Gamma_t^i} (\kappa^i)^2 \, ds$$

$$= -2 \sum_{i=1}^3 \gamma^i \left\{ \int_{\Gamma_t^i} (\kappa_s^i)^2 \, ds + h^i (\kappa^i)^2 \big|_{s=r^i} \right\} + \sum_{i=1}^3 \gamma^i \int_{\Gamma_t^i} (\kappa^i)^4 \, ds$$

$$+ \sum_{i=1}^3 \gamma^i (\kappa^i)^2 v^i \big|_{s=0}.$$

Since $V^i = \kappa^i$ the proof of the lemma follows. $\qquad \square$

Let us define a bilinear form I as

$$I[\varphi, \varphi] = \sum_{i=1}^3 \gamma^i \left\{ \int_0^{r^i} (\varphi_s^i)^2 \, ds + h^i (\varphi^i)^2 \big|_{s=r^i} \right\}$$

for $\varphi \in \mathcal{E}(\Gamma)$ where

$$\mathcal{E}(\Gamma) = \left\{ (\varphi^1, \varphi^2, \varphi^3) \in H^1(0, r^1) \times H^1(0, r^2) \times H^1(0, r^3) \, \bigg| \, \sum_{i=1}^3 \gamma^i \varphi^i(0) = 0 \right\}.$$

Since $\boldsymbol{V} \in \mathcal{E}(\Gamma)$ we can rewrite the statement ii) of Lemma 6.1 as

$$\frac{d}{dt}\|\boldsymbol{\kappa}\|_{L^2}^2 + 2I[\boldsymbol{V}, \boldsymbol{V}] = \|\boldsymbol{\kappa}\|_{L^4}^4 + \sum_{i=1}^{3} \gamma^i (\kappa^i)^2 v^i \big|_{s=0}. \tag{6.7}$$

The following lemmata are crucial in the derivation of a priori estimates.

6.2. Higher order estimates for the curvature

We define the averaged curvature along the curve Γ_t^i as $\kappa_{av}^i = \frac{1}{L[\Gamma_t^i]}\int_{\Gamma_t^i}\kappa^i ds$.

Lemma 6.2 *The following estimates for a C^3-curve Γ_t^i hold true.*

(i) $\|\kappa^i - \kappa_{av}^i\|_{L^\infty} \leq (L[\Gamma_t^i])^{\frac{1}{2}}\|\kappa_s^i\|_{L^2}$, $|\kappa_{av}^i| \leq \frac{1}{(L[\Gamma_t^i])^{\frac{1}{2}}}\|\kappa^i\|_{L^2}$,

(ii) $\int_{\Gamma_t^i}(\kappa^i)^4\,ds \leq 2\big\{L[\Gamma_t^i]\|\kappa_s^i\|_{L^2}^2 + \frac{1}{L[\Gamma_t^i]}\|\kappa^i\|_{L^2}^2\big\}\|\kappa^i\|_{L^2}^2$,

(iii) *There are $m_1 = m_1(L[\Gamma_t^i])$ and $m_2 = m_2(1/L[\Gamma_t^i])$ such that*

$$\big|(\kappa^i|_{s=0})^3\big| \leq m_1\|\kappa^i\|_{L^2}\|\kappa_s^i\|_{L^2}^2 + m_2\|\kappa^i\|_{L^2}^3.$$

Proof. The estimates in (i) are established in a standard way and we do not present details here. By the estimates in (i), we have

$$\|\kappa^i\|_{L^\infty} \leq \big\|\kappa^i - \kappa_{av}^i\big\|_{L^\infty} + |\kappa_{av}^i| \leq (L[\Gamma_t^i])^{\frac{1}{2}}\|\kappa_s^i\|_{L^2} + \frac{1}{(L[\Gamma_t^i])^{\frac{1}{2}}}\|\kappa^i\|_{L^2}.$$

It implies that

$$\left|\int_{\Gamma_t^i}(\kappa^i)^4\,ds\right| \leq \|\kappa^i\|_{L^\infty}^2\|\kappa^i\|_{L^2}^2$$

$$\leq \left\{(L[\Gamma_t^i])^{\frac{1}{2}}\|\kappa_s^i\|_{L^2} + \frac{1}{(L[\Gamma_t^i])^{\frac{1}{2}}}\|\kappa^i\|_{L^2}\right\}^2\|\kappa^i\|_{L^2}^2.$$

The statement (ii) now follows from the elementary inequality $(a + b)^2 \leq 2(a^2 + b^2)$.

Let r_0^i be such that $\kappa^i(r_0^i, t) - \kappa_{av}^i(t) = 0$. We then obtain

$$(\kappa^i - \kappa^i_{av})^3 \big|_{s=0} = -3 \int_0^{r_0^i} \kappa^i_s (\kappa^i - \kappa^i_{av})^2 \, ds.$$

This implies that

$$\left| (\kappa^i - \kappa^i_{av})^3 \big|_{s=0} \right| \le 3 \int_{\Gamma_t^i} |\kappa^i_s||\kappa^i - \kappa^i_{av}|^2 \, ds$$

$$\le 3 \|\kappa^i - \kappa^i_{av}\|_{L^\infty} \|\kappa^i_s\|_{L^2} \|\kappa^i - \kappa^i_{av}\|_{L^2}$$

$$\le 3 \left(L[\Gamma_t^i] \right)^{\frac{1}{2}} \|\kappa^i_s\|^2_{L^2} \left\{ \|\kappa^i\|_{L^2} + (L[\Gamma_t^i])^{\frac{1}{2}} |\kappa^i_{av}| \right\}$$

$$\le 6 \left(L[\Gamma_t^i] \right)^{\frac{1}{2}} \|\kappa^i\|_{L^2} \|\kappa^i_s\|^2_{L^2}.$$

Note that $(\kappa^i)^3 = (\kappa^i - \kappa^i_{av})^3 + 3\kappa^i \kappa^i_{av}(\kappa^i - \kappa^i_{av}) + (\kappa^i_{av})^3$. Then it follows that

$$\left\| \kappa^i \kappa^i_{av}(\kappa^i - \kappa^i_{av}) \right\|_{L^\infty}$$

$$\le \left\| \kappa^i_{av}(\kappa^i - \kappa^i_{av})^2 \right\|_{L^\infty} + \left\| (\kappa^i_{av})^2 (\kappa^i - \kappa^i_{av}) \right\|_{L^\infty}$$

$$\le L[\Gamma_t^i]|\kappa^i_{av}|\|\kappa^i_s\|^2_{L^2} + \left(L[\Gamma_t^i] \right)^{\frac{1}{2}} |\kappa^i_{av}|^2 \|\kappa^i_s\|_{L^2}$$

$$\le \left(L[\Gamma_t^i] \right)^{\frac{1}{2}} \|\kappa^i\|_{L^2} \|\kappa^i_s\|^2_{L^2} + \left(L[\Gamma_t^i] \right)^{\frac{1}{2}} |\kappa^i_{av}| \cdot \frac{1}{2} \left(|\kappa^i_{av}|^2 + \|\kappa^i_s\|^2_{L^2} \right)$$

$$\le \left(L[\Gamma_t^i] \right)^{\frac{1}{2}} \|\kappa^i\|_{L^2} \|\kappa^i_s\|^2_{L^2} + \frac{1}{2} \|\kappa^i\|_{L^2} \left(\frac{1}{L[\Gamma_t^i]} \|\kappa^i\|^2_{L^2} + \|\kappa^i_s\|^2_{L^2} \right)$$

$$= \left\{ \left(L[\Gamma_t^i] \right)^{\frac{1}{2}} + \frac{1}{2} \right\} \|\kappa^i\|_{L^2} \|\kappa^i_s\|^2_{L^2} + \frac{1}{2L[\Gamma_t^i]} \|\kappa^i\|^3_{L^2}.$$

Thus we have

$$\left| (\kappa^i|_{s=0})^3 \right| \le \left| (\kappa^i - \kappa^i_{av})^3 \big|_{s=0} \right| + 3 \left\| \kappa^i \kappa^i_{av}(\kappa^i - \kappa^i_{av}) \right\|_{L^\infty} + |\kappa^i_{av}|^3$$

$$\le \left\{ 9 \left(L[\Gamma_t^i] \right)^{\frac{1}{2}} + \frac{3}{2} \right\} \|\kappa^i\|_{L^2} \|\kappa^i_s\|^2_{L^2} + \left\{ \frac{3}{2L[\Gamma_t^i]} + \frac{1}{(L[\Gamma_t^i])^{3/2}} \right\} \|\kappa^i\|^3_{L^2}.$$

Hence the proof is complete. $\qquad\qquad\square$

Lemma 6.3 *For smooth solutions of* (5.10), (5.11) *we have*

$$(v^1, v^2, v^3)^T = Q(V^1, V^2, V^3)^T,$$

for any $t \in (0, T]$, where the matrix Q was defined in Lemma 2.1.

Proof. At the triple junction $p(t)$, we have, for all $i, j \in \{1, 2, 3\}$,

$$\frac{dp}{dt} = v^i T^i + V^i N^i = v^j T^j + V^j N^j.$$

Taking the inner product with T^i we obtain

$$v^i = v^j (T^i, T^j) + V^j (T^i, N^j).$$

By (5.8), $(T^i, T^j) = \cos \theta^k$ and $(T^i, N^j) = -\sin \theta^k$. Thus we derive

$$v^i - c^k v^j = -s^k V^j.$$

If we solve this with respect to $(v^1, v^2, v^3)^T$, we are led to the desired result.
□

By Lemma 6.3 and $V^i = \kappa^i$, we have

$$\left| \sum_{i=1}^{3} \gamma^i (\kappa^i)^2 v^i \big|_{s=0} \right| \le C \left\{ \sum_{i=1}^{3} \gamma^i (\kappa^i)^2 \big|_{s=0} \right\} \left(\sum_{i=1}^{3} |\kappa^i| \big|_{s=0} \right)$$

$$\le \hat{C} \sum_{i=1}^{3} \gamma^i |\kappa^i|^3 \big|_{s=0}. \tag{6.8}$$

6.3. Structural stability of the bilinear form

The aim of this subsection is to show that positivity of the bilinear form I_* is invariant with respect to small perturbations of the curve parametrization ρ^i. More precisely, we will show that the bilinear form I is positive definite provided that I_* is positive definite and $\boldsymbol{\rho} = (\rho^1, \rho^2, \rho^3)$ is small in the C^1-norm. Taking into account Corollary 4.2 and the continuity of embedding $H^2 \hookrightarrow C^1$ the positive definiteness of the bilinear form is preserved if the L^2-norm of the curvature $\boldsymbol{\kappa} = (\kappa^1, \kappa^2, \kappa^3)$ is small.

According to Lemma 3.4, I_* is positive provided the maximal eigenvalue for the linearized problem is negative. The following lemma is a direct consequence of [13, Lemma 3.1 and Prop. 3.3].

Lemma 6.4

(i) *Let λ be the maximal eigenvalue of the linearized problem. For $\varepsilon > 0$ there exists a $\delta > 0$ such that, for any perturbation $|h^i - h^i_*| < \delta$ and $|L[\Gamma^i] - L[\Gamma^i_*]| < \delta$, $i = 1, 2, 3$, we have*

$$I[\boldsymbol{\varphi}, \boldsymbol{\varphi}] > (-\lambda - \varepsilon)\|\boldsymbol{\varphi}\|^2_{L^2} \quad \text{for } \boldsymbol{\varphi} \in \mathcal{E}(\Gamma).$$

(ii) *There exists a $c > 0$ such that*

$$c\,\|\boldsymbol{\varphi}_s\|^2_{L^2} \leq I[\boldsymbol{\varphi}, \boldsymbol{\varphi}] + \|\boldsymbol{\varphi}\|^2_{L^2} \quad \text{for } \boldsymbol{\varphi} \in \mathcal{E}(\Gamma).$$

Using Lemma 6.4, we obtain the existence of constants $\delta_* > 0$ and $c_* > 0$ such that

$$I[\boldsymbol{\varphi}, \boldsymbol{\varphi}] > -\frac{\lambda}{2}\|\boldsymbol{\varphi}\|^2_{L^2} + c_*\|\boldsymbol{\varphi}_s\|^2_{L^2} \quad \text{for } \boldsymbol{\varphi} \in \mathcal{E}(\Gamma_t) \tag{6.9}$$

provided that, for $i = 1, 2, 3$,

$$|h^i - h^i_*| < \delta_*, \quad \left|L[\Gamma^i] - L[\Gamma^i_*]\right| < \delta_*. \tag{6.10}$$

Lemma 6.5   *We have the following estimates.*

(i) *There exist constants δ_2, $C > 0$ such that $|h^i - h^i_*| \leq C\|\rho^i\|_{C^0(\mathcal{I}^i)}$ provided that $\|\rho^i\|_{C^0(\mathcal{I}^i)} < \delta_2$.*

(ii) *There exist constants δ_3, $C > 0$ such that*

$$\left|L[\Gamma^i] - L[\Gamma^i_*]\right| \leq C\|\boldsymbol{\rho}\|_{C^1} \quad \text{and} \quad |p| \leq C\|\boldsymbol{\rho}\|_{C^0}$$

provided that $\|\boldsymbol{\rho}\|_{C^1} < \delta_3$, where p is the triple junction of $\Gamma = \bigcup_{i=1}^3 \Gamma^i$.

Here we have denoted $\|\boldsymbol{\rho}\|_{C^{k+\alpha}} = \sum_{i=1}^3 \|\rho^i\|_{C^{k+\alpha}(\mathcal{I}^i)}$ for $k \in \mathbb{N} \cup \{0\}$, $\alpha \in [0, 1)$.

Proof.   To prove (i), we recall that $\kappa_{\partial\Omega}(X^i)$ is represented by

$$\kappa_{\partial\Omega}(X^i) = -\frac{1}{|\nabla\psi(X^i)|}\big([D^2\psi(X^i)]T_{\partial\Omega}(X^i), T_{\partial\Omega}(X^i)\big)_{\mathbb{R}^2}. \tag{6.11}$$

Since the right hand side does not depend on derivatives of ρ^i, the mean

value theorem implies the second inequality of (i).

In order to prove (ii), we have to analyze properties of the function $\mu^i_{\partial\Omega}$ introduced in Section 2. From the definition it follows that $\mu^i_{\partial\Omega}(0) = l^i$ and $p_* + \mu^i_{\partial\Omega}(q)T^i_* + qN^i_* \in \partial\Omega$. Therefore $\psi(p_* + \mu^i_{\partial\Omega}(q)T^i_* + qN^i_*) = 0$ and so we can compute the derivative of $\mu^i_{\partial\Omega}(q)$ as

$$\frac{d}{dq}\mu^i_{\partial\Omega}(q) = -\frac{\left(\nabla\psi(p_* + \mu^i_{\partial\Omega}(q)T^i_* + qN^i_*), N^i_*\right)_{\mathbb{R}^2}}{\left(\nabla\psi(p_* + \mu^i_{\partial\Omega}(q)T^i_* + qN^i_*), T^i_*\right)_{\mathbb{R}^2}}.$$

Since $(\nabla\psi, N^i_*)_{\mathbb{R}^2} = 0$ on $\partial\Omega$ we obtain $\frac{d}{dq}\mu^i_{\partial\Omega}(0) = 0$. Now, by taking the second derivative of $\mu^i_{\partial\Omega}$ and taking into account the expression for the curvature h^i_* at the intersection of $\partial\Omega$ and Γ^i_* we obtain $\frac{d^2}{dq^2}\mu^i_{\partial\Omega}(0) = h^i_*$. Thus

$$\mu^i_{\partial\Omega}(q) = l^i + \frac{1}{2}h^i_* q^2 + o(q^2) \quad \text{as } q \to 0.$$

We recall that the parameterization Φ^i of the curve Γ^i is given by

$$\Phi^i(\sigma) = p_* + \left[\mu^i + \frac{\sigma}{l^i}\left\{\mu^i_{\partial\Omega}(\rho^i(\sigma)) - \mu^i\right\}\right]T^i_* + \rho^i(\sigma)N^i_*.$$

Using the above property of the function $\mu^i_{\partial\Omega}$ and the fact $\boldsymbol{\mu}^T = Q\boldsymbol{\rho}(0)^T$ we obtain

$$\left|L[\Gamma^i] - L[\Gamma^i_*]\right| = \left|\int_0^{l^i}\left(|\Phi^i_\sigma(\sigma)| - 1\right)d\sigma\right| \le C\|\boldsymbol{\rho}\|_{C^1}.$$

Similarly, as $p = \mu^i T^i_* + \rho^i(0)N^i_*$ we obtain $|p| \le C\|\boldsymbol{\rho}\|_{C^0}$. With this all statements of the lemma have been shown. $\qquad\square$

6.4. Exponential stabilization of the solution

Lemma 6.6 *Let λ be the maximal eigenvalue of the linearized problem. Assume that λ is negative. Then there exists a $\delta_3 > 0$ such that*

$$\|\boldsymbol{\kappa}(t)\|^2_{L^2} \le e^{\lambda t/2}\|\boldsymbol{\kappa}(0)\|^2_{L^2} \quad \text{for } t \in [0, T],$$

$$\int_0^T \|\boldsymbol{\kappa}_s(\tau)\|^2_{L^2}\,d\tau \le \frac{1}{c_*}\|\boldsymbol{\kappa}(0)\|^2_{L^2}$$

provided that $\|\boldsymbol{\kappa}(t)\|_{L^2}^2 < \delta_4$ *holds on* $[0, T]$ *where* $c_* > 0$ *is a constant as in* (6.9).

Proof. According to Corollary 4.2, there exists a $C > 0$ such that $\|\boldsymbol{\rho}\|_{H^2} \leq C\|\boldsymbol{\kappa}\|_{L^2}$ for $\|\boldsymbol{\kappa}\|_{L^2}^2 < \delta_1$. By (6.7) we have

$$\frac{d}{dt}\|\boldsymbol{\kappa}\|_{L^2}^2 + 2I[\boldsymbol{V}, \boldsymbol{V}] = \|\boldsymbol{\kappa}\|_{L^4}^4 + \sum_{i=1}^{3} \gamma^i (\kappa^i)^2 v^i \big|_{s=0}.$$

Let us first choose $\delta_2 \in (0, \delta_*) \cap (0, \delta_1)$. Then, it follows from Lemmata 6.2 and 6.5 and the inequalities (6.8), (6.9), and (6.10) that there are $C > 0$ such that

$$\frac{d}{dt}\|\boldsymbol{\kappa}\|_{L^2}^2 + (-\lambda)\|\boldsymbol{V}\|_{L^2}^2 + 2c_*\|\boldsymbol{V}_s\|_{L^2}^2 \leq C\big(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{\kappa}\|_{L^2}^2\big)\big(\|\boldsymbol{\kappa}\|_{L^2}^2 + \|\boldsymbol{\kappa}_s\|_{L^2}^2\big).$$

Since $V^i = \kappa^i$, we are led to

$$\frac{d}{dt}\|\boldsymbol{\kappa}\|_{L^2}^2 + \big\{(-\lambda) - C\big(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{\kappa}\|_{L^2}^2\big)\big\}\|\boldsymbol{\kappa}\|_{L^2}^2$$

$$+ \big\{2c_* - C\big(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{\kappa}\|_{L^2}^2\big)\big\}\|\boldsymbol{\kappa}_s\|_{L^2}^2 \leq 0. \qquad (6.12)$$

Then, we choose a constant $\delta_4 > 0$ satisfying

$$0 < \delta_4 < \min\left\{1, \frac{-\lambda}{4C}, \left(\frac{-\lambda}{4C}\right)^2, \frac{c_*}{2C}, \left(\frac{c_*}{2C}\right)^2\right\}.$$

If we assume $\|\boldsymbol{\kappa}(t)\|_{L^2}^2 < \delta_4$ for $t \in [0, T]$ we then have

$$\frac{d}{dt}\|\boldsymbol{\kappa}(t)\|_{L^2}^2 + \frac{(-\lambda)}{2}\|\boldsymbol{\kappa}(t)\|_{L^2}^2 + c_*\|\boldsymbol{\kappa}_s(t)\|_{L^2}^2 \leq 0. \qquad (6.13)$$

Using the Gronwall inequality we obtain the desired result. $\qquad \square$

6.5. Higher order energy inequalities

So far we have shown the exponential decay of the L^2-norm of the curvature κ^i. In order to prove stabilization of the curvature in the stronger $C^{1+\alpha}$-norm we need to derive higher order energy type inequalities. These estimates will enable us to conclude convergence of the curvature to zero in

the $C^{1+\alpha}$-norm. In order to derive higher order estimates we differentiate the curvature equation (see Lemma 5.2) with respect to t and derive an energy estimate for κ_t^i. To this end, let us denote

$$w^i = \kappa_t^i.$$

Then differentiating the curvature equation $\kappa_t^i = \kappa_{ss}^i + (\kappa^i)^3 + v^i \kappa_s^i$ with respect to t and taking into account the commutation relation $\partial_t \partial_s = \partial_s \partial_t$ we obtain

$$w_t^i = w_{ss}^i + 3(\kappa^i)^2 w^i + v^i w_s^i + v_t^i \kappa_s^i$$

for $i = 1, 2, 3$. Multiplying the above equation with w^i and integrating over Γ_t^i yields

$$
\frac{1}{2} \frac{d}{dt} \int_{\Gamma_t^i} (w^i)^2 \, ds
$$

$$
= \int_{\Gamma_t^i} w^i w_{ss}^i \, ds + 3 \int_{\Gamma_t^i} (\kappa^i)^2 (w^i)^2 \, ds + \int_{\Gamma_t^i} \left(v^i w^i w_s^i + v_t^i w^i \kappa_s^i \right) ds
$$

$$
= \left[w^i w_s^i \right]_{s=0}^{s=r^i} - \int_{\Gamma_t^i} (w_s^i)^2 \, ds + 3 \int_{\Gamma_t^i} (\kappa^i)^2 (w^i)^2 \, ds
$$

$$
+ \int_{\Gamma_t^i} \left(v^i w^i w_s^i + v_t^i w^i \kappa_s^i \right) ds. \tag{6.14}
$$

In what follows, we analyze the boundary term $[w^i w_s^i]_{s=0}^{s=r^i}$ appearing in the right hand side of (6.14). First we analyze the boundary term at the triple junction position $s = 0$. Differentiating (5.12) with respect to t, we obtain

$$
\sum_{i=1}^{3} \gamma^i w^i = 0 \tag{6.15}
$$

at the triple junction point $p(t)$. It follows from (5.13) that there exists a function $G(t)$ such that

$$
\kappa_s^i(0, t) + \kappa^i(0, t) v^i(0, t) = G(t) \quad \text{for } t \geq 0 \text{ and } i = 1, 2, 3.
$$

Differentiating this equation with respect to t, we conclude

$$w_s^i + w^i v^i + \kappa^i v_t^i = G'(t).$$

Therefore we obtain, by using (6.15),

$$\sum_{i=1}^{3} \gamma^i w^i w_s^i = G'(t) \sum_{i=1}^{3} \gamma^i w^i - \sum_{i=1}^{3} \gamma^i \{(w^i)^2 v^i + \kappa^i w^i v_t^i\}$$

$$= -\sum_{i=1}^{3} \gamma^i \{(w^i)^2 v^i + \kappa^i w^i v_t^i\}. \tag{6.16}$$

By Lemma 6.3, we can express the term v^i as a time independent linear combination of curvatures κ^i ($i = 1, 2, 3$) evaluated at the triple junction $p(t)$ and so v_t^i can be expressed as a time independent linear combination of $\kappa_t^i = w^i$ ($i = 1, 2, 3$). Therefore there exists a constant $C > 0$ such that

$$\left| \sum_{i=1}^{3} \gamma^i w^i w_s^i \big|_{s=0} \right| \leq C \|\boldsymbol{w}\|_{L^\infty}^2 \|\boldsymbol{\kappa}\|_{L^\infty}. \tag{6.17}$$

Next we proceed with the estimation of the boundary term at the point $X^i \in \Gamma^i \cap \partial\Omega$, i.e. we consider $s = r^i(t)$. Notice that r^i is no longer constant and its dependence on time t has to be taken into account. We will differentiate the boundary condition (5.14)

$$\kappa_s^i(r^i(t), t) + h^i(X^i(r^i(t), t))\kappa^i(r^i(t), t) = 0$$

with respect to t. Since $\frac{d}{dt}X^i(r^i(t), t) = X_t^i + X_s^i(r^i)' = \kappa^i N^i + \{v^i + (r^i)'\}T^i = \kappa^i N^i$ (see Lemma 5.3) and $\kappa_{ss}^i = \kappa_t^i - (\kappa^i)^3 - v^i \kappa_s^i = w^i - (\kappa^i)^3 - v^i \kappa_s^i$ we obtain

$$w_s^i + h^i w^i = -\left(\kappa_{ss}^i + h^i \kappa_s^i\right)(r^i)' - \left(\nabla h^i, \frac{d}{dt}X^i\right)_{\mathbb{R}^2} \kappa^i$$

$$= \{w^i - (\kappa^i)^3 + (h^i - v^i)\kappa_s^i\}v^i - (\kappa^i)^2(\nabla h^i, N^i)_{\mathbb{R}^2}$$

$$= \{w^i - (\kappa^i)^3 - (h^i - v^i)h^i \kappa^i\}v^i - (\kappa^i)^2(\nabla h^i, N^i)_{\mathbb{R}^2}. \tag{6.18}$$

Here we have used the equations $\kappa^i_s + h^i \kappa^i = 0$ and $v^i + (r^i)' = 0$ and expressed h_i by the right hand side of (6.11). We now denote by Ξ^i the right hand side of (6.18). Then we get

$$\sum_{i=1}^{3} \gamma^i w^i w^i_s \big|_{s=r^i} = -\sum_{i=1}^{3} \gamma^i h^i (w^i)^2 \big|_{s=r^i} + \sum_{i=1}^{3} \gamma^i w^i \Xi^i \big|_{s=r^i}. \qquad (6.19)$$

As the outer boundary $\partial \Omega$ is assumed to be C^3 smooth we obtain that the terms h^i, ∇h^i are uniformly bounded. Hence the remainder term $\sum_{i=1}^{3} \gamma^i w^i \Xi^i$ can be estimated as

$$\left| \sum_{i=1}^{3} \gamma^i w^i \Xi^i \big|_{s=r^i} \right| \leq C \|\boldsymbol{w}\|_{L^\infty} \big(\|\boldsymbol{\kappa}\|^2_{L^\infty} + \|\boldsymbol{w}\|_{L^\infty} \|\boldsymbol{v}\|_{L^\infty} + \|\boldsymbol{\kappa}\|^3_{L^\infty} \|\boldsymbol{v}\|_{L^\infty}$$
$$+ \|\boldsymbol{v}\|^2_{L^\infty} \|\boldsymbol{\kappa}\|_{L^\infty} + \|\boldsymbol{v}\|_{L^\infty} \|\boldsymbol{\kappa}\|_{L^\infty} \big). \quad (6.20)$$

In order to complete our estimates we have to derive L^∞- estimates on the tangential velocity v^i and its time derivative v^i_t. Since $v^i_s = (\kappa^i)^2$ we have

$$v^i(s,t) = v^i(0,t) + \int_0^s |\kappa^i(\zeta,t)|^2 \, d\zeta. \qquad (6.21)$$

By Lemma 6.3, we can express $v^i(0,t)$ as a time independent linear combination of κ^i ($i = 1, 2, 3$) evaluated at the triple junction $p(t)$. Therefore there exists a constant $C > 0$ such that

$$\|\boldsymbol{v}\|_{L^\infty} \leq C \big(\|\boldsymbol{\kappa}\|_{L^\infty} + \|\boldsymbol{\kappa}\|^2_{L^2} \big) \leq C \|\boldsymbol{\kappa}\|_{L^\infty} \qquad (6.22)$$

for $\|\boldsymbol{\kappa}\|_{L^2} \leq 1$. Analogously, as $v^i_t(s,t) = v^i_t(0,t) + 2 \int_0^s \kappa^i(\zeta,t) \kappa^i_t(\zeta,t) \, d\zeta$ and $v^i_t(0,t)$ is a time independent linear combination of $\kappa^i_t = w^i$ ($i = 1, 2, 3$) evaluated at the triple junction position $p(t)$, we conclude

$$\|\boldsymbol{v}_t\|_{L^\infty} \leq C(\|\boldsymbol{w}\|_{L^\infty} + \|\boldsymbol{\kappa}\|_{L^2} \|\boldsymbol{w}\|_{L^2}) \leq C \|\boldsymbol{w}\|_{L^\infty} \qquad (6.23)$$

for $\|\boldsymbol{\kappa}\|_{L^2} \leq 1$.

Summarizing we have shown the following equality

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}\gamma^i \int_{\Gamma_t^i}(w^i)^2\,ds + I[\boldsymbol{w},\boldsymbol{w}]$$

$$= \sum_{i=1}^{3}\gamma^i\left\{w^i\Xi^i\big|_{s=r^i} - w^i w^i_s\big|_{s=0} + 3\int_{\Gamma_t^i}(\kappa^i)^2(w^i)^2\,ds\right.$$

$$\left. + \int_{\Gamma_t^i}\left(v^i w^i w^i_s + v^i_t w^i \kappa^i_s\right)ds\right\}$$

and, consequently, the estimate

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_{L^2}^2 + I[\boldsymbol{w},\boldsymbol{w}]$$

$$\leq C\Big\{\big(\|\boldsymbol{\kappa}\|_{L^\infty}^2 + \|\boldsymbol{\kappa}\|_{L^\infty}^3 + \|\boldsymbol{\kappa}\|_{L^\infty}^4\big)\|\boldsymbol{w}\|_{L^\infty} + \|\boldsymbol{\kappa}\|_{L^\infty}\|\boldsymbol{w}\|_{L^\infty}^2 + \|\boldsymbol{\kappa}\|_{L^\infty}^2\|\boldsymbol{w}\|_{L^2}^2$$

$$+ \|\boldsymbol{\kappa}\|_{L^\infty}\|\boldsymbol{w}\|_{L^2}\|\boldsymbol{w}_s\|_{L^2} + \|\boldsymbol{w}\|_{L^\infty}\|\boldsymbol{w}\|_{L^2}\|\boldsymbol{\kappa}_s\|_{L^2}\Big\}. \tag{6.24}$$

The application of the above inequality will be twofold. At first, we utilize it in order to prove a bound on $\|\boldsymbol{\kappa}_{ss}(t)\|_{L^2}$ uniformly for $t \in [0, T)$ where $T > 0$ is the maximal time of existence of a $C^{2+\alpha}$ solution $\boldsymbol{\rho}$. This implies together with Theorem 2.3 the possibility of global continuation of the $C^{2+\alpha}$ solution $\boldsymbol{\rho}$ up to the maximal time of existence $T = +\infty$ and hence the global existence of a $C^{2+\alpha}$ solution will follow. As a second application of the above inequality we will prove exponential stabilization of a solution in the H^2-norm of the curvature yielding the exponential stabilization $\rho(t)$ in its phase-space $C^{2+\alpha}$-norm.

To accomplish this goal, we have to establish bounds for $\|\boldsymbol{\kappa}\|_{L^\infty}$ in terms of the norms $\|\boldsymbol{w}\|_{L^2}$ and $\|\boldsymbol{\kappa}\|_{L^2}$. This can be done by taking into account the equation $w^i \equiv \kappa^i_t = \kappa^i_{ss} + (\kappa^i)^3 + v^i\kappa^i_s$. From this equation we have, for $i = 1, 2, 3$,

$$\big\|\kappa^i_{ss}\big\|_{L^2} \leq C\big(\|w^i\|_{L^2} + \|(\kappa^i)^3\|_{L^2} + \|v^i\kappa^i_s\|_{L^2}\big)$$

$$\leq C\big(\|w^i\|_{L^2} + \|\kappa^i\|_{L^6}^3 + \|\kappa^i\|_{L^\infty}\|\kappa^i_s\|_{L^2}\big). \tag{6.25}$$

Let us denote by $\|\cdot\|_{H^k}$ the following Sobolev norm of the Sobolev space $H^k = W^{k,2}$

$$\|\boldsymbol{\varphi}\|_{H^k} = \|\boldsymbol{\varphi}\|_{L^2} + \left\|\partial_s^k \boldsymbol{\varphi}\right\|_{L^2}.$$

Due to the continuity of embeddings $H^2 \hookrightarrow H^1$ and $H^2 \hookrightarrow L^\infty$ and using Gagliardo-Nirenberg interpolation inequalities (cf. [1, Lemma 5.18 and Theorem 4.17]), we infer the existence of a constant $C_0 > 0$ such that

$$\begin{cases} \|\boldsymbol{\kappa}\|_{L^\infty} \le C_0 \|\boldsymbol{\kappa}\|_{H^2}^{\frac{1}{4}} \|\boldsymbol{\kappa}\|_{L^2}^{\frac{3}{4}}, & \|\boldsymbol{\kappa}_s\|_{L^2} \le C_0 \|\boldsymbol{\kappa}\|_{H^2}^{\frac{1}{2}} \|\boldsymbol{\kappa}\|_{L^2}^{\frac{1}{2}}, \\ \|\boldsymbol{w}\|_{L^\infty} \le C_0 \|\boldsymbol{w}\|_{H^1}^{\frac{1}{2}} \|\boldsymbol{w}\|_{L^2}^{\frac{1}{2}}. \end{cases} \tag{6.26}$$

By the Young inequality $ab \le a^p/p + b^q/q$ with $p = 4/3$, $q = 4$, we have, for any $\varepsilon > 0$,

$$\|\boldsymbol{\kappa}\|_{L^\infty} \|\boldsymbol{\kappa}_s\|_{L^2} \le C_0^2 \|\boldsymbol{\kappa}\|_{H^2}^{\frac{3}{4}} \|\boldsymbol{\kappa}\|_{L^2}^{\frac{5}{4}} \le \varepsilon \|\boldsymbol{\kappa}\|_{H^2} + C_\varepsilon \|\boldsymbol{\kappa}\|_{L^2}^5$$

and, analogously, $\|\boldsymbol{\kappa}\|_{L^6}^3 \le C \|\boldsymbol{\kappa}\|_{L^\infty}^3 \le C_0 C \|\boldsymbol{\kappa}\|_{H^2}^{\frac{3}{4}} \|\boldsymbol{\kappa}\|_{L^2}^{\frac{9}{4}} \le \varepsilon \|\boldsymbol{\kappa}\|_{H^2} + C_\varepsilon \|\boldsymbol{\kappa}\|_{L^2}^9$. By taking $0 < \varepsilon \ll 1$ small enough we obtain from (6.25)

$$\|\boldsymbol{\kappa}_{ss}\|_{L^2} \le C(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{w}\|_{L^2}) \quad \text{for } \|\boldsymbol{\kappa}\|_{L^2} \le 1.$$

Consequently,

$$\|\boldsymbol{\kappa}_s\|_{L^2} \le C_0 \|\boldsymbol{\kappa}\|_{H^2}^{\frac{1}{2}} \|\boldsymbol{\kappa}\|_{L^2}^{\frac{1}{2}} \le \frac{C_0}{2} (\|\boldsymbol{\kappa}\|_{H^2} + \|\boldsymbol{\kappa}\|_{L^2}) \le C(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{w}\|_{L^2})$$

for $\|\boldsymbol{\kappa}\|_{L^2} \le 1$. Similarly

$$\|\boldsymbol{\kappa}\|_{L^\infty} \le C(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{w}\|_{L^2})$$

for $\|\boldsymbol{\kappa}\|_{L^2} \le 1$ where $C > 0$ is a generic positive constant. Due to the continuity of embedding $H^1 \hookrightarrow L^\infty$ we have $\|\boldsymbol{w}\|_{L^\infty} \le C \|\boldsymbol{w}\|_{H^1}$.

We proceed by estimating the right hand side of (6.24). From (6.26) we have

$$\|\boldsymbol{\kappa}\|_{L^\infty}^4 \le C_0 \|\boldsymbol{\kappa}\|_{H^2} \le C(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{w}\|_{L^2}) \le C(1 + \|\boldsymbol{w}\|_{L^2})$$

for $\|\boldsymbol{\kappa}\|_{L^2} \le 1$. Consequently, by using the Young inequality, we obtain

$$\|\boldsymbol{\kappa}\|_{L^\infty}^2 + \|\boldsymbol{\kappa}\|_{L^\infty}^3 + \|\boldsymbol{\kappa}\|_{L^\infty}^4 \le C(1 + \|\boldsymbol{w}\|_{L^2})$$

for $\|\boldsymbol{\kappa}\|_{L^2} \leq 1$. From the embedding $\|\boldsymbol{w}\|_{L^\infty} \leq C\|\boldsymbol{w}\|_{H^1}$ and Young's inequality it follows that

$$\big(\|\boldsymbol{\kappa}\|_{L^\infty}^2 + \|\boldsymbol{\kappa}\|_{L^\infty}^3 + \|\boldsymbol{\kappa}\|_{L^\infty}^4\big)\|\boldsymbol{w}\|_{L^\infty} \leq \varepsilon\|\boldsymbol{w}\|_{H^1}^2 + C_\varepsilon\big(1 + \|\boldsymbol{w}\|_{L^2}^2\big)$$

for $\|\boldsymbol{\kappa}\|_{L^2} \leq 1$. Using the Gagliardo-Nirenberg inequality (6.26) and Young's inequality, we can estimate the second summand in (6.24) as

$$\|\boldsymbol{\kappa}\|_{L^\infty}\|\boldsymbol{w}\|_{L^\infty}^2 \leq C\|\boldsymbol{\kappa}\|_{L^\infty}\|\boldsymbol{w}\|_{H^1}\|\boldsymbol{w}\|_{L^2} \leq \varepsilon\|\boldsymbol{w}\|_{H^1}^2 + C_\varepsilon\|\boldsymbol{\kappa}\|_{L^\infty}^2\|\boldsymbol{w}\|_{L^2}^2.$$

Then, by means of $\|\boldsymbol{\kappa}\|_{L^\infty}^2 \leq \|\boldsymbol{\kappa}\|_{L^2}^2 + \|\boldsymbol{\kappa}_s\|_{L^2}^2 \leq 1 + \|\boldsymbol{\kappa}_s\|_{L^2}^2$ for $\|\boldsymbol{\kappa}\|_{L^2} \leq 1$, we have

$$\|\boldsymbol{\kappa}\|_{L^\infty}^2\|\boldsymbol{w}\|_{L^2}^2 \leq \big(1 + \|\boldsymbol{\kappa}_s\|_{L^2}^2\big)\|\boldsymbol{w}\|_{L^2}^2,$$

and so

$$\|\boldsymbol{\kappa}\|_{L^\infty}\|\boldsymbol{w}\|_{L^\infty}^2 \leq \varepsilon\|\boldsymbol{w}\|_{H^1}^2 + C_\varepsilon\big(1 + \|\boldsymbol{\kappa}_s\|_{L^2}^2\big)\|\boldsymbol{w}\|_{L^2}^2.$$

The remaining terms in (6.24) can be easily estimated with help of Young's inequality as

$$\|\boldsymbol{\kappa}\|_{L^\infty}\|\boldsymbol{w}\|_{L^2}\|\boldsymbol{w}_s\|_{L^2} \leq \varepsilon\|\boldsymbol{w}\|_{H^1}^2 + C_\varepsilon\big(1 + \|\boldsymbol{\kappa}_s\|_{L^2}^2\big)\|\boldsymbol{w}\|_{L^2}^2,$$

$$\|\boldsymbol{w}\|_{L^\infty}\|\boldsymbol{w}\|_{L^2}\|\boldsymbol{\kappa}_s\|_{L^2} \leq \varepsilon\|\boldsymbol{w}\|_{H^1}^2 + C_\varepsilon\|\boldsymbol{\kappa}_s\|_{L^2}^2\|\boldsymbol{w}\|_{L^2}^2$$

for $\|\boldsymbol{\kappa}\|_{L^2} \leq 1$. Let us introduce $\eta(t) := 1 + \|\boldsymbol{\kappa}_s(t)\|_{L^2}^2$. Then, by choosing $\varepsilon > 0$ sufficiently small and taking into account the positivity of the bilinear form I, we obtain

$$I[\boldsymbol{w}, \boldsymbol{w}] \geq \delta\|\boldsymbol{w}\|_{H^1}^2.$$

Therefore the function $\|\boldsymbol{w}(t)\|_{L^2}^2$ satisfies the differential inequality

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}(t)\|_{L^2}^2 \leq C_1 + C_2\eta(t)\|\boldsymbol{w}(t)\|_{L^2}^2. \tag{6.27}$$

According to Lemma 6.6, the function η is integrable on the interval $(0, T)$ and

$$\int_0^T \eta(t)\, dt \le T + \int_0^{+\infty} \|\boldsymbol{\kappa}_s(t)\|_{L^2}^2\, dt < \infty$$

provided that $T < \infty$. A Gronwall lemma type of argument applied to the differential inequality (6.27) yields the existence of a C_T, which is monotone increasing and bounded as long as T is bounded, such that

$$\sup_{0 \le t < T} \|\boldsymbol{w}(t)\|_{L^2}^2 \le C_T < +\infty.$$

By means of Lemma 6.6, we see that $\|\boldsymbol{\kappa}\|_{L^2}$ is small if $\|\boldsymbol{\rho}(0)\|_{C^{2+\alpha}}$ is small enough. Furthermore, $\|\boldsymbol{\rho}\|_{C^{1+\alpha}}$ ($0 < \alpha < 1/2$) is small provided when $\|\boldsymbol{\kappa}\|_{L^2}$ is small. In addition, using $\|\boldsymbol{\kappa}_{ss}\|_{L^2} \le C(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{w}\|_{L^2})$ and the fact that the norm $\|\boldsymbol{\rho}\|_{C^{2+\alpha}}$ can be estimated by $\|\boldsymbol{\kappa}\|_{H^2}$, we just have shown the following conclusion.

Theorem 6.7   *The local solution of Theorem 2.3 can be extended to the time interval $[0, \infty)$ provided that ρ_0^i is small enough in the $C^{2+\alpha}$-norm.*

7.   Exponential stability of stationary solutions

In this section we combine all the previous results to prove exponential stabilization of a solution to the triple junction problem which have initial data close to a stationary stable solution.

Theorem 7.1   *Let the assumptions of Theorem 2.3 and Theorem 6.7 hold and let Γ_* be such that the bilinear form I_* is positive. Then there exist constants C, ω, $\delta > 0$ such that*

$$\|\boldsymbol{\rho}(t)\|_{H^2} \le C e^{-\omega t} \|\boldsymbol{\kappa}(0)\|_{L^2}$$

for any $t \ge 0$ and $\|\boldsymbol{\kappa}(0)\|_{L^2} < \delta$.

Proof.   The proof directly follows from Lemma 6.6 and Corollary 4.2.   □

Since the H^2-norm of ρ dominates its $C^{1+\alpha}$-norm and the $C^{2+\alpha}$-norm majorizes L^2-norm of κ we can state the following consequence of the previous theorem.

Corollary 7.2   *Under the assumptions of Theorem 7.1 there exist constants $C, \omega, \delta > 0$ such that*

$$\|\boldsymbol{\rho}(t)\|_{C^{1+\alpha}} \le Ce^{-\omega t}\|\boldsymbol{\rho}(0)\|_{C^{2+\alpha}}$$

for any $t \ge 0$ and $\|\boldsymbol{\rho}(0)\|_{C^{2+\alpha}} < \delta$.

Finally, we are able to prove exponential decay in stronger norms. As it was already indicated in the previous section, we will utilize the higher energy estimate (6.24) once more in order to prove exponential stabilization in the H^2-norm of the curvature κ.

Recall that, for $p \ge 2$, we have

$$\begin{aligned}
\|\boldsymbol{\kappa}\|_\infty^p \|\boldsymbol{w}\|_{L^\infty} &\le C^p(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{w}\|_{L^2})^p\|\boldsymbol{w}\|_{L^\infty} \\
&\le C^p(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{w}\|_{L^2})^{p-1}(\|\boldsymbol{\kappa}\|_{L^2}\|\boldsymbol{w}\|_{L^\infty} + \|\boldsymbol{w}\|_{L^2}\|\boldsymbol{w}\|_{L^\infty}) \\
&\le C(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{w}\|_{L^2})(\|\boldsymbol{\kappa}\|_{L^2}^2\|\boldsymbol{w}\|_{H^1}^2) \\
&\le C\|\boldsymbol{\kappa}\|_{L^2}^2 + C(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{w}\|_{L^2})\|\boldsymbol{w}\|_{H^1}^2 \quad\quad (7.1)
\end{aligned}$$

provided that $\|\boldsymbol{\kappa}\|_{L^2} \le 1$ and $\|\boldsymbol{w}\|_{L^2} \le 1$. Since $\|\boldsymbol{w}\|_{L^2}\|\boldsymbol{w}_s\|_{L^2} \le \|\boldsymbol{w}\|_{H^1}^2$ and $\|\boldsymbol{w}\|_{L^\infty}\|\boldsymbol{w}\|_{L^2} \le C\|\boldsymbol{w}\|_{H^1}^2$ we conclude from (6.24), (6.25), (7.1)

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_{L^2}^2 + I[\boldsymbol{w}, \boldsymbol{w}] \le C\|\boldsymbol{\kappa}\|_{L^2}^2 + C(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{w}\|_{L^2})\|\boldsymbol{w}\|_{H^1}^2$$

for some positive constant provided that $\|\boldsymbol{\kappa}\|_{L^2} \le 1$ and $\|\boldsymbol{w}\|_{L^2} \le 1$.

Similarly as in the proof of exponential decay of $\|\boldsymbol{\kappa}\|_{L^2}$ we use the fact that the full Sobolev norm $\|\boldsymbol{w}\|_{H^1}$ can be estimated by the bilinear form $I(w, w)$ as follows:

$$\delta\|\boldsymbol{w}\|_{H^1}^2 \le I[\boldsymbol{w}, \boldsymbol{w}]$$

for some positive constant $\delta > 0$. Taking $\|\boldsymbol{\kappa}\|_{L^2}$ and $\|\boldsymbol{w}\|_{L^2}$ sufficiently small such that $C(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{w}\|_{L^2}) \le \delta/2$ we end up with the inequality

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_{L^2}^2 + \frac{\delta}{2}\|\boldsymbol{w}\|_{H^1}^2 \le C\|\boldsymbol{\kappa}\|_{L^2}^2.$$

Defining $y = \|\boldsymbol{w}\|_{L^2}^2$ and using $\|\boldsymbol{w}\|_{L^2} \le \|\boldsymbol{w}\|_{H^1}$ we have

$$\frac{dy}{dt} + \delta y \le 2CMe^{-\omega t}$$

where M, $\omega > 0$ are the modulus and rate of exponential decay of $\|\boldsymbol{\kappa}\|_{L^2}^2$, i.e. $\|\boldsymbol{\kappa}(.,t)\|_{L^2}^2 \leq Me^{-\omega t}$. Solving the above differential inequality with respect to $y = y(t)$ we end up with the following estimate

$$y(t) \leq y(0)e^{-\delta t} + \frac{2CM}{|\omega - \delta|}|e^{-\omega t} - e^{-\delta t}|.$$

It means that the norm $\|\boldsymbol{w}\|_{L^2}^2$ exponentially decays with the rate $\min(\delta, \omega)$. Since $\|\boldsymbol{\kappa}_{ss}\|_{L^2} \leq C(\|\boldsymbol{\kappa}\|_{L^2} + \|\boldsymbol{w}\|_{L^2})$ and the full Sobolev norm $\|\boldsymbol{\kappa}\|_{H^1}$ dominates $\|\boldsymbol{\kappa}\|_{L^\infty}$ as well as the $\|\boldsymbol{\kappa}_s\|_{L^2}$-norm we obtain the following convergence result:

Theorem 7.3 *There exist constants δ, M, $\omega > 0$ such that the solution of Theorem* 6.7 *fulfills*

$$\|\boldsymbol{\kappa}(t)\|_{H^2}^2 \leq Me^{-\omega t}$$

for all $t \geq 0$ provided that $\|\boldsymbol{\kappa}(0)\|_{H^2}^2 \leq \delta$.

References

[1] Adams R. A., *Sobolev Spaces*. Academic Press, New York, San Francisco, London 1975.

[2] Amann H., *Existence and regularity for semilinear parabolic evolution equations*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **11**(4) (1984), 593–676.

[3] Angenent S., *Parabolic equations for curves on surfaces. I. Curves with p-integrable curvature*. Ann. of Math. **132**(3) (1990), 451–483.

[4] Bronsard L. and Reitich F., *On three-phase boundary motion and the singular limit of a vector valued Ginzburg–Landau equation*. Arch. Rat. Mech. **124** (1993), 355–379.

[5] Elliott C. M. and Garcke H., *Existence results for diffusive surface motion laws*. Adv. Math. Sci. Appl. **7**(1) (1997), 465–488.

[6] Gage M. and Hamilton R. S., *The heat equation shrinking convex plane curves*. J. Differential Geom. **23**(1) (1986), 69–95.

[7] Garcke H., Ito K. and Kohsaka Y., *Linearized stability analysis of stationary solutions for surface diffusion with boundary conditions*. SIAM J. Math. Anal. **36**(4) (2005), 1031–1056.

[8] Garcke H., Ito K. and Kohsaka Y., *Nonlinear stability of stationary solutions for surface diffusion with boundary conditions*. SIAM J. Math. Anal. **40**(2) (2008), 491–515.

[9] Garcke H., Ito K. and Kohsaka Y., *Surface diffusion with triple junctions: A stability criterion for stationary solutions*, (in preparation).

[10] Giga Y., *Surface evolution equations. A level set approach*. Monographs in Mathematics 99, Birkhäuser Verlag, Basel. 2006.

[11] Grayson M. A., *The heat equation shrinks embedded plane curves to round points*. J. Differential Geom. **26**(2) (1987), 285–314.

[12] Gurtin M. E., *Thermomechanics of Evolving Phase Boundaries in the Plane*. Oxford University Press, 1993.

[13] Ikota R. and Yanagida E., *A stability criterion for stationary curves to the curvature-driven motion with a triple junction*. Differential Integral Equations **16** (2003), 707–726.

[14] Ikota R. and Yanagida E., *Stability of stationary interfaces of binary-tree type*. Calc. Var. Partial Differ. Equ. **22**(4) (2005), 375–389.

[15] Ito K. and Kohsaka Y., *Three phase boundary motion by surface diffusion: Stability of a mirror symmetric stationary solution*. Interfaces Free Bound. **3** (2001), 45–80.

[16] Kinderlehrer D. and Liu Ch., *Evolution of grain boundaries*. Math. Models Methods Appl. Sci., **11**(4) (2001), 713–729.

[17] Lunardi A., *Analytic semigroups and optimal regularity in parabolic problems*. Birkhäuser Verlag, Basel, 1995.

[18] Mantegazza C., Novaga M. and Tortorelli V. M., *Motion by curvature of planar networks*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3**(2) (2004), 235–324.

[19] Mikula K. and Ševčovič D., *Evolution of plane curves driven by a nonlinear function of curvature and anisotropy*. SIAM J. Appl. Math. **61** (2001), 1473–1501.

[20] Schnürer O. and Schulze F., *Self-similarly expanding networks to curve shortening flow*. Ann. Scula Norm. Sup. Pisa Cl. Sci. (5) **6**(4) (2007), 511–528.

[21] Sternberg P. and Ziemer W. P., *Local minimisers of a three-phase partition problem with triple junctions*. Proc. Roy. Soc. Edinburgh Sect. A **124**(6) (1994), 1059–1073.

[22] Terreni B., *Hölder regularity results for nonhomogeneous parabolic initial-*

boundary value linear problems. Lecture Notes in Pure and Appl. Math., 116, Semigroup theory and applications (Trieste, 1987), 387–401, Dekker, New York.

[23] Young T., *An essay on the cohesion of fluids.* Phil. Trans. Roy. Soc. London **95** (1805), 65–87.

[24] Zeidler E., *Nonlinear Functional Analysis and its Applications I.* Springer Verlag 1986.

H. Garcke
NWF-I Mathematik
Universität Regensburg
93040 Regensburg, Germany

Y. Kohsaka
Muroran Institute of Technology
Muroran 050-8585, Japan
E-mail: kohsaka@mmm.muroran-it.ac.jp

D. Ševčovič
Department of Applied Mathematics and Statistics
Faculty of Mathematics, Physics, and Informatics
Comenius University
842 48 Bratislava, Slovak Republic