Hokkaido Mathematical Journal Vol. 38 (2009) p. 557–561

An example of a solid von Neumann algebra

Narutaka Ozawa

(Received June 24, 2008)

Abstract. We prove that the group-measure-space von Neumann algebra $L^\infty(\mathbb{T}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$ is solid. The proof uses topological amenability of the action of $\mathrm{SL}(2,\mathbb{Z})$ on the Higson corona of $\mathbb{Z}^2$.

Key words: solid von Neumann algebra, amenable action.

1. Introduction

Let $\mathrm{SL}(2,\mathbb{Z}) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$ act by linear transformations on the 2-torus $\mathbb{T}^2$ with the Haar measure, and $L^\infty(\mathbb{T}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$ be the crossed product von Neumann algebra. Recall that a finite von Neumann algebra is called *solid* if every diffuse subalgebra has an amenable relative commutant. The main result of this paper is the following, which strengthens a result in [Oz1], [Oz2]. See [CI] for some application of this result to ergodic theory.

Theorem *The von Neumann algebra $L^\infty(\mathbb{T}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$ is solid.*

For the proof of Theorem, we take $L^\infty(\mathbb{T}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$ as the group von Neumann algebra of the semidirect product $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ of $\mathbb{Z}^2$ by the linear action of $\mathrm{SL}(2,\mathbb{Z})$, and study the behavior of the action at infinity. This involves the notion of amenability for a group action on a topological space, which we recall briefly. We refer the reader to [AR1, AR2, BO] for detailed accounts of amenable actions. For a discrete group Γ, we denote by

$$\mathcal{P}(\Gamma) = \{\mu \in \ell_1(\Gamma) : \mu \geq 0,\ \|\mu\| = 1\}$$

the space of probability measures on Γ, equipped with the norm topology (which coincides with the pointwise-convergence topology). The group Γ acts on $\mathcal{P}(\Gamma)$ by left translations: $(g\mu)(h) = \mu(g^{-1}h)$ for $g, h \in \Gamma$ and $\mu \in \mathcal{P}(\Gamma)$.

2000 Mathematics Subject Classification : Primary 46L35; Secondary 43A07, 37A20.

Definition Let Γ be a countable discrete group and X be a compact topological space on which Γ acts as homeomorphisms. We say the Γ-action (or the Γ-space X) is *amenable* if there is a sequence of continuous maps $\mu_n \colon X \to \mathcal{P}(\Gamma)$ such that

$$\forall g \in \Gamma, \quad \lim_{n\to\infty} \sup_{x\in X} \|\mu_n(gx) - g\mu_n(x)\| = 0.$$

We consider the linear action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^2$. Since the stabilizer subgroups of non-zero elements are all cyclic (amenable), it is easy to show the action of $\mathrm{SL}(2,\mathbb{Z})$ on the Stone-Čech remainder $\beta\mathbb{Z}^2 \setminus \mathbb{Z}^2$ of $\mathbb{Z}^2$ is amenable. We will prove a stronger proposition. The Higson corona $\partial\mathbb{Z}^2$ is defined to be the maximal quotient of $\beta\mathbb{Z}^2 \setminus \mathbb{Z}^2$, on which $\mathbb{Z}^2$ acts trivially:

$$C(\partial\mathbb{Z}^2) = \Big\{ f \in \ell_\infty(\mathbb{Z}^2) : \forall a \in \mathbb{Z}^2,\ \lim_{x\to\infty} |f(x+a) - f(x)| = 0 \Big\} / c_0(\mathbb{Z}^2).$$

The $\mathrm{SL}(2,\mathbb{Z})$-action on $\mathbb{Z}^2$ naturally gives rise to an $\mathrm{SL}(2,\mathbb{Z})$-action on $\partial\mathbb{Z}^2$.

Proposition *The* $\mathrm{SL}(2,\mathbb{Z})$*-action on* $\partial\mathbb{Z}^2$ *is amenable.*

2. Proof of Proposition

We consider the group $\mathrm{SL}(2,\mathbb{R}) = \big\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : a,b,c,d \in \mathbb{R},\ ad - bc = 1 \big\}$ acting on the real projective line $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ by linear fractional transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : t \mapsto \frac{at+b}{ct+d}.$$

The stabilizer of the point $\infty \in \widehat{\mathbb{R}}$ is the subgroup P of upper triangular matrices. Since P is a closed amenable subgroup of $\mathrm{SL}(2,\mathbb{R})$, the linear fractional action of $\mathrm{SL}(2,\mathbb{Z})$ on $\widehat{\mathbb{R}} \cong \mathrm{SL}(2,\mathbb{R})/P$ is amenable. For the proof of this fact, see Example 3.9 in [AR1] or Section 5.4 in [BO]. Now, we observe that the map $\varphi \colon \mathbb{Z}^2 \setminus \{0\} \to \widehat{\mathbb{R}}$, defined by $\varphi\left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right) = m/n$, is $\mathrm{SL}(2,\mathbb{Z})$-equivariant and satisfies

$$\lim_{x\to\infty} d(\varphi(x+a), \varphi(x)) = 0$$

for every $a \in \mathbb{Z}^2$, where d is a fixed metric on $\widehat{\mathbb{R}}$ which agrees with the topology. By considering $\varphi^*\colon C(\widehat{\mathbb{R}}) \to \ell_\infty(\mathbb{Z}^2)$, one sees that φ gives rise to an $\mathrm{SL}(2,\mathbb{Z})$-equivariant continuous map $\tilde{\varphi}\colon \partial\mathbb{Z}^2 \to \widehat{\mathbb{R}}$. It is clear from the definition that amenability of $\widehat{\mathbb{R}}$ implies that of $\partial\mathbb{Z}^2$. □

3. Proof of Theorem

The proof of Theorem is almost a verbatim translation of Section 4 of [Oz2], and we give it rather sketchily. For another approach, we refer the reader to Chapter 15 of [BO].

We follow the notations used in Section 4 of [Oz2] and plug $C^*_\lambda(\mathbb{Z}^2)$ into A and $\mathrm{SL}(2,\mathbb{Z})$ into Γ. We note that Γ is virtually-free and hence $\Gamma \in \mathcal{S}$, i.e., the left-and-right translation action of $\Gamma \times \Gamma$ on the Stone-Čech remainder $\beta\Gamma \setminus \Gamma$ of Γ is amenable. It is proved in [Oz2] that $\Gamma \ltimes \Lambda \in \mathcal{S}$ if $\Gamma \in \mathcal{S}$, Λ is amenable, and there is a map $\zeta\colon \Lambda \to \mathcal{P}(\Gamma)$ such that

$$\lim_{y\to\infty} \big(\|g\zeta(y) - \zeta(gy)\| + \|\zeta(xyx') - \zeta(y)\|\big) = 0$$

for all $g \in \Gamma$ and $x, x' \in \Lambda$. Indeed, for Corollary 4.5 in [Oz2], the only specific property we require of $\Lambda = \Delta_\Gamma$ is the existence of $\xi = \zeta^{1/2}$ in the proof of Proposition 4.4 in [Oz2]. From now on, let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and $\Lambda = \mathbb{Z}^2$ and view them as abstract multiplicative groups. It is left to construct $\zeta\colon \Lambda \to \mathcal{P}(\Gamma)$ satisfying the above condition. Although this can be done by modifying Proposition 4.1 in [Oz2], we give an alternative proof here. By (the proof of) Proposition, there is a sequence of maps $\zeta_n\colon \Lambda \to \mathcal{P}(\Gamma)$ such that

$$\limsup_{y\to\infty} \big(\|\zeta_n(gy) - g\zeta_n(y)\| + \|\zeta_n(xyx') - \zeta_n(y)\|\big) < 1/n$$

for all $n \in \mathbb{N}$, $g \in \Gamma$ and $x, x' \in \Lambda$. (Indeed, let $\zeta_n(x) = \mu_n(\varphi(x))$ for a suitable $\mu_n\colon \widehat{\mathbb{R}} \to \mathcal{P}(\mathrm{SL}(2,\mathbb{Z}))$ that verifies amenability of $\widehat{\mathbb{R}}$.) For $g \in \Gamma$, $x, x' \in \Lambda$, we define finite subsets $D_n(g; x, x') \subset \Lambda$ by

$$D_n(g;x,x') = \{y \in \Lambda : \|\zeta_n(gy) - g\zeta_n(y)\| + \|\zeta_n(xyx') - \zeta_n(y)\| \geq 1/n\}.$$

Take an increasing sequence $\{1\} = E_0 \subset E_1 \subset \cdots \subset \Gamma$ of finite symmetric subsets such that $\bigcup E_n = \Gamma$ and likewise for $\{1\} = F_0 \subset F_1 \subset \cdots \subset \Lambda$. We define finite subsets $\{1\} = \Omega_0 \subset \Omega_1 \cdots$ of Λ inductively by

$$\Omega_n = \bigcup_{g \in E_n,\, x, x' \in F_n,\, y \in \Omega_{n-1}} \big(D_n(g; x, x') \cup \{gy, xyx'\} \big)$$

for $n \geq 1$. We define $l(y) = \min\{n : y \in \Omega_n\}$ and define $\zeta \colon \Lambda \to \mathcal{P}(\Gamma)$ by

$$\zeta(y) = \frac{1}{l(y)} \sum_{n=0}^{l(y)-1} \zeta_n(y).$$

(The value of ζ at the unit 1 does not matter.) Let $g \in \Gamma$ and $x, x' \in \Lambda$ be given arbitrary and take k such that $g \in E_k$ and $x, x' \in F_k$. We observe that $|l(gy) - l(y)| \leq 1$ and $|l(xyx') - l(y)| \leq 1$ for every y with $l(y) > k$; and that $\|\zeta_n(gy) - g\zeta_n(y)\| + \|\zeta_n(xyx') - \zeta_n(y)\| < 1/n$ for every n with $k \leq n < l(y)$. It follows that

$$\lim_{l(y) \to \infty} \big(\|g\zeta(y) - \zeta(gy)\| + \|\zeta(xyx') - \zeta(y)\| \big) = 0,$$

which verifies the required condition. This proves $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z}) \in \mathcal{S}$, and hence the von Neumann algebra $L^\infty(\mathbb{T}^2) \rtimes \mathrm{SL}(2, \mathbb{Z}) \cong \mathcal{L}(\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z}))$ is solid by Theorem 6 in [Oz1]. □

Acknowledgment The essential part of this work was done during the author's stay at Banff International Research Station for the workshop "Topics in von Neumann Algebras." The author would like to thank the institute and the organizers for their kind hospitality, and Professor Asger Törnquist for raising a question during the workshop that dispelled the author's misconception. The travel was supported by Japan Society for the Promotion of Science.

References

[AR1] Anantharaman-Delaroche C. and Renault J., Amenable groupoids. *Groupoids in analysis, geometry, and physics (Boulder, CO, 1999)*, 35–46. Contemp. Math., 282. Amer. Math. Soc., Providence, RI, 2001.

[AR2] Anantharaman-Delaroche C. and Renault J., *Amenable groupoids*. With a foreword by Georges Skandalis and Appendix B by E. Germain. Monographies de L'Enseignement Mathématique, 36. Geneva, 2000.

[BO] Brown N. and Ozawa N., *C^*-algebras and finite-dimensional approximations*. Grad. Stud. Math., 88. Amer. Math. Soc., Providence, RI, 2008.

[CI] Chifan I. and Ioana A., *Ergodic Subequivalence Relations Induced by a Bernoulli Action*. Preprint arXiv:0802.2353.

[Oz1] Ozawa N., *Solid von Neumann algebras*. Acta Math. **192** (2004), 111–117.

[Oz2] Ozawa N., *A Kurosh type theorem for type II_1 factors*. Int. Math. Res. Not. **2006**, Art. ID 97560, 21 pp.

Department of Mathematical Sciences
University of Tokyo
Tokyo 153-8914
E-mail: narutaka@ms.u-tokyo.ac.jp