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# On a result of Saeki-Takahashi and a theorem of Bochner

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**Abstract.** Saeki extended the F. and M. Riesz theorem to $\mathbb{R}^N$ ($N \geqq 2$), and Takahashi extended Saeki's result to a LCA group. We give a result, which is relevant to theirs. We also give a strong version of Bochner's generalization of the F. and M. Riesz theorem.

*Key words*: LCA group, measure, Fourier transform, quasi-invariant.

## 1. Introduction

Let $G$ be a LCA group with the dual group $\hat{G}$. Let $L^1(G)$ and $M(G)$ be the group algebra and the measure algebra, respectively. We denote by $m_G$ the Haar measure of $G$. For $\mu$ in $M(G)$, $\hat{\mu}$ denotes the Fourier-Stieltjes transform of $\mu$, i.e., $\hat{\mu}(\gamma) = \int_G (-x, \gamma)\, d\,\mu(x)$ for $\gamma \in \hat{G}$. For a closed subset $E$ of $\hat{G}$, $M_E(G)$ denotes the space of measures in $M(G)$ whose Fourier-Stieltjes transform vanish off $E$, and $E$ is called a Riesz set if $M_E(G) \subset L^1(G)$. For a closed subgroup $H$ of $G$, $H^{\perp}$ stands for the annihilator of $H$.

Saeki [10] obtained the following theorem as an extension of the F. and M. Riesz theorem on $\mathbb{R}$.

**Theorem A** ([10, Theorem 2]) *Suppose $N \geqq 2$, and let $\mathbb{R}^N$ be the $N$-dimensional Euclidean space. Suppose $\mu \in M(\mathbb{R}^N)$ satisfies the following two conditions*:

( i ) $\hat{\mu}(t) = 0$ *for all* $t = (t_1, \ldots, t_N) \in \mathbb{R}^N$ *with* $t_1 \leqq 0$, *and*
( ii ) *for each* $t_1 > 0$, $\hat{\mu}(t_1, \cdot)$ *is the Fourier trnsform of some* $f_{t_1} \in L^1(\mathbb{R}^{N-1})$.

*Then $\mu$ is absolutely continuous with respect to $m_{\mathbb{R}^N}$.*

As an application of this theorem, he gave an alternative proof of a theorem of Bochner. Moreover, Takahashi [12] extended Theorem A to a LCA group as follows.

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**Theorem B** ([12, Theorem 2]) *Let $G$ be a LCA group, and let $H$ be a closed subgroup of $G$. Let $\tilde{E}$ be a Riesz set in $\hat{G}/H^\perp$, and put $E = \pi^{-1}(\tilde{E})$, where $\pi : \hat{G} \to \hat{G}/H^\perp$ is the natural homomorphism. Suppose $\mu \in M(G)$ satisfies the following two conditions*:

( i ) $\mu \in M_E(G)$, *and*
( ii ) *for each* $\gamma \in E$, $\alpha(\overline{\gamma}\mu) \in L^1(G/H)$, *where* $\alpha : G \to G/H$ *is the natural homomorphism and* $\overline{\gamma}$ *denotes the complex conjugate of* $\gamma$.

*Then $\mu$ is absolutely continuous with respect to $m_G$.*

On the other hand, Glicksberg obtained the following.

**Glicksberg's result** (cf. [3]) Suppose $\mu \in M(\mathbb{R}^2)$ satisfies the following two conditions:

( i ) $\hat{\mu}(t) = 0$ for all $t = (t_1, t_2) \in \mathbb{R}^2$ with $t_1 \leqq 0$, and
( ii ) $\int_{\mathbb{R}} |\hat{\mu}(t_1, s)|\, dm_{\mathbb{R}}(s) < \infty$ for a dense set of $t_1$.

Then $\mu$ is absolutely continuous with respect to $m_{\mathbb{R}^2}$.

We have a slight extension of Theorem B, which includes Glicksberg's result.

**Theorem C** *Under the assumption in Theorem B, let $\tilde{E}$ be a Riesz set in $\hat{G}/H^\perp$ and $\tilde{D}$ a dense subset of $\tilde{E}$. Put $E = \pi^{-1}(\tilde{E})$ and $D = \pi^{-1}(\tilde{D})$. Suppose $\mu \in M(G)$ satisfies the following two conditions*:

( i ) $\mu \in M_E(G)$, *and*
( ii ) $\alpha(\overline{\gamma}\mu) \in L^1(G/H)$ *for all* $\gamma \in D$.

*Then $\mu$ is absolutely continuous with respect to $m_G$.*

We prove Theorem C in the next section.

The F. and M. Riesz theorem on $\mathbb{R}$ states that if $\mu \in M(\mathbb{R})$ and $\hat{\mu}(t) = 0$ for $t < 0$, then $\mu$ is absolutely continuous. However, the following holds.

(1.1) If $\mu$ is a nonzero measure in $M(\mathbb{R})$ and $\hat{\mu}(t) = 0$ for $t < 0$, then $\mu$ and $m_{\mathbb{R}}$ are mutually absolutely continuous.

From the point of view of (1.1), we give a result, which is relevant to Theorem C. We also give a strong version of Bochner's generalization of the F. and M. Riesz theorem.

## 2. Notation and results

Let $G$ be a LCA group with the dual group $\hat{G}$. For $x \in G$, $\delta_x$ denotes the point mass at $x$. We denote by $Trig(G)$ the set of trigonometric polynomials on $G$. Let $C_o(G)$ be the Banach space of continuous functions on $G$ which vanish at infinity. Then $M(G)$ is identified with the dual space of $C_o(G)$. Let $M^+(G)$ be the set of nonnegative measures in $M(G)$. For $\mu \in M(G)$ and $f \in L^1(|\mu|)$, we often use the notation $\mu(f)$ as $\int_G f(x)d\mu(x)$.

**Definition 2.1** Let $G$ be a LCA group, and let $\mu \in M(G)$. $\mu$ is said to be quasi-invariant if $|\mu| * \delta_x \ll |\mu|$ for all $x \in G$.

**Remark 2.1** (cf. [14, Remark 4.1]) If there exists a nonzero measure $\mu \in M(G)$ that is quasi-invariant, then $G$ is $\sigma$-compact.

**Remark 2.2** (cf. [14, Proposition 4.1]) Let $G$ be a LCA group, and let $\mu$ be a nonzero measure in $M(G)$. Then the following are equivalent.

( i ) $\mu$ is quasi-invariant.
( ii ) $|\mu|$ and $m_G$ are mutually absolutely continuous.

**Definition 2.2** Let $G$ be a LCA group, and let $E$ be a closed subset of $\hat{G}$. We say that $E$ satisfies condition $(*)$ if the following holds.

$(*)$ For $\mu \in M_E(G)$, $\mu$ is quasi-invariant.

We state our results.

**Theorem 2.1** *Let $G$ be a $\sigma$-compact, LCA group, and let $H$ be a closed subgroup of $G$. Let $\tilde{E}$ be a closed set in $\hat{G}/H^\perp$ that satisfies condition $(*)$, and let $\tilde{D}$ be a dense subset of $\tilde{E}$. Put $E = \pi^{-1}(\tilde{E})$ and $D = \pi^{-1}(\tilde{D})$, where $\pi : \hat{G} \to \hat{G}/H^\perp$ is the natural homomorphism. Suppose a nonzero measure $\mu \in M(G)$ satisfies the following two conditions*:

( i ) *$\mu \in M_E(G)$, and*
( ii ) *for $\gamma \in D$ with $\alpha(\overline{\gamma}\mu) \neq 0$, $\alpha(\overline{\gamma}\mu)$ and $m_{G/H}$ are mutually absolutely continuous, where $\alpha : G \to G/H$ is the natural homomorphism.*

*Then $\mu$ and $m_G$ are mutually absolutely continuous.*

From this theorem, the following corollary follows immediately.

**Corollary 2.1** *Suppose $N \geqq 2$ and a nonzeo measure $\mu \in M(\mathbb{R}^N)$ satisfies the following two conditions*:

( i ) *$\hat{\mu}(t) = 0$ for all $t = (t_1, \ldots, t_N) \in \mathbb{R}^N$ with $t_1 \leqq 0$, and*
( ii ) *for $t_1 > 0$ with $\hat{\mu}(t_1, \cdot) \neq 0$, there exists $f_{t_1} \in L^1(\mathbb{R}^{N-1})$, with the property that $f_{t_1}$ and $m_{\mathbb{R}^{N-1}}$ are mutually absolutely continuous, such that $\hat{\mu}(t_1, s) = \hat{f_{t_1}}(s)$ for all $s \in \mathbb{R}^{N-1}$.*

*Then $\mu$ and $m_{\mathbb{R}^N}$ are mutually absolutely continuous.*

**Remark 2.3** An analogue of Corollary 2.1 holds for the $N$-dimensional torus $\mathbb{T}^N$.

For $x$, $y \in \mathbb{R}^N$, $\langle x, y\rangle$ stands for the inner product. We denote by $S$ the set of unit vectors in $\mathbb{R}^N$. For $a \in S$, let $\Omega_a$ be a set of closed sets $E$ in $\mathbb{R}^N$ which satisfy the following two conditions:

$$E \subset \{x \in \mathbb{R}^N : \langle x, a\rangle \geqq 0\}, \tag{2.1}$$

$$\text{for each } t > 0,\ E \cap \{x \in \mathbb{R}^N : \langle a, x - ta\rangle \leqq 0\} \text{ is a compact set.} \tag{2.2}$$

Set $\Omega = \bigcup_{a\in S,\, b\in\mathbb{R}^N} (\Omega_a + b)$, where $\Omega_a + b = \{E + b : E \in \Omega_a\}$. Evidently, proper cones in $\mathbb{R}^N$ belong to $\Omega$.

**Remark 2.4** Let $a$ be a unit vector in $\mathbb{R}^N$, and let $\theta$ be a rotation of $\mathbb{R}^N$. If $E \in \Omega_a$, then $\theta(E) \in \Omega_{\theta(a)}$.

**Corollary 2.2** *Let $E \in \Omega$, and let $\mu$ be a nonzero measure in $M_E(\mathbb{R}^N)$. Then $\mu$ and $m_{\mathbb{R}^N}$ are mutually absolutely continuous.*

An analogue of Corollary 2.2 holds for $\mathbb{T}^N$.

**Corollary 2.3** (cf. [1, Theorem 5]) *Let $E \in \Omega$, and let $\nu$ be a nonzero measure in $M_{E\cap\mathbb{Z}^N}(\mathbb{T}^N)$. Then $\nu$ and $m_{\mathbb{T}^N}$ are mutually absolutely continuous.*

**Example 2.1** (1) Let $f$ and $g$ be functions on $[0, \infty)$ such that $g(s) \leqq f(s)$ for all $s \in [0, \infty)$. Put $E = \{(s, t) \in \mathbb{R}^2 : s \geqq 0,\ g(s) \leqq t \leqq f(s)\}$. It follows from Corollary 2.1 that $E$ satisfies condition $(*)$.
(2) Let $\mathbb{Z}^+$ be the set of nonnegative integers, and let $f$ and $g$ be functions on $\mathbb{Z}^+$ such that $g(n) \leqq f(n)$ for all $n \in \mathbb{Z}^+$. Put $F = \{(n, m) \in \mathbb{Z}^2 : n \in \mathbb{Z}^+,\ g(n) \leqq m \leqq f(n)\}$. Then $F$ satisfies condition $(*)$, by Remark 2.3.

Before we close this section, we prove Theorem C. The following lemma is well-known.

**Lemma 2.1** *Let $G$ be a LCA group, and let $\mu \in M(G)$. Let $\gamma \in \hat{G}$, and let $\{\gamma_\alpha\}$ be a net, with $\gamma_\alpha \in \hat{G}$, such that $\lim_\alpha \gamma_\alpha = \gamma$. Then $\lim_\alpha \|\gamma_\alpha\mu - \gamma\mu\| = 0$.*

Now we prove Theorem C. It is easy to verify that $D$ is dense in $E$. Let $\gamma_0 \in E$. Since $D$ is dense in $E$, there exists $\gamma_\beta \in D$ such that $\lim_\beta \gamma_\beta = \gamma_0$. It follwos from Lemma 2.1 that

$$\lim_\beta \|\overline{\gamma}_\beta\mu - \overline{\gamma}_0\mu\| = 0,$$

which yields

$$\lim_\beta \|\alpha(\overline{\gamma}_\beta\mu) - \alpha(\overline{\gamma}_0\mu)\| = 0. \tag{2.3}$$

Since $\gamma_\beta \in D$, the hypothesis (ii) implies that $\alpha(\overline{\gamma}_\beta\mu) \in L^1(G/H)$. Thus we have, by (2.3),

$$\alpha(\overline{\gamma}_0\mu) \in L^1(G/H).$$

Since $\gamma_0$ is any element in $E$, it follows from Theorem B that $\mu \ll m_G$. This completes the proof of Theorem C.

## 3. Proofs of Theorem 2.1 and corollaries

In this section, we give the proof of Theorem 2.1. We also prove Corollaries 2.2 and 2.3. Following Takahashi [12], we use the theory of disintegration of measures.

**Proposition 3.1** *Theorem* 2.1 *holds for a $\sigma$-compact, metrizable LCA group $G$.*

*Proof.* By Theorem C, we have

(1) $\mu \ll m_G$.

Since $\mu$ is a nonzero measure in $M_E(G)$ and $D$ is dense in $E$, there exists $\gamma_0 \in D$ such that $\hat{\mu}(\gamma_0) \neq 0$. We note that $\alpha(\bar{\gamma_0}\mu) \neq 0$. Hence the hypothesis (ii) in the theorem implies that $m_{G/H} \ll |\alpha(\bar{\gamma_0}\mu)|$, which yields

(2) $m_{G/H} \ll \alpha(|\mu|)$.

It follows from (1) that

(3) $\alpha(|\mu|) \ll m_{G/H}$.

Thus,

(4) $\alpha(|\mu|)$ and $m_{G/H}$ are mutually absolutely continuous.

Put $\eta = \alpha(|\mu|)$. By the theory of disintegration of measures, there exists a family $\{\lambda_{\dot{x}}\}_{\dot{x}\in G/H}$ of measures in $M^+(G)$ with the following properties:

(5) $\dot{x} \to \lambda_{\dot{x}}(f)$ is a Borel measurable function on $G/H$ for each bounded Borel function $f$ on $G$,
(6) $\|\lambda_{\dot{x}}\| = 1$,
(7) $\mathrm{supp}(\lambda_{\dot{x}}) \subset \alpha^{-1}(\{\dot{x}\})$,
(8) $|\mu|(f) = \int_{G/H} \lambda_{\dot{x}}(f) d\eta(\dot{x})$ for each bounded Borel function $f$ on $G$.

Let $h$ be a bounded Borel function on $G$ such that $|h| = 1$ and $\mu = h|\mu|$. We define meaures $\mu_{\dot{x}} \in M(G)$ by $\mu_{\dot{x}} = h\lambda_{\dot{x}}$. Then $|\mu_{\dot{x}}| = \lambda_{\dot{x}}$, and we have the following:

(9) $\|\mu_{\dot{x}}\| = 1$,
(10) $\dot{x} \to \mu_{\dot{x}}(f)$ is a Borel measurable function on $G/H$ for each bounded Borel function $f$ on $G$,
(11) $\mathrm{supp}(\mu_{\dot{x}}) \subset \alpha^{-1}(\{\dot{x}\})$,
(12) $\mu(f) = \int_{G/H} \mu_{\dot{x}}(f) d\eta(\dot{x})$ for each bounded Borel function $f$ on $G$.

It follows from (11) that there exists $x \in \alpha^{-1}(\{\dot{x}\})$ and $\xi_{\dot{x}} \in M(H)$ such that

(13) $\mu_{\dot{x}} = \xi_{\dot{x}} * \delta_x$.

We note that $\lambda_{\dot{x}} = |\mu_{\dot{x}}| = |\xi_{\dot{x}}| * \delta_x$. Since $\mathrm{supp}(\hat{\mu}) \subset E(= \pi^{-1}(\tilde{E}))$, we have, by the proof of Lemma 4.2 in [11],

$$\xi_{\dot{x}} \in M_{\tilde{E}}(H) \quad \eta - \text{a.a.}\, \dot{x} \in G/H.$$

Hence the hypothesis that $\tilde{E}$ satisfies the condition $(*)$ implies that

(14) $|\xi_{\dot{x}}| \ll m_H,\, m_H \ll |\xi_{\dot{x}}| \quad \eta - \text{a.a.}\, \dot{x} \in G/H$.

Let $F$ be a Borel set in $G$ such that $|\mu|(F) = 0$. We have, by (8),

$$0 = |\mu|(F) = \int_{G/H} \lambda_{\dot{x}}(F) d\eta(\dot{x}).$$

Thus there exists a Borel set $\tilde{B}$ in $G/H$ such that

(15) $\eta(\tilde{B}) = 0$, and
(16) $\{\dot{x} \in G/H : \lambda_{\dot{x}}(F) > 0\} \subset \tilde{B}$.

It follows from (4) and (15) that $m_{G/H}(\tilde{B}) = 0$. Hence we have

(17)
$$\begin{aligned} m_G(F) &= \int_G \chi_F(x) dm_G(x) \\ &= \int_{G/H} \int_H \chi_F(x+y) dm_H(y) dm_{G/H}(\dot{x}) \\ &= \int_{\tilde{B}} \int_H \chi_F(x+y) dm_H(y) dm_{G/H}(\dot{x}) \\ &\quad + \int_{\tilde{B}^c} \int_H \chi_F(x+y) dm_H(y) dm_{G/H}(\dot{x}) \\ &= 0 + \int_{\tilde{B}^c} \int_H \chi_F(x+y) dm_H(y) dm_{G/H}(\dot{x}), \end{aligned}$$

where $\chi_F$ denotes the characteristic function of $F$. If $\dot{x} \notin \tilde{B}$, we have, by (16),

$$\begin{aligned} 0 = \lambda_{\dot{x}}(F) &= |\xi_{\dot{x}}| * \delta_x(F) \\ &= |\xi_{\dot{x}}|(F - x). \end{aligned}$$

This, together with (14), yields

$$\int_H \chi_F(x+y) dm_H(y) = m_H(F - x) = 0 \quad \eta - \text{a.a.}\, \dot{x} \in \tilde{B}^c,$$

which implies

$$\int_{\tilde{B}^c} \int_H \chi_F(x+y) dm_H(y) dm_{G/H}(\dot{x}) = 0.$$

Thus we have that $m_G(F) = 0$, by (17). This shows that

(18) $m_G \ll |\mu|$.

It follows from (1) and (18) that $|\mu|$ and $m_G$ are mutually absolutely continuous. This completes the proof. □

Before we prove Theorem 2.1, we state several lemmas. The following two lemmas are obtained in [14].

**Lemma 3.1** (cf. [14, Lemma 4.1]) *Let $G$ be a LCA group, and let $E$ be a closed subset of $\hat{G}$ that satisfies the condition $(*)$. Then, for any open subroup $\Gamma$ of $\hat{G}$, the following $(*)_\Gamma$ holds:*

$(*)_\Gamma$ *For any nonzero measure $\zeta \in M_{E\cap\Gamma}(G/\Gamma^\perp)$, $|\zeta|$ and $m_{G/\Gamma^\perp}$ are mutually absolutely continuous.*

**Lemma 3.2** (cf. [14, Proposition 4.2]) *Let $G$ be a LCA group, and let $\Gamma$ be an open subgroup of $\hat{G}$. Let $E$ be a closed subset of $\hat{G}$ contained in $\Gamma$. Suppose that $E$ satisfies the condition $(*)$ in $\Gamma$. Then $E$ satisfies the condition $(*)$ in $\hat{G}$.*

**Lemma 3.3** *Let $G$ be a $\sigma$-compact, LCA group, and let $\Gamma$ be an open subgroup of $\hat{G}$. Let $\mu$ be a nonzero measure in $M(G)$ with* $\mathrm{supp}(\hat{\mu}) \subset \Gamma$. *Then the following are equvalent.*

( i ) *$\mu$ is quasi-invariant.*
( ii ) *$\alpha_{\Gamma^\perp}(\mu)$ is quasi-invariant, where $\alpha_{\Gamma^\perp} : G \to G/\Gamma^\perp$ is the natural homomorphism.*

*Proof.* (i) $\Rightarrow$ (ii): Suppose that $\mu$ is quasi-invariant. Then $\mu \in L^1(G)$, and so

(1) $\alpha_{\Gamma^\perp}(\mu) \in L^1(G/\Gamma^\perp)$.

$\Gamma^\perp$ is a compact subgroup of $G$, and we have, by [5, (28.54) Theorem and (28.55) Theorem],

(2) $g \circ \alpha_{\Gamma^\perp} \in L^1(G)$ and $\int_{G/\Gamma^\perp} g(\dot{x})dm_{G/\Gamma^\perp}(\dot{x}) = \int_G g(\alpha_{\Gamma^\perp}(x))dm_G(x)$

for all $g \in L^1(G/\Gamma^\perp)$. We define a map $J : L^1(G/\Gamma^\perp) \to L^1(G)$ by

$$J(g) = g \circ \alpha_{\Gamma^\perp}.$$

For $g \in L^1(G/\Gamma^\perp)$, we have

$$J(g)^\wedge(\gamma) = \begin{cases} \hat{g}(\gamma) & \text{for } \gamma \in \Gamma \\ 0 & \text{for } \gamma \in \hat{G} \backslash \Gamma, \end{cases}$$

which, together with the hypothesis that $\mathrm{supp}(\hat{\mu}) \subset \Gamma$, yields

(3) $J(\alpha_{\Gamma^\perp}(\mu)) = \mu.$

Since $\mu$ is quasi-invariant, $|\mu|$ and $m_G$ are mutually absolutely continuous. Hence

(4) $\alpha_{\Gamma^\perp}(|\mu|)$ and $m_{G/\Gamma^\perp}$ are mutually absolutely continuous.

By the definition of $J$,

$$J(|\alpha_{\Gamma^\perp}(\mu)|) = |J(\alpha_{\Gamma^\perp}(\mu))|,$$

which, combined with (3), yields

$$|\mu| = J(|\alpha_{\Gamma^\perp}(\mu)|).$$

Since $|\alpha_{\Gamma^\perp}(\mu)| = \alpha_{\Gamma^\perp}(J(|\alpha_{\Gamma^\perp}(\mu)|)) = \alpha_{\Gamma^\perp}(|\mu|)$, it follows from (4) that $|\alpha_{\Gamma^\perp}(\mu)|$ and $m_{G/\Gamma^\perp}$ are mutually absolutely continuous. Hence $\alpha_{\Gamma^\perp}(\mu)$ is quasi-invariant.
(ii) $\Rightarrow$ (i): This can be proved by an argument similar to that in the proof of Proposition 4.2 in [14]. □

Now prove Theorem 2.1. Let $\mu$ be a nonzero measure in $M(G)$ that satisfies the conditions (i) and (ii) in the theorem. By Theorem C, we have

(1) $\mu \ll m_G$.

It is sufficient to prove that

(2) $m_G \ll |\mu|$.

Since $\mu$ is a nonzero measure in $M_E(G)$ and $D$ is dense in $E$, there exists $\gamma_0 \in D$ such that $\hat{\mu}(\gamma_0) \neq 0$. Then $\alpha(\overline{\gamma_0}\mu) \neq 0$. Suppose (2) does not hold. By (1), $\hat{\mu}$ belongs to $C_o(\hat{G})$. Hence there exists a $\sigma$-compact, open subgroup $\Gamma$ of $\hat{G}$, with $\gamma_0 \in \Gamma$, which satisfies the following:

(3) $\mathrm{supp}(\hat{\mu}) \subset \Gamma$, and
(4) $\alpha_{\Gamma^\perp}(m_G)$ $(= m_{G/\Gamma^\perp})$ is not absolutely continuous with respect to $\alpha_{\Gamma^\perp}(|\mu|)$, where $\alpha_{\Gamma^\perp} : G \to G/\Gamma^\perp$ is the natural homomorphism.

Put $E_o = E \cap \Gamma$. Let $\pi|_{\Gamma+H^\perp} : \Gamma + H^\perp \to (\Gamma + H^\perp)/H^\perp$ and $\beta : \Gamma \to \Gamma/\Gamma \cap H^\perp$ be the natural homomorphisms, respectively. Let $\tau : (\Gamma + H^\perp)/H^\perp \to \Gamma/\Gamma \cap H^\perp$ be a map defined by

$$\tau(\gamma + H^\perp) = \gamma + \Gamma \cap H^\perp \quad (\gamma \in \Gamma).$$

Then $\tau$ is a topological isomorphism (cf. [5, (5.33) Theorem]). We claim that

(5) $\beta(E_o)$ is a closed subst of $\Gamma/\Gamma \cap H^\perp$ which satisfies the condition $(*)$.

In fact, since $\beta^{-1}(\beta(E_o)) = E_o + \Gamma \cap H^\perp = E_o$, $\beta(E_o)$ is closed. Hence $\pi|_{\Gamma+H^\perp}(E_o) = \tau^{-1}(\beta(E_o))$ is also a closed subset of $(\Gamma + H^\perp)/H^\perp$. Since $\pi|_{\Gamma+H^\perp}(E_o) \subset \pi(E) = \tilde{E}$, Lemma 3.1 implies that $\pi|_{\Gamma+H^\perp}(E_o)$ satisfies the condition $(*)$ in $(\Gamma + H^\perp)/H^\perp$. Thus $\beta(E_o) = \tau(\pi|_{\Gamma+H^\perp}(E_o))$ satisfies the condition $(*)$ in $\Gamma/\Gamma \cap H^\perp$. This establishes the claim in (5).

Put $\tilde{G} = \alpha_{\Gamma^\perp}(G)$ and $\tilde{H} = \alpha_{\Gamma^\perp}(H)$. Then $\tilde{G}$ is a $\sigma$-compact, metrizable LCA group, and the annihilator of $\Gamma \cap H^\perp$ in $\tilde{G}$ coincides with $\tilde{H}$. Since $\gamma_0 \in \Gamma$ and $\alpha_{\Gamma^\perp}(\mu)^\wedge(\gamma_0) = \hat{\mu}(\gamma_0) \neq 0$,

(6) $\alpha_{\Gamma^\perp}(\mu)$ is a nonzero measure in $M_{E_o}(\tilde{G})$.

Put $D_o = D \cap \Gamma$. Let $\tilde{\pi} : \hat{\tilde{G}}(\cong \Gamma) \to \hat{\tilde{G}}/\tilde{H}^\perp(\cong \Gamma/\Gamma \cap H^\perp)$ be the natural homomorphism, and put $\tilde{D}_o = \tilde{\pi}(D_o)$ and $\tilde{E}_o = \tilde{\pi}(E_o)$. Then we have

(7) $\tilde{\pi}^{-1}(\tilde{D}_o) = D_o, \quad \tilde{\pi}^{-1}(\tilde{E}_o) = E_o.$

Since $D$ is dense in $E$, $D_o$ is dense in $E_o$. Hence $\tilde{D}_o$ is dense in $\tilde{E}_o$. Moreover, the following holds.

(8) For $\gamma \in D_o$ with $\tilde{\alpha}(\overline{\gamma}\alpha_{\Gamma^\perp}(\mu)) \neq 0$, $\tilde{\alpha}(\overline{\gamma}\alpha_{\Gamma^\perp}(\mu))$ and $m_{\tilde{G}/\tilde{H}}$ are mutually absolutely continuous, where $\tilde{\alpha} : \tilde{G} \to \tilde{G}/\tilde{H}$ is the natural homomorphism.

In fact, let $\gamma$ be an element in $D_o$ such that $\tilde{\alpha}(\overline{\gamma}\alpha_{\Gamma^\perp}(\mu)) \neq 0$. For $\omega \in \Gamma \cap H^\perp(\cong \tilde{G}\hat{/}\tilde{H})$, we have

$$\begin{aligned}\tilde{\alpha}(\overline{\gamma}\alpha_{\Gamma^\perp}(\mu))^\wedge(\omega) = (\overline{\gamma}\alpha_{\Gamma^\perp}(\mu))^\wedge(\omega) &= \hat{\mu}(\omega + \gamma) \\ &= \alpha(\overline{\gamma}\mu)^\wedge(\omega),\end{aligned}$$

which yields

(9) $\alpha(\overline{\gamma}\mu) \neq 0.$

Hence, by the condition (ii) of the theorem, we have

(10) $\alpha(\overline{\gamma}\mu)$ and $m_{G/H}$ are mutually absolutely continuous.

We note that $\widehat{G/H} \cong H^{\perp}$ and $\Gamma \cap H^{\perp}$ is an open subgroup of $H^{\perp}$. Moreover, we have, by (3) and the fact that $\gamma \in \Gamma$,

$$\mathrm{supp}(\alpha(\overline{\gamma}\mu)^{\wedge}) \subset \Gamma \cap H^{\perp}.$$

Thus (10) and Lemma 3.3 imply that

(11) $\sigma(\alpha(\overline{\gamma}\mu))$ and $m_{\tilde{G}/\tilde{H}}$ are mutually absolutely continuous,

where $\sigma : G/H \to \tilde{G}/\tilde{H}$ is a continuous homomorphism defined by

$$\sigma(x+H) = \alpha_{\Gamma^{\perp}}(x) + \tilde{H} \quad (x \in G).$$

For $\omega \in \Gamma \cap H^{\perp}$, we have, by an argument used above (9),

$$\begin{aligned} \sigma(\alpha(\overline{\gamma}\mu))^{\wedge}(\omega) &= \alpha(\overline{\gamma}\mu)^{\wedge}(\omega) \\ &= \tilde{\alpha}(\overline{\gamma}\alpha_{\Gamma^{\perp}}(\mu))^{\wedge}(\omega), \end{aligned}$$

which yields

(12) $\sigma(\alpha(\overline{\gamma}\mu)) = \tilde{\alpha}(\overline{\gamma}\alpha_{\Gamma^{\perp}}(\mu)).$

By (11) and (12), $\tilde{\alpha}(\overline{\gamma}\alpha_{\Gamma^{\perp}}(\mu))$ and $m_{\tilde{G}/\tilde{H}}$ are mutually absolutely continuous. This shows that (8) holds.

We note that $\tilde{E}_o$ satisfies the condition $(*)$, by (5). Since $\tilde{G}$ is $\sigma$-compact and metrizable, it follows from (6), (7), (8) and Proposition 3.1 that $\alpha_{\Gamma^{\perp}}(\mu)$ and $m_{\tilde{G}}(= m_{G/\Gamma^{\perp}})$ are mutually absolutely continuous. Hence, in particular,

$$m_{G/\Gamma^{\perp}} \ll |\alpha_{\Gamma^{\perp}}(\mu)| \ll \alpha_{\Gamma^{\perp}}(|\mu|),$$

which contradicts (4). This completes the proof of Theorem 2.1.

Let $k(t) = \frac{1}{\pi} \cdot \frac{1-\cos t}{t^2}$. Then $\hat{k}(s) = \int_{-\infty}^{\infty} k(t)e^{-ist}dt = \max(1 - |s|, 0)$. We define functions $w(t)$ and $\Delta(x)$ on $\mathbb{R}^N$ as follows:

$$w(t) = \prod_{k=1}^{N} \frac{1}{\pi} \cdot \frac{1 - \cos t_k}{t_k^2} \quad (t = (t_1, \ldots, t_N) \in \mathbb{R}^N);$$

$$\Delta(x) = \prod_{k=1}^{N} \max(1 - |x_k|, 0) \quad (x = (x_1, \ldots, x_N) \in \mathbb{R}^N).$$

We note that $\hat{w}(x) = \Delta(x)$ $(x \in \mathbb{R}^N)$.

For $\mu \in M(\mathbb{T}^N)$, let $\tilde{\mu}$ be the periodic extension of $\mu$ to $\mathbb{R}^N$, i.e., for a Borel set $E \subset [0, 2\pi) \times \cdots \times [0, 2\pi) + 2\pi n$ $(n \in \mathbb{Z}^N)$,

$$\tilde{\mu}(E) = \mu(E - 2\pi n).$$

Then $w\tilde{\mu}$ belongs to $M(\mathbb{R}^N)$. We define a map $J : M(\mathbb{T}^N) \to M(\mathbb{R}^N)$ by

$$J(\mu) = w\tilde{\mu}. \tag{3.1}$$

We need the following lemma to prove our corollaries.

**Lemma 3.4** (cf. [8, Lemma 1]) *For $\mu \in M(\mathbb{T}^N)$, we have*

$$J(\mu)^\wedge(x) = \sum_{n \in \mathbb{Z}^N} \hat{\mu}(n)\Delta(x - n). \tag{3.2}$$

*Furthermore $J$ is an isometry, and the following hold.*

( i ) *$J(\mu) \geqq 0$ if and only if $\mu \geqq 0$.*
( ii ) *$J(\mu) \in L^1(\mathbb{R}^N)$ if and only if $\mu \in L^1(\mathbb{T}^N)$.*
(iii) *$J(\mu)$ is quasi-invariant if and only if $\mu$ is quasi-invariant.*

*Proof.* (3.2), (i)–(ii) and the fact that $J$ is an isometry follow from Lemma 1 in [8]. Considering zero points of $w(t)$, we obtain (iii) by the definition of $J$. This completes the proof. □

**Lemma 3.5** *Let $e_1 = (1, 0, \cdots, 0)$ be the unit vector in $\mathbb{R}^N$, and let $E \in \Omega_{e_1}$. Let $\mu$ be a nonzero measure in $M_E(\mathbb{R}^N)$. Then $\mu$ and $m_{\mathbb{R}^N}$ are mutually absolutely continuous.*

*Proof.* For $t_1 > 0$, suppose $\hat{\mu}(t_1, \cdot) \neq 0$. Then there exists a nonzero measure $\nu_{t_1} \in M(\mathbb{R}^{N-1})$ such that

$$\hat{\nu_{t_1}}(s) = \hat{\mu}(t_1, s) \quad \text{for all } s \in \mathbb{R}^{N-1}.$$

Since $\mu \in M_E(\mathbb{R}^N)$ and $E$ belongs to $\Omega_{e_1}$, $\mathrm{supp}(\hat{\nu_{t_1}})$ is a compact set in $\mathbb{R}^{N-1}$. Compact sets in $\mathbb{R}^{N-1}$ satisfy the condition $(*)$; hence the lemma follows from Corollary 2.1. $\square$

Now we prove Corollary 2.2. Considering translation of $E$, we may assume that $E \in \Omega_a$ for some unit vector $a$ in $\mathbb{R}^N$. Let $\theta$ be a rotation of $\mathbb{R}^N$ such that $\theta(e_1) = a$. Let $\mu_\theta = \theta^{-1}(\mu)$, the continuous image of $\mu$ under $\theta^{-1}$. Then $\hat{\mu_\theta} = \hat{\mu} \circ \theta$, which yields

$$\mu_\theta \in M_{\theta^{-1}(E)}(\mathbb{R}^N). \tag{3.3}$$

It follwos from from Remark 2.4 that $\theta^{-1}(E) \in \Omega_{\theta^{-1}(a)} = \Omega_{e_1}$. Hence we have, by Lemma 3.5,

$$\mu_\theta \ll m_{\mathbb{R}^N}, \quad m_{\mathbb{R}^N} \ll |\mu_\theta|.$$

Thus $\mu$ and $m_{\mathbb{R}^N}$ are mutually absolutely continuous, and the proof is complete.

Next we prove Corollary 2.3. By translaion of $E \cap \mathbb{Z}^N$ by an element in $\mathbb{Z}^N$, we may assume that $E \in \Omega_a$ for some unit vector $a$ in $\mathbb{R}^N$. Since $\nu \in M_{E\cap\mathbb{Z}^N}(\mathbb{T}^N)$, we have

$$\mathrm{supp}(J(\nu)^\wedge) \subset Q + E, \tag{3.4}$$

where $Q = [-1, 1]^N$. Since $Q + E$ is a set in $\Omega$ and $J(\nu)$ is a nonzero measure in $M_{Q+E}(\mathbb{R}^N)$, it follows from Corollary 2.2 that $J(\nu)$ and $m_{\mathbb{R}^N}$ are mutually absolutely continuous; hence $\nu$ and $m_{\mathbb{T}^N}$ are so, by Lemma 3.4. This completes the proof.

## References

[ 1 ] Bochner S., *Boundary values of analytic functions in several variables and of almost periodic functions.* Ann. of Math. **45** (1944), 708–722.

[ 2 ] DeLeeuw K. and Glicksberg I., *Quasi-invariance and analyticity of measures on compact groups*. Acta Math. **109** (1963), 179–205.

[ 3 ] Glicksberg I., *Fourier-Stieltjes transforms with small supports*. Illinois J. Math. **9** (1965), 418–427.

[ 4 ] Graham C.C., *Fourier-Stieltjes transforms with small supports*. Illinois J. Math. **18** (1974), 532–534.

[ 5 ] Hewitt E. and Ross K.A., Abstract Harmonic Analysis, Volumes I and II, New York-Heidelberg-Berlin, Springer, 1963 and 1970.

[ 6 ] MacLean H.A., *Riesz sets and a theorem of Bochner*. Pacific J. Math. **113** (1984), 115–135.

[ 7 ] Pigno L. and Saeki S., *Fourier-Stieltjes transforms which vanish at infinity*. Math. Z. **141** (1975), 83–91.

[ 8 ] Pigno L. and Saeki S., *Constructions of singular measures with remarkable convolution properties*. Studia Math. **LXIX** (1980), 133–141.

[ 9 ] Rudin W., Fourier Analysis on Groups, New York, Interscience, 1962.

[10] Saeki S., *The F. and M. Riesz theorem and singular measures*. Proc. Amer. Math. Soc. **90** (1984), 391–396.

[11] Takahashi Y. and Yamaguchi H., *On measures which are continuous by certain translation*. Hokkaido Math. J. **13** (1984), 109–117.

[12] Takahashi Y., *On a theorem of S. Saeki*. Hokkaido Math. J. **15** (1986), 157–161.

[13] Yamaguchi H., *On the product of a Riesz set and a small p set*. Proc. Amer. Math. Soc. **81** (1981), 273–278.

[14] Yamaguchi H., *The F. and M. Riesz theorem on certain transformation groups*, Hokkaido Math. J. **17** (1988), 289–332.

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