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Mean iterations derived from transformation formulas for the hypergeometric function

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Abstract. From Goursat's transformation formulas for the hypergeometric function $F(\alpha, \beta, \gamma; z)$, we derive several double sequences given by mean iterations and express their common limits by the hypergeometric function. Our results are analogies of the fact that the arithmetic-geometric mean of 1 and $x \in (0, 1)$ can be expressed as the reciprocal of $F\left(\frac{1}{2}, \frac{1}{2}, 1; 1 - x^2\right)$.

Key words: hypergeometric function, mean iteration.

1. Introduction

Let m_1 and m_2 be the arithmetic mean and the geometric mean:

$$m_1(x, y) = \frac{x + y}{2}, \quad m_2(x, y) = \sqrt{xy}.$$

For $0 < b < a$, we give a double sequence $\{a_n\}$ and $\{b_n\}$ by the iteration of two means m_1 and m_2 with initial (a, b):

$$(a_0, b_0) = (a, b), \quad (a_{n+1}, b_{n+1}) = (m_1(a_n, b_n), m_2(a_n, b_n)).$$

This double sequence converges and has a common limit, which is called the arithmetic-geometric mean of a and b, or the compound $m_1 \diamond m_2(a, b)$. It is known that the arithmetic-geometric mean can be expressed by the hypergeometric function, that is

$$m_1 \diamond m_2(a, b) = \frac{a}{F\left(\frac{1}{2}, \frac{1}{2}, 1; 1 - \left(\frac{b}{a}\right)^2\right)},$$

where

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$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n (1)_n} z^n.$$

We remark that the Gauss quadratic transformation formula for the hypergeometric function implies this fact, refer to Section 3 for details.

In this paper, from Goursat's transformation formulas in [G] instead of Gaussian, we induce several double sequences given by mean iterations and express their common limits by the hypergeometric function $F(\alpha, \beta, \gamma; z)$. We list pairs of means and their common limits induced from quadratic transformations in Theorem 2 and those from cubic ones in Theorem 3. It turns out that the parameters (α, β, γ) of the hypergeometric function in Theorem 2 satisfy

$$\left\{ \frac{1}{|1-\gamma|}, \frac{1}{|\gamma - \alpha - \beta|}, \frac{1}{|\alpha - \beta|} \right\} = \{2, 2, \infty\}, \ \{2, 4, 4\},$$

and that those in Theorem 3 satisfy

$$\left\{ \frac{1}{|1-\gamma|}, \frac{1}{|\gamma - \alpha - \beta|}, \frac{1}{|\alpha - \beta|} \right\} = \{2, 3, 6\}.$$

B.C. Carlson considers in [C] the twelve double sequences given by the iteration of means m_i and m_j $(1 \le i, j \le 4, i \ne j)$, where

$$m_3(x, y) = \sqrt{x \frac{x+y}{2}}, \quad m_4(x, y) = \sqrt{\frac{x+y}{2} y}.$$

They converge and their common limits $m_i \diamond m_j(a, b)$ admit integral representations of Euler type. Theorem 2 can be obtain by these results together with some functional equations for the hypergeometric function in Lemma 2.

J.M. and P.B. Borwein study in [BB1] two double sequences given by the iteration of m_5 and m_6 and by that of m_7 and m_8, where

$$m_5(x, y) = \frac{x+2y}{3}, \quad m_6(x, y) = \sqrt[3]{y \frac{x^2+xy+y^2}{3}},$$

$$m_7(x, y) = \frac{x+3y}{4}, \quad m_8(x, y) = \sqrt{y \frac{x+y}{2}}.$$

They converge and their common limits $m_5 \diamond m_6(a,b)$ and $m_7 \diamond m_8(a,b)$ can be expressed as

$$\frac{a}{F\left(\frac{1}{3},\frac{2}{3},1;1-\left(\frac{b}{a}\right)^3\right)}, \quad \frac{a}{F\left(\frac{1}{4},\frac{3}{4},1;1-\left(\frac{b}{a}\right)^2\right)^2},$$

respectively. We remark that Theorem 3 is independent of the results in [BB1] and [C].

The above expressions of common limits of double sequences in [BB1] are extended to those of multiple sequences by the hypergeometric function F_D of multi variables, refer to [KS], [KM] and [MO]. Similar extensions of some results in Theorem 2 are studied in [M].

A list of transformation formulas for the generalized hypergeometric function ${}_3F_2\left({\alpha_0,\alpha_1,\alpha_2 \atop \beta_1,\beta_2};z\right)$ is given in [K]. We attempt to find double sequences whose common limits can be expressed by ${}_3F_2$. It turns out that we can not get proper expressions of common limits by ${}_3F_2$ because of the reduction and the Clausen formula for ${}_3F_2$, refer to Section 6.

2. Mean iterations

In this section, we formalize the notion of mean iterations, for which we refer to Section 8 in [BB2].

Let $\mathbb{R}_+^*$ be the multiplicative group of positive real numbers. A *mean* is a continuous function $m : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ satisfying

$$\begin{aligned} &\min(x,y) \le m(x,y) \le \max(x,y), \\ &m(tx,ty) = tm(x,y) \end{aligned}$$

for any $x, y, t \in \mathbb{R}_+^*$. A mean $m(x,y)$ is *strict* if

$$m(x,y) = x \ \text{ or } \ m(x,y) = y \quad \Rightarrow \quad x = y.$$

For two means m_1 and m_2 and two positive real numbers a and b, we define a double sequence $\{a_n\}$ and $\{b_n\}$ with initial $(a_0, b_0) = (a, b)$ by

$$(a_{n+1}, b_{n+1}) = (m_1(a_n, b_n), m_2(a_n, b_n)).$$

This double sequence is called *the* (m_1, m_2)*-sequence with initial* (a, b). If the (m_1, m_2)-sequence with initial (a, b) converges and has a common limit, this value is called *the compound of* m_1 *and* m_2 *with initial* (a, b) and denoted by $m_1 \diamond m_2(a, b)$.

If $a \geq b$ and two means m_1 and m_2 satisfy

$$m_1(x, y) \geq m_2(x, y), \quad \text{for any } (x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+^*, \tag{1}$$

or

$$(x - y)(m_1(x, y) - m_2(x, y)) \geq 0, \quad \text{for any } (x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+^*, \tag{2}$$

then the (m_1, m_2)-sequence with initial (a, b) satisfies

$$b_0 \leq b_1 \leq b_2 \leq \cdots \leq b_n \leq a_n \leq \cdots \leq a_2 \leq a_1 \leq a_0.$$

If $a \geq b$ and two means m_1 and m_2 satisfy

$$(x - y)(m_1(x, y) - m_2(x, y)) \leq 0, \quad \text{for any } (x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+^*, \tag{3}$$

then the (m_1, m_2)-sequence with initial (a, b) satisfies

$$b_0 \leq a_1 \leq b_2 \leq \cdots \leq b_{2n} \leq a_{2n+1} \leq b_{2n+1} \leq a_{2n} \leq \cdots \leq a_2 \leq b_1 \leq a_0.$$

Lemma 1 *Suppose that two means* m_1 *and* m_2 *satisfy* (1) *or* (2) *or* (3). *If either* m_1 *or* m_2 *is strict, then the* (m_1, m_2)*-sequence with initial* (a, b) *converges and has a common limit, and the compound* $m_1 \diamond m_2$ *becomes a mean. Moreover, the convergence is uniform on any compact subset of* $\mathbb{R}_+^* \times \mathbb{R}_+^*$.

Proof. For a proof of the cases (1) and (2), refer to [BB2]. Suppose that (3) is satisfied. We may assume that $a \geq b$. Let $\{a_n\}$ and $\{b_n\}$ be the (m_1, m_2)-sequence with initial (a, b). Then both of four sequences $\{a_{2n}\}$, $\{a_{2n+1}\}$, $\{b_{2n}\}$ and $\{b_{2n+1}\}$ are monotonous and bounded. Thus they converge; we set

$$\lim_{n\to\infty} a_{2n} = \alpha_0, \quad \lim_{n\to\infty} a_{2n+1} = \alpha_1, \quad \lim_{n\to\infty} b_{2n} = \beta_0, \quad \lim_{n\to\infty} b_{2n+1} = \beta_1.$$

Let $n \to \infty$ for the inequalities

$$a_{2n-1} \leq b_{2n} \leq a_{2n+1} \leq b_{2n+1} \leq a_{2n} \leq b_{2n-1},$$

we have

$$\alpha_1 \le \beta_0 \le \alpha_1 \le \beta_1 \le \alpha_0 \le \beta_1, \quad \text{i.e.,} \quad \beta_0 = \alpha_1, \quad \beta_1 = \alpha_0, \quad \alpha_1 \le \alpha_0.$$

Let $n \to \infty$ for the equalities

$$a_{2n+1} = m_1(a_{2n}, b_{2n}), \quad b_{2n+1} = m_2(a_{2n}, b_{2n}),$$

we have

$$\alpha_1 = m_1(\alpha_0, \beta_0) = m_1(\alpha_0, \alpha_1), \quad \alpha_0 = \beta_1 = m_2(\alpha_0, \beta_0) = m_2(\alpha_0, \alpha_1).$$

Since either m_1 or m_2 is strict, α_0 should be equal to α_1.

Let us show that $\mu = m_1 \diamond m_2$ is a mean. In order to show that μ is continuous, take any $(a,b) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, any $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that

$$|a_n(a,b) - \mu(a,b)| < \varepsilon$$

for any $n > N$. We fix a natural number n satisfying $2n > N$. We can regard a_{2n} and a_{2n+1} as continuous functions of the initial terms (a,b). Thus there exists $\delta > 0$ such that

$$\begin{matrix} |x-a| < \delta, \\ |y-b| < \delta, \end{matrix} \quad \Rightarrow \quad \begin{matrix} |a_{2n}(x,y) - a_{2n}(a,b)| < \varepsilon, \\ |a_{2n+1}(x,y) - a_{2n+1}(a,b)| < \varepsilon. \end{matrix}$$

If $|x-a| < \delta$, $|y-b| < \delta$ and $x \ge y$ then we have

$$\mu(x,y) \le a_{2n}(x,y) < a_{2n}(a,b) + \varepsilon < \mu(a,b) + 2\varepsilon,$$

$$\mu(x,y) \ge a_{2n+1}(x,y) > a_{2n+1}(a,b) - \varepsilon > \mu(a,b) - 2\varepsilon;$$

i.e.,

$$|\mu(x,y) - \mu(a,b)| < 2\varepsilon.$$

If $|x-a| < \delta$, $|y-b| < \delta$ and $x \le y$ then we have

$$\mu(x,y) \le a_{2n+1}(x,y) < a_{2n+1}(a,b) + \varepsilon < \mu(a,b) + 2\varepsilon,$$

$$\mu(x,y) \ge a_{2n}(x,y) > a_{2n}(a,b) - \varepsilon > \mu(a,b) - 2\varepsilon;$$

i.e.,

$$|\mu(x,y) - \mu(a,b)| < 2\varepsilon.$$

Hence μ is continuous at (a,b). It is clear that

$$\min(x,y) \le \mu(x,y) \le \max(x,y),$$
$$\mu(tx,ty) = t\mu(x,y)$$

for any $x, y, t \in \mathbb{R}_+^*$.

Let K be any compact subset of $\mathbb{R}_+^* \times \mathbb{R}_+^*$, and K_+ and K_- be closed subsets of K given as $\{(x,y) \in K \mid \pm(x-y) \ge 0\}$, respectively. Since μ is continuous on $\mathbb{R}_+^* \times \mathbb{R}_+^*$ and the sequences $\{a_{2n+1}\}$ and $\{a_{2n}\}$ are monotonous on K_+ (resp. K_-), they uniformly converge to μ on the compact subset K_+ (resp. K_-) by Dini's theorem. Thus $\{a_n\}$ uniformly converges to μ on the compact subset K. □

The key observation about $m_1 \diamond m_2$ is the following fact in [BB2].

Fact 1 (Invariant principle) *Suppose that the compound $m_1 \diamond m_2$ of two means m_1 and m_2 exists. Then $m_1 \diamond m_2$ is the unique mean μ satisfying*

$$\mu(m_1(a,b), m_2(a,b)) = \mu(a,b)$$

for any $a, b \in \mathbb{R}_+^*$.

3. The hypergeometric function and mean iterations

The hypergeometric function $F(\alpha,\beta,\gamma;z)$ with parameters (α,β,γ) is defined as

$$F(\alpha,\beta,\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n} z^n,$$

where the variable z is in $\{z \in \mathbb{C} \mid |z| < 1\}$, $\gamma \ne 0, -1, -2, \ldots$, and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$. This function admits an integral representation of Euler type

$$F(\alpha,\beta,\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha}(1-t)^{\gamma-\alpha}(1-zt)^{-\beta}\frac{dt}{t(1-t)},$$

and satisfies the hypergeometric differential equation

$$z(1-z)\frac{d^2F}{dz^2}+[\gamma-(\alpha+\beta+1)z]\frac{dF}{dz}-\alpha\beta F=0.$$

Theorem 1 *Suppose that the compound $m_1\diamond m_2$ of two means m_1 and m_2 exists. If m_1 and m_2 satisfy $m_2(a,b)^p<2m_1(a,b)^p$ and*

$$\frac{m_1(a,b)}{F\Big(\alpha,\beta,\gamma;1-\Big(\frac{m_2(a,b)}{m_1(a,b)}\Big)^p\Big)^q}=\frac{a}{F\big(\alpha,\beta,\gamma;1-\big(\frac{b}{a}\big)^p\big)^q} \tag{4}$$

for some $\alpha,\beta,\gamma,p,q\in\mathbb{R}$ and for any $a,b\in\mathbb{R}_+^*$ with $b^p<2a^p$, then we have

$$m_1\diamond m_2(a,b)=\frac{a}{F\big(\alpha,\beta,\gamma;1-\big(\frac{b}{a}\big)^p\big)^q}. \tag{5}$$

Proof. Let $\{a_n\}$ and $\{b_n\}$ be the (m_1,m_2)-sequence with initial (a,b). The equality (4) implies that

$$\begin{aligned}\frac{a_0}{F\big(\alpha,\beta,\gamma;1-\big(\frac{b_0}{a_0}\big)^p\big)^q}&=\frac{a_1}{F\big(\alpha,\beta,\gamma;1-\big(\frac{b_1}{a_1}\big)^p\big)^q}\\&=\frac{a_2}{F\big(\alpha,\beta,\gamma;1-\big(\frac{b_2}{a_2}\big)^p\big)^q}=\cdots=\frac{a_n}{F\big(\alpha,\beta,\gamma;1-\big(\frac{b_n}{a_n}\big)^p\big)^q}.\end{aligned}$$

Let $n\to\infty$, then we have

$$\frac{a}{F\big(\alpha,\beta,\gamma;1-\big(\frac{b}{a}\big)^p\big)^q}=\frac{\lim\limits_{n\to\infty}a_n}{F\Big(\alpha,\beta,\gamma;1-\lim\limits_{n\to\infty}\big(\frac{b_n}{a_n}\big)^p\Big)^q}=m_1\diamond m_2(a,b),$$

since $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=m_1\diamond m_2(a,b)$ and $F(\alpha,\beta,\gamma;0)=1$. $\square$

Corollary 1 *Suppose that the compound $m_1\diamond m_2$ of two means m_1 and m_2 exists and that it satisfies (5) for $a,b\in\mathbb{R}_+^*$ such that b/a is sufficiently near to 1. If*

$$m'_1(x,y) = m_1(x^r,y^r)^{(s-t)/r} m_2(x^r,y^r)^{t/r} x^{1-s+t} y^{-t},$$
$$m'_2(x,y) = m_1(x^r,y^r)^{(s-t-1)/r} m_2(x^r,y^r)^{(t+1)/r} x^{1-s+t} y^{-t}$$

are means for given $r(\neq 0), s, t \in \mathbb{R}$, and the compound $m'_1 \diamond m'_2$ exists for such $a, b \in \mathbb{R}^*_+$, then we have

$$m'_1 \diamond m'_2(a,b) = \frac{a^{t+1}}{b^t F\left(\alpha,\beta,\gamma; 1-\left(\frac{b}{a}\right)^{pr}\right)^{qs/r}}.$$

Proof. By Fact 1, we have the equality (4). Since

$$\frac{m'_2(a,b)}{m'_1(a,b)} = \frac{m_2(a^r,b^r)^{1/r}}{m_1(a^r,b^r)^{1/r}},$$

we can easily obtain

$$\frac{m'_1(a,b)^{t+1}}{m'_2(a,b)^t F\left(\alpha,\beta,\gamma; 1-\left(\frac{m'_2(a,b)}{m'_1(a,b)}\right)^{pr}\right)^{qs/r}} = \frac{a^{t+1}}{b^t F\left(\alpha,\beta,\gamma; 1-\left(\frac{b}{a}\right)^{pr}\right)^{qs/r}}.$$

Fact 1 implies this theorem. □

Remark 1 Though $m'_1(x,y)$ and $m'_2(x,y)$ do not satisfy the condition

$$\min(x,y) \leq m'_i(x,y) \leq \max(x,y) \quad (i=1,2)$$

for some r, s, t in Corollary 1, it occurs that the double sequence $\{a_n\}$ and $\{b_n\}$ obtained by $m'_1(x,y)$ and $m'_2(x,y)$ has a non-zero common limit expressed by the hypergeometric function.

Corollary 2 *Suppose that the compound $m_1 \diamond m_2$ of two means m_1 and m_2 exists and that it satisfies (5) for $a, b \in \mathbb{R}^*_+$ such that b/a is sufficiently near to 1. If the compound $m'_1 \diamond m'_2$ of $m'_1(x,y) = m_2(y,x)$ and $m'_2(x,y) = m_1(y,x)$ exists for such $a, b \in \mathbb{R}^*_+$, then we have*

$$\begin{aligned} m'_1 \diamond m'_2(a,b) &= \frac{a}{\left(\frac{b}{a}\right)^{pq\alpha-1} F\left(\gamma-\beta,\alpha,\gamma; 1-\left(\frac{b}{a}\right)^p\right)^q} \\ &= \frac{a}{\left(\frac{b}{a}\right)^{pq\beta-1} F\left(\gamma-\alpha,\beta,\gamma; 1-\left(\frac{b}{a}\right)^p\right)^q}. \end{aligned}$$

Proof. It is shown in [IKSY], p. 38 that

$$F(\alpha,\beta,\gamma;z) = (1-z)^{-\alpha} F\left(\gamma-\beta,\alpha,\gamma;\frac{z}{z-1}\right)$$
$$= (1-z)^{-\beta} F\left(\gamma-\alpha,\beta,\gamma;\frac{z}{z-1}\right)$$

for $z \in \mathbb{C}$ satisfying $|z| < 1$ and $\mathrm{Re}(z) < \frac{1}{2}$. By the first equality for $z = 1 - b^p/a^p$ and for $z = 1 - m_2(a,b)^p/m_1(a,b)^p$, we rewrite (4) as

$$\frac{m_2(a,b)}{\left(\frac{m_2(a,b)}{m_1(a,b)}\right)^{1-pq\alpha} F\left(\gamma-\beta,\alpha,\gamma;1-\left(\frac{m_1(a,b)}{m_2(a,b)}\right)^p\right)^q}$$
$$= \frac{b}{\left(\frac{b}{a}\right)^{1-pq\alpha} F\left(\gamma-\beta,\alpha,\gamma;1-\left(\frac{a}{b}\right)^p\right)^q}.$$

Recall that we give m_1' and m_2' by changing the role of x, y and that of m_1, m_2. Fact 1 for m_1' and m_2' implies

$$m_1' \diamond m_2'(a,b) = \frac{a}{\left(\frac{b}{a}\right)^{pq\alpha-1} F\left(\gamma-\beta,\alpha,\gamma;1-\left(\frac{b}{a}\right)^p\right)^q}.$$

Similarly we can get the second expression of $m_1' \diamond m_2'(a,b)$. $\square$

Let us explain how to utilize Theorem 1 and Corollary 1. The Gauss quadratic transformation formula is as follows:

$$(1+z)^{2\alpha} F\left(\alpha,\alpha-\beta+\frac{1}{2},\beta+\frac{1}{2};z^2\right) = F\left(\alpha,\beta,2\beta;\frac{4z}{(1+z)^2}\right), \quad (6)$$

where z is in a small neighbourhood of 0, and the value of $(1+z)^{2\alpha}$ is 1 at $z = 0$. By substituting

$$\frac{b}{a} = \frac{1-z}{1+z}, \quad \alpha = \beta = \frac{1}{2}$$

into the equality (6), we have

$$\frac{(a+b)/2}{F\left(\frac{1}{2},\frac{1}{2},1;1-\left(\frac{2\sqrt{ab}}{a+b}\right)^2\right)}=\frac{a}{F\left(\frac{1}{2},\frac{1}{2},1;1-\left(\frac{b}{a}\right)^2\right)}.$$

Let m_1 be the arithmetic mean and m_2 the geometric mean. It is easy to show that the double sequence $\{a_n\}$ and $\{b_n\}$ defined by $(a_0,b_0)=(a,b)$, and

$$(a_{n+1},b_{n+1})=(m_1(a_n,b_n),m_2(a_n,b_n))=\left(\frac{a_n+b_n}{2},\sqrt{a_nb_n}\right)$$

has a common limit $\mu(a,b)$, which is called the arithmetic-geometric mean of a and b. Theorem 1 implies a well-known formula

$$\mu(a,b)=\frac{a}{F\left(\frac{1}{2},\frac{1}{2},1;1-\left(\frac{b}{a}\right)^2\right)} \tag{7}$$

for $0<b\le a$. By applying Corollary 1 for $(r,s,t)=(2,1,0)$ to (7), we have

$$m_1'\diamond m_2'(a,b)=\frac{a}{\sqrt{F\left(\frac{1}{2},\frac{1}{2},1;1-\left(\frac{b}{a}\right)^4\right)}}$$

for $a\ge b>0$ and two means

$$m_1'(x,y)=\sqrt{\frac{x^2+y^2}{2}},\quad m_2'(x,y)=\sqrt{xy}.$$

By applying Corollary 1 for $(r,s,t)=(1,\frac{1}{2},0)$ to (7), we have

$$m_1'\diamond m_2'(a,b)=\frac{a}{\sqrt{F\left(\frac{1}{2},\frac{1}{2},1;1-\left(\frac{b}{a}\right)^2\right)}}$$

for $a\ge b>0$ and two means

$$m_1'(x,y)=\sqrt{x\frac{x+y}{2}},\quad m_2'(x,y)=\sqrt{x\frac{2xy}{x+y}}.$$

4. Compounds of means by quadratic transformation formulas

In 1881 Goursat gave a list of transformation formulas of the form

$$F(\alpha,\beta,\gamma;z)=\varphi(z)F(\alpha',\beta',\gamma';\psi(z)),$$

in [G], where $\varphi(z)$ and $\psi(z)$ are algebraic functions with values 1 and 0 at $z=0$, respectively. In this section, we give a list of the compound means expressed by the hypergeometric function derived from Theorem 1 and quadratic transformation formulas G(25), ..., G(52) in [G].

It turns out that parameters (α,β,γ) of the hypergeometric function satisfy

$$\left\{\frac{1}{|1-\gamma|},\frac{1}{|\gamma-\alpha-\beta|},\frac{1}{|\alpha-\beta|}\right\}=\{2,2,\infty\},\ \text{or}\ \{2,4,4\},\ \text{or}\ \{\infty,\infty,\infty\},$$

for our consideration. We classify our results by these data. For the case $\{\infty,\infty,\infty\}$, we have the classical arithmetic-geometric mean explained in the previous section.

Theorem 2 *We have the following tables.*

$$\{2,2,\infty\}$$

No.	$m_1(a,b)$	$m_2(a,b)$	type	$m_1\diamond m_2(a,b)$
Q(1)	$\sqrt{ab}$	$\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})}{2}$	(M)	$a/F\big(1,1,\frac{3}{2};1-\frac{b}{a}\big)$
Q(2)	$\frac{\sqrt{a}(\sqrt{a}+\sqrt{b})}{2}$	$\sqrt{ab}$	(M)	$a/F\big(1,\frac{1}{2},\frac{3}{2};1-\frac{b}{a}\big)$
Q(3)	$\sqrt{\frac{a(a+b)}{2}}$	$\frac{a+b}{2}$	(M)	$a/F\big(\frac{1}{2},\frac{1}{2},\frac{3}{2};1-\big(\frac{b}{a}\big)^2\big)$
Q(4)	$\sqrt[4]{\frac{2ab}{a+b}a^2b}$	$\sqrt[4]{\frac{a+b}{2}ab^2}$	(M)	$a/F\big(\frac{1}{2},\frac{1}{2},\frac{1}{2};1-\big(\frac{b}{a}\big)^2\big)^{\frac{1}{2}}=\sqrt{ab}$
Q(5)	$\frac{2ab}{a+b}$	$\frac{a+b}{2}$	(M)	$a/F\big(\frac{1}{2},1,1;1-\frac{b}{a}\big)=\sqrt{ab}$

$\{2,4,4\}$

No.	$m_1(a,b)$	$m_2(a,b)$	type	$m_1 \diamond m_2(a,b)$
Q(6)	$\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})}{2}$	$\sqrt{ab}$	(A)	$a/F\left(1,\frac{3}{4},\frac{5}{4};1-\frac{b}{a}\right)$
Q(7)	$\sqrt{ab}$	$\frac{\sqrt{a}(\sqrt{a}+\sqrt{b})}{2}$	(A)	$a/F\left(1,\frac{1}{2},\frac{5}{4};1-\frac{b}{a}\right)$
Q(8)	$\sqrt{\frac{b(a+b)}{2}}$	$\frac{a+b}{2}$	(A)	$a/F\left(\frac{1}{4},\frac{3}{4},\frac{5}{4};1-\left(\frac{b}{a}\right)^2\right)^2$
Q(9)	$\frac{a+b}{2}$	$\sqrt{\frac{a(a+b)}{2}}$	(A)	$a/F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4};1-\left(\frac{b}{a}\right)^2\right)^2$
Q(10)	$\sqrt[4]{\frac{2ab}{a+b}ab^2}$	$\sqrt[4]{\frac{a+b}{2}a^2b}$	(A)	$a/F\left(\frac{1}{4},\frac{3}{4},\frac{3}{4};1-\left(\frac{b}{a}\right)^2\right)=\sqrt{ab}$
Q(11)	$\sqrt[4]{\frac{a+b}{2}ab^2}$	$\sqrt[4]{\frac{2ab}{a+b}a^2b}$	(A)	$a/F\left(\frac{1}{2},\frac{3}{4},\frac{3}{4};1-\left(\frac{b}{a}\right)^2\right)^{\frac{1}{2}}=\sqrt{ab}$

Here b/a is sufficiently near to 1, the type (M) means the (m_1,m_2)-sequence is monotonous, i.e., they satisfy

$$b_n \le b_{n+1} \le a_{n+1} \le a_n \quad \textit{or} \quad b_n \ge b_{n+1} \ge a_{n+1} \ge a_n;$$

the type (A) means the (m_1,m_2)-sequence is alternative, i.e., they satisfy

$$b_0 \le a_1 \le b_2 \le \cdots \le b_{2n} \le a_{2n+1} \le b_{2n+1} \le a_{2n} \le \cdots \le a_2 \le b_1 \le a_0.$$

Proof. We show Q(3). The quadratic transformation formula G(41) in [G] is

$$\begin{aligned}F(\alpha,1-\alpha,\gamma;z) &= (1-z)^{\gamma-1}F\left(\frac{\gamma-\alpha}{2},\frac{\gamma+\alpha-1}{2},\gamma;4z(1-z)\right)\\ &= (1-z)^{\gamma-1}(1-2z)F\left(\frac{\gamma+\alpha}{2},\frac{\gamma+1-\alpha}{2},\gamma;4z(1-z)\right).\end{aligned}$$

Substitute

$$\alpha=\frac{1}{2},\quad \gamma=\frac{3}{2},\qquad \frac{b}{a}=1-2z$$

into the first row of this formula, then we have

$$\frac{m_1(a,b)}{F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};1-\left(\frac{m_2(a,b)}{m_1(a,b)}\right)^2\right)} = \frac{a}{F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};1-\left(\frac{b}{a}\right)^2\right)},$$

where

$$m_1(a,b) = \sqrt{\frac{a(a+b)}{2}}, \quad m_2(a,b) = \frac{a+b}{2}.$$

We can easily show that the (m_1, m_2)-sequence converges and has a common limit by Lemma 1. Theorem 1 implies that

$$m_1 \diamond m_2(a,b) = \frac{a}{F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};1-\left(\frac{b}{a}\right)^2\right)}.$$

We list used formulas in [G] and substitutions to prove this proposition.

Q(1)	$\alpha=\beta=1,\ b/a=(1-2z)^2,$
G(38):	$F(\alpha,\beta,\frac{\alpha+\beta+1}{2};z)=(1-2z)F(\frac{\alpha+1}{2},\frac{\beta+1}{2},\frac{\alpha+\beta+1}{2};4z(1-z))$
Q(2)	$\alpha=1/2,\ \gamma=3/2, \quad b/a=1-z,$
G(35):	$F(\alpha,\alpha+\frac{1}{2},\gamma;z)=\left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha}F\left(2\alpha,2\alpha+1-\gamma,\gamma;\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)$
Q(3)	$\alpha=1/2,\ \gamma=3/2, \quad b/a=1-2z,$
G(41):	$F(\alpha,1-\alpha,\gamma;z)=(1-z)^{\gamma-1}F\left(\frac{\gamma-\alpha}{2},\frac{\gamma+\alpha-1}{2},\gamma;4z(1-z)\right)$
Q(4)	$\alpha=1/2,\ \gamma=1/2, \quad b/a=1-2z,$
G(41):	$F(\alpha,1-\alpha,\gamma;z)=\frac{1-2z}{(1-z)^{1-\gamma}}F\left(\frac{\gamma+\alpha}{2},\frac{\gamma+1-\alpha}{2},\gamma;4z(1-z)\right)$
Q(5)	$\alpha=1,\ \beta=1/2, \quad b/a=1-z,$
G(44):	$F(\alpha,\beta,2\beta;z)=\frac{1-\frac{z}{2}}{(1-z)^{(\alpha+1)/2}}F\left(\beta+\frac{1-\alpha}{2},\frac{1+\alpha}{2},\beta+\frac{1}{2};\frac{z^2}{4(z-1)}\right)$

Q(6)	$\alpha = 1,\ \beta = 3/4, \quad b/a = (1+z)^2/(1-z)^2,$
G(49):	$F(\alpha,\beta,\alpha-\beta+1;z) = \frac{1+z}{(1-z)^{\alpha+1}} F\big(\frac{\alpha+1}{2}, \frac{\alpha}{2}+1-\beta, \alpha-\beta+1; \frac{-4z}{(1-z)^2}\big)$
Q(7)	$\alpha = 1,\ \beta = 1/2, \quad b/a = 1/(1-2z)^2,$
G(39):	$F\big(\alpha,\beta,\frac{\alpha+\beta+1}{2};z\big) = (1-2z)^{-\alpha} F\big(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\alpha+\beta+1}{2}; \frac{4z(z-1)}{(2z-1)^2}\big)$
Q(8)	$\alpha = 3/4,\ \gamma = 5/4, \quad b/a = 1/(1-2z),$
G(42):	$F(\alpha,1-\alpha,\gamma;z) = \frac{(1-z)^{\gamma-1}}{(1-2z)^{\gamma-\alpha}} F\big(\frac{\gamma-\alpha}{2}, \frac{\gamma+1-\alpha}{2}, \gamma; \frac{-4z(1-z)}{(1-2z)^2}\big)$
Q(9)	$\alpha = 1/2,\ \beta = 1/4, \quad b/a = (1+z)/(1-z),$
G(48):	$F(\alpha,\beta,\alpha-\beta+1;z) = (1-z)^{-\alpha} F\big(\frac{\alpha}{2}, \frac{\alpha+1-2\beta}{2}, \alpha-\beta+1; \frac{-4z}{(1-z)^2}\big)$
Q(10)	$\alpha = 1/4,\ \gamma = 3/4, \quad b/a = 1/(1-2z),$
G(42):	$F(\alpha,1-\alpha,\gamma;z) = \frac{(1-z)^{\gamma-1}}{(1-2z)^{\gamma-\alpha}} F\big(\frac{\gamma-\alpha}{2}, \frac{\gamma+1-\alpha}{2}, \gamma; \frac{-4z(1-z)}{(1-2z)^2}\big)$
Q(11)	$\alpha = 1/2,\ \beta = 3/4, \quad b/a = (1+z)/(1-z),$
G(49):	$F(\alpha,\beta,\alpha-\beta+1;z) = \frac{1+z}{(1-z)^{\alpha+1}} F\big(\frac{\alpha+1}{2}, \frac{\alpha}{2}+1-\beta, \alpha-\beta+1; \frac{-4z}{(1-z)^2}\big)$

Here we remark that the formulas (G38), (G41) and (G42) consist of some equalities. □

Lemma 2 *We have*

$$F\left(\frac{\gamma-\alpha}{2}, \frac{\gamma+\alpha-1}{2}, \gamma; 1-t^2\right) = tF\left(\frac{\gamma+\alpha}{2}, \frac{\gamma+1-\alpha}{2}, \gamma; 1-t^2\right),$$

$$F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \beta+\frac{1}{2}; 1-t^2\right) = t^{2(\beta-\alpha)} F\left(\beta-\frac{\alpha}{2}, \beta+\frac{1-\alpha}{2}, \beta+\frac{1}{2}; 1-t^2\right),$$

for t *in a small neighbourhood of* 1. *Especially,*

$$F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1-t^2\right) = tF\left(1, 1, \frac{3}{2}; 1-t^2\right),$$

$$F\left(\frac{1}{2}, \frac{1}{4}, \frac{5}{4}, 1-t^2\right) = tF\left(\frac{3}{4}, 1, \frac{5}{4}, 1-t^2\right),$$

$$F\left(\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, 1-t^4\right) = tF\left(\frac{1}{2}, 1, \frac{5}{4}, 1-t^4\right).$$

Proof. By substituting $t = 1 - 2z$ into the formula (G41), and $t = \frac{2\sqrt{1-z}}{2-z}$ into (G45), we obtain the first and second equalities in this lemma, respectively. In order to get the rest, put $(\alpha, \gamma) = (1/2, 3/2)$ and $(\alpha, \gamma) = (1/4, 5/4)$ in the first equality, and $(\alpha, \beta) = (1/2, 3/4)$, $\sqrt{t} = t'$ in the second. $\square$

Remark 2 Carlson studied in [C] compound means of two means taken from the following four means:

$$m_1(x,y) = \frac{x+y}{2}, \quad m_2(x,y) = \sqrt{xy},$$

$$m_3(x,y) = \sqrt{x\frac{x+y}{2}}, \quad m_4(x,y) = \sqrt{\frac{x+y}{2}y}.$$

Refer also to Section 8.5 in [BB2] for these results. Note that the compound mean $m_1 \diamond m_2$ is the classical arithmetic-geometric mean. It is shown that the compound means $m_3 \diamond m_4(a,b)$ and $m_4 \diamond m_3(a,b)$ are expressed as

$$\sqrt{\frac{a^2 - b^2}{2\log(a/b)}},$$

which is called Carlson's log expression. The other compound means $m_i \diamond m_j$ can be expressed by the hypergeometric function by Theorem 2, Corollary 1 and Lemma 2.

For example, Q(3) in Theorem 2 coincides with the expression of $m_3 \diamond m_1$ shown in [BB2] and [C]. The compound mean $m_2 \diamond m_4$ is expressed as

$$\frac{a}{\sqrt{\frac{a}{b}}F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1 - \left(\frac{b}{a}\right)^2\right)^{1/2}}$$

by Exercises 1 of Section 8.5 in [BB2]. This result is obtained by Q(1) in Theorem 2, Corollary 1 for $(r,s,t) = (2,1,0)$ and Lemma 2.

Carlson's log expression can be obtained by the following functional equation for the hypergeometric function.

Lemma 3 *For $\alpha, n \in \mathbb{C}$ and x in a small neighbourhood of* 1*, we have*

$$n(1-x)F(n(\alpha-1)+1, 1, 2; 1-x) = (1-x^n)F(\alpha, 1, 2; 1-x^n) = \frac{x^{(1-\alpha)n} - 1}{\alpha - 1},$$

where the value of x^n is 1 at $x = 1$. Especially, if $\alpha = 1$ and $n \in \mathbb{N}$ then it reduces

$$F(1,1,2;1-x) = \left(\frac{1+x+x^2+\cdots+x^{n-1}}{n}\right) F(1,1,2;1-x^n) = \frac{\log x}{x-1}.$$

Proof. It is easy to show that the functions $n(1-x)F(n(\alpha-1)+1,1,2;1-x)$ and $(1-x^n)F(\alpha,1,2;1-x^n)$ satisfy the differential equation

$$\frac{d^2\varphi}{dx^2} = -\frac{n(\alpha-1)+1}{x}\frac{d\varphi}{dx}$$

with initial conditions $\varphi(1) = 0$ and $\frac{d\varphi}{dx}(1) = -n$. Thus these functions coincide with $(x^{(1-\alpha)n} - 1)/(\alpha - 1)$. Note that this function converges to $-n \log x$ as $\alpha \to 1$. $\square$

By Lemma 3 for $\alpha = 1$, $n = 2$ and $b/a = x$, we have

$$\frac{a}{\sqrt{F\left(1,1,2;1-\left(\frac{b}{a}\right)^2\right)}} = \frac{m_3(a,b)}{\sqrt{F\left(1,1,2;1-\left(\frac{m_4(a,b)}{m_3(a,b)}\right)^2\right)}} = \frac{m_4(a,b)}{\sqrt{F\left(1,1,2;1-\left(\frac{m_3(a,b)}{m_4(a,b)}\right)^2\right)}}.$$

Theorem 1 implies Carlson's log expression.

5. Compounds of means by cubic transformation formulas

We give a list of the compound means expressed by the hypergeometric function derived from cubic transformation formulas (G78), ..., (G125) in [G], Theorem 1 and Corollary 2.

Theorem 3 *We have the following table*:

No.	$m_1(a,b)$	$m_2(a,b)$	type	$m_1 \diamond m_2(a,b)$
C(1)	$b^{\frac{2}{3}}X_1$	$b^{\frac{2}{3}}X_2$	(A)	$a/F\big(\frac{1}{2},1,\frac{7}{6};1-\big(\frac{b}{a}\big)^2\big)$
C(2)	$X_1X_2^2$	X_2^3	(A)	$a/F\big(\frac{1}{6},\frac{2}{3},\frac{7}{6};1-\big(\frac{b}{a}\big)^2\big)^3$
C(3)	$X_1X_2^2$	$X_2^2X_3$	(A)	$a/F\big(\frac{1}{2},\frac{2}{3},\frac{3}{2};1-\big(\frac{b}{a}\big)^2\big)$
C(4)	$b^{\frac{1}{3}}X_1X_2$	$b^{\frac{1}{3}}X_2X_3$	(A)	$a/F\big(\frac{5}{6},1,\frac{3}{2};1-\big(\frac{b}{a}\big)^2\big)^{\frac{1}{2}}$
C(5)	$b^{\frac{2}{3}}X_1$	$b^{\frac{2}{3}}X_3$	(A)	$a/F\big(\frac{1}{3},\frac{1}{2},\frac{1}{2};1-\big(\frac{b}{a}\big)^2\big)=\sqrt[3]{ab^2}$
C(6)	$a^{\frac{2}{3}}Y_2$	$a^{\frac{2}{3}}Y_1$	(A)	$a/F\big(\frac{1}{6},\frac{1}{2},\frac{7}{6};1-\big(\frac{b}{a}\big)^2\big)$
C(7)	Y_2^3	$Y_1Y_2^2$	(A)	$a/\big[\frac{b}{a}F\big(\frac{2}{3},1,\frac{7}{6};1-\big(\frac{b}{a}\big)^2\big)\big]^3$
C(8)	$Y_2^2Y_3$	$Y_1Y_2^2$	(A)	$a/F\big(\frac{1}{2},\frac{5}{6},\frac{3}{2};1-\big(\frac{b}{a}\big)^2\big)$
C(9)	$a^{\frac{1}{3}}Y_2Y_3$	$a^{\frac{1}{3}}Y_1Y_2$	(A)	$a/F\big(\frac{2}{3},1,\frac{3}{2};1-\big(\frac{b}{a}\big)^2\big)^{\frac{1}{2}}$
C(10)	$a^{\frac{2}{3}}Y_3$	$a^{\frac{2}{3}}Y_1$	(A)	$a/F\big(\frac{1}{6},\frac{1}{2},\frac{1}{2};1-\big(\frac{b}{a}\big)^2\big)=\sqrt[3]{a^2b}$

where b/a *is sufficiently near to* 1,

$$X_1=\frac{\xi_1^{\frac{1}{3}}+\xi_2^{\frac{1}{3}}}{2},\quad X_2=\sqrt{\frac{\xi_1^{\frac{2}{3}}+\xi_1^{\frac{1}{3}}\xi_2^{\frac{1}{3}}+\xi_2^{\frac{2}{3}}}{3}},\quad X_3=\sqrt{\xi_1^{\frac{2}{3}}-\xi_1^{\frac{1}{3}}\xi_2^{\frac{1}{3}}+\xi_2^{\frac{2}{3}}},$$

(ξ_1,ξ_2) *is the preimage of* (a,b) *under the arithmetic and geometric means*:

$$\frac{\xi_1+\xi_2}{2}=a,\quad \sqrt{\xi_1\xi_2}=b,\qquad \{\xi_1,\xi_2\}=\big\{a\pm\sqrt{a^2-b^2}\big\},$$

and $-\frac{\pi}{6}<\arg(\xi_i^{\frac{1}{3}})<\frac{\pi}{6}$ $(i=1,2)$, $\xi_1^{\frac{1}{3}}\xi_2^{\frac{1}{3}}=b^{\frac{2}{3}}\in\mathbb{R}_+^*$;

$$Y_1=\frac{\eta_1^{\frac{1}{3}}+\eta_2^{\frac{1}{3}}}{2},\quad Y_2=\sqrt{\frac{\eta_1^{\frac{2}{3}}+\eta_1^{\frac{1}{3}}\eta_2^{\frac{1}{3}}+\eta_2^{\frac{2}{3}}}{3}},\quad Y_3=\sqrt{\eta_1^{\frac{2}{3}}-\eta_1^{\frac{1}{3}}\eta_2^{\frac{1}{3}}+\eta_2^{\frac{2}{3}}},$$

(η_1,η_2) *is the preimage of* (a,b) *under the geometric and arithmetic means*:

$$\sqrt{\eta_1\eta_2}=a,\quad \frac{\eta_1+\eta_2}{2}=b,\qquad \{\eta_1,\eta_2\}=\big\{b\pm\sqrt{b^2-a^2}\big\},$$

and $-\frac{\pi}{6}<\arg(\eta_i^{\frac{1}{3}})<\frac{\pi}{6}$ $(i=1,2)$, $\eta_1^{\frac{1}{3}}\eta_2^{\frac{1}{3}}=a^{\frac{2}{3}}\in\mathbb{R}_+^*$.

Remark 3 Parameters (α,β,γ) of the hypergeometric function in Theorem 3 satisfy

$$\left\{\frac{1}{|1-\gamma|},\frac{1}{|\gamma-\alpha-\beta|},\frac{1}{|\alpha-\beta|}\right\}=\{2,3,6\}.$$

We give two lemmas in order to prove Theorem 3.

Lemma 4 *If $b<a$ then*

$$b<b^{\frac{2}{3}}X_1<b^{\frac{2}{3}}X_2<b^{\frac{2}{3}}X_3<X_2^2X_3<a,$$
$$b<Y_2^2Y_3<a^{\frac{2}{3}}Y_3<a^{\frac{2}{3}}Y_2<a^{\frac{2}{3}}Y_1<a;$$

if $a<b$ then

$$a<X_2^2X_3<b^{\frac{2}{3}}X_3<b^{\frac{2}{3}}X_2<b^{\frac{2}{3}}X_1<b,$$
$$a<a^{\frac{2}{3}}Y_1<a^{\frac{2}{3}}Y_2<a^{\frac{2}{3}}Y_3<Y_2^2Y_3<b.$$

Proof. Suppose that $b<a$. Since ξ_1,ξ_2 are real and $a=\frac{(\xi_1^{\frac{1}{3}})^3+(\xi_2^{\frac{1}{3}})^3}{2}$, it is easy to show that

$$b<b^{\frac{2}{3}}X_1<b^{\frac{2}{3}}X_2<b^{\frac{2}{3}}X_3<X_2^2X_3$$

and

$$a^2-X_2^4X_3^2=\frac{1}{36}\left(5\xi_1^{\frac{2}{3}}+11\xi_1^{\frac{1}{3}}\xi_2^{\frac{1}{3}}+5\xi_2^{\frac{2}{3}}\right)\left(\xi_1^{\frac{2}{3}}-\xi_1^{\frac{1}{3}}\xi_2^{\frac{1}{3}}+\xi_2^{\frac{2}{3}}\right)\left(\xi_1^{\frac{1}{3}}-\xi_2^{\frac{1}{3}}\right)^2>0.$$

In order to show the other inequalities, we assume that $a=1$ by the homogeneity. Note that η_i do not belong to $\mathbb{R}$ and that

$$|\eta_i|=1,\quad \mathrm{Re}(\eta_i)=b,\quad -\frac{\pi}{2}<\arg(\eta_i)<\frac{\pi}{2}.$$

If we take branches of $\eta_i^{\frac{1}{3}}$ so that $-\frac{\pi}{6}<\arg\left(\eta_i^{\frac{1}{3}}\right)<\frac{\pi}{6}$, then we have

$$\eta_1^{\frac{1}{3}}\eta_2^{\frac{1}{3}}=1,\quad \frac{\sqrt{3}}{2}<\frac{\eta_1^{\frac{1}{3}}+\eta_2^{\frac{1}{3}}}{2}=Y_1<1.$$

Since

$$b = 4Y_1^3 - 3Y_1, \quad Y_2 = \sqrt{\frac{4Y_1^2 - 1}{3}}, \quad Y_3 = \sqrt{4Y_1^2 - 3},$$

we have

$$\begin{aligned}
Y_1^2 - Y_2^2 &= \frac{1}{3}(1 - Y_1^2) > 0, \\
Y_2^2 - Y_3^2 &= \frac{8}{3}(1 - Y_1^2) > 0, \\
Y_3 - Y_2^2 Y_3 &= Y_3(1 - Y_2^2) > Y_3(Y_1^2 - Y_2^2) > 0, \\
Y_2^4 Y_3^2 - b^2 &= \frac{1}{9}(1 - Y_1^2)(1 + 20Y_1^2)(4Y_1^2 - 3) > 0
\end{aligned}$$

for $\frac{\sqrt{3}}{2} < Y_1 < 1$. □

Lemma 5 *For any $a, b \in \mathbb{R}_+^*$, we have*

$$\frac{2\sqrt{2}}{3} < \frac{X_2}{X_1} < \frac{2\sqrt{3}}{3}, \quad 0 < \frac{X_3}{X_1} < 2, \quad \frac{\sqrt{3}}{2} < \frac{Y_1}{Y_2} < \frac{3\sqrt{2}}{4}, \quad \frac{1}{2} < \frac{Y_1}{Y_3} < \infty.$$

Proof. By the homogeneity of X_i, we normalize $b = 1$. Note that

$$\frac{\sqrt{3}}{2} < X_1 < \infty,$$

and that

$$\frac{X_2}{X_1} = \sqrt{\frac{4X_1^2 - 1}{3X_1^2}}, \quad \frac{X_3}{X_1} = \sqrt{\frac{4X_1^2 - 3}{X_1^2}}$$

are monotonous as functions of X_1. Consider their limits as $X_1 \to \frac{\sqrt{3}}{2}$ and as $X_1 \to \infty$. Normalize $a = 1$ to show the inequalities for Y_1/Y_i. □

Proof of Theorem 3. Lemmas 1 and 4 imply that the (m_1, m_2)-sequence alternatively converges for (m_1, m_2) in Theorem 3 and for any $a, b \in \mathbb{R}_+^*$. We show C(1). Substitute $\alpha = 1/6$ into the formula G(112):

$$F\left(\alpha,\alpha+\frac{1}{2},2\alpha+\frac{5}{6};z\right)=(1-z)^{\frac{1}{3}}F\left(\alpha+\frac{1}{3},\alpha+\frac{5}{6},2\alpha+\frac{5}{6};z\right)$$
$$=(1-9t)^{2\alpha}F\left(3\alpha,3\alpha+\frac{1}{2},2\alpha+\frac{5}{6};t\right),$$

where $27t(1-t)^2+(1-9t)^2z=0$. We have

$$F\left(\frac{1}{2},1,\frac{7}{6};t\right)=\frac{3t+1}{1-9t}F\left(\frac{1}{2},1,\frac{7}{6};1-\frac{(3t+1)^3}{(1-9t)^2}\right).$$

Put $u=(3t+1)/(1-9t)$, then $t=(u-1)/(3(1+3u))$ and

$$F\left(\frac{1}{2},1,\frac{7}{6};1-\frac{4(1+2u)}{3(1+3u)}\right)=uF\left(\frac{1}{2},1,\frac{7}{6};1-\frac{4u^3}{1+3u}\right).$$

Solve the equation

$$\left(\frac{b}{a}\right)^2=\frac{4u^3}{1+3u}$$

with the variable u for given $a,b>0$. Then we have

$$u=\frac{1}{a}b^{\frac{2}{3}}X_1,\quad \frac{1}{X_1}=\frac{X_3^2}{a},$$
$$\frac{4}{1+3u}\frac{1+2u}{3}=\left(\frac{b}{a}\right)^2\frac{1}{u^2}\frac{1/u+2}{3}=\frac{1}{X_1^2}\frac{X_3^2+2b^{\frac{2}{3}}}{3}=\frac{X_2^2}{X_1^2},$$

and

$$\frac{m_1(a,b)}{F\left(\frac{1}{2},1,\frac{7}{6};1-\left(\frac{m_2(a,b)}{m_1(a,b)}\right)^2\right)}=\frac{a}{F\left(\frac{1}{2},1,\frac{7}{6};1-\left(\frac{b}{a}\right)^2\right)}$$

for

$$m_1(a,b)=b^{\frac{2}{3}}X_1,\quad m_2(a,b)=b^{\frac{2}{3}}X_2.$$

Theorem 1 implies

$$m_1 \diamond m_2(a,b) = \frac{a}{F\big(\frac{1}{2}, 1, \frac{7}{6}; 1 - \big(\frac{b}{a}\big)^2\big)}.$$

In order to get (C2), $\dots$, (C5), we use the following.

C(2)	$\alpha = 1/6, \quad u = 2(1+3z)/(1-9z),\ (b/a)^2 = u^3/(3u+2)$
G(119):	$(1-z)^{\frac{1}{3}-4\alpha} F\big(\frac{1}{3}-\alpha, \frac{5}{6}-\alpha, 2\alpha+\frac{5}{6}; z\big)$
	$= (1-9z)^{-2\alpha} F\big(\alpha, \alpha+\frac{1}{2}, 2\alpha+\frac{5}{6}; \frac{-27z(1-z)^2}{(1-9z)^2}\big)$
C(3)	$\alpha = 0, \quad t = 1 - 4/(1+3u),\ (b/a)^2 = 4u^3/(1+3u),$
G(87):	$F\big(\alpha+\frac{1}{2}, \frac{2}{3}-\alpha, \frac{3}{2}; z\big) = \frac{9(1-t)^{2\alpha+1}}{9-t} F\big(3\alpha+\frac{1}{2}, \alpha+\frac{2}{3}, \frac{3}{2}; t\big)$
	$(t-9)^2 t + 27(1-t)^2 z = 0$
C(4)	$\alpha = 1/6, \quad t = 1 - 4/(1+3u),\ (b/a)^2 = 4u^3/(1+3u),$
G(87):	$(1-z)^{\frac{1}{3}} F\big(1-\alpha, \alpha+\frac{5}{6}, \frac{3}{2}; z\big) = \frac{9(1-t)^{2\alpha+1}}{9-t} F\big(3\alpha+\frac{1}{2}, \alpha+\frac{2}{3}, \frac{3}{2}; t\big)$
	$(t-9)^2 t + 27(1-t)^2 z = 0$
C(5)	$\alpha = 1/6, \quad t = 1 - 4/(1+3u),\ (b/a)^2 = 4u^3/(1+3u),$
G(86):	$(1-z)^{\frac{1}{3}} F\big(\frac{1}{2}-\alpha, \alpha+\frac{1}{3}, \frac{1}{2}; z\big) = (1-t)^{2\alpha} F\big(3\alpha, \alpha+\frac{1}{6}, \frac{1}{2}; t\big)$
	$(t-9)^2 t + 27(1-t)^2 z = 0$

We remark that the formulas G(86), G(87) and G(119) consist of some equalities. The equality C$(5+k)$ is obtained by Corollary 2 for C(k) ($k = 1, \dots, 5$). $\square$

6. Compounds of means by transformation formulas for ${}_3F_2$

The generalized hypergeometric function ${}_3F_2$ is defined as

$${}_3F_2\left(\begin{matrix}\alpha_0, \alpha_1, \alpha_2 \\ \beta_1, \beta_2\end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n(\alpha_2)_n}{(1)_n(\beta_1)_n(\beta_2)_n} z^n,$$

where $\beta_1, \beta_2 \neq 0, -1, -2, \dots$, and $|z| < 1$. Note that this function reduces to the hypergeometric function $F(\alpha_0, \alpha_1, \beta_1; z)$ when $\alpha_2 = \beta_2$. In this section, we attempt to find pairs of means whose compounds can be expressed by ${}_3F_2$ by using transformation formulas for ${}_3F_2$ in [K].

Proposition 1 *We have functional equations of the form*

$$ {}_3F_2\left(\begin{matrix}\alpha_0,\alpha_1,\alpha_2\\ \beta_1,\beta_2\end{matrix};z\right)=\varphi(z)\,{}_3F_2\left(\begin{matrix}\alpha_0,\alpha_1,\alpha_2\\ \beta_1,\beta_2\end{matrix};\psi(z)\right), \tag{8}$$

where $\{\alpha_0,\alpha_1,\alpha_2\}$, $\{\beta_1,\beta_2\}$, $\varphi(z)$ *and* $\psi(z)$ *are given as follows.*

No.	$\{\alpha_0,\alpha_1,\alpha_2\}$	$\{\beta_1,\beta_2\}$	$\varphi(z)$	$\psi(z)$
K(1)	$\left\{\frac{1}{2},\frac{3}{4},1\right\}$	$\left\{\frac{5}{4},\frac{3}{2}\right\}$	$\frac{1}{1-z}$	$1-\left(\frac{1+z}{1-z}\right)^2$
K(2)	$\left\{\frac{1}{4},\frac{1}{2},\frac{3}{4}\right\}$	$\left\{\frac{3}{4},\frac{5}{4}\right\}$	$\frac{1}{\sqrt{1-z}}$	$1-\left(\frac{1+z}{1-z}\right)^2$
K(3)	$\left\{\frac{1}{3},\frac{2}{3},1\right\}$	$\left\{\frac{7}{6},\frac{4}{3}\right\}$	$\frac{1}{1-4z}$	$1-\frac{(1-z)(1+8z)^2}{(1-4z)^3}$
K(4)	$\left\{\frac{1}{6},\frac{1}{2},\frac{5}{6}\right\}$	$\left\{\frac{5}{6},\frac{7}{6}\right\}$	$\frac{1}{\sqrt{1-4z}}$	$1-\frac{(1-z)(1+8z)^2}{(1-4z)^3}$

Proof. We can easily show these functional equations by the formulas (2.1) and (2.2) in [K]. □

Remark 4 Non-trivial functional equations of the form (8) can not directly obtained any more by the formulas (2.1), . . . , (2.5) in [K].

Note that each ${}_3F_2$ in the functional equations K(2) and K(4) has a common parameter in the sets $\{\alpha_0,\alpha_1,\alpha_2\}$ and $\{\beta_1,\beta_2\}$. Thus these functional equations reduce to

$$ F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4};z\right)=\frac{1}{\sqrt{1-z}}F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4};1-\left(\frac{1+z}{1-z}\right)^2\right), \tag{9}$$

$$ F\left(\frac{1}{6},\frac{1}{2},\frac{7}{6};z\right)=\frac{1}{\sqrt{1-4z}}F\left(\frac{1}{6},\frac{1}{2},\frac{7}{6};1-\frac{(1-z)(1+8z)}{(1-4z)^3}\right), \tag{10}$$

which appear when we study Q(9) and C(6), respectively.

By the Clausen formula

$$ 3F_2\left(\begin{matrix}2\alpha,2\beta,\alpha+\beta\\ 2\alpha+2\beta,\alpha+\beta+1/2\end{matrix};z\right)=F(\alpha,\beta,\alpha+\beta+1/2;z)^2,$$

we have

$$ {}_3F_2\left(\begin{matrix}1/2,3/4,1\\5/4,3/2\end{matrix};z\right)=F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4};z\right)^2, $$

$$ {}_3F_2\left(\begin{matrix}1/3,2/3,1\\7/6,4/3\end{matrix};z\right)=F\left(\frac{1}{6},\frac{1}{2},\frac{7}{6};z\right)^2. $$

Thus the functional equations K(1) and K(3) reduce to (9) and (10), respectively.

Hence we conclude that proper expressions of compounds of means by ${}_3F_2$ can not directly obtained by transformation formulas for ${}_3F_2$ in [K].

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