

Logarithmic vector fields along smooth divisors in projective spaces

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Abstract. We show that a smooth divisor in a projective space can be reconstructed from the isomorphism class of the sheaf of logarithmic vector fields along it if and only if its defining equation is of Sebastiani–Thom type.

Key words: logarithmic vector fields, vector bundles on projective spaces.

1. Introduction

Let D be a smooth divisor in \mathbb{P}^n defined by a homogeneous polynomial f of degree k over the field of complex numbers. We say that f is of *Sebastiani–Thom type* if f can be represented as the sum

$$f(x_0, \dots, x_n) = f_1(x_0, \dots, x_l) + f_2(x_{l+1}, \dots, x_n)$$

for a choice of a homogeneous coordinate $(x_i)_{i=0}^n$ of \mathbb{P}^n and some $0 \leq l \leq n - 1$.

We study the Torelli problem for logarithmic vector fields in the sense of Dolgachev and Kapranov [1]. For a divisor D in the projective space \mathbb{P}^n , the sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ of logarithmic vector fields along D is defined as the subsheaf of the tangent sheaf $\mathcal{T}_{\mathbb{P}^n}$ whose section consists of vector fields tangent to D . It is the sheafification of

$$D_0(-\log f) = \{\delta \in \text{Der}_A \mid \delta f = 0\},$$

where A is a homogeneous coordinate ring of \mathbb{P}^n and $f \in A$ is the defining polynomial of D . A divisor D is said to be *Torelli* if the isomorphism class of $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ as an $\mathcal{O}_{\mathbb{P}^n}$ -module determines D uniquely. The main theorem of Dolgachev and Kapranov [1] is a condition for an arrangement of sufficiently many hyperplanes to be Torelli.

The main result in this paper is the following:

Theorem 1 *A smooth divisor in a projective space is Torelli if and only if its defining equation is not of Sebastiani–Thom type.*

The strategy for the proof is the following:

1. The Jacobi ideal of a smooth divisor D of degree k is determined by the set of divisors E of degree $k - 1$ such that the dimension of $H^0(\mathcal{T}_{\mathbb{P}^n}(-\log D)(-1)|_E)$ jumps up.
2. A smooth divisor is determined by its Jacobi ideal if it is not of Sebastiani–Thom type.
3. A divisor D is not Torelli if its defining equation is of Sebastiani–Thom type.

As a corollary of Theorem 1, we give another proof of the main theorem of [2] that a smooth plane cubic curve is Torelli if and only if its j -invariant does not vanish.

2. Jacobi ideals from logarithmic vector fields

Let D be a smooth divisor of degree k in \mathbb{P}^n defined by a homogeneous polynomial f , and $\mathcal{T}_{\mathbb{P}^n}(-\log D) \subset \mathcal{T}_{\mathbb{P}^n}$ be the sheaf of logarithmic vector fields along D . We have an exact sequence

$$0 \rightarrow \mathcal{T}_{\mathbb{P}^n}(-\log D) \rightarrow \mathcal{T}_{\mathbb{P}^n} \rightarrow \mathcal{N}_{D/\mathbb{P}^n} \rightarrow 0,$$

where $\mathcal{N}_{D/\mathbb{P}^n}$ is the normal bundle of D in \mathbb{P}^n , and an isomorphism

$$\begin{array}{ccc} df : \mathcal{N}_{D/\mathbb{P}^n} & \xrightarrow{\sim} & \mathcal{O}_D(k) \\ \downarrow & & \downarrow \\ X & \mapsto & Xf \end{array}$$

of $\mathcal{O}_{\mathbb{P}^n}$ -modules. By the Euler sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \rightarrow \mathcal{T}_{\mathbb{P}^n}(-1) \rightarrow 0,$$

the space $H^0(\mathcal{T}_{\mathbb{P}^n}(-1))$ of global sections of $\mathcal{T}_{\mathbb{P}^n}(-1)$ is spanned by $\{\partial/\partial x_i\}_{i=0}^n$. The image of the map

$$H^0(\mathcal{T}_{\mathbb{P}^n}(-1)) \rightarrow H^0(\mathcal{O}_D(k-1))$$

induced by the composition

$$\mathcal{T}_{\mathbb{P}^n}(-1) \rightarrow \mathcal{N}_{D/\mathbb{P}^n}(-1) \rightarrow \mathcal{O}_D(k-1)$$

is the restriction to D of the degree $k-1$ part

$$J(f)_{k-1} = \text{span}\{\partial f / \partial x_i\}_{i=0}^n$$

of the Jacobi ideal $J(f)$ of f .

Lemma 2 *For a divisor E of \mathbb{P}^n of degree $k-1$, the dimension of $H^0(\mathcal{T}_{\mathbb{P}^n}(-\log D)(-1)|_E)$ jumps up if and only if the defining equation of E is contained in the Jacobi ideal of D .*

Proof. Since D is smooth,

$$\text{Tor}_1^{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_D, \mathcal{O}_E) = 0$$

and we have an exact sequence

$$0 \rightarrow \mathcal{T}_{\mathbb{P}^n}(-\log D)(-1)|_E \rightarrow \mathcal{T}_{\mathbb{P}^n}(-1)|_E \rightarrow \mathcal{O}_{D \cap E}(k-1) \rightarrow 0,$$

from which follows the long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{T}_{\mathbb{P}^n}(-\log D)(-1)|_E) &\rightarrow H^0(\mathcal{T}_{\mathbb{P}^n}(-1)|_E) \rightarrow H^0(\mathcal{O}_{D \cap E}(k-1)) \\ &\rightarrow H^1(\mathcal{T}_{\mathbb{P}^n}(-\log D)(-1)|_E) \rightarrow \dots \end{aligned}$$

Note that the image of the map

$$H^0(\mathcal{T}_{\mathbb{P}^n}(-1)|_E) \rightarrow H^0(\mathcal{O}_{D \cap E}(k-1))$$

is the restriction to $D \cap E$ of the degree $k-1$ part of the Jacobi ideal of D . Since the dimension of $H^0(\mathcal{T}_{\mathbb{P}^n}(-1)|_E)$ does not depend on E , the dimension of $H^0(\mathcal{T}_{\mathbb{P}^n}(-\log D)(-1)|_E)$ jumps up if and only if the defining equation of E is contained in the Jacobi ideal of D . \square

3. Divisors from their Jacobi ideals

We prove the following in this section:

Lemma 3 *If two smooth distinct divisors in \mathbb{P}^n have identical Jacobi ideals, their defining equations are of Sebastiani–Thom type.*

Proof. We divide the proof into steps. Let f and g be homogeneous polynomials of degree k defining distinct smooth hypersurfaces such that their Jacobi ideals $J(f)$ and $J(g)$ are identical.

Step 1 The pencil over f and g contains a polynomial F such that $\partial_0 F = \cdots = \partial_l F = 0$ and $\{\partial_i F\}_{i=l+1}^n$ is linearly independent for some integer l and a suitable choice of a homogeneous coordinate $(x_i)_{i=0}^n$ of \mathbb{P}^n .

Indeed, any pencil of projective hypersurfaces contains a singular element F , and the assumption $J(f) = J(g)$ implies that $\partial_0 F, \dots, \partial_n F$ are linearly dependent. Let l be n minus the dimension of the linear span of $\{\partial_i F\}_{i=0}^n$. Then we can choose a homogeneous coordinate so that $\partial_i F = 0$ for $i = 0, \dots, l$ and $\{\partial_i F\}_{i=l+1}^n$ is linearly independent. Note that one has $l < n$ since the divisors defined by f and g are distinct.

Step 2 There exists a matrix $(a_{ij})_{i,j=0}^n$ such that $\det(a_{i,j})_{i,j=l+1}^n \neq 0$ and

$$\frac{\partial F}{\partial x_i} = \sum_{j=0}^n a_{ij} \frac{\partial f}{\partial x_j}$$

for $i = 0, \dots, n$.

The existence of the matrix $(a_{ij})_{i,j=0}^n$ follows from the inclusion $J(F) \subset J(f)$. We will show that if $\det(a_{i,j})_{i,j=l+1}^n$ vanishes, then the hypersurface defined by f is singular. Indeed, the vanishing of $\det(a_{i,j})_{i,j=l+1}^n$ and linear independence of $\{\partial_i F\}_{i=l+1}^n$ and of $\{\partial_i f\}_{i=0}^n$ imply that some linear combination of $\{\partial_i f\}_{i=0}^l$ is a linear combination of $\{\partial_i F\}_{i=l+1}^n$. Then one can choose a homogeneous coordinate so that $\partial_0 f$ is a linear combination of $\{\partial_j F\}_{j=l+1}^n$. Assume that $\deg f \geq 2$, since any linear form is of Sebastiani–Thom type. Note that F does not depend on $\{x_i\}_{i=0}^l$ since $\partial_i F = 0$ for $i = 0, \dots, l$. It follows that $[1 : 0 : \cdots : 0] \in \mathbb{P}^n$ is a singular point of the hypersurface defined by f , since $f(x_0, \dots, x_n)$ is the sum of x_0 times some linear combination of $\{\partial_j F\}_{j=l+1}^n$ and terms which are zero at

$$x_1 = \cdots = x_n = 0.$$

Step 3 There is a homogeneous coordinate $(X_i)_{i=0}^n$ such that $\partial_i F = 0$ for $i = 0, \dots, l$ and $\partial_i f \in J(F)$ for $i = l + 1, \dots, n$.

Since the $(n - l) \times (n - l)$ matrix $(a_{ij})_{i,j=l+1}^n$ is invertible, one can find an $(n - l) \times (n + 1)$ matrix (b_{ij}) such that

$$\sum_{j=0}^l b_{ij} \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_i} \in J(F)$$

for $i = l + 1, \dots, n$. Now make the projective coordinate transformation from $(x_i)_{i=0}^n$ to $(X_i)_{i=0}^n$ defined by

$$x_j = \begin{cases} X_j + \sum_{i=l+1}^n b_{ij} X_i & 0 \leq j \leq l, \\ X_j & l + 1 \leq j \leq n. \end{cases}$$

Then one has

$$\begin{aligned} \frac{\partial f}{\partial X_i} &= \sum_{j=0}^n \frac{\partial x_j}{\partial X_i} \frac{\partial f}{\partial x_j} \\ &= \begin{cases} \frac{\partial f}{\partial x_i} & i = 0, \dots, l, \\ \sum_{j=0}^n b_{ij} \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_i} & i = l + 1, \dots, n. \end{cases} \end{aligned}$$

This implies

$$\frac{\partial F}{\partial X_i} = \frac{\partial F}{\partial x_i} = 0$$

for $0 \leq i \leq l$ and

$$\frac{\partial f}{\partial X_i} \in J(F)$$

for $l + 1 \leq i \leq n$.

Step 4 f is of Sebastiani–Thom type.

The fact

$$\frac{\partial F}{\partial X_0} = \cdots = \frac{\partial F}{\partial X_l} = 0$$

and

$$\frac{\partial f}{\partial X_i} \in J(F)$$

for $l + 1 \leq i \leq n$ shows

$$\frac{\partial^2 f}{\partial X_i \partial X_j} = 0$$

for $0 \leq i \leq l$ and $l + 1 \leq j \leq n$. This implies that f is of Sebastiani–Thom type. \square

Since the isomorphism class of the sheaf of logarithmic vector fields along the divisor defined by $\mu F(x_0, \dots, x_l) + \nu G(x_{l+1}, \dots, x_n)$ does not depend on the choice of $(\mu, \nu) \in (\mathbb{C}^\times)^2$, a divisor is not Torelli if its defining equation is of Sebastiani–Thom type.

4. Smooth plane cubic curves

Theorem 1 immediately yields the following:

Theorem 4 ([2, Theorem 7]) *A smooth plane cubic curve is Torelli if and only if its j -invariant does not vanish.*

Proof. Since a smooth plane cubic curve has a vanishing j -invariant if and only if it is defined by the Fermat polynomial

$$x^3 + y^3 + z^3$$

for a suitable choice of a homogeneous coordinate, it suffices to show that any cubic polynomial of the form

$$f(x) + g(y, z)$$

can be brought to the Fermat polynomial by a projective linear coordinate change, which is obvious. \square

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