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Chu Correspondences

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Abstract. The concept of Chu correspondences between formal contexts is introduced. The construction of formal concepts induces a functor *Gal* from the category of Chu correspondences to the category of sup-preserving maps between complete lattices. It turns out that the category of Chu correspondences has a $*$-autonomous category structure and the functor *Gal* is shown to preserve the $*$-autonomous category structure.

Key words: Chu space, formal concept analysis, $*$-autonomous category.

Introduction

In spite of the extensional philosophy behind modern mathematics as is expressed by the adoption of set theory as the foundation, quite a few mathematical structures carry certain dualism, in the sense that their structures are described by relationships between two entities quite different in nature. For example, in classical geometry, the "structure" of a plane is described by the incidence relation between points and lines.

In modern geometry, a space has its "locus" and its "function algebra" and the structure is often encoded either in the evaluation map.

Most of these "dualism" are captured by the mathematical structure called formal contexts (A, X, R), where

$$R : A \times X \to V$$

is a map. A formal concept (A, X, R) is called *binary* if $V = \{0, 1\}$. A binary formal context is give by a relation $R \subset A \times X$ defined by $R = 1$.

For classical plane geometry, A is the set of points and X is the set of lines and $R(a, x) = 1$ means the point a is on the line x. For modern geometry, A is the set of points of a manifold and X is the set of admissible functions on A and $R(x, a) = x(a)$. A topological space is a formal context $(A, \mathcal{A}, R)$, where $\mathcal{A}$ is the set of closed subsets and $R(a, F) = 1$ iff $a \in F$. For the model theory of a first order theory T, A is a model of T and X is

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the set of first order formula with one free variable and $a \models \varphi$ means $\varphi(a)$ is true on A (See §A).

The homomorphisms between mathematical structures with formal context descriptions $(A_i,\, X_i,\, \models_i)$ $(i = 1,\, 2)$, usually induce Chu maps, namely a pair of maps

$$\ell\colon A_1 \to A_2 \quad r\colon X_2 \to X_1$$

satisfying the relation

$$R(\ell(a_1),\, x_2) = R(a_1,\, r(x_2)),$$

for $a_1 \in A_1$ and $x_2 \in X_2$.

For example, a continuous map between topological spaces $(A_i,\, X_i,\, \models_i$ $)$ $(i = 1,\, 2)$, is a map $\ell\colon A_1 \to A_2$ satisfying $\ell^{-1}(X_2) \subset X_1$ so that $(\ell,\, r)$ with $r := \ell^{-1}\colon X_2 \to X_1$ is a Chu map. Conversely, a Chu map $(\ell,\, r)$ from $(X_1,\, A_1,\, \models_1)$ to $(X_2,\, A_2,\, \models_2)$ comes from a continuous map ℓ.

For many mathematical structures, hence, the category of structure preserving maps is also described as the subcategory of Chu maps where the objects are restricted to those formal contexts arising from them. The category of Chu maps has rich methods of construction of objects and for example is a $*$-autonomous category. Although, for example, the dual $(\mathcal{A},\, A)$ of the formal context $(A,\, \mathcal{A})$ of a topological space is not a topological space in the usual sense, we can extend the second factor A of $(\mathcal{A},\, A)$ in a natural way to a topological structure on the huge set $\mathcal{A}$.

The mechanism which enables the formal context to describe mathematical structure concisely is *the formal concept lattice construction.*

The formal concept lattice of a binary formal context $(A,\, X,\, R)$ is the intersection closed family $\mathcal{A} \subset \mathbf{pow}(A)$ generated by the polar sets

$$x^* := \{a \in A \mid R(a,\, x) = 1\},$$

which is a complete lattice. This lattice is anti-isomorphic to the intersection closed family $\mathcal{X} \subset \mathbf{pow}(X)$ generated by the polar sets

$$a^* := \{x \in X \mid R(a,\, x) = 1\}.$$

More generally, if the value space V is a Heyting algebra, then the formal concept lattice is defined.

The formal concept lattice of the binary formal context (V, V^*, R), where V is a linear space and $R(v,\, \phi) = 1$ iff $\phi(v) = 0$ is the space of linear subspaces of V, which is the disjoint union of various Grassmann manifolds. The meet is the set theoretical intersection and the join is the sum of linear subspaces. The formal concept lattice of the binary formal context $(A, \mathcal{A}, R)$ of a topological space is anti-isomorphic to $\mathcal{A}$.

In fact, we have a functor, called Galois functor [13], from the category of Chu maps to the category of join preserving maps between complete lattices, which preserve the $*$-autonomous category.

The Galois functor is neither faithful nor full in general. For example, if $\mathbf{C}_i = (V_i, V_i^*, \models)$ $(i = 1, 2)$ are formal contexts of linear spaces, the Chu map (f, f^*) which corresponds to a linear map $f\colon V_1 \to V_2$ corresponds to the same join preserving map as the Chu map $(cf, (cf)^*)$ does if $c \neq 0$. Hence the Galois functor is not faithful. On the other hand, suppose $V_2 = V_1 \times W$ and consider the join preserving map

$$\kappa\colon Gal(\mathbf{C}_1) \to Gal(\mathbf{C}_2)$$

which maps a subspace $U \subset V_1$ to $U \times W \subset V_2$. Then κ corresponds to no linear maps from V_1 to V_2, whence the Galois functor is not full.

The *Chu correspondences* fill the gaps between Chu maps and the join preserving map. In fact, the "Galois functor" is defined also on the category of Chu correspondences and turns out to be full and faithful.

The definition of Chu correspondences is as follows. Let $\mathbf{C}_i = (A_i, X_i, R_i)$ $(i = 1, 2)$ be binary formal contexts. *A Chu correspondence from* $\mathbf{C}_1$ *to* $\mathbf{C}_2$ *in the weak sense* is a pair of maps $L\colon A_1 \to \mathbf{pow}(A_2)$ and $R\colon X_2 \to \mathbf{pow}(X_1)$ satisfying

$$R_2(La_1, x_2) = R_1(a_1, Rx_2),$$

where

$$R\colon \mathbf{pow}(A) \times \mathbf{pow}(X) \to \{0, 1\}$$

is defined by

$$R(B, Y) = \inf_{b \in B,\, y \in Y} R(b, y).$$

Hence $R(B, Y) = 1$ iff $R(b, y) = 1$ for all $b \in B$ and $y \in Y$.

A Chu correspondence (L, R) is a Chu correspondence in the weak sense satisfying the condition that $Lx_1 \subset X_2$ and $Ra_2 \subset A_1$ are closed for all $x_1 \in X_1$ and $a_2 \in A_2$.

The category of Chu correspondences is equivalent to the category of join preserving maps. Not only the concept of formal contexts gives us a method of describing concisely complete lattices, but also the concept of Chu correspondence often enables us to describe concisely join preserving maps, whose description generally tends to be complicated in finite mathematics.

There are natural operations such as tensor products, internal homs, duals in the category of Chu correspondences, which corresponds to those in the category of join preserving maps.

In fact, we show that the category of Chu correspondences has a structure of $*$-autonomous category, for which the Galois functor is a $*$-autonomous functor.

In Section one, we recall a few facts from lattice theory and formal concept analysis to fix terminologies and notations. In Section two, we define the concept of Chu correspondence and study its basic properties and give basic bijection between Chu correspondences and bonds (Theorem 38). Section three gives examples of Chu correspondences, both concrete and conceptual. Most examples show there are much more Chu correspondences than Chu maps between formal contexts in general. In Section four, we introduce the category $ChuCors$ of Chu correspondences and define the Galois functor Gal from $ChuCors$ to the category $\mathcal{S}lat$ of join preserving maps between complete lattices. It turns out that the functor Gal is full and faithfull (Theorem 65) and in fact is an equivalence of categories (Theorem 73). In Section Five, we give an explicit description (Theorem 98) of the structure of the $*$-autonomous category of $ChuCors$ induced from that of $\mathcal{S}lat$ via the equivalence functor Gal. In the appendex A, we apply the concept of Chu correspondences to model theory which we hope shows its potential usefulness.

By the paper [8] we notice that some of our results are already obtained in the works of W. Xia [16], Ganter and Wille[9]. However our article has intention and scope which differ considerably from them and seems to put the theme in more appropriate context and we hope that it widens its applicability to other domains of researches.

1. Preliminaries

1.1. Galois pairs

We recall briefly basic facts on Galois pairs between complete lattices. The proofs of propositions are mostly omitted since they are more or less well-known. For details, see [4, 6] for example.

Let L, M be posets. A Galois pair from L to M is a pair

$$\varphi = (\varphi_*\colon L \to M,\ \varphi^*\colon M \to L)$$

of maps satisfying

$$\varphi_*\ell \leq m \Longleftrightarrow \ell \leq \varphi^* m$$

for $\ell \in L$, $m \in M$.

Proposition 1 *Let φ be a Galois pair. Then*

(i) *φ_* preserves the join and φ^* the meet, whence if L and M are lattices both are order preserving.*

(ii) *The operator $C_\varphi := \varphi^*\varphi_*\colon L \to L$ is order increasing, namely, $C_\varphi \ell \geq \ell$ for $\ell \in L$ and the operator $C^\varphi := \varphi_*\varphi^*\colon M \to M$ is order decreasing.*

Example 1 Let $L_i = \mathbf{pow} A_i$ $(i = 1, 2)$ and $f\colon A_1 \to A_2$ be a map. Then, for $M_i \subset A_i$ $(i = 1, 2)$,

$$f(M_1) \subset M_2 \Longleftrightarrow M_1 \subset f^{-1}M_2,$$

whence (f, f^{-1}) is a Galois pair. We have in fact

$$f\Big(\bigcup_i M_i\Big) = \bigcup_i f(M_i), \quad f^{-1}\Big(\bigcap_i M_i\Big) = \bigcap_i f^{-1}(M_i).$$

Note that f^{-1} preserves also the join because it has also right adjoint (See (3)).

$f^{-1}f(M_1)$ is the saturation of M_1 with respect to the equivalence relation $a \sim b \overset{\text{def}}{\Longleftrightarrow} f(a) = f(b)$ and $ff^{-1}M_2 = M_2 \bigcap Im(f)$.

Proposition 2 *If L_i $(i = 1, 2)$ are complete, then the components φ_* and φ^* of the Galois pair φ determine each other by*

$$\varphi^* m = \bigvee_{\varphi_*\ell \leq m} \ell \tag{1}$$

and

$$\varphi_*\ell = \bigwedge_{\ell \le \varphi^* m} m.$$

Proposition 3 *Every join preserving map $\varphi_*\colon L \to M$ defines a unique Galois pair $\varphi = (\varphi_*, \varphi^*)$, where the second component is defined by* (1)*.*

Proposition 4 *If a join preserving map φ_* is bijective, then it is an order isomorphism and*

$$\varphi^* = (\varphi_*)^{-1}.$$

Proposition 5 *For $m \in M$ and $\ell \in L$,*

$$\varphi_*\varphi^*\varphi_*\ell = \varphi_*\ell,$$
$$\varphi^*\varphi_*\varphi^* m = \varphi^* m$$

Proposition 6 *C_φ is a closure operator in the sense that it is order preserving, increasing and idempotent.*

C^φ is a coclosure operator in the sense that it is order preserving, decreasing and idempotent.

Let C be a closure operator on a complete lattice L and let L^C be the set of all the closed sets, namely, the set of C-fixed subsets.

Lemma 7 (i) *The subset $L^C \subset L$ is meet-closed and hence a complete lattice with the join of $N \subset L^C$ is given by*

$$C\Big(\bigvee_{x \in N} x\Big).$$

(ii) *Let $\iota\colon L^C \subset L$ be the inclusion. Then the pair (C, ι) is a Galois pair from L to L^C, namely,*

$$\ell \le \iota m \Leftrightarrow C\ell \le m$$

for $\ell \in L$ and $m \in L^C$.

(iii) *In particular, $\varphi\colon L \to L^\varphi$ is join-preserving and for $N \subset L$*

$$C\Big(\bigvee N\Big) = C\Big(\bigvee_{x \in N} Cx\Big).$$

(iv) *A map $f\colon X \to L$ is uniquely extended to a join-preserving map*

$$f_*\colon \mathbf{pow}(X) \to L^C$$

by

$$f_*(N) = C\Big(\bigvee_{n\in N} f(n)\Big).$$

Proof. (i). Let $\ell_i \in L^C$ $(i \in I)$. From $\bigwedge_j \ell_j \le \ell_i$, it follows $C\bigwedge_j \ell_j \le C\ell_i = \ell_i$ for all $i \in I$, whence

$$C\bigwedge_j \ell_j \le \bigwedge_i \ell_i.$$

On the other hand, since C is increasing,

$$C\bigwedge_j \ell_j \ge \bigwedge_i \ell_i.$$

Hence $\bigwedge_i \ell_i \in L^C$.

(ii). Let $m \in L^C$ and $\ell \in L$. If $\ell \le \iota m$, then $C\ell \le Cm = m$. On the other hand, if $C\ell \le m$, then $\ell \le C\ell \le m = \iota m$.

(iii). Obvious since, in L^C, the join of CN is given by the right hand side. □

Hence we have

Proposition 8 *The fixed point set $L_\varphi := \{\ell \in L \mid C_\varphi \ell = \ell\}$ is meet closed and $M^\varphi := \{m \in L \mid C^\varphi m = m\}$ is join closed.*

The correspondence φ_* induces an isomorphism

$$L_\varphi \overset{\simeq}{\to} M^\varphi,$$

whose inverse is φ^*.

Let $G_\varphi := \{(\ell, m) \mid \varphi_*\ell = m \text{ and } \varphi^* m = \ell\}$ with the product order. The order is defined $(\ell_1, m_1) \le (\ell_2, m_2)$ if and only if $\ell_1 \le \ell_2$. In fact $\ell_1 \le \ell_2$ implies $m_1 = \varphi_*\ell_1 \le \varphi_*\ell_2 = m_2$.

Proposition 9 *The poset G_φ is a complete lattice with*

$$\bigwedge_i (\ell_i, m_i) = \Big(\bigwedge_i \ell_i,\ \varphi_* \bigwedge_i \ell_i\Big)$$

$$\bigvee_i (\ell_i, m_i) = \Big(\varphi^* \bigvee_i m_i, \bigvee_i m_i\Big).$$

Proof. For $\ell \in C_\varphi$,

$$\begin{aligned} &(\ell, \varphi_*\ell) \leq (\ell_i, m_i) \text{ for all } i \in I \\ \iff &\ell \leq \ell_i \text{ for all } i \in I \\ \iff &\ell \leq \bigwedge_{i\in I} \ell_i \\ \iff &(\ell, \varphi_*\ell) \leq \Big(\bigwedge_{i\in I} \ell_i, \varphi_* \bigwedge_{i\in I} \ell_i\Big) \end{aligned}$$

□

Define $\pi_L : G_\varphi \to L$ and $\pi_M : G_\varphi \to M$ respectively by $\pi_L(\ell, m) = \ell$ and $\pi_M(\ell, m) = m$. Then π_L is meet preserving injection with the image L_φ and π_M is join preserving injection with the image M^φ.

1.2. The category *Slat* of join preserving maps

We recall briefly the $*$-autonomous category *Slat* of join preserving maps between complete lattices, which is briefly touched in [1, 10]. See for the detail in [13]. See [11, 2] for basic terminologies of category theory and [7, 1, 2, 5] for basic facts on autonomous categories and $*$-autonomous categories.

The complete lattice $\mathbf{2} = \{0, 1\}$ with $0 < 1$ is the unit object with $\mathcal{S}lat(\mathbf{2}, L) \simeq L$. The tensor $L_1 \otimes L_2$ is defined to be the set of its bi-ideals, namely, the down-closed subsets $T \subset L_1 \times L_2$ which and join-biclosed, in the sense that, for $A_i \subset L_i$ and $\ell_i \in L_i$ $(i = 1, 2)$

$$A_1 \times \{\ell_2\} \subset T \text{ implies } \Big(\bigvee A_1, \ell_2\Big) \in T$$

and

$$\{\ell_1\} \times A_2 \subset T \text{ implies } \Big(\ell_1, \bigvee A_2\Big) \in T.$$

The bi-ideals $\ell_1 \otimes \ell_2$ $(\ell_i \in L_i)$ which is the smallest bi-ideal containing (ℓ_1, ℓ_2) forms a dense subset of $L_1 \otimes L_2$. The tensor bifunctor $L_1, L_2 \mapsto L_1 \otimes L_2$ and natural isomorphisms

- $a(L_1, L_2, L_3) : (L_1 \otimes L_2) \otimes L_3 \to L_1 \otimes (L_2 \otimes L_3)$,
- $\ell_L : \mathbf{2} \otimes L \to L$,
- $r_L : L \otimes \mathbf{2} \to L$,

- $s(L_1, L_2)\colon L_1 \otimes L_2 \to L_2 \otimes L_1$

give the category $\mathcal{S}lat$ a symmetric monoidal category structure [13], [10].

Moreover this category is closed. In fact, if we denote by $L_1 \multimap L_2$ the homset $\mathcal{S}lat(L_1, L_2)$ regarded as a complete lattice by the partial order defined pointwise, then the functor $L \multimap (-)$ is a right adjoint to $(-) \otimes L$, namely, there are natural isomorphisms for $A, B \in \mathcal{S}lat$,

$$\mathcal{S}lat(A \otimes L, B) \simeq \mathcal{S}lat(A, L \multimap B).$$

If we put $A = \mathbf{2}$, since $\mathcal{S}lat(\mathbf{2}, L) \simeq L$, we have in particular a natural isomorphism

$$\mathcal{S}lat(\mathbf{2}, L \multimap B) \simeq \mathcal{S}lat(\mathbf{2} \otimes L, B) \simeq \mathcal{S}lat(L, B),$$

and the category $\mathcal{S}lat$ is enriched over itself, namely, there are composition arrows

$$c(L_1, L_2, L_3)\colon (L_2 \multimap L_3) \otimes (L_1 \multimap L_3) \to (L_1 \multimap L_3)$$

which give the compositions of $\mathcal{S}lat$. See [13], [5].

The category $\mathcal{S}lat$ is in fact a $*$-autonomous category. First, it is self-dual, namely, there is an isomorphic functor

$$(-)^*\colon \mathcal{S}lat \to \mathcal{S}lat^{\mathrm{op}},$$

which maps L to the dual L^* and $\varphi_*\colon L_1 \to L_2$ to $\varphi^*\colon L_2 \to L_1$, which is meet preserving and hence join preserving from L_2^* to L_1^*. In particular, there are natural isomorphisms

$$\mathcal{S}lat(L_1, L_2) \simeq \mathcal{S}lat(L_2^*, L_1^*). \tag{2}$$

Moreover $\mathbf{2}$ is a dualizing object of $\mathcal{S}lat$. In fact, putting $L_1 = L$, $L_2 = \mathbf{2}$ in (2),

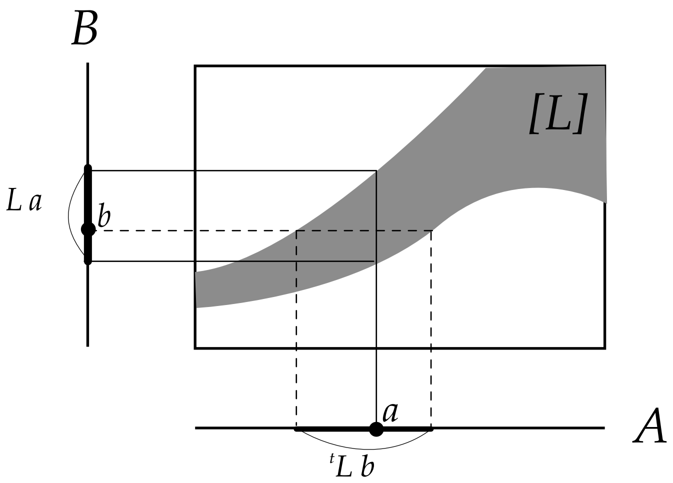
$$\mathcal{S}lat(L, \mathbf{2}) \simeq \mathcal{S}lat(\mathbf{2}^*, L^*) \simeq \mathcal{S}lat(\mathbf{2}, L^*) \simeq L^*,$$

since $\mathbf{2}$ is self-dual. Hence $\mathbf{2}$ is a dualizing object in $\mathcal{S}lat$.

Finally, the tensors and the internal homomorphisms are related by the isomorphism

$$L_1 \otimes L_2 \simeq (L_1 \multimap L_2^*)^*$$

where $\ell_1 \otimes \ell_2$ is mapped to $f_{\ell_1 \otimes \ell_2}\colon L_1 \to L_2^*$, which maps $\bot$ to $\top$, $(\ell_1 \downarrow) \setminus \{\bot\}$ to ℓ_2 and other elements to $\bot$ [13].



1.3. Operators induced from correspondences

We recall basic facts on correspondences between sets mainly to fix notations. A correspondence from a set A to B is a map

$$L\colon A \to \mathbf{pow}(B),$$

and will be denoted as

$$L\colon A \rightsquigarrow B.$$

Its graph $[L] \subset A \times B$ is defined by

$$[L] = \{(a,\, b) \mid b \in La\},$$

and its transpose

$$^tL\colon B \to \mathbf{pow}(A)$$

is defined by

$$^tLb = \{a \mid b \in La\}.$$

We identify a map $f\colon A \to B$ with the correspondence from A to B which maps a to the singleton set $\{f(a)\}$.

Denote by $Cors(A,\, B)$ the poset of all the correspondences from A to

B, with $L_1 \leq L_2$ be defined by

$$L_1 a \subset L_2 a \text{ for all } a \in A.$$

Define

$$\Big(L_1 \bigcap L_2\Big) a = L_1 a \bigcap L_2 a.$$

Then, we have obviously

Proposition 10 *The correspondences $L \leftrightarrow [L] \leftrightarrow {}^t L$ define poset isomorphisms*:

$$Cors(A,\, B) \simeq \mathbf{pow}(A \times B) \simeq Cors(B,\, A).$$

Moreover

$$\Big[L_1 \bigcap L_2\Big] = [L_1] \bigcap [L_2],$$

$${}^t\Big(L_1 \bigcap L_2\Big) = {}^t L_1 \bigcap {}^t L_2.$$

Identifying $a_1 \in A_1$ with the singleton $\{a_1\}$, the complete lattice $\mathbf{pow}(A_1)$ is a free sup-lattice generated by A_1. Hence a map $L\colon A_1 \to \mathbf{pow}(A_2)$ induces two join preserving maps

$$L_*\colon \mathbf{pow}(A_1) \to \mathbf{pow}(A_2)$$

and

$$L_\circ\colon \mathbf{pow}(A_1) \to \mathbf{pow}(A_2)^{\mathrm{op}},$$

which are defined respectively by

$$L_* K_1 = \bigcup_{a \in K_1} La$$

and

$$L_\circ K_1 = \bigcap_{a \in K_1} La,$$

for $K_1 \subset A_1$.

The adjoint $L^*\colon \mathbf{pow}(A_2) \to \mathbf{pow}(A_1)$ of L_* is characterized by

$$L_* K_1 \subset K_2 \Longleftrightarrow K_1 \subset L^* K_2$$

whence

$$L^*K_2 = \bigcup_{L_*K_1 \subset K_2} K_1 = \{a_1 \in A_1 \mid La_1 \subset K_2\},$$

for $K_2 \subset A_2$.

Similarly, the adjoint $L^\circ \colon \mathbf{pow}(A_2)^{\mathrm{op}} \to \mathbf{pow}(A_1)$ of $L_\circ$ is characterized by

$$L_\circ K_1 \supset K_2 \Longleftrightarrow K_1 \subset L^\circ K_2.$$

From this we have

Proposition 11 (i) *For $a_2 \in A_2$ and $K_1 \subset A_1$,*

$$a_2 \in L_\circ K_1 \Longleftrightarrow K_1 \subset {}^tLa_2.$$

(ii) *For $K_i \subset A_i$ $(i = 1, 2)$,*

$$L_\circ K_1 \supset K_2 \Longleftrightarrow K_1 \times K_2 \subset [L] \Longleftrightarrow L^\circ K_2 \supset K_1.$$

(iii)

$$L^\circ = ({}^tL)_\circ.$$

Proof. The first assertion follows from the following equivalence.

$$\begin{aligned} a_2 \in L_\circ K_1 &\Leftrightarrow a_2 \in La_1 \text{ for all } a_1 \in K_1 \\ &\Leftrightarrow a_1 \in {}^tLa_2 \text{ for all } a_1 \in K_1 \\ &\Leftrightarrow K_1 \subset {}^tLa_2. \end{aligned}$$

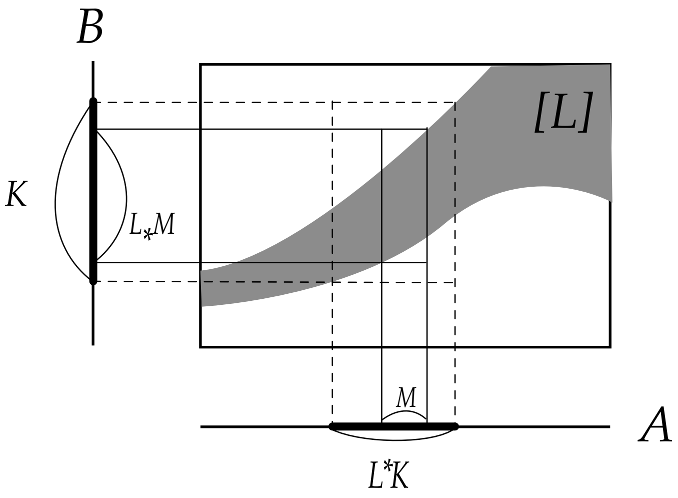
The second follows from the following.

$$\begin{aligned} K_1 \times K_2 \subset [L] &\Leftrightarrow K_1 \times \{a_2\} \subset [L] \quad \text{for all } a_2 \in K_2 \\ &\Leftrightarrow K_1 \subset {}^tLa_2 \quad \text{for all } a_2 \in K_2 \\ &\Leftrightarrow a_2 \in L_\circ K_1 \quad \text{for all } a_2 \in K_2 \\ &\Leftrightarrow K_2 \subset L_\circ K_1. \end{aligned}$$

Since L° is join preserving when regarded as a map from $\mathbf{pow}(A_2)$ to $\mathbf{pow}(A_1)^{\mathrm{op}}$ it suffices to show $L^\circ\{a_2\} = {}^tLa_2$, which follows from

$$\begin{aligned} a_1 \in L^\circ\{a_2\} &\Leftrightarrow a_2 \in L_\circ\{a_1\} = La_1 \\ &\Leftrightarrow a_1 \in {}^tLa_2, \end{aligned}$$

for $a_1 \in A_1$. □



Proposition 12 *The pairs* (L_*, L^*) *and* $(L_\circ, L^\circ)$ *are Galois pairs respectively from* $\mathbf{pow}A_1$ *to* $\mathbf{pow}A_2$ *and from* $\mathbf{pow}A_1$ *to* $\mathbf{pow}A_2^{\mathrm{op}}$.

By Proposition 1, we have

Corollary 13 *The operator* L_* *preserves the unions and* L^* *the intersections. The operators* $L_\circ$ *and* L° *convert unions to intersections.*

Remark If a map $f\colon A_1 \to A_2$ is considered as a correspondence L_f by $L_f a = \{fa\}$, then for $K_i \subset A_i$ $(i = 1, 2)$

$$(L_f)_* K_1 = f(K_1),$$
$$(L_f)^* K_2 = f^{-1} K_2 = ({}^t L_f)_* K_2,$$
$$({}^t L_f)^* K_1 = f_!(K_1) := \{a_2 \in A_2 \mid f^{-1} a_2 \subset K_1\}.$$

In this case,

$$L_{f*} \dashv L_f^* = ({}^t L_f)_* \dashv ({}^t L_f)^*, \tag{3}$$

whence $(L_f)_*$ preserves the union, $({}^t L_f)^*$ the intersection, and $({}^t L_f)_* = (L_f)^*$ both.

In the rest of this section, we prove a few properties of L^*.

We have the following expression for L^*.

Proposition 14 $L^* K = \big(({}^t L)_* K^c\big)^c$.

Proof. Define $R\colon \mathbf{pow}(A_2) \to \mathbf{pow}(A_1)$ by

$$RK_2 = (L^* K_2^c)^c.$$

Since $(-)^c$ converts joins to meets, L^* meets to meets, and $(-)^c$ meets to joints, it follows that both R and $({}^tL)^*$ preserves the joins. Hence it suffices to prove $R\{a_2\} = {}^tLa_2$, which is seen as follows.

$$\begin{aligned} a_1 \in R\{a_2\} &\Longleftrightarrow a_1 \notin (L^*)\{a_2\}^c \\ &\Longleftrightarrow La_1 \not\subset \{a_2\}^c \\ &\Longleftrightarrow a_2 \in La_1 \\ &\Longleftrightarrow a_1 \in {}^tLa_2. \end{aligned}$$

□

Corollary 15 *The correspondence $L \to L^*$ reverses the order.*

The correspondences form a category with the usual composition definted as follows. The composition $L_2 \circ L_1$ of $L_i\colon A_i \to A_{i+1}$ $(i = 1, 2)$ is defined by

$$[L_2 \circ L_1] := \left\{ (a_1, a_3) \in A_1 \times A_3 \;\middle|\; L_1 a_1 \bigcap {}^tL_2 a_3 \neq \emptyset \right\}.$$

Obviously the correspondence $L \mapsto L^*$ is functorial. Namely we have

Proposition 16 *Let $L_i\colon A_i \to A_{i+1}$ $(i = 1, 2)$ be correspondences. Then*

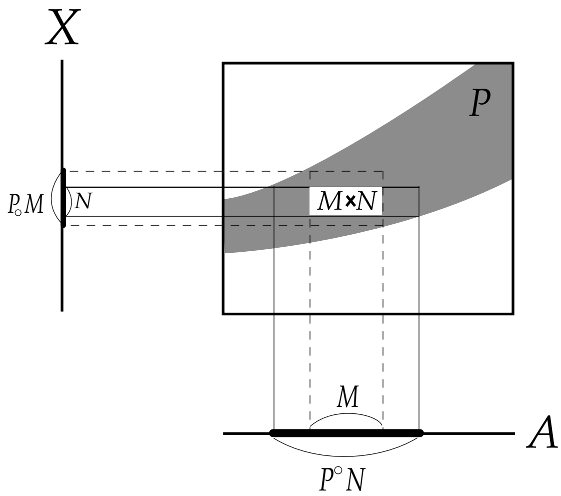
$$(L_2 \circ L_1)_* = (L_2)_* \circ (L_1)_*.$$
$$(L_2 \circ L_1)^* = (L_1)^* \circ (L_2)^*.$$

Proof. The former assertion is obvious, since both sides preserve joins and maps $a_1 \in A_1$ to $(L_2)_* L_1 a_1$.

The latter assertion follows from the former since, for $K_i \subset A_i$ $(i = 1, 3)$

$$\begin{aligned} K_1 \leq (L_1)^*(L_2)^* K_3 &\Longleftrightarrow (L_1)_* K_1 \leq (L_2)^* K_3 \\ &\Longleftrightarrow (L_2)_*(L_1)_* K_1 \leq K_3 \\ &\Longleftrightarrow (L_2 \circ L_1)_* K_1 \leq K_3 \\ &\Longleftrightarrow K_1 \leq (L_2 \circ L_1)^* K_3. \end{aligned}$$

□



1.4. Basic concepts of formal concept analysis

We recall basic framework of formal concept analysis. See [9, 6] for the detail.

A formal context is a triple (A, X, P) where P is a correspondence from a set A to a set X. When P is clear from the context, we write $x \in Pa$ as $a \models x$ and denote the formal context simply as $(A, X, \models)$. The transpose $(X, A, {}^tP)$ will be denoted by $(A, X, P)^*$ and called the dual of (A, X, P).

The correspondence induces join preserving maps, called the *polar maps*,

$$P_\circ \colon \mathbf{pow}(A) \to \mathbf{pow}(X)^{\mathrm{op}}$$

and

$$P^\circ \colon \mathbf{pow}(X) \to \mathbf{pow}(A)^{\mathrm{op}}.$$

A formal context is called *extensional* and *intentional* when P° and $P_\circ$ are injective respectively.

By Proposition 11,

$$N \subset P_\circ M \leftrightarrow M \times N \subset P \leftrightarrow M \subset P^\circ N.$$

Since the pair $(P_\circ, P^\circ)$ is a Galois pair from $\mathbf{pow}A$ to $(\mathbf{pow}B)^{\mathrm{op}}$,

$$C_P := P^\circ P_\circ$$

is a closure operator of the complete lattice $\mathbf{pow}A$ and

$$C^P := P_\circ P^\circ$$

is a coclosure operator of the complete lattice $(\mathbf{pow}B)^{\mathrm{op}}$ and hence is a closure operator of $\mathbf{pow}B$.

The fixed points of C_P forms the intersection closed sublattice which is anti-isomorphic to the fixed points sets of C^P.

We give the following specific terminologies and notations for this special case. When P is clear from the context, we write $M' = P_\circ M$ and $N' = P^\circ N$. For a subset M of A, the subset M'', also written as $\overline{M}$, is called *the closure of* M. When $M'' = M$, the subset M is called *closed*. We use similar terminologies for subsets of X. The set of all the closed subsets of A and X are denoted respectively by $\mathcal{A}(\mathbf{C})$ and $\mathcal{X}(\mathbf{C})$.

A formal concept of $\mathbf{C}$ is a pair $(M, N) \in \mathbf{pow}A \times \mathbf{pow}X$ satisfying $M = N'$ and $N = M'$. A formal concept is usually written either as (M, M') or as (N', N), with closed M and N.

The set of all formal concepts is denoted by $Gal(\mathbf{C})$ with the order

$$(M_1, N_1) \le (M_2, N_2) \overset{\text{def}}{\iff} M_1 \subset M_2,$$

which is equivalent to $N_1 \supset N_2$ since the polar map is order reversing.

By Section 1.1, the polar map defines isomorphisms.

$$\mathcal{A}(\mathbf{C}) \simeq \mathcal{X}(\mathbf{C})^{\mathrm{op}} \simeq Gal(\mathbf{C}), \tag{4}$$

and the join and meet of $(M_i, N_i)_{i\in I}$ are given respectively by

$$\bigvee_i (M_i, N_i) = \Big(\Big(\bigcap_i N_i\Big)', \bigcap_i N_i\Big) = \Big(\overline{\bigcup_i M_i}, \Big(\bigcup_i M_i\Big)'\Big)$$

and

$$\bigwedge_i (M_i, N_i) = \Big(\bigcap_i M_i, \Big(\bigcap_i M_i\Big)'\Big) = \Big(\Big(\bigcup_i N_i\Big)', \overline{\bigcup_i N_i}\Big).$$

Note that each $a \in A$ defines a formal concept $(\overline{a}, a')$ called a *token-based* concept. Similarly, each $x \in X$ defines a formal concept $(x', \overline{x})$ called a *type-based* concept.

Since every concept (E, F) is written either as the join

$$(E, F) = \bigvee_{a\in E} (\overline{a}, a')$$

and as the meet

$$(E, F) = \bigwedge_{x \in F} (x', \overline{x}),$$

we have the following theorem.

Theorem 17 *The token-based concepts are $\bigvee$-dense in $Gal(\mathbf{C})$ and the type-based ones are $\bigwedge$-dense in $Gal(\mathbf{C})$.*

Example 2 A set A defines a formal context $\mathbf{P}(A) := (A, \mathbf{pow}(A), \in)$, called the power context of the set A. Then

$$Gal(\mathbf{P}(A)) = \mathbf{pow}(A).$$

Example 3 Let $\top = \mathbf{P}(\{*\})$. Then

$$Gal(\top) \simeq \mathbf{pow}(\{*\}) \simeq \mathbf{2}.$$

Similarly $\bot = \top^*$ has the concept lattice $\mathbf{pow}(\{*\})^{\mathrm{op}} \simeq \mathbf{2}^* \simeq \mathbf{2}$.

From a formal context $\mathbf{C} = (A, X, \models)$, we define anthor one

$$\mathbf{powC} := (\mathbf{pow}A, \mathbf{pow}X, \models)$$

by

$$M \models N \overset{\text{def}}{\iff} m \models n \quad \text{for all } m \in M \text{ and } n \in N.$$

The following obvious lemma will be used frequently.

Lemma 18 *For $M \subset A$ and $N \subset X$,*

$$M \models N \iff \overline{M} \models N \iff M \models \overline{N}.$$

Proof.

$$M \models N \iff M \subset N' \iff \overline{M} \subset N' \iff \overline{M} \models N.$$

The other equivalence is proved similarly. □

Proposition 19 *The map $(M, N) \mapsto (\mathbf{pow}M, \mathbf{pow}N)$ induces a bijection:*

$$\iota\colon Gal(\mathbf{C}) \overset{\approx}{\to} Gal(\mathbf{powC}).$$

Proof. Let $M \subset A$. Denote by M^* the polar of $M \in \mathbf{pow}M$ with respect to the context $\mathbf{powC}$. Then $N \in M^*$ means $M \models N$ which is equivalent to $N \subset M'$. Thus

$$M^* = \mathbf{pow}(M').$$

Suppose now $\mathcal{M} = \{M_i \mid i \in I\} \subset \mathbf{pow}A$. Then

$$\mathcal{M}^* = \bigcap_{i \in I} M_i^* = \bigcap_{i \in I} \mathbf{pow}(M_i') = \mathbf{pow}\Big(\bigcup_{i \in I} M_i\Big)'.$$

This proves $\mathcal{M}^* = \mathbf{pow}(\bigcup \mathcal{M})'$. Suppose $(\mathcal{M}, \mathcal{N})$ is a formal concept of $\mathbf{powC}$. Then

$$\mathcal{M} = \mathcal{M}^{**} = \mathbf{pow}(\overline{\bigcup \mathcal{M}}) = \mathbf{pow}(M),$$

where $M = \overline{\bigcup \mathcal{M}}$. Similarly

$$\mathcal{N} = \mathbf{pow}(N),$$

with $N = \overline{\bigcup \mathcal{N}}$. Moreover, $\mathbf{pow}(M) = \mathbf{pow}(N')$ implies $M = N'$. Similarly $N = M'$ holds. Hence (M, N) is a formal concept for the context K and $\iota(M, N) = (\mathcal{M}, \mathcal{N})$. $\square$

Let $\mathcal{A}$ be an intersection closed family of subsets of A.

Proposition 20

$$Gal(A, \mathcal{A}, \in) \simeq \mathcal{A},$$
$$Gal(\mathcal{A}, A, \ni) \simeq \mathcal{A}^{\mathrm{op}}.$$

Proof. Denote $A_1 = A$ and $X_1 = \mathcal{A}$. The polar of $x \in X_1 = \mathcal{A}$ is $x \in \mathbf{pow}(A)$ itself and since $\mathcal{A}$ is intersection closed, the polar of $F \subset \mathcal{A}$ is written as $\bigcap F \in \mathcal{A}$, whence the set of closed sets of A coincides with $\mathcal{A}$. Hence the formal concept can be written as $(E, E\uparrow)$ $(E \in \mathcal{A})$, where $E\uparrow = \{F \in \mathcal{A} \mid E \subset F\}$.

The second follows from the first one, since the formal concepts are $(E\uparrow, E)$ $(E \in \mathcal{A})$. $\square$

1.5. Chu maps

A Chu map from a formal context $\mathbf{C}_1 = (A_1, X_1, \models)$ to $\mathbf{C}_2 = (A_2, X_2, \models)$ is a pair of maps $(f \colon A_1 \to A_2,\, g \colon X_2 \to X_1)$ satisfying, for all $(a_1, x_2) \in$

$A_1 \times X_2$,

$$f(a_1) \models x_2 \Longleftrightarrow a_1 \models g(x_2).$$

Usually the concept of Chu maps is considered to be the correct concept of morphism between formal concepts [1].

The set of Chu maps from $\mathbf{C}_1$ to $\mathbf{C}_2$ is denoted by $ChuMaps(\mathbf{C}_1, \mathbf{C}_2)$.

Example 4 A map $f\colon A_1 \to A_2$ induces a Chu map

$$(f, f^{-1})\colon (A_1, \mathbf{pow}(A_1), \in) \to (A_2, \mathbf{pow}(A_2), \in),$$

since

$$f(a_1) \in B \Leftrightarrow a_1 \in f^{-1}B,$$

by the definition of f^{-1}.

In fact, we have the following.

Proposition 21 *The correspondence*

$$Map(A_1, A_2) \ni f \mapsto (f, f^{-1}) \in ChuMaps(\mathbf{P}(A_1), \mathbf{P}(A_2))$$

is a bijection.

Proof. If

$$(f, g)\colon (A_1, \mathbf{pow}(A_1), \in) \to (A_2, \mathbf{pow}(A_2), \in)$$

is a Chu map, then $g = f^{-1}$, since the condition

$$a_1 \in g(B) \Leftrightarrow f(a_1) \in B$$

implies $g(B) = f^{-1}B$. □

For a Chu map $\gamma = (f, g)$, define $\gamma_*\colon Gal(\mathbf{C}_1) \to Gal(\mathbf{C}_2)$ by

$$\gamma_*(M, M') = (\overline{f(M)}, f(M)')$$

and

$$\gamma^*\colon Gal(\mathbf{C}_2) \to Gal(\mathbf{C}_1)$$

by

$$\gamma^*(L', L) = (g(L)', \overline{g(L)}).$$

Then we have

$$\gamma_*(M, M') \leq (L', L) \Longleftrightarrow (M, M') \leq \gamma^*(L', L), \tag{5}$$

from which it follows that γ_* preserves the join and γ^* the meet.

Not that the equivalence (5) is rewritten as

$$f(M)' \supset L \Longleftrightarrow M \subset g(L)'$$

which is equivalent to

$$f(M) \models L \Longleftrightarrow M \models g(L)$$

which follows directly from the defining property of the Chu maps.

We have other expressions of γ_* and γ^*.

Proposition 22

$$\gamma_*((N', N)) = ((g^{-1}N)', g^{-1}N),$$
$$\gamma^*((K, K')) = (f^{-1}K, (f^{-1}K)').$$

Proof. It suffices to show that $f(N')' = g^{-1}N$, which follows from

$$\begin{aligned} y \in f(N')' &\Longleftrightarrow f(N') \models y \\ &\Longleftrightarrow N' \models g(y) \\ &\Longleftrightarrow g(y) \in \overline{N} = N \\ &\Longleftrightarrow y \in g^{-1}N. \end{aligned}$$

The other equality follows similarly. □

2. Chu correspondence

There may be no Chu maps between formal contexts although there are many join preserving maps between their Galois lattices. We extend the concept of Chu maps to Chu correspondences which provide us rich relations between formal contexts.

2.1. Definition

Let $\mathbf{C}_i = (A_i, X_i, P_i)$ $(i = 1, 2)$ be formal contexts. A pair $\varphi = (L_\varphi, R_\varphi)$ is called *a correspondence from* $\mathbf{C}_1$ *to* $\mathbf{C}_2$ if L_φ and R_φ are correspondences respectively from A_1 to A_2 and from X_2 to X_1. L_φ and R_φ are called *the extent* and *the intent parts* of φ respectively.

We use the notational conventions introduced for correspondences in §1.3. For example, $L_{\varphi*}$ denotes the unique extension of L_φ to the join preserving map from $\mathbf{pow}(A_1)$ to $\mathbf{pow}(A_2)$.

Definition 1 A correspondence φ from $\mathbf{C}_1$ to $\mathbf{C}_2$ is called a *Chu correspondence in the weak sense* if for every $a_1 \in A_1$ and $x_2 \in X_2$

$$L_\varphi a_1 \models x_2 \Leftrightarrow a_1 \models R_\varphi x_2.$$

Definition 2 A Chu correspondence φ in the weak sense from $\mathbf{C}_1$ to $\mathbf{C}_2$ is called simply a Chu correspondence if both $L_\varphi a_1 \subset A_1$ and $R_\varphi x_2 \subset X_1$ are closed for every $a_1 \in A_1$ and $x_2 \in X_2$.

Proposition 23 *Let φ be a Chu correspondence in the weak sense from $\mathbf{C}_1$ to $\mathbf{C}_2$. Define a correspondence $\overline{\varphi}$ from $\mathbf{C}_1$ to $\mathbf{C}_2$ by*

$$L_{\overline{\varphi}} a_1 = \overline{L_\varphi a_1}$$

and

$$R_{\overline{\varphi}} x_2 = \overline{R_\varphi x_2}.$$

Then $\overline{\varphi}$ is a Chu correspondence.

Proof. Let $a_1 \in A_1$ and $x_2 \in X_2$. Then

$$\begin{aligned} L_{\overline{\varphi}} a_1 \models x_2 \Longleftrightarrow \overline{L_\varphi a_1} \models x_2 \Longleftrightarrow L_\varphi a_1 \models x_2 \Longleftrightarrow a_1 \models R_\varphi x_2 \\ \Longleftrightarrow a_1 \models \overline{R_\varphi x_2} \Longleftrightarrow a_1 \models R_{\overline{\varphi}} x_2. \end{aligned}$$

□

A Chu correspondence $\varphi\colon \mathbf{C}_1 \to \mathbf{C}_2$ is called **a strong isomorphism** if there are bijections

$$f\colon A_1 \to A_2, \quad g\colon X_2 \to X_2$$

satisfying

$$L_\varphi a_1 = \overline{\{f(a_1)\}}, \quad R_\varphi x_2 = \overline{\{g(x_2)\}}$$

for $a_1 \in A_1$ and $x_2 \in X_2$. (f, g) is called *a generator of* the the strong isomorphism φ. We note that there may be generally many generators of a Chu correspondence φ.

Example 5 Let $\mathbf{C}_1 = \mathbf{C}_2 = (V, V^*, \perp)$, where V is a finite-dimensional linear space over a field k and $v \perp w$ means $w(v) = 0$. Then the Chu correspondences

$$(a\,\mathrm{id}_V,\, b\,\mathrm{id}_{V^*}) \qquad a,\, b \in k \setminus \{0\}$$

in the weak sense have the same closures.

Lemma 24 *Let (f, g) be a generator of a strong isomorphism φ. Then (f^{-1}, g^{-1}) is a Chu correspondence in the weak sense and its closure is a strong isomorphism.*

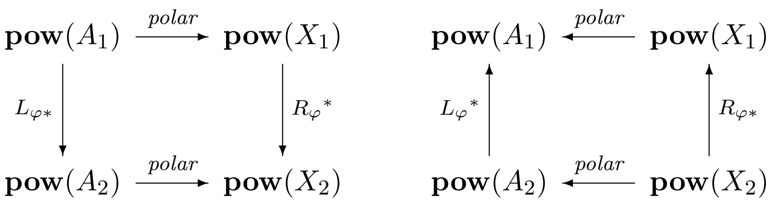
2.2. Basic properties of Chu correspondences

Proposition 25 *Let φ be a Chu correspondence. Then for $N_1 \subset A_1$ and $M_2 \subset X_2$,*

$$(L_{\varphi *} N_1)' = R_\varphi{}^* N_1',$$

$$(R_{\varphi *} M_2)' = L_\varphi{}^* M_2'.$$

In other words, the following diagrams commute:



Proof. Write $L = L_\varphi$ and $R = R_\varphi$ for brevity. Then $x_2 \in (L_* N_1)'$ if and only if $L_* N_1 \models x_2$ if and only if $N_1 \models R x_2$ if and only if $R x_2 \subset N_1'$ if and only if $x_2 \in R^* N_1'$.

The other assertion is proved similarly. □

Conversely, we have

Proposition 26 *A correspondence φ is a Chu correspondence if*

$$(L_{\varphi *} N_1)' = R_\varphi{}^* N_1' \quad \textit{for every } N_1 \subset A_1.$$

Similarly, φ is a Chu correspondence if

$$(R_{\varphi *} M_2)' = L_\varphi{}^* M_2' \quad \textit{for every } M_2 \subset X_2.$$

Proof. Let $a_1 \in A_1$ and $x_2 \in X_2$.

$$\begin{aligned} L_\varphi a_1 \models x_2 &\Longleftrightarrow x_2 \in (L_\varphi a_1)' = R_\varphi{}^*(a_1') \\ &\Longleftrightarrow R_\varphi x_2 \subset a_1' \\ &\Longleftrightarrow a_1 \models R_\varphi x_2, \end{aligned}$$

Hence φ is a Chu correspondence and the first assertion holds.
The latter can be proved similarly. □

Hence we have the following propositions.

Proposition 27 *A correspondence φ is a Chu correspondence in the weak sense if and only if*

$$(L_\varphi a_1)' = R_\varphi{}^* a_1' \quad \textit{for all } a_1 \in A_1$$

if and only if

$$(R_\varphi x_2)' = L_\varphi{}^*(x_2') \quad \textit{for all } x_2 \in X_2.$$

Proposition 28 *If φ is a Chu correspondence from $\mathbf{C}_1$ to $\mathbf{C}_2$, then for $a_1 \in A_1$*

$$L_\varphi a_1 = (R_\varphi{}^* a_1')'$$

and for $x_2 \in X_2$

$$R_\varphi x_2 = (L_\varphi{}^* x_2')'.$$

In particular, L_φ and R_φ determine each other.

Note that $L_{\varphi *} N_1$ might not be closed even if φ is a Chu correspondence and $N_1 \subset A_1$ is closed.

The following property of the join-preserving operator

$$L_{\varphi *} \colon \mathbf{pow}(A_1) \to \mathbf{pow}(A_2)$$

will be used frequently.

Proposition 29 *If φ is a Chu correspondence in the weak sense from $\mathbf{C}_1$ to $\mathbf{C}_2$, then*

$$\overline{L_{\varphi *} \overline{N_1}} = \overline{L_{\varphi *} N_1}$$

for $N_1 \subset A_1$*. In particular*

$$\overline{L_{\varphi *}\overline{a_1}} = \overline{L_\varphi a_1}$$

for $a_1 \in A_1$*. Hence if* φ *is a Chu correspondence then*

$$\overline{L_{\varphi *}\overline{a_1}} = L_\varphi a_1.$$

Proof. Suffices to show

$$(L_{\varphi *}\overline{N_1})' = (L_{\varphi *}N_1)'.$$

Let $x_1 \in X_1$. Then

$$\begin{aligned} x_1 \in (L_{\varphi *}\overline{N_1})' &\Longleftrightarrow L_{\varphi *}\overline{N_1} \models x_1 \\ &\Longleftrightarrow \overline{N_1} \models R_\varphi x_1 \\ &\Longleftrightarrow N_1 \models R_\varphi x_1 \\ &\Longleftrightarrow L_{\varphi *}N_1 \models x_1 \Longleftrightarrow x_1 \in (L_{\varphi *}N_1)'. \end{aligned}$$

□

The right adjoint

$$L_\varphi{}^*\colon \mathbf{pow}(A_1) \to \mathbf{pow}(A_2)$$

has the following basic properties.

Proposition 30 (i) *If* $N_2 \subset A_2$ *is closed then* $L_\varphi{}^* N_2 \subset A_1$ *is closed.*
(ii) *The operator* $L_\varphi{}^*$ *preserves intersection.*
(iii) *For* $E \subset A_2$*,*

$$\overline{L_\varphi{}^* E} \subset L_\varphi{}^* \overline{E}.$$

Proof. Let $M_2 = (N_2)'$ so that $N_2 = M_2'$. Then by Proposition 25

$$L_\varphi{}^* N_2 = L_\varphi{}^*(M_2)' = (R_{\varphi *}M_2)',$$

whence $L_\varphi{}^* N_2$ is closed, whence the assertion (i).

The assertion (ii) holds by basic properties Corollary 13 of Galois pairs.

Since $L_\varphi{}^*\overline{E}$ is closed and includes $L_\varphi{}^* E$, we have the assertion (iii).

□

Definition 3 Let $\mathbf{C}_i$ $(i = 1, 2)$ be formal contexts. A correspondence $L\colon A_1 \to \mathbf{pow}A_2$ is called *a continuous extent correspondence from* $\mathbf{C}_1$ *to*

$\mathbf{C}_2$ if

$$L^*\colon \mathbf{pow}A_2 \to \mathbf{pow}A_1$$

preserves the closed sets.

Similarly $R\colon X_2 \to \mathbf{pow}(X_1)$ is called a continuous intent correspondence from $\mathbf{C}_1$ to $\mathbf{C}_2$ if R^* preserves the closed sets.

If φ is a Chu correspondence, L_φ is a continuous extent relation and R_φ is a continuous intent relation from $\mathbf{C}_1$ to $\mathbf{C}_2$. In fact the converse holds.

Theorem 31 *Suppose $L\colon A_1 \to \mathbf{pow}A_2$ is a continuous extent relation from $\mathbf{C}_1$ to $\mathbf{C}_2$. Then there is a correspondence $R\colon X_2 \to \mathbf{pow}X_2$ with (L, R) being a Chu correspondence from $\mathbf{C}_1$ to $\mathbf{C}_2$.*

Proof. Define R by

$$Rx_2 = (L^*x_2')' \quad \text{for } x_2 \in X_2.$$

Then, for $a_1 \in A_1$ and $x_2 \in X_2$,

$$\begin{aligned} a_1 \models Rx_2 &\Longleftrightarrow a_1 \in (Rx_2)' = \overline{L^*x_2'} = L^*x_2' \\ &\Longleftrightarrow La_1 \subset x_2' \Longleftrightarrow La_1 \models x_2, \end{aligned}$$

whence (L, R) is a Chu correspondence. $\square$

Similarly, we have the following:

Theorem 32 *Suppose $R\colon X_2 \to \mathbf{pow}X_1$ is a continuous intent correspondence from $\mathbf{C}_1$ to $\mathbf{C}_2$. Then there is a correspondence $L\colon A_1 \to \mathbf{pow}A_2$ with (L, R) being a Chu correspondence from $\mathbf{C}_1$ to $\mathbf{C}_2$.*

2.3. Description by Bonds

Chu correspondences are described by bonds introduced by Ganter and Wille [9]. Let $\mathbf{C}_i = (A_i, X_i, \models)$ $(i = 1, 2)$ be formal contexts.

Definition 4 A bond from $\mathbf{C}_1$ to $\mathbf{C}_2$ is a correspondence Z from A_1 to X_2 satisfying the condition that both Za_1 and ${}^tZx_2$ be closed for $x_2 \in X_2$ and $a_1 \in A_1$.

Example 6 If (f, g) is a Chu map from $\mathbf{C}_1$ to $\mathbf{C}_2$, then the correspondence Z from A_1 to X_2 defined by $Za_1 := f(a_1)'$, which is also determined by ${}^tZx_2 = g(x_2)'$ is a bond.

Example 7 If $E_1 \subset A_1$ and $F_2 \subset X_2$, the subset $Z = E_1 \times F_2 \subset A_1 \times X_2$ is a bond if and only if E_1 and F_2 are closed.

Example 8 If $\mathbf{C} = (A, X, P)$ is a formal context, then $P \subset A \times X$ is obviously a bond from $\mathbf{C}$ to $\mathbf{C}$, called the tautological bond.

We denote by $Bond(\mathbf{C}_1, \mathbf{C}_2)$ the set of all the bonds from $\mathbf{C}_1$ to $\mathbf{C}_2$, equipped with the partial order $Z_1 \leq Z_2$ defined by $[Z_1] \subset [Z_2]$. (See Section 1.3)

Proposition 33 *$Bond(\mathbf{C}_1, \mathbf{C}_2)$ is the intersection closed subset of $\mathbf{pow}(A_1 \times X_2)$. In particular, it is a complete lattice with the meet operation given by the intersection.*

Proof. Suppose B_i $(i \in I)$ are bonds from $\mathbf{C}_1$ to $\mathbf{C}_2$. Then $B := \bigcap_{i\in I} B_i$ defined by

$$Ba_1 = \bigcap_{i\in I} B_i a_1$$

for $a_1 \in A_1$ is also a Bond. In fact Ba_1 is closed and by Proposition 10

$${}^tBx_2 = \bigcap_{i\in I} {}^tB_i x_2$$

is also closed. □

We show that there is an anti-isomorphic correspondence between the complete lattice of Chu correspondences and that of bonds.

2.3.1. Bonds defines Chu correspondences First we show that bonds define Chu correspondences.

Proposition 34 *Let $Z\colon \mathbf{C}_1 \to \mathbf{C}_2$ be a bond. Define a correspondence $\varphi_Z\colon \mathbf{C}_1 \to \mathbf{C}_2$ by*

$$(L_{\varphi_Z})a_1 = (Za_1)' \subset A_2 \quad \text{for } a_1 \in A_1$$

and

$$(R_{\varphi_Z})x_2 = ({}^tZx_2)' \subset X_1 \quad \text{for } x_2 \in X_2.$$

Then φ_Z is a Chu correspondence from $\mathbf{C}_1$ to $\mathbf{C}_2$.

Proof. Let $(a_1, x_2) \in A_1 \times X_2$. Then

$$\begin{aligned} L_\varphi a_1 \models x_2 &\Leftrightarrow (Z\ a_1)' \models x_2 \\ &\Leftrightarrow x_2 \in \overline{Za_1} = Za_1 \\ &\Leftrightarrow a_1 \in {}^t Zx_2 = \overline{Zx_2} \\ &\Leftrightarrow a_1 \models (Zx_2)' = R_\varphi x_2. \end{aligned}$$

Hence φ is a Chu correspondence. □

Example 9 Suppose $E_1 \subset A_1$ and $F_2 \subset X_2$ are closed. The Chu correspondence φ corresponding to the bond $E_1 \times F_2$ satisfies

$$L_\varphi a_1 = \begin{cases} F_2' & \text{if } a_1 \in E_1 \\ A_2 & \text{otherwise.} \end{cases}$$

whence

$$[L_\varphi] = E_1 \times F_2' \bigcup (E_1)^c \times A_2.$$

Similarly

$$[R_\varphi] = E_1' \times F_2 \bigcup (A_1) \times (A_2)^c.$$

Example 10 The tautological bond defines the identity Chu correspondence.

2.3.2. Bonds defined by Chu correspondences Conversely let φ be a Chu correspondence from $\mathbf{C}_1$ to $\mathbf{C}_2$. Define a correspondence Z_φ from A_1 to X_2 by

$$Z_\varphi a_1 := (L_\varphi a_1)'.$$

Example 11 The identity Chu correspondence defines the tautological bond.

Proposition 35 *Z_φ is a bond from $\mathbf{C}_1$ to $\mathbf{C}_2$.*

Proof. Put $Z = Z_\varphi$ for brevity. Then for $a_1 \in A_1$, $Z_\varphi a_1$ is obviously closed by definition.

For $x_2 \in X_2$,

$${}^t Zx_2 = \{a_1 \mid x_2 \in Za_1\} = \{a_1 \mid x_2 \in (L_\varphi a_1)'\}.$$

Since

$$x_2 \in (L_\varphi a_1)' \iff L_\varphi a_1 \models x_2 \iff a_1 \models R_\varphi x_2, \iff a_1 \in (R_\varphi x_2)',$$

we have

$$^t Z x_2 = (R_\varphi x_2)',$$

which implies $^t Z x_2$ is closed. □

Proposition 36 *The correspondences $\varphi \mapsto Z_\varphi$ and $Z \mapsto \varphi_Z$ are inverse to each other.*

Proof. Let Z be a bond. Since $Z_{\varphi_Z} a_1 = (L_{\varphi_Z} a_1)' = \overline{Z a_1} = Z a_1$, we have $Z_{\varphi_Z} = Z$.

On the other hand, let φ be a Chu correspondence. Then

$$L_{\varphi_{Z_\varphi}} a_1 = (Z_\varphi a_1)' = \overline{L_\varphi a_1} = L_\varphi a_1,$$

whence $L_{\varphi_{Z_\varphi}} = L_\varphi$, which implies $\varphi_{Z_\varphi} = \varphi$. □

Proposition 37 *Let $\mathbf{C}_i = (A_i, X_i, \models)$ $(i = 1, 2)$ be formal contexts and $Z\colon A_1 \to X_2$ be a bond from $\mathbf{C}_1$ and $\mathbf{C}_2$.*

(i) *For $N_1 \subset A_1$,*

$$((L_\varphi)_* N_1)' = Z_\circ N_1.$$

(ii) *The subset $Z_\circ N_1 \subset X_2$ is closed for $N_1 \subset A_1$ and $Z^\circ M_2 \subset A_1$ is also closed for $M_2 \subset X_2$.*

Proof.

$$\begin{aligned} ((L_\varphi)_* N_1))' &= \bigcap_{m \in (L_\varphi)_* N_1} m' \\ &= \bigcap_{n \in N_1} \bigcap_{m \in L_\varphi n} m' \\ &= \bigcap_{n \in N_1} (L_\varphi n)' \\ &= \bigcap_{n \in N_1} Zn \\ &= Z_\circ N_1. \end{aligned}$$

The assertion (ii) follows from (i). It is however obvious since $Z_\circ N = \bigcap_{n \in N} Zn$ and Zn's are closed by definition. □

2.4. The complete lattice of Chu correspondences

Let $\mathbf{C}_i = (A_i, X_i, \models)$ $((i = 1, 2))$ be formal contexts. Let $ChuCors(\mathbf{C}_1, \mathbf{C}_2)$ denotes the set of Chu correspondences from $\mathbf{C}_1$ to $\mathbf{C}_2$ with the order defined by $\varphi_1 \leq \varphi_2$ if and only if $L_{\varphi_1} \subset L_{\varphi_2}$. Note that by Proposition 28 and Corollary 15, this is equivalent to $R_{\varphi_1} \subset R_{\varphi_2}$.

By Proposition 36, we have

Theorem 38 *The correspondence which assigns to each Chu correspondence φ the bond Z_φ is a bijection between $ChuCors(\mathbf{C}_1, \mathbf{C}_2)$ to $Bond(\mathbf{C}_1, \mathbf{C}_2)$. In fact, as complete lattices, we have*

$$ChuCors(\mathbf{C}_1, \mathbf{C}_2) \simeq Bond(\mathbf{C}_1, \mathbf{C}_2)^*.$$

Proof. It remains to check that the bijection reverse the order. Suppose $\alpha \leq \beta$ as Chu correspondences. Then, for $a_1 \in A_1$,

$$Z_\alpha a_1 = (L_\alpha a_1)' \supset (L_\beta a_1)' = Z_\beta a_1.$$

Hence $Z_\alpha \geq Z_\beta$. □

We write

$$\mathbf{C}_1 \bowtie \mathbf{C}_2 := \{[Z] \mid Z \in Bond(\mathbf{C}_1, \mathbf{C}_2^*)\}, \tag{6}$$

which consists of the graph of correspondences $Z\colon A_1 \to A_2$ for which both Za_1 and ${}^tZa_2$ are closed for $a_i \in A_i$ $(i = 1, 2)$.

From Theorem 38 and Proposition 33

Proposition 39 *The poset $ChuCors(\mathbf{C}_1, \mathbf{C}_2)$ is complete.*

For a Chu correspondence φ from $\mathbf{C}_1$ to $\mathbf{C}_2$, define a Chu correspondence φ^* from $\mathbf{C}_2^*$ to $\mathbf{C}_1^*$ by

$$L_{\varphi^*} = R_\varphi, \quad R_{\varphi^*} = L_\varphi.$$

Obviously we have

Proposition 40 *The correspondence $\varphi \mapsto \varphi^*$ defines a poset isomorphism*

$$ChuCors(\mathbf{C}_1, \mathbf{C}_2) \simeq ChuCors(\mathbf{C}_2^*, \mathbf{C}_1^*).$$

Proposition 41

$$ChuCors(\mathbf{C}, \perp) \simeq \mathcal{A}(\mathbf{C}),$$
$$ChuCors(\top, \mathbf{C}) \simeq \mathcal{X}(\mathbf{C}).$$

Proof. By Theorem 38, it suffices to show

$$Bond(\mathbf{C},\, \perp) \simeq \mathcal{A}(\mathbf{C}).$$

A bond Z from $\mathbf{C}$ to $\perp$ is a subset of $A \times \{\,*\,\}$, which corresponds to a closed subset of A, whence the former assertion. The latter is proved similarly. □

We see that the above isomorphisms are natural.

Proposition 42 *If $\varphi\colon \mathbf{C}_1 \to \mathbf{C}_2$ is a Chu correspondence, then the following diagram commutes*:

$$\begin{array}{ccc} ChuCors(\mathbf{C}_1, \perp) & \xrightarrow{\simeq} & \mathcal{A}(\mathbf{C}_1) \\ {\scriptstyle ChuCors(\varphi,1)}\big\uparrow & & \big\uparrow{\scriptstyle L_\varphi{}^*} \\ ChuCors(\mathbf{C}_2, \perp) & \xrightarrow{\simeq} & \mathcal{A}(\mathbf{C}_2) \end{array} \qquad \begin{array}{ccc} ChuCors(\perp, \mathbf{C}_1) & \xrightarrow{\simeq} & \mathcal{X}(\mathbf{C}_1) \\ {\scriptstyle ChuCors(1,\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle R_\varphi{}^*} \\ ChuCors(\perp, \mathbf{C}_2) & \xrightarrow{\simeq} & \mathcal{X}(\mathbf{C}_2) \end{array}$$

Proof. Describe $\perp$ as $(\{0,\, 1\},\, \{\,*\,\},\, \{(1,\, *\,)\})$. Let $\psi\colon \mathbf{C}_2 \to \perp$. The bond Z_ψ is given by

$$Z_\psi = \{(a_2,\, *\,) \mid a_2 \in A_2,\, L_\psi a_2 \models *\},$$

which correspondes to $L_\psi{}^*\{1\} \subset A_2$.

On the other hand,

$$Z_{\psi\circ\varphi} = \{(a_1,\, *\,) \mid a_1 \in A_1,\, L_{\psi\circ\varphi} a_1 \models *\,\}.$$

Since

$$\begin{aligned} L_{\psi\circ\varphi} a_1 \models * &\Leftrightarrow L_{\psi\circ\varphi} a_1 \subset \{1\} \\ &\Leftrightarrow a_1 \in L_{\psi\circ\varphi}{}^*\{1\} = (L_\psi L_\varphi)^*\{1\} = L_\varphi{}^* L_\psi{}^*\{1\}. \end{aligned}$$

The last equality follows from Proposition 16. This proves the commutativity of the left diagram. The comutativity of the right diagram can be proved similarly. □

3. Examples of Chu correspondences

3.1. Simple examples

Let $\mathbf{C} = (B_1,\, B_2,\, \leq)$, where $B_1 = B_2 = \mathbf{B} = \{0,\, 1\}$. Then the set of closed sets of B_1 and B_2 are respectively $\{0,\, 01\}$ and $\{1,\, 01\}$. In particular there are four correspondences L_i $(1 \le i \le 4)$ from B_1 to B_1 whose images are closed sets, namely,

L	$L0$	$L1$
L_1	0	0
L_2	0	$\mathbf{B}$
L_3	$\mathbf{B}$	0
L_4	$\mathbf{B}$	$\mathbf{B}$

.

To check if L is a continuous intent map, we compute L^*K for closed $K =$ 0, 01.

L	L^*0	$L^*\mathbf{B}$
L_1	$\mathbf{B}$	$\mathbf{B}$
L_2	0	$\mathbf{B}$
L_3	1	$\mathbf{B}$
L_4	$\emptyset$	$\mathbf{B}$

,

whence only L_1 and L_2 are the extent parts of Chu correspondences. By Proposition 28, the intent parts are computed for example as follows:

$$R_2 0 = ({L_2}^* 0')' = ({L_2}^* 0)' = 0' = \mathbf{B}$$

and

$$R_2 1 = ({L_2}^* 1')' = ({L_2}^* \mathbf{B})' = (\mathbf{B})' = 1.$$

By similar calculation, we have

R	$R0$	$R1$
R_1	1	1
R_2	$\mathbf{B}$	1

.

The bonds corresponding to the Chu correspondences $\varphi_i = (L_i, R_i)$ $(i =$ 1, 2) is described as follows:

Z_{φ_1} :

	0	1
0	1	1
1	1	1

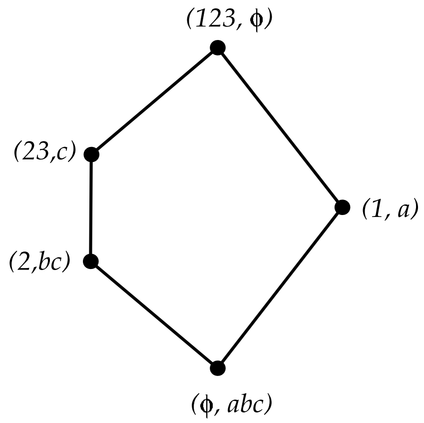
Z_{φ_2} :

	0	1
0	1	1
1	0	1

.

3.2. Chu correspondences which are not Chu maps

We give an example of a formal context with Chu auto correspondences which are not Chu maps.



Let $\mathbf{C} = (A, X, R)$ with $A = \{1, 2, 3\}$, $X = \{a, b, c\}$ and R is given by the following table:

	a	b	c
1	1	0	0
2	0	1	1
3	0	0	1

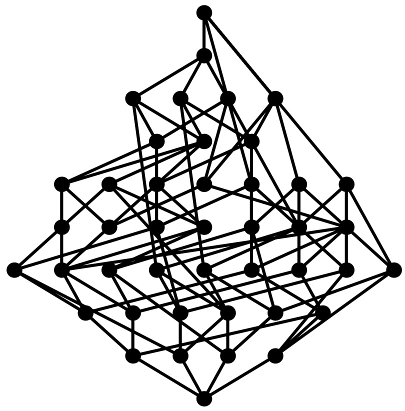
The Galois lattice of $\mathbf{C}$ is described as follows:

There are 43 Chu correspondences given as follows, where a correspondence φ is described by the triple $((L_\varphi 1)', (L_\varphi 2)', (L_\varphi 3)')$: $(0, 0, 0)$, $(0, 0, c)$, $(0, 0, bc)$, $(0, 0, a)$, $(0, 0, abc)$, $(0, c, c)$, $(0, c, bc)$, $(0, c, abc)$, $(0, bc, bc)$, $(0, bc, abc)$, $(0, a, a)$, $(0, a, abc)$, $(0, abc, abc)$, $(c, 0, 0)$, $(c, 0, a)$, (c, c, c), (c, c, bc), (c, c, abc), (c, bc, bc), (c, bc, abc), (c, a, a), (c, abc, abc), $(bc, 0, 0)$, $(bc, 0, a)$, (bc, c, c), (bc, bc, bc), (bc, bc, abc), (bc, a, a), (bc, abc, abc), $(a, 0, 0)$, $(a, 0, c)$, $(a, 0, bc)$, (a, c, c), (a, c, bc), (a, bc, bc), (a, a, a), (a, a, abc), (a, abc, abc), $(abc, 0, 0)$, (abc, c, c), (abc, bc, bc), (abc, a, a), (abc, abc, abc).

Among these Chu correspondences, only the three (a, bc, bc), (a, c, bc), (bc, a, a) come from Chu maps.

3.3. Chu maps as Chu correspondences

Proposition 43 *Let $\mathbf{C}_i = (A_i, X_i, \models)$ $(i = 1, 2)$ be formal contexts and (f, g) be a pair of maps $f\colon A_1 \to A_2$ and $g\colon X_2 \to X_2$. Then (f, g) is a Chu map if only if (f, g) is a Chu correspondence in the weak sense when*



regarded as a correspondence from $\mathbf{C}_1$ *to* $\mathbf{C}_2$*. In particular, its closure* φ *defined by*

$$L_\varphi(a_1) = \overline{f(a_1)} \quad R_\varphi(x_2) = \overline{g(x_2)}$$

is a Chu correspondence.

Proof. Suppose (f, g) is a Chu map. Let $a_1 \in A_1$ and $x_2 \in X_2$. Then

$$a_1 \models \{g(x_2)\} \Leftrightarrow a_1 \models g(x_2) \Leftrightarrow f(a_1) \models x_2 \Leftrightarrow \{f(a_1)\} \models x_2,$$

whence (f, g) is a Chu correspondence in the weak sense.

Conversely if (f, g) is a Chu correspondence in the weak sense, then

$$a_1 \models g(x_2) \Leftrightarrow a_1 \models \{gx_2\} \Leftrightarrow \{f(a_1)\} \models x_2 \Leftrightarrow f(a_1) \models x_2,$$

whence (f, g) is a Chu map. $\square$

Remark The following example shows that Chu maps are very few compared with Chu correspondences. Let

$$\mathbf{C} = (\{1, 2, 3\}, \{1, 2, 3\}, P)$$

where P is defined by $P(i, j) = 1 - \delta_{ij}$, where δ is the Kronecker's symbol.

Then $Gal(\mathbf{C}) = \{(A, A^c) \mid A \subset \{1, 2, 3\}\} \simeq \mathbf{pow}(\{1, 2, 3\})$.

Since every subset of $\{1, 2, 3\}$ is closed, any relation $L \subset \{1, 2, 3\}^2$ is continuous extent relation from $\mathbf{C}$ to itself and hence there are 2^9 Chu correspondences. On the other hand there are only 6 Chu endomaps of $\mathbf{C}$.

In fact, suppose $(f,\, g)$ be a Chu map. Suppose f is not a bijection. Then there is a $k \notin \mathrm{Im}(f)$. This k satisfies $f(i) \models k$ for all i, whence $i \models g(k)$ for all i, which is impossible. Hence f must be a bijection. Conversely, suppose we have a bijection f from $\{1,\, 2,\, 3\}$ to itself. Define $g(k)$ to be the unique element of the polar set of $\{j \mid f(j) \neq k\}$ which consists of two elements. Then obviously $(f,\, g)$ is a Chu map. Hence the set of Chu maps is bijective to the set of bijective auto-maps of $\{1,\, 2,\, 3\}$.

However, between the formal contexts associated with complete lattices, all Chu correspondences are induced from Chu maps. Consider the formal contexts

$$\mathbf{C}_i = (L_i,\, L_i,\, \leq)$$

where L_i are complete lattices $(i = 1,\, 2)$. Then a Chu map $(f,\, g)$ from $\mathbf{C}_1$ to $\mathbf{C}_2$ is precisely a Galois pair $f\colon L_1 \to L_2$ and $g\colon L_2 \to L_1$ satisfying

$$f(a) \leq b \Longleftrightarrow a \leq g(b).$$

When this Chu map is regarded as a Chu correspondence in the weak sense, then its closure φ is given by

$$L_\varphi a_1 = f(a_1)\downarrow \quad \text{and} \quad R_\varphi a_2 = g(a_2)\uparrow$$

for $a_i \in A_i$ $(i = 1,\, 2)$. Conversely, suppose φ is a Chu correspondence from $\mathbf{C}_1$ to $\mathbf{C}_2$. Define

$$f(a_1) = \bigvee L_\varphi a_1 \quad \text{and} \quad g(a_2) = \bigvee R_\varphi a_2.$$

Then $(f,\, g)$ is a Chu map. Hence Chu correspondences and Chu maps from $\mathbf{C}_1$ to $\mathbf{C}_2$ corresponds one to the other bijectively.

3.4. Chu correspondences between powercontexts

Let A_i $(i = 1,\, 2)$ be sets and

$$\mathbf{P}(A_i) = (A_i,\, \mathbf{pow}A_i,\, \in)$$

$(i = 1,\, 2)$ be their power contexts. Recall that there is a bijection

$$Map(A_1,\, A_2) \xrightarrow{\simeq} ChuMaps(\mathbf{P}(A_1),\, \mathbf{P}(A_2)).$$

for sets A_i $(i = 1,\, 2)$, by Proposition 21.

We show that set theoretical correspondences induce Chu correspondences between power contexts.

Proposition 44 *Let* $T\colon A_1 \to \mathbf{pow}A_2$ *be a correspondence. Define*

$$L_T = T$$

and

$$R_T\colon \mathbf{pow}A_2 \to \mathbf{powpow}A_1$$

by

$$R_T N_2 = \{N_1 \subset X_1 \mid L^* N_2 \subset N_1\} = L^* N_2 \uparrow,$$

where $B \uparrow$ *denotes the family of subsets including* B*. Then* $\widetilde{T} = (L_T, R_T)$ *is a Chu correspondence from* $\mathbf{P}(A_1)$ *to* $\mathbf{P}(A_2)$*.*

Proof. We write $R = R_T$, $L = L_T$ for brevity. Let $a_1 \in A_1$ and $N_2 \subset A_2$. We show

$$La_1 \models N_2 \Longleftrightarrow a_1 \models RN_2$$

First note that

$$La_1 \models N_2 \Longleftrightarrow a_1 \in L^* N_2,$$

since

$$La_1 \models N_2 \Leftrightarrow La_1 \subset N_2 \Leftrightarrow a_1 \in L^* N_2.$$

Note also that

$$a_1 \models RN_2 \Longleftrightarrow a_1 \in \bigcap RN_2,$$

since $a_1 \models RN_2$ means $a_1 \in N_1$ for all N_1 with $N_1 \in RN_2$.

Since $RN_2 = L^* N_2 \uparrow$,

$$\bigcap RN_2 = L^* N_2.$$

Hence $La_1 \models N_2$ if and only if $a_1 \in L^* N_2$ if and only if $a_1 \in \bigcap RN_2$ if and only if $a_1 \models RN_2$. Hence, (L, R) is a Chu correspondence. □

Corollary 45 *Let*

$$\varphi\colon \mathbf{P}(A_1) \to \mathbf{P}(A_2)$$

be a Chu correspondence. Then

$$\varphi = \widetilde{L_\varphi}.$$

In particular, the correspondence T *to* $\widetilde{T}$ *defines a bijection*

$$Cor(A_1,\, A_1) \simeq ChuCors(\mathbf{P}(A_1),\, \mathbf{P}(A_2)).$$

Proposition 46 *Let* $f\colon A_1 \to A_2$ *be a map considered as a correspondence. Then*

$$[R_f] = \{(N_1,\, N_2) \mid f^{-1}N_2 \subset N_1\} \subset \mathbf{pow}(A_1) \times \mathbf{pow}(A_2).$$

Proof. Put $L = f$. It suffices to show that $L^*N_2 = f^{-1}N_2$, which follows directly from

$$L^*N_2 = \{a_1 \mid La_1 \subset N_2\} = \{a_1 \mid f(a_1) \in N_2\} = f^{-1}N_2.$$

□

Remark Hence there are much more Chu correspondences from $\mathbf{P}(A_1)$ to $\mathbf{P}(A_2)$ than Chu maps. In fact, if $n_i = |A_i|$ $(i = 1,\, 2)$, then there are $2^{n_1 \times n_2} = (2^{n_2})^{n_1}$ Chu correspondences and $n_2^{n_1}$ Chu maps. As n_2 increases, the ratio of the number of Chu correspondences against that of Chu maps increases rapidly.

3.5. Chu correspondences as Chu maps

Chu correspondences correspond to Chu maps between the power contexts.

Lemma 47 *A correspondence* φ *from* $\mathbf{C}_1$ *to* $\mathbf{C}_2$ *is a Chu correspondence if and only if*

$$L_{\varphi *}N_1 \models M_2 \Longleftrightarrow N_1 \models R_{\varphi *}M_2$$

for all $N_1 \subset A_1$ *and* $M_2 \subset X_2$*.*

Proof. $L_{\varphi *}N_1 \models M_2$ if and only if $L_\varphi n_1 \models m_2$ for all $n_1 \in N_1$ and $m_2 \in M_2$ if and only if $n_1 \models R_\varphi m_2$ for all $n_1 \in N_1$ and $m_2 \in M_2$ if and only if $N_1 \models R_{\varphi *}M_2$.

Conversely, suppose the latter condition holds. Then taking $N_1 = \{a_1\}$ and $M_2 = \{x_2\}$, we have

$$L_\varphi a_1 \models x_2 \Longleftrightarrow a_1 \models R_\varphi x_2.$$

□

This can be rephrased as follows:

Theorem 48 *A correspondence φ from $\mathbf{C}_1$ to $\mathbf{C}_2$ is a Chu correspondence if and only if $(L_{\varphi *}, R_{\varphi *})$ is a Chu map from $\mathbf{pow}(\mathbf{C}_1)$ to $\mathbf{pow}(\mathbf{C}_2)$.*

3.6. Chu relation in the sense of Pratt

V. Pratt introduced a concept called "Chu relation" [15]. A correspondence (L, R) is called a "Chu relation" if for all $a_i \in A_i$ and $x_i \in X_i$ $(i = 1, 2)$, the condition $(a_1, a_2) \in [L]$ and $(x_1, x_2) \in [R]$ imply the equivalence of the conditions $a_1 \models x_1$ and $a_2 \models x_2$.

If (L, R) is a "Chu relation" in the sense of Pratt, then it is a Chu correspondence in our sense. In fact, for $a_1 \in A_1$ and $x_2 \in X_2$, for all $a_2 \in La_1$ and $x_1 \in Rx_2$, we have $x_1 \models a_1$ iff $a_2 \models x_2$. Hence $La_1 \models x_2$ implies $a_2 \models x_2$ for all $a_2 \in La_1$, which implies $a_1 \models x_1$ for all $x_1 \in Rx_2$, namely, $a_1 \models Rx_2$. The other implication is proved similarly, whence (L, R) is a Chu correspondence in our sense. Note that the above arguments also show that if a correspondence (L, R) from $\mathbf{C}_1$ to $\mathbf{C}_2$ is a "Chu relation" (L, R) in the sense of Pratt, the correspondence $({}^tL, {}^tR)$ form $\mathbf{C}_2$ to $\mathbf{C}_1$ is also a Chu correspondence since it satisfies also the condition ${}^tLa_2 \models x_1$ iff $a_2 \models {}^tRx_1$ for $x_1 \in X_1$ and $a_2 \in A_2$.

For the following formal concepts $\mathbf{C}_i$ $(i = 1, 2)$, there are 569 Chu correspondences from $\mathbf{C}_1$ to $\mathbf{C}_2$ and 578 ones from $\mathbf{C}_2$ to $\mathbf{C}_1$, which means that our Chu correspondence is strictly more general than the one defined by Pratt.

$$\mathbf{C}_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \mathbf{C}_2 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The Galois lattices are as follows.

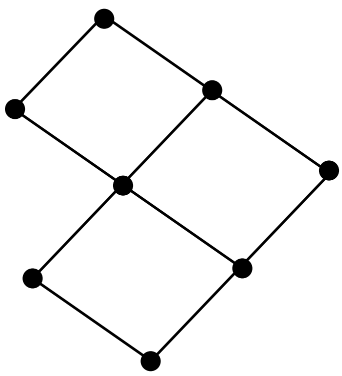
4. Category of Chu correspondences

The Chu correspondences form a category by a natural composition.

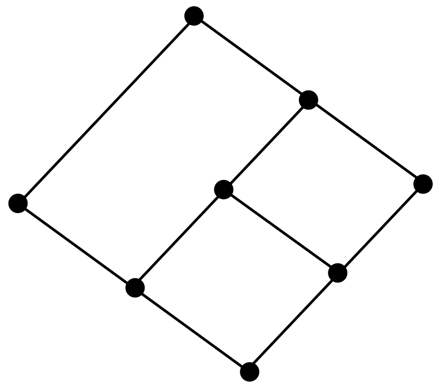
4.1. Definition

Let *ChuCors* be the category whose objects are extensional and intensional formal contexts and whose arrows are Chu correspondences.

The identity Chu correspondence of a formal context $\mathbf{C}$ is the closure of the identity Chu map of $\mathbf{C}$ considered as a Chu correspondence in the



galois(C_1)



galois(C_2)

weak sense as in Section 3.3.

The composition is defined as follows. If φ and ϕ are Chu correspondences respectively from $\mathbf{C}_1$ to $\mathbf{C}_2$ and $\mathbf{C}_2$ to $\mathbf{C}_3$, their composition $\phi \circ \varphi$ is defined by

$$L_{\phi\circ\varphi}a_1 = \overline{L_{\phi *}(L_\varphi a_1)}$$

for $a_1 \in A_1$ and

$$R_{\phi\circ\varphi}x_3 = \overline{R_{\varphi *}(R_\phi x_3)}$$

for $x_3 \in X_3$.

Proposition 49 *The correspondence $\phi \circ \varphi$ from $\mathbf{C}_1$ to $\mathbf{C}_3$ is a Chu correspondence.*

Proof. Let $a_1 \in A_1$, $x_3 \in X_3$. Then

$$\begin{aligned}\overline{L_{\phi *}(L_\varphi a_1)} \models x_3 &\Longleftrightarrow L_{\phi *}(L_\varphi a_1) \models x_3 \quad \text{by Lemma 18}\\ &\Longleftrightarrow L_\varphi a_1 \models R_\phi x_3\\ &\Longleftrightarrow a_1 \models R_{\varphi *}(R_\phi x_3) \quad \text{by Lemma 47}\\ &\Longleftrightarrow a_1 \models \overline{R_{\varphi *}R_\phi x_3}.\end{aligned}$$

This proves that $\phi \circ \varphi$ is a Chu correspondence. □

The identity axiom follows from the following.

Proposition 50 *For a Chu correspondence φ from $\mathbf{C}_1$ to $\mathbf{C}_2$, the following equalities hold.*

$$\begin{aligned}
\overline{L_{\varphi*}(L_{\mathrm{id}}a_1)} &= L_\varphi a_1\\
\overline{R_{\mathrm{id}\,*}(R_\varphi x_2)} &= R_\varphi x_2\\
\overline{L_{\mathrm{id}\,*}(L_\varphi a_1)} &= L_\varphi a_1\\
\overline{R_{\varphi*}(R_{\mathrm{id}}x_2)} &= R_\varphi x_2
\end{aligned}$$

Proof. We prove the first and the third equalities. The others are proved similarly.

Let $a_1 \in A_1$.

$$\begin{aligned}
\overline{L_{\mathrm{id}\,*}(L_\varphi a_1)} &= \overline{\bigcup_{a_2\in L_\varphi a_1} \overline{a_2}}\\
&= \overline{\bigcup_{a_2\in L_\varphi a_1} a_2} \quad \text{by Lemma 7}\\
&= \overline{L_\varphi a_1} = L_\varphi a_1.
\end{aligned}$$

Let $a_1 \in A_1$. Then by Proposition 29

$$\overline{L_{\varphi*}L_{\mathrm{id}}a_1} = \overline{L_{\varphi*}\overline{a_1}} = L_\varphi a_1,$$

by whence $L_{\varphi\circ\mathrm{id}} = L_\varphi$. □

To show the the associativity, we need the following lemma.

Lemma 51 *For $N_1 \subset A_1$,*

$$\overline{L_{\phi\circ\varphi}N_1} = \overline{L_{\phi*}L_{\varphi*}N_1}.$$

Proof.

$$\begin{aligned}
\overline{L_{\phi\circ\varphi}N_1} &= \overline{\bigcup_{x\in N_1} L_{\phi\circ\varphi}x}\\
&= \overline{\bigcup_{x\in N_1} \overline{L_{\phi*}L_\varphi x}}\\
&= \overline{\bigcup_{x\in N_1} L_{\phi*}L_\varphi x} \quad \text{by Lemma 7}\\
&= \overline{L_{\phi*}L_{\varphi*}N_1}.
\end{aligned}$$

□

The following proposition shows the associativity of the composition.

Proposition 52 *Let $\varphi_i \colon \mathbf{C}_i \to \mathbf{C}_{i+1}$ $(i = 1, 2, 3)$ be Chu correspondences. Then, for $a_1 \in A_1$,*

$$\overline{L_{(\varphi_1 \circ \varphi_2) \circ \varphi_3} a_1} = \overline{L_{\varphi_1 \circ (\varphi_2 \circ \varphi_3)} a_1}.$$

Proof.

$$\begin{aligned} \overline{L_{(\varphi_1 \circ \varphi_2) \circ \varphi_3} a_1} &= \overline{L_{(\varphi_1 \circ \varphi_2)*} L_{\varphi_3} a_1} \\ &= \overline{L_{\varphi_1 *} L_{\varphi_2 *} L_{\varphi_3} a_1} \quad \text{by Proposition 52.} \end{aligned}$$

On the other hand,

$$\begin{aligned} \overline{L_{\varphi_1 \circ (\varphi_2 \circ \varphi_3)} a_1} &= \overline{L_{\varphi_1 *} L_{\varphi_2 \circ \varphi_3} a_1} \\ &= \overline{L_{\varphi_1 *} \overline{L_{\varphi_2 *} L_{\varphi_3} a_1}} \quad \text{by Proposition 52} \\ &= \overline{L_{\varphi_1 *} L_{\varphi_2 *} L_{\varphi_3} a_1} \quad \text{by Proposition 29.} \end{aligned}$$

□

By Proposition 39, the homset $ChuCors(\mathbf{C}_1, \mathbf{C}_2)$ is a complete lattice. We note that the category $ChuCors$ has a structure of $\mathcal{S}lat$-enriched category.

4.2. Functor from the category of Chu maps

Let $ChuMaps$ be the category whose arrows are Chu maps. Define the functor

$$\iota \colon ChuMaps \to ChuCors$$

which is identity on objects and for a Chu map (f, g), $\iota(f, g)$ is the closure of (f, g) regarded as a Chu correspondence in the weak sense by Proposition 43.

Proposition 53 *ι is a functor.*

Proof. By definition $\iota(\mathrm{id}_{\mathbf{C}})$ is the identity Chu correspondence of the formal context $\mathbf{C}$.

Let (f_i, g_i) be Chu maps from $\mathbf{C}_i$ to $\mathbf{C}_{i+1}$ $(i = 1, 2)$ and Put $\varphi_i := \iota(f_i, g_i)$ $(i = 1, 2)$. Define $(f, g) = (f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_1 \circ g_2)$ and $\varphi = \iota(f, g)$. Then

$$L_{\varphi_2 \circ \varphi_1} a_1 = \overline{L_{\varphi_2 *} L_{\varphi_1} a_1}$$

$$
\begin{aligned}
&=\overline{L_{\varphi_2 *}\overline{f_1(a_1)}}\\
&=\overline{L_{\varphi_2 *}f_1(a_1)} \quad \text{by Proposition 29}\\
&=\overline{\overline{f_2(f_1(a_1))}} = \overline{f(a_1)} = L_\varphi a_1.
\end{aligned}
$$

Hence $\varphi = \varphi_2 \circ \varphi_1$. □

The strong isomorphisms are exactly the image of Chu isomorphisms, whence

Proposition 54 *Let $\varphi\colon \mathbf{C}_1 \to \mathbf{C}_2$ be a strong isomorphism with a generator: (f, g). Then it is an isomorphism whose inverse is the closure of (f^{-1}, g^{-1}).*

We write $\mathbf{C}_1 \cong \mathbf{C}_2$ if there is a Chu isomorphism from $\mathbf{C}_1$ to $\mathbf{C}_2$.

4.3. Galois functor

We have defined the complete lattice $Gal(\mathbf{C})$ of formal concepts of a formal context $\mathbf{C}$. This induces the Galois functor

$$Gal\colon ChuCors \to \mathcal{S}lat$$

in the following way.

Let $\mathbf{C}_i = (A_i, X_i, \models)$ $(i = 1, 2)$ be formal contexts and

$$\varphi\colon \mathbf{C}_1 \to \mathbf{C}_2$$

be a Chu correspondence.

Define $\varphi_*\colon Gal(\mathbf{C}_1) \to Gal(\mathbf{C}_2)$ by

$$\varphi_*(M_1, M_1') = \big(\overline{L_{\varphi *}M_1}, (L_{\varphi *}M_1)'\big) \tag{7}$$

and $\varphi^*\colon Gal(\mathbf{C}_2) \to Gal(\mathbf{C}_1)$ by

$$\varphi^*(N_2', N_2) = ((R_{\varphi *}N_2)', \overline{R_{\varphi *}N_2}). \tag{8}$$

Proposition 55 *The pair (φ_*, φ^*) is a galois pair, namely, for closed $M_1 \subset A_1$ and $N_2 \subset X_2$,*

$$\varphi_*(M_1, M_1') \leq (N_2', N_2) \Longleftrightarrow (M_1, M_1') \leq \varphi^*(N_2', N_2). \tag{9}$$

Proof. The condition (9) is equivalent to

$$(L_{\varphi *}M_1)' \supset N_2 \Longleftrightarrow M_1 \subset (R_{\varphi *}N_2)',$$

i.e. to

$$L_{\varphi*}M_1 \models N_2 \Longleftrightarrow M_1 \models R_{\varphi*}N_2,$$

which holds by Lemma 47. □

Corollary 56 *φ_* preserves the joins and φ^* preserves the meets.*

We define

$$Gal(\varphi) := \varphi_* \colon Gal(\mathbf{C}_1) \to Gal(\mathbf{C}_2).$$

Proposition 57 *Gal is a functor from ChuCors to $\mathcal{S}lat$.*

Proof. First $Gal(\mathrm{id}_{\mathbf{C}}) = \mathrm{id}_{Gal(\mathbf{C})}$ follows from

$$\overline{L_{\mathrm{id}}M} = \overline{\bigcup_{a \in M} \overline{a}} = \overline{M} = M$$

for closed $M \subset A$.

Let $\varphi_1 \colon \mathbf{C}_1 \to \mathbf{C}_2$ and $\varphi_2 \colon \mathbf{C}_2 \to \mathbf{C}_3$ be Chu correspondences.

Since $Gal(\mathbf{C}_1)$ is $\bigvee$-generated by $(\overline{a_1},\, a_1')$ $(a_1 \in A_1)$, it suffices to show

$$(\varphi_2 \circ \varphi_1)_*(\overline{a_1}, a_1') = \varphi_{2*}(\varphi_{1*}(\overline{a_1}, a_1')).$$

The first component of the left hand side is

$$\begin{aligned} L_{\varphi_2 \circ \varphi_1} a_1 &= \overline{L_{\varphi_2 *} L_{\varphi_1} a_1} \\ &= \overline{L_{\varphi_2 *} \overline{L_{\varphi_1} a_1}} \quad \text{by Proposition 29} \\ &= \overline{L_{\varphi_2 *} \overline{L_{\varphi_1 *} \overline{a_1}}} \quad \text{by Proposition 29} \end{aligned}$$

which is the first component of the right hand side. □

By Proposition 25, the map φ_* can be described also as follows.

Proposition 58

$$\varphi_*(N_1', N_1) = \big((R_\varphi{}^* N_1)',\, R_\varphi{}^* N_1\big).$$

Proof. By Proposition 26,

$$(L_{\varphi*}N_1')' = R_\varphi{}^* \overline{N_1} = R_\varphi{}^* N_1,$$

for closed $N_1 \subset X_1$, whence

$$\varphi_*(N_1',\, N_1) = (\overline{L_{\varphi*}N_1'},\, (L_{\varphi*}N_1')')$$

$$= ((R_{\varphi}{}^{*}\overline{N_1})',\, R_{\varphi}{}^{*}\overline{N_1}) = ((R_{\varphi}{}^{*}N_1)',\, R_{\varphi}{}^{*}N_1)$$

□

Note that this proposition also proves that φ_* preserves the join, since the second component of the join is the set theoretical intersection and $R_{\varphi}{}^{*}$ preserves the intersection by Corollary 13.

Note that the correspondences $\mathbf{C} \mapsto \mathcal{A}(\mathbf{C}),\, \mathcal{X}(\mathbf{C})$ of § 1.4 are functors from *ChuCors* to $\mathcal{S}lat$ and $\mathcal{S}lat^{\mathrm{op}}$ respectively if we define

$$\mathcal{A}(\varphi)\colon \mathcal{A}(\mathbf{C}_1) \to \mathcal{A}(\mathbf{C}_2)$$
$$\mathcal{X}(\varphi)\colon \mathcal{X}(\mathbf{C}_2) \to \mathcal{X}(\mathbf{C}_1)$$

for $\varphi\colon \mathbf{C}_1 \to \mathbf{C}_2$ by

$$\mathcal{A}(\varphi)(N_1) = \overline{L_{\varphi *}N_1},$$

for $N_1 \subset A_1$ and

$$\mathcal{X}(\varphi)(M_2) = \overline{R_{\varphi *}M_2},$$

for $M_2 \subset X_2$ respectively.

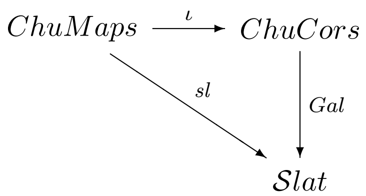
The following proposition follows directly from the definition.

Proposition 59 *There are natural isomorphisms among functors*:

$$\mathcal{A} \simeq \mathcal{X}^{\mathrm{op}} \simeq Gal.$$

Recall that the Chu maps induce join preserving maps between the complete lattices of formal concepts [13]. The following shows that they coincide with those induced when the Chu maps are considered as Chu correspondences.

Proposition 60 *If* $(f,\, g)\colon \mathbf{C}_1 \to \mathbf{C}_2$ *is a Chu map. Then* $Gal(\iota(f,\, g))$ *maps* $(M_1,\, M_1')$ *to* $(\overline{f(M_1)},\, f(M_1)')$. *In particular, the following diagram of functors commutes.*



Proof. Let $\varphi = \iota(f, g)$. Then

$$\varphi_*(M_1, M_1') = (\overline{L_{\varphi *}M_1}, (L_{\varphi *}M_1)').$$

The assertion follows from

$$\overline{L_{\varphi *}M_1} = \overline{\bigcup_{a\in M_1} L_\varphi a} = \overline{\bigcup_{a\in M_1} \overline{f(a)}} = \overline{\bigcup_{a\in M_1} f(a)} = \overline{f(M_1)},$$

where the third equality follows from Lemma 7. □

The action of the functor *Gal* on Chu correspondences has the following alternative descriptions, either by bonds or by powercontexts.

First we describe the Galois functor by bonds.

Proposition 61 *Suppose a bond $Z \in Bond(\mathbf{C}_1, \mathbf{C}_2)$ corresponds to a Chu correspondence φ. Then $Gal(\varphi)$ maps (N, N') to $((Z_\circ N)', Z_\circ N)$.*

Proof. By definition, (N, N') corresponds to $(\overline{(L_\varphi)_* N}, ((L_\varphi)_* N))')$. By Proposition 37,

$$((L_\varphi)_* N))' = Z_\circ N.$$

□

Now we describe the Galois functor by powercontexts. Let $\mathbf{C}_i = (A_i, X_i, \models)$ $(i = 1, 2)$ be formal contexts and $\varphi\colon \mathbf{C}_1 \to \mathbf{C}_2$ be a Chu correspondence. Let $T\colon \mathbf{pow}(\mathbf{C}_1) \to \mathbf{pow}(\mathbf{C}_2)$ be the Chu map associated with it defined in Section 3.5.

Proposition 62 *The composition κ_T*

$$Gal(\mathbf{C}_1) \stackrel{\simeq}{\to} Gal(\mathbf{pow}\mathbf{C}_1) \stackrel{Gal(T)}{\longrightarrow} Gal(\mathbf{pow}\mathbf{C}_2) \stackrel{\simeq}{\to} Gal(\mathbf{C}_2)$$

is given by

$$\kappa_T(N', N) = ((R_\varphi)^* N)', (R_\varphi)^* N),$$

and hence

$$\kappa_T = Gal(\varphi).$$

Proof. By definition, the second component K of $\kappa_T(N', N)$ is characterized by the property

$$(R_{\varphi *})^{-1}\mathbf{pow}(N) = \mathbf{pow}(K).$$

Suppose $L \subset X_2$ does not satisfy $R_{\varphi *}(L) \in \mathbf{pow}(N)$, i.e. $R_{\varphi *}(L) \not\subset N$. This is equivalent to

$$R_{\varphi *}(L) \bigcap N^c \neq \emptyset,$$

to $(N^c \times L) \bigcap R \neq \emptyset$ and hence to $L \bigcap R_* N^c \neq \emptyset$. Hence

$$R_{\varphi *}(L) \subset N \Longleftrightarrow L \subset (R_\varphi)^* N.$$

Hence $K = (R_\varphi)^* N$. □

By Propositions 41, 42, 59, we see that $\top$ represents the functor *Gal*.

Proposition 63 *There are natural isomorphisms*:

$$ChuCors(\mathbf{C}, \perp) \simeq Gal(\mathbf{C})^*,$$
$$ChuCors(\top, \mathbf{C}) \simeq Gal(\mathbf{C}).$$

Proof. The former isomorphism is the composition of that of Proposition 41 and the isomorphism $\mathcal{A}^{\mathrm{op}} \simeq Gal(\mathbf{C})^*$.

The latter is proved similarly. □

4.4. Bifunctor of bonds

There is a bifunctor

$$ChuCors^{\mathrm{op}} \times ChuCors \to \mathcal{S}lat$$

which maps $(\mathbf{C}_1, \mathbf{C}_2)$ to $Bond(\mathbf{C}_1, \mathbf{C}_2)$. The action of arrows is defined as follows. Let $\psi_1 \colon \mathbf{D}_1 \to \mathbf{C}_1$ and $\psi_2 \colon \mathbf{C}_2 \to \mathbf{D}_2$ be Chu correspondences. Let $Z \in Bond(\mathbf{C}_1, \mathbf{C}_2)$. Define a correspondence

$$\psi_2 \circ Z \circ \psi_1 \colon B_1 \rightsquigarrow Y_2$$

by its graph

$$[\psi_2 \circ Z \circ \psi_1] = \{(b_1, y_2) \in B_1 \times Y_2 \mid L_{\psi_1} b_1 \times R_{\psi_2} y_2 \subset Z\},$$

where $\mathbf{D}_i = \{B_i, Y_i, \models_i\}$ $(i = 1, 2)$

Lemma 64 *The correspondence* $\psi_2 \circ Z \circ \psi_1 \colon B_1 \rightsquigarrow Y_2$ *is a bond.*

Proof. Let $b_1 \in B_1$ and $y_2 \in Y_2$. It suffices to show that both

$$\{y \in Y_2 \mid L_{\psi_1} b_1 \times R_{\psi_2} y \subset Z\} \subset Y_2,$$

and

$$\{b \in B_1 \mid L_{\psi_1} b \times R_{\psi_2} y_2 \subset Z\} \subset B_1$$

are closed. The condition $L_{\psi_1} b_1 \times R_{\psi_2} y \subset Z$ is equivalent to

$$R_{\psi_2} y \subset Z_\circ L_{\psi_1} b_1,$$

and to

$$y \in R^*_{\psi_2} Z_\circ L_{\psi_1} b_1.$$

The right hand side is closed By Propositions 30 and 37, since $L_{\psi_1} b_1$ is closed.

The latter assertion is proved similarly. □

These data define a bifunctor:

$$Bond(-,-) : ChuCors^{\mathrm{op}} \times ChuCors \to \mathcal{S}lat.$$

4.5. Fullness and faithfullness of *Gal*

Theorem 65 *The functor Gal is full and faithful, namely,*

$$Gal\colon ChuCors(\mathbf{C}_1, \mathbf{C}_2) \to \mathcal{S}lat(Gal(\mathbf{C}_1), Gal(\mathbf{C}_2))$$

is a bijection.

We prove the theorem by showing that the map

$$\lambda\colon \mathcal{S}lat(Gal(\mathbf{C}_1), Gal(\mathbf{C}_2)) \to ChuCors(\mathbf{C}_1, \mathbf{C}_2)$$

defined below is the inverse map of *Gal*. Let

$$\phi\colon Gal(\mathbf{C}_1) \to Gal(\mathbf{C}_2)$$

be a join preserving map and ϕ^* be its order adjoint. Define a correspondence (L, R) from $\mathbf{C}_1$ to $\mathbf{C}_2$ as follows. For $a_1 \in A_1$, define $La_1 \subset A_2$ to be the first component of the formal concept $\phi(\overline{a_1}, a_1')$, and for $x_2 \in X_2$, define $Rx_2 \subset X_1$ to be the second component of $\phi^*(x_2', \overline{x_2})$. Then

Lemma 66

$$\lambda(\phi) := (L, R)$$

is a Chu correspondence from $\mathbf{C}_1$ *to* $\mathbf{C}_2$.

Proof. In fact, for $a_1 \in A_1$ and $x_2 \in X_2$,

$$\begin{aligned} La_1 \models x_2 &\Longleftrightarrow La_1 \subset x_2' \\ &\Longleftrightarrow (La_1,\, (La_1)') \le (x_2',\, \overline{x_2}) \\ &\Longleftrightarrow \phi(\overline{a_1},\, a_1') \le (x_2',\, \overline{x_2}) \\ &\Longleftrightarrow (\overline{a_1},\, a_1') \le \phi^*(x_2',\, \overline{x_2}) = ((Rx_2)',\, Rx_2) \\ &\Longleftrightarrow a_1' \supset Rx_2 \\ &\Longleftrightarrow a_1 \models Rx_2. \end{aligned}$$

□

Lemma 67

$$\lambda \circ Gal = \mathrm{id}\,.$$

Proof. Let $\varphi\colon \mathbf{C}_1 \to \mathbf{C}_2$ be a Galois correspondence and put $\psi = \lambda(\varphi_*)$. The subset

$$L_\psi a_1 \subset A_2$$

is the first component of $\varphi_*(\overline{a_1},\, a_1')$, namely the closure of $L_{\varphi*}\overline{a_1}$, which is $L_\varphi a_1$ by Proposition 29. Hence $L_\psi = L_\varphi$. □

Lemma 68 *For $\varphi\colon Gal(\mathbf{C}_1) \to Gal(\mathbf{C}_2)$,*

$$Gal(\lambda(\varphi)) = \varphi.$$

Proof. Put $\psi = \lambda(\varphi)$. By definition,

$$\psi_*((\overline{a_1},\, a_1')) = \big(\overline{L_{\psi*}\overline{a_1}},\, (L_{\psi*}\overline{a_1})'\big).$$

By Proposition 29, $(L_{\psi*}\overline{a_1})' = (L_\psi a_1)'$, whence

$$\psi_*((\overline{a_1},\, a_1')) = (\overline{L_\psi a_1},\, (L_\psi a_1)') = (L_\psi a_1,\, (L_\psi a_1)') = \varphi(\overline{a_1},\, a_1'),$$

since $L_\psi a_1$ is the first component of $\varphi(\overline{a_1},\, a_1')$ by definition.

Since $Gal(\mathbf{C}_1)$ is join generated by $\{(\overline{a_1},\, a_1') \mid a_1 \in A_1\}$, it follows

$$\psi_* = \varphi.$$

□

Hence we have proved that λ is the inverse of Gal and the proof of Theorem 65 is completed.

Corollary 69 *A Chu correspondence φ is an isomorphism if $Gal(\varphi)$ is bijective.*

We use often the following proposition which follows from Corollary 69.

Proposition 70 *Let $(f, g)\colon \mathbf{C}_1 \to \mathbf{C}_2$ be a Chu map. Suppose $f\colon A_1 \to A_2$ is a bijection and preserves the closure operators in the sense that $f(\overline{B}) = \overline{f(B)}$ for $B \subset A_1$. Then $\iota(f, g)$ is an isomorphism.*

Proof. It suffices to show that $F := Gal(\iota(f, g))$ is an isomorphism by Corollary 69. By Proposition 60,

$$F((M_1, M_1')) = (\overline{f(M_1)}, f(M_1)') = (f(M_1), f(M_1)').$$

Since $M_1 \mapsto f(M_1)$ is a bijection from $\mathcal{A}_1$ to $\mathcal{A}_2$, we conclude F is an isomorphism. Here $\mathcal{A}_i$ is the set of closed subsets of A_i ($i = 1, 2$). □

4.6. Equivalence of *ChuCors* and *Slat*

We show that the functor Gal is in fact an equivalence of categories between $ChuCors$ and $\mathcal{S}lat$.

Recall that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ with natural isomorphisms

$$1_{\mathcal{C}} \simeq G \circ F \quad 1_{\mathcal{D}} \simeq F \circ G.$$

A functor G satisfying these conditions is called *a weak inverse of F*.

Define a functor $r\colon \mathcal{S}lat \to ChuCors$ by

$$r(L) = (L, L, \leq)$$

and for each join preserving map

$$\phi\colon L_1 \to L_2,$$

the pair (ϕ, ϕ^*) is a Chu map from $r(L_1)$ to $r(L_2)$, where ϕ^* is the order adjoint of ϕ. We denote this Chu map regarded as a Chu correspondence by $r(\phi)$.

In the following we show that r is a weak inverse of the functor Gal.

Lemma 71 *The maps*

$$\iota_K\colon K \to Gal(r(K))$$

defined, for each complete lattice K, by

$$\iota(k) = (k\downarrow,\, k\uparrow)$$

define natural isomorphisms.

Proof. Obvious, since the formal concepts of the context $(K, K, \leq)$ are written uniquely as $(k\downarrow, k\uparrow)$ with $k \in K$. $\square$

Lemma 72 *For formal contexts $\mathbf{C}$, there are natural isomorphisms*

$$\varpi_{\mathbf{C}}\colon \mathbf{C} \overset{\simeq}{\to} r(Gal(\mathbf{C})).$$

Proof. By Theorem 65, there is a unique isomorphism $\varpi_{\mathbf{C}}$ in $ChuCors$, which corresponds under the Galois functor to the isomorphism

$$\iota_{Gal(\mathbf{C})}\colon Gal(\mathbf{C}) \to Gal(rGal(\mathbf{C})).$$

of Lemma 71. Then using the faithfullness of Gal, it is easy to see that $\varpi_{\mathbf{C}}$'s form a natural transformation. $\square$

Hence we have proved the equivalence of categories.

Theorem 73 *The Galois functor is an equivalence between the category of the Chu correspondences and the category of join preserving maps.*

Remark The isomorphism $\varpi_C\colon \mathbf{C} \to rGal(\mathbf{C})$ of Lemma 72 is described explicitely as follows.

The formal context $rGal(\mathbf{C}) = (Gal(\mathbf{C}), Gal(\mathbf{C}), \leq)$ is described also as $(\mathcal{A}(\mathbf{C}), \mathcal{X}(\mathbf{C}), \models)$ since

$$(E_1, F_1) \leq (E_2, F_2) \Longleftrightarrow E_1 \subset E_2 = (F_2)' \Longleftrightarrow E_1 \models F_2.$$

Define a correspondence φ from $\mathbf{C}$ to $rGal(\mathbf{C})$ by by

$$L_\varphi a = \overline{a}\downarrow := \{M \in \mathcal{A}(\mathbf{C}) \mid M \subset \overline{a}\}$$

for $a \in A$ and

$$R_{\varphi *} F = F$$

for closed $F \subset X$. Then it is a Chu correspondence and is in fact an isomorphism.

Remark The Theorem 73 follows from Corrollary 112 of [9] and Proposition 36.

4.7. Completeness and cocompleteness of ChuCors

The category *ChuCors* is complete, since it is equivalent to the complete category $\mathcal{S}lat$. Being selfdual, *ChuCors* is also cocomplete.

We give explicitly products and equalizers.

For formal contexts $\mathbf{C}_i = (A_i, X_i, \models_i)$ $(i = 1, 2)$ define a formal context

$$\mathbf{C}_1 \times \mathbf{C}_2 := (A_1 \times A_2, X_1 \coprod X_2, \models),$$

by $(a_1, a_2) \models x$ if and only if $a_i \models_i x$ when $x \in X_i$ $(i = 1, 2)$.

Define a Chu correspondence, for $i = 1, 2$,

$$\pi_i \colon \mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{C}_i$$

which corresponds to the Chu map (L_i, R_i) with the standard projection

$$L_i \colon A_1 \times A_2 \to A_i$$

and the standard inclusion

$$R_i \colon X_i \to X_1 \coprod X_2.$$

It is straightforward to show the following.

Proposition 74 *The diagram*

$$\mathbf{C}_1 \xleftarrow{\pi_1} \mathbf{C}_1 \times \mathbf{C}_2 \xrightarrow{\pi_2} \mathbf{C}_2$$

is a product of $\mathbf{C}_i$ $(i = 1, 2)$*. The dual of this product diagram for the duals* $\mathbf{C}_i^*$ $(i = 1, 2)$*, namely,*

$$\mathbf{C}_1 \to (\mathbf{C}_1^* \times \mathbf{C}_2^*)^* \leftarrow \mathbf{C}_2,$$

is a coproduct of $\mathbf{C}_i$ $(i = 1, 2)$*.*

Now we describe the equalizer. Let $\kappa_i \colon \mathbf{C}_1 \to \mathbf{C}_2$ $(i = 1, 2)$ be Chu correspondences. Put $L_i = L_{\kappa_i}$ and $R_i = R_{\kappa_i}$ $(i = 1, 2)$ for brevity. Define a set A and a correspondence $L \colon A \to \mathcal{A}(\mathbf{C}_1) \subset \mathbf{pow}(A_1)$ by the equalizer diagram

$$A \xrightarrow{L} \mathcal{A}(\mathbf{C}_1) \overset{(L_1)_*}{\underset{(L_2)_*}{\rightrightarrows}} \mathcal{A}(\mathbf{C}_2)$$

in the category of sets and maps. Define a correspondence $R \colon X_1 \to X$ by

$Rx_1 = F(\overline{x_1})$ where

$$X \xleftarrow{F} \mathcal{X}(\mathbf{C}_1) \underset{(R_2)_*}{\overset{(R_1)_*}{\leftleftarrows}} \mathcal{X}(\mathbf{C}_2)$$

is a coequalizer in the category of sets and maps. For $a \in A$ and $N_1 \in \mathcal{X}(\mathbf{C}_1)$, Define $a \models F(N_1)$ by $La \models N_1$. It is easily seen that this is well-defined and we put

$$\mathbf{C} = (A,\, X,\, \models).$$

Then $\kappa = (L,\, R)$ is a Chu correspondence from $\mathbf{C}$ to $\mathbf{C}_1$. It is straightforward to show the following.

Proposition 75 *The following diagram is an equalizer in ChuCors.*

$$\mathbf{C} \xrightarrow{\kappa} \mathbf{C}_1 \underset{\kappa_2}{\overset{\kappa_1}{\rightrightarrows}} \mathbf{C}_2$$

Note that the equalizer diagram

$$\mathbf{C}_1^* \underset{\kappa_1^*}{\overset{\kappa_2^*}{\leftleftarrows}} \mathbf{C}_2^* \xleftarrow[\lambda]{} \mathbf{K}$$

goes to a coequalizer diagram

$$\mathbf{C}_1 \underset{\kappa_2}{\overset{\kappa_1}{\rightrightarrows}} \mathbf{C}_2 \xrightarrow{\lambda^*} \mathbf{K}^*\ .$$

4.8. Canonical forms of formal contexts

In the category of *ChuCors*, a formal context $\mathbf{C}$ is canonically isomorphic to

$$cf(\mathbf{C}) := (A,\, \mathcal{A},\, \in),$$

where $\mathcal{A}$ is the family of closed subsets of A.

In fact, define a correspondence

$$\lambda(\mathbf{C})\colon \mathbf{C} \to cf(\mathbf{C})$$

by

$$L_\lambda a = \overline{a}, \quad R_\lambda B = B'$$

for $a \in A$ and $B \in \mathcal{A}$. Then λ is a Chu correspondence since

$$\begin{aligned} L_\lambda a \models B &\Longleftrightarrow \overline{a} \models B \\ &\Longleftrightarrow \overline{a} \subset B' = R_\lambda B \\ &\Longleftrightarrow a \in R_\lambda B \Longleftrightarrow a \models R_\lambda B. \end{aligned}$$

Note that cf is an endo functor by defining

$$cf(\varphi)\colon cf(\mathbf{C}_1) \to cf(\mathbf{C}_2)$$

for a Chu correspondence $\varphi\colon \mathbf{C}_1 \to \mathbf{C}_2$ by

$$L_{cf(\varphi)} = L_\varphi, \quad R_{cf(\varphi)} = L_\varphi{}^*.$$

Proposition 76 *The Chu correspondences $\lambda(-)$ is a natural isomorphism from the indentity functor of $ChuCors$ to the functor cf.*

Proof. We show that $Gal(\lambda)$ is an isomorphism. Let $(M, M') \in Gal(\mathbf{C})$, where $M \subset A$ is closed. Since the closure operator on A corresponding to $\mathbf{C}$ is the same as that corresponding to $(A, \mathcal{A}, \in)$, we have

$$\lambda_*(M, M') = (\overline{L_\lambda M}, (L_\lambda M)').$$

But, using Lemma 7,

$$\overline{L_\lambda M} = \overline{\bigcup_{a\in M} \overline{a}} = \overline{\bigcup_{a\in M} a} = \overline{M} = M,$$

whence λ_* is an isomorphism.

Since the extent part of λ is the closure of the identity, the naturality follows immediately. □

5. Structures in ChuCors

5.1. Internal hom functor

Let $\mathbf{C}_i$ $(i = 1, 2)$ be formal contexts. Define a new formal context $\mathbf{C}_1 \multimap \mathbf{C}_2$ by

$$\mathbf{C}_1 \multimap \mathbf{C}_2 := (ChuCors(\mathbf{C}_1, \mathbf{C}_2), A_1 \times X_2, \models)$$

where

$$\varphi \models (a_1, x_2) \overset{\text{def}}{\Longleftrightarrow} L_\varphi a_1 \models x_2.$$

Note that since φ is a Chu correspondence, the condition of the right hand side is equivalent to $a_1 \models R_\varphi x_2$.

Lemma 77 *Let* $\varphi \in ChuCors(\mathbf{C}_1, \mathbf{C}_2)$. *Then*

$$\varphi' = [Z_\varphi].$$

Proof. Let $a_1 \in A_1$ and $x_2 \in X_2$. Then

$$\begin{aligned}\varphi \models (a_1, x_2) &\Leftrightarrow L_\varphi a_1 \models x_2\\ &\Leftrightarrow x_2 \in (L_\varphi a_1)' = Z_\varphi a_1\\ &\Leftrightarrow (a_1, x_2) \in [Z_\varphi].\end{aligned}$$

□

From this, we obtain the following isomorphism.

Theorem 78 *The formal concepts of the formal context* $\mathbf{C}_1 \multimap\!\!\bullet\, \mathbf{C}_2$ *are written uniquely as*

$$(\varphi\downarrow, [Z_\varphi]),$$

with a Chu correspondence φ. *In particular, We have isomorphisms*

$$\mu(\mathbf{C}_1, \mathbf{C}_2)\colon Gal(\mathbf{C}_1 \multimap\!\!\bullet\, \mathbf{C}_2) \simeq Gal(\mathbf{C}_1) \multimap Gal(\mathbf{C}_2).$$

Proof. Since $Bond(\mathbf{C}_1, \mathbf{C}_2) \subset \mathbf{pow}(A_1 \times X_2)$ is intersection closed family by Proposition 33, and the polar $\varphi \mapsto [Z_\varphi]$ is bijective by Proposition 36, we have

$$Gal(\mathbf{C}_1 \multimap\!\!\bullet\, \mathbf{C}_2) = \{(\varphi\downarrow, [Z_\varphi]) \mid \varphi \in ChuCors(\mathbf{C}_1, \mathbf{C}_2)\}.$$

Hence we have an isomorphism

$$Gal(\mathbf{C}_1 \multimap\!\!\bullet\, \mathbf{C}_2) \simeq ChuCors(\mathbf{C}_1, \mathbf{C}_2).$$

By Theorem 65,

$$\begin{aligned}ChuCors(\mathbf{C}_1, \mathbf{C}_2) &\simeq \mathcal{S}lat(Gal(\mathbf{C}_1), Gal(\mathbf{C}_2))\\ &\simeq Gal(\mathbf{C}_1) \multimap Gal(\mathbf{C}_2).\end{aligned}$$

The composition of these isomorphism define the isomorphism

$$\mu(\mathbf{C}_1, \mathbf{C}_2)\colon Gal(\mathbf{C}_1 \multimap\!\!\bullet\, \mathbf{C}_2) \to Gal(\mathbf{C}_1) \multimap Gal(\mathbf{C}_2).$$

By definition,

$$\mu(\mathbf{C}_1, \mathbf{C}_2)((\varphi\downarrow, [Z_\varphi])) = Gal(\varphi).$$

□

To define the bi-functor

$$(-) \multimap (-)\colon ChuCors^{\mathrm{op}} \times ChuCors \to ChuCors,$$

let $\psi_2\colon \mathbf{C}_2 \to \mathbf{D}_2$ and $\psi_1\colon \mathbf{D}_1 \to \mathbf{C}_1$ be Chu correspondences. Let

$$\mathbf{C}_i = (A_i, X_i, \models), \quad \mathbf{D}_i = (B_i, Y_i, \models),$$

$(i = 1, 2)$. Then

$$ChuCors(\psi_1, \psi_2)\colon (\mathbf{C}_1 \multimap \mathbf{C}_2) \to (\mathbf{D}_1 \multimap \mathbf{D}_2)$$

is given by (L, R), where

$$L\varphi = (\psi_2 \circ \varphi \circ \psi_1)\downarrow$$

and

$$R(b_1, y_2) = L_{\psi_1} b_1 \times R_{\psi_2} y_2.$$

Then (L, R) is in fact a Chu correspondence. To see this, let

$$\varphi \in ChuCors(\mathbf{C}_1, \mathbf{C}_2) \quad \text{and} \quad b_1 \in B_1 \quad \text{and} \quad y_2 \in Y_2.$$

Then

$$\begin{aligned} L\varphi \models (b_1, y_2) &\iff (\psi_2 \circ \varphi \circ \psi_1)\downarrow \models (b_1, y_2) \\ &\iff \psi_2 \circ \varphi \circ \psi_1 \models (b_1, y_2) \\ &\iff b_1 \models R_{\psi_2\circ\varphi\circ\psi_1} y_2 \\ &\iff b_1 \models R_{\psi_1} R_\varphi R_{\psi_2} y_2. \end{aligned}$$

On the other hand

$$\begin{aligned} \varphi \models R(b_1, y_2) &\iff \varphi \models L_{\psi_1} b_1 \times R_{\psi_2} y_2 \\ &\iff L_\varphi L_{\psi_1} b_1 \models R_{\psi_2} y_2 \\ &\iff b_1 \models R_{\psi_1} R_\varphi R_{\psi_2} y_2, \end{aligned}$$

whence (L, R) is a Chu correspondence.

Proposition 79 *Let* $\mathbf{C}_i, \mathbf{D}_i$ $(i = 1, 2)$ *be formal contexts and let* $\psi_2 \colon \mathbf{C}_2 \to \mathbf{D}_2$ *and* $\psi_1 \colon \mathbf{D}_1 \to \mathbf{C}_1$ *be Chu correspondences. Then the following diagram commutes.*

$$\begin{array}{ccc}
Gal(\mathbf{C}_1 \relbar\!\bullet\, \mathbf{C}_2) & \xrightarrow{\mu} & Gal(\mathbf{C}_1) \multimap Gal(\mathbf{C}_2) \\
\Big\downarrow {\scriptstyle Gal(\psi_2 \relbar\!\bullet \psi_1)} & & \Big\downarrow {\scriptstyle Gal(\psi_1) \multimap Gal(\psi_2)} \\
Gal(\mathbf{D}_1 \relbar\!\bullet\, \mathbf{D}_2) & \xrightarrow{\mu} & Gal(\mathbf{D}_1) \multimap Gal(\mathbf{D}_2)
\end{array}$$

Proof. Let $\xi = (\varphi\downarrow, [Z_\varphi]) \in Gal(\mathbf{C}_1 \relbar\!\bullet\, \mathbf{C}_2)$. Then

$$\begin{aligned}
(Gal(\psi_2) \multimap Gal(\psi_1))\mu(\xi) &= (Gal(\psi_2) \multimap Gal(\psi_1))Gal(\varphi) \\
&= Gal(\psi_2) \circ Gal(\varphi) \circ Gal(\psi_1) \\
&= Gal(\psi_2 \circ \varphi \circ \psi_1).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mu(Gal(\psi_2 \relbar\!\bullet\, \psi_1)\xi) &= \mu(\psi_2 \circ \varphi \circ \psi_1 \downarrow, [Z_{\psi_2 \circ \varphi \circ \psi_1}]) \\
&= Gal(\psi_2 \circ \varphi \circ \psi_1).
\end{aligned}$$

□

Proposition 80 *There are natural strong isomorphisms*:

$$\mathbf{C}_1 \relbar\!\bullet\, \mathbf{C}_2^* \cong \mathbf{C}_2 \relbar\!\bullet\, \mathbf{C}_1^*.$$

Proof. Note that

$$\mathbf{C}_1 \relbar\!\bullet\, \mathbf{C}_2^* = (ChuCors(\mathbf{C}_1, \mathbf{C}_2^*), A_1 \times A_2)),$$

and

$$\mathbf{C}_2 \relbar\!\bullet\, \mathbf{C}_1^* = (ChuCors(\mathbf{C}_2, \mathbf{C}_1^*), A_2 \times A_1)).$$

Let L be the bijection

$$L \colon ChuCors(\mathbf{C}_1, \mathbf{C}_2^*) \to ChuCors(\mathbf{C}_2, \mathbf{C}_1^*)$$

of Proposition 40 regarded as a correspondence and R the transpose bijection $A_2 \times A_1 \to A_1 \times A_2$ regarded as a correspondence. Then the closure of the Chu correspondence (L, R) is an isomorphism. □

The isomorphisms of Theorem 38 between Chu correspondences and the bonds give natural isomorphism.

Proposition 81 *The bifunctors* $ChuCors(-,-)$ *and* $Bond(-,-)$ *are naturally isomorphic by the map* $\varphi \mapsto Z_\varphi$.

Proof. Let $\mathbf{C}_i$ and $\mathbf{D}_i$ ($i = 1, 2$) be formal contexts and $\psi_1 \colon \mathbf{D}_1 \to \mathbf{C}_1$ and $\psi_2 \colon \mathbf{C}_2 \to \mathbf{D}_2$ be Chu correspondences. Let $\varphi \colon \mathbf{C}_1 \to \mathbf{C}_2$ be a Chu correspondence. Then

$$ChuCors(\psi_1, \psi_2)\varphi = \psi_2 \circ \varphi \circ \psi_1,$$

whereas

$$Bond(\psi_1, \psi_2)Z_\varphi = \psi_2 \circ Z_\varphi \circ \psi_1.$$

It suffices to show that

$$Z_{\psi_2 \circ \varphi \circ \psi_1} = \psi_2 \circ Z_\varphi \circ \psi_1.$$

Let $(b_1, y_2) \in B_1 \times Y_2$. Then

$$\begin{aligned}
(b_1, y_2) \in [Z_{\psi_2 \circ \varphi \circ \psi_1}] &\Leftrightarrow y_2 \in (L_{\psi_2 \circ \varphi \circ \psi_1} b_1)' \\
&\Leftrightarrow y_2 \in ((L_{\psi_2})_* L_{\varphi \circ \psi_1} b_1)' \\
&\Leftrightarrow y_2 \in R^*_{\psi_2}(L_{\varphi \circ \psi_1} b_1)' \\
&\Leftrightarrow R_{\psi_2} y_2 \subset (L_{\varphi \circ \psi_1} b_1)' = (Z_\varphi)_\circ (L_{\psi_1} b_1) \\
&\Leftrightarrow L_{\psi_1} b_1 \times R_{\psi_2} y_2 \subset [Z_\varphi]. \\
&\Leftrightarrow (b_1, y_2) \in [\psi_2 \circ Z_\varphi \circ \psi_1].
\end{aligned}$$

□

Similar natural isomorphisms exists for bifunctors for the $ChuCors$-valued bifunctors. Define a formal context by

$$\mathbf{C}_1 \multimap_{\diamond} \mathbf{C}_2 = (Bond(\mathbf{C}_1, \mathbf{C}_2), A_1 \times X_2, \ni).$$

By the proof of Lemma 77, the isomorphism L of Theorem 38 and the identity map of $A_1 \times X_2$ induce the following strong isomorphism

Proposition 82 *We have natural isomorphisms*

$$\mathbf{C}_1 \multimap_{\bullet} \mathbf{C}_2 \cong \mathbf{C}_1 \multimap_{\diamond} \mathbf{C}_2.$$

5.2. Self-duality

The category *ChuCors* is self-dual with the dualizing functor defined by $\mathbf{C} \mapsto \mathbf{C}^*$ and for a Chu correspondence φ from $\mathbf{C}_1$ to $\mathbf{C}_2$, φ^* from $\mathbf{C}_2^*$ to $\mathbf{C}_1^*$ as is defined in Section 2.

Theorem 83 *There are natural strong isomorphisms*:

$$\delta_{\mathbf{C}} \colon Gal(\mathbf{C}^*) \cong Gal(\mathbf{C})^*.$$

Proof. Define

$$\delta_{\mathbf{C}}(E,\, F) = (F,\, E),$$

which is obviously a bijective order reversing correspondence.

To show the naturality, let $\varphi \colon \mathbf{C}_1 \to \mathbf{C}_2$ be a Chu correspondence. Then

$$\varphi^* \colon \mathbf{C}_2 \to \mathbf{C}_1$$

is defined by $L_{\varphi^*} = R_\varphi$ and $R_{\varphi^*} = L_\varphi$. We show the commutativity of the following diagram

$$\begin{array}{ccc} Gal(\mathbf{C}_1^*) & \xrightarrow{\delta_{\mathbf{C}_1}} & Gal(\mathbf{C}_1)^* \\ {\scriptstyle Gal(\varphi^*)} \uparrow & & \uparrow {\scriptstyle Gal(\varphi)^*} \\ Gal(\mathbf{C}_2^*) & \xrightarrow[\delta_{\mathbf{C}_2}]{} & Gal(\mathbf{C}_2)^* \end{array}$$

For $(N_2,\, N_2') \in Gal(\mathbf{C}_2^*)$ with $N_2 \subset X_2$ closed, we have by (7),

$$\begin{aligned} \delta_{\mathbf{C}_1} Gal(\varphi^*)(N_2,\, N_2') &= \delta_{\mathbf{C}_1}(\overline{L_{\varphi^*} N_2},\, (L_{\varphi^*} N_2)') \\ &= \delta_{\mathbf{C}_1}(\overline{R_\varphi N_2},\, (R_\varphi N_2)') \\ &= ((R_\varphi N_2)',\, \overline{R_\varphi N_2}). \end{aligned}$$

On the other hand, by (8),

$$Gal(\varphi)^* \delta_{\mathbf{C}_2}(N_2,\, N_2') = Gal(\varphi)^*(N_2',\, N_2) = ((R_\varphi N_2)',\, \overline{R_\varphi N_2}).$$

□

In the category of Chu maps, the formal context $\perp$ of Example 3 is a dualizing object [13] in the sense that

$$\mathbf{C} \multimap \perp \simeq \mathbf{C}^*.$$

In fact $\perp$ is a dualizing object also in the category of Chu correspondences.

Theorem 84 *We have natural isomorphisms*

$$d(\mathbf{C})\colon \mathbf{C} \multimapdot \perp \simeq \mathbf{C}^*.$$

Proof. By Theorem 38, Proposition 41, the self duality of *ChuCors* and Proposition 76, there are isomorphisms:

$$\begin{aligned} \mathbf{C} \multimapdot \perp &\cong (Bond(\mathbf{C}, \perp), A \times \{*\}, \in) \\ &\cong (\mathcal{A}(\mathbf{C}), A, \in) \\ &\cong cf(\mathbf{C})^*. \\ &\simeq \mathbf{C}^*, \end{aligned}$$

where the first three ones are strong isomorphisms. Here $cf(\mathbf{C})$ is the canonical form of $\mathbf{C}$ (§4.8).

Since the first components of the formal contexts appearing above are all bijective to A and, under these bijections, the intent parts of the isomorphisms corresponds to the closures of the identity of A, the above isomorphisms are natural. □

It is straightforward to prove the following.

Proposition 85 *The following diagram commutes.*

$$\begin{array}{ccc} Gal(\mathbf{C} \multimapdot \perp) & \xrightarrow{Gal(d(C))} & Gal(\mathbf{C}^*) \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \delta_{\mathbf{C}}} \\ Gal(\mathbf{C}) \multimap \mathbf{2} & \xrightarrow{\simeq} & Gal(\mathbf{C})^* \end{array},$$

where ν is the composition of the isomorphisms

$$Gal(\mathbf{C} \multimapdot \perp) \simeq (Gal(\mathbf{C}) \multimap Gal(\perp)) \simeq (Gal(\mathbf{C}) \multimap \mathbf{2}).$$

5.3. Tensor

In the $*$-autonomous category,

$$A \otimes B \simeq (A \multimap B^*)^*.$$

Since we have already the internal hom-functor and the self duality, we define

$$\mathbf{C}_1 \boxtimes \mathbf{C}_2 := (\mathbf{C}_1 \multimap\!\!\bullet\, \mathbf{C}_2^*)^* = (A_1 \times A_2,\, ChuCors(\mathbf{C}_1,\, \mathbf{C}_2^*)).$$

Proposition 86 $\mathbf{C}_1 \boxtimes \mathbf{C}_2 = (\mathbf{C}_1 \multimap\!\!\bullet\, \mathbf{C}_2^*)^*$.

The structure of the bifunctor $(-)\boxtimes(-)$ is described explicitly using the $\mathcal{S}lat$-valued bifunctor $(-) \bowtie (-)$ defined in Section 4.8 as follows. Define

$$\mathbf{C}_1 \otimes \mathbf{C}_2 := (A_1 \times A_2,\, \mathbf{C}_1 \bowtie \mathbf{C}_2,\, \in).$$

and make $(-)\otimes(-)$ a bifunctor as follows. Let $\psi_i \colon \mathbf{C}_i \to \mathbf{D}_i$ $(i = 1, 2)$ be Chu correspondences and define

$$\psi_1 \otimes \psi_2 \colon \mathbf{C}_1 \otimes \mathbf{C}_2 \to \mathbf{D}_1 \otimes \mathbf{D}_2$$

to be (L, R), where

$$L \colon A_1 \times A_2 \to \mathbf{pow}(B_1 \times B_2)$$

is defined by

$$L(a_1,\, a_2) := \overline{L_{\psi_1} a_1 \times L_{\psi_2} a_2}$$

for $a_i \in A_i$ $(i = 1, 2)$, and

$$R \colon \mathbf{D}_1 \bowtie \mathbf{D}_2 \to \mathbf{pow}(\mathbf{C}_1 \bowtie \mathbf{C}_2)$$

by

$$RZ = \{(a_1,\, a_2) \mid L_{\psi_1} a_1 \times L_{\psi_2} a_2 \subset Z\} \downarrow .$$

Then we have

Theorem 87 *There are natural strong isomorphisms*:

$$\mathbf{C}_1 \boxtimes \mathbf{C}_2 \cong \mathbf{C}_1 \bowtie \mathbf{C}_2.$$

By Theorem 78 and Proposition 83, we obtain

Theorem 88 *There are natural isomorphisms*

$$\kappa(\mathbf{C}_1,\, \mathbf{C}_2) \colon Gal(\mathbf{C}_1 \boxtimes \mathbf{C}_2) \simeq Gal(\mathbf{C}_1) \otimes Gal(\mathbf{C}_2).$$

If $\mathbf{C}_i = (A_i,\, X_i,\, \models_i)$ $(i = 1, 2)$ *and* $Gal(\mathbf{C}_i)$ *is identified with the family* $\mathcal{A}_i$

of closed subsets of A_i $(i = 1, 2)$, then $\kappa(\mathbf{C}_1, \mathbf{C}_2)$ maps $E_1 \times E_2$ to $E_1 \otimes E_2$, where $E_i \in \mathcal{A}_i$ $(i = 1, 2)$.

Proof. By Theorem 87, $\mathbf{C}_1 \boxtimes \mathbf{C}_2$ is naturally and strongly isomorphic to

$$\mathbf{C}_1 \bigotimes \mathbf{C}_2 = (A_1 \times A_2,\, Bond(\mathbf{C}_1,\, \mathbf{C}_2^*),\, \in).$$

By Example 9, the bond $E_1 \times E_2 \subset A_1 \times A_2$ correspondes to the Chu correspondence $\varphi\colon \mathbf{C}_1 \to \mathbf{C}_2^*$ defined by

$$L_\varphi(a_1) = \begin{cases} E_2', & \text{if } a_1 \in E_1 \\ X_2, & \text{otherwise.} \end{cases}$$

This correspondes to the join preserving map f_{E_1,E_2} (cf. Section 1.2)

$$Gal(\mathbf{C}_1) \to Gal(\mathbf{C}_2)^*,$$

where E_i are regarded as in $Gal(\mathbf{C}_i)$ $(i = 1, 2)$, which correspondes to $E_1 \otimes E_2$ of $Gal(\mathbf{C}_1) \otimes Gal(\mathbf{C}_2)$. □

The tensor is associative.

Theorem 89 *For formal contexts $\mathbf{C}_i$ $(i = 1, 2, 3)$, there are strong isomorphisms*

$$a(\mathbf{C}_1,\, \mathbf{C}_2,\, \mathbf{C}_3)\colon (\mathbf{C}_1 \boxtimes \mathbf{C}_2) \boxtimes \mathbf{C}_3 \to \mathbf{C}_1 \boxtimes (\mathbf{C}_2 \boxtimes \mathbf{C}_3),$$

whose extent part is given by the bijection

$$((a_1,\, a_2),\, a_3) \mapsto \overline{(a_1,\, (a_2,\, a_3))}.$$

Proof. The assertion follows from the observation that both $(\mathbf{C}_1 \boxtimes \mathbf{C}_2) \boxtimes \mathbf{C}_3$ and $\mathbf{C}_1 \boxtimes (\mathbf{C}_2 \boxtimes \mathbf{C}_3)$ are strongly isomorphic to

$$(A_1 \times A_2 \times A_3,\, W)$$

where W is the set of subsets $Z \subset A_1 \times A_2 \times A_2$, which satisfies the condition that for each $a_i \in A_i$ $(i = 1, 2, 3)$ the subsets $Z(a_1,\, a_2,\, -) \subset A_3$, $Z(a_1,\, -,\, a_3) \subset A_2$, and $Z(-,\, a_2,\, a_3) \subset A_1$ are closed. Here

$$Z(a_1,\, a_2,\, -) := \{a_3 \in A_3 \mid (a_1,\, a_2,\, a_3) \in Z\}$$

etc.. □

Proposition 90 *The following diagram commutes.*

$$\begin{array}{ccc} Gal((\mathbf{C}_1 \boxtimes \mathbf{C}_2) \boxtimes \mathbf{C}_3) & \xrightarrow{a_1} & Gal(\mathbf{C}_1 \boxtimes (\mathbf{C}_2 \boxtimes \mathbf{C}_3)) \\ \Big\downarrow u_1 & & \Big\downarrow u_2 \\ (Gal(\mathbf{C}_1) \otimes Gal(\mathbf{C}_2)) \otimes Gal(\mathbf{C}_3) & \xrightarrow[a_2]{} & Gal(\mathbf{C}_1) \otimes (Gal(\mathbf{C}_2) \otimes Gal(\mathbf{C}_3)) \end{array}$$

where

$$a_1 = Gal(a(\mathbf{C}_1,\, \mathbf{C}_2,\, \mathbf{C}_3))$$

and

$$a_2 = a(Gal(\mathbf{C}_1),\, Gal(\mathbf{C}_2),\, Gal(\mathbf{C}_3)),$$

and u_1 *is the composition of the isomorphisms*

$$\begin{aligned} Gal((\mathbf{C}_1 \boxtimes \mathbf{C}_2) \boxtimes \mathbf{C}_3) &\overset{\kappa}{\simeq} Gal((\mathbf{C}_1 \boxtimes \mathbf{C}_2)) \otimes Gal(\mathbf{C}_3) \\ &\overset{\kappa\otimes 1}{\simeq} (Gal(\mathbf{C}_1) \otimes Gal(\mathbf{C}_2)) \otimes Gal(\mathbf{C}_3), \end{aligned}$$

and the isomorphism u_2 *is defined similarly.*

Proof. Let $a_i \in A_i$ $(i = 1, 2, 3)$. Since $Gal((\mathbf{C}_1 \boxtimes \mathbf{C}_2) \boxtimes \mathbf{C}_3)$ is join generated by $(\overline{((a_1,\, a_2),\, a_3)},\, ((a_1,\, a_2),\, a_3)')$, it suffices to show that these go to the same elements of

$$Gal(\mathbf{C}_1) \otimes (Gal(\mathbf{C}_2) \otimes Gal(\mathbf{C}_3)).$$

It is easily seen that $(\overline{((a_1,\, a_2),\, a_3)},\, ((a_1,\, a_2),\, a_3)')$ goes to

$$(\overline{a_1},\, a_1') \otimes ((\overline{a_2},\, a_2') \otimes (\overline{a_3},\, a_3'))$$

in either way. □

Obviously the tensor $(-) \boxtimes (-)$ is symmetric.

Theorem 91 *There is a strong isomorphism*

$$s(\mathbf{C}_1,\, \mathbf{C}_2) \colon \mathbf{C}_1 \boxtimes \mathbf{C}_2 \cong \mathbf{C}_2 \boxtimes \mathbf{C}_1,$$

whose extent part maps $(a_1,\, a_2)$ *to* $\overline{(a_2,\, a_1)}$.

Proposition 92 *The following diagram commutes.*

$$\begin{array}{ccc} Gal(\mathbf{C}_1 \boxtimes \mathbf{C}_2) & \xrightarrow{Gal(s)} & Gal(\mathbf{C}_2 \boxtimes \mathbf{C}_1) \\ \downarrow \simeq & & \downarrow \simeq \\ Gal(\mathbf{C}_1) \otimes Gal(\mathbf{C}_2) & \xrightarrow[s]{} & Gal(\mathbf{C}_2) \otimes Gal(\mathbf{C}_1) \end{array} .$$

Finally, we note that the functor $\mathbf{C} \multimap (-)$ is right adjoint to the functor $(-) \boxtimes \mathbf{C}$.

Theorem 93

$$ChuCors(\mathbf{C}_1 \boxtimes \mathbf{C},\, \mathbf{C}_2) \simeq ChuCors(\mathbf{C}_1,\, \mathbf{C} \multimap \mathbf{C}_2).$$

This follows from Proposition 41 and the following

Theorem 94 *There are natural strong isomorphisms*

$$(\mathbf{C}_1 \boxtimes \mathbf{C}) \multimap \mathbf{C}_2 \cong \mathbf{C}_1 \multimap (\mathbf{C} \multimap \mathbf{C}_2).$$

Proof. We have the following strong natural isomorphisms.

$$\begin{aligned} (\mathbf{C}_1 \boxtimes \mathbf{C}) \multimap \mathbf{C}_2 &\cong ((\mathbf{C}_1 \boxtimes \mathbf{C}) \boxtimes \mathbf{C}_2^*)^* \\ &\cong (\mathbf{C}_1 \boxtimes (\mathbf{C} \boxtimes \mathbf{C}_2^*))^* \\ &\cong (\mathbf{C}_1 \boxtimes (\mathbf{C} \multimap \mathbf{C}_2)*)^* \\ &\cong \mathbf{C}_1 \multimap (\mathbf{C} \multimap \mathbf{C}_2). \end{aligned}$$

□

5.4. Structure of *-autonomous category

We have introduced in $ChuCors$ the ingredients of a $*$-autonomous category [7], [5], namely, the unit object "$\top$", the tensor bifunctor "$\boxtimes$", the internal hom functor $\multimap$, the selfduality $\mathbf{C}^{\mathrm{op}} \simeq \mathbf{C}$ and the dualizing object $\perp$.

The Galois functor preserves these operators in the following sense. By Example 3 and Propositions 88, 78, 83,

Proposition 95 *There are following natural isomorphisms*:

$$\kappa(\mathbf{C}_1, \mathbf{C}_2) \colon Gal(\mathbf{C}_1 \boxtimes \mathbf{C}_2) \simeq Gal(\mathbf{C}_1) \otimes Gal(\mathbf{C}_2),$$

$\mu(\mathbf{C}_1, \mathbf{C}_2) : Gal(\mathbf{C}_1 \multimap\!\!\bullet\ \mathbf{C}_2) \simeq Gal(\mathbf{C}_1) \multimap Gal(\mathbf{C}_2),$
$Gal(\top) \simeq \mathbf{2},$
$\delta\mathbf{C}\colon Gal(\mathbf{C}^*) \simeq Gal(\mathbf{C})^*.$

We define the structural natural isomorphisms as follows.
By Theorem 87,

$$\top \boxtimes \mathbf{C} \cong (\{\,*\,\} \times A,\ \{B \subset \{\,*\,\} \times A \mid B \text{ is closed in } A\},\ \in)$$

whence there is a strong isomorphism

$$\top \boxtimes \mathbf{C} \cong cf(\mathbf{C}).$$

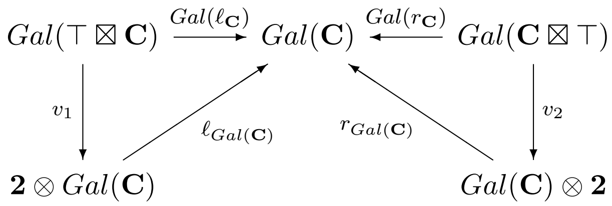
Since $cf(\mathbf{C}) \simeq \mathbf{C}$ by Proposition 76, we have natural isomorphisms

$$\ell_{\mathbf{C}}\colon \top \boxtimes \mathbf{C} \simeq \mathbf{C}.$$

Similarly, we define

$$r_{\mathbf{C}}\colon \mathbf{C} \boxtimes \top \simeq \mathbf{C}.$$

Proposition 96 *The following diagram commutes*



where v_1 *is the isomorphism defined by the composition of*

$$Gal(\top \boxtimes \mathbf{C}) \overset{\kappa}{\simeq} Gal(\top) \otimes Gal(\mathbf{C}) \simeq \mathbf{2} \otimes Gal(\mathbf{C}),$$

and the isomorphism v_2 *is the similar composition.*

Proof. Since $Gal(\top \times \mathbf{C})$ is join-generated by $(\overline{*} \times \overline{a},\ (*,\ a))$, it suffices to show the commutativity of the right triangle that they go to the same element by either way.

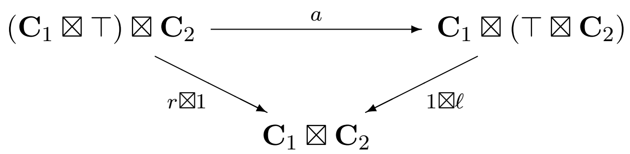
By Theorem 88, $(\overline{*} \times \overline{a},\ (*,\ a))$ is mapped to $1 \otimes (\overline{a},\ a') \in \mathbf{2} \times Gal(\mathbf{C})$ and then to $(\overline{a},\ a')$ in the left round way.

On the other hand, it is mapped by $Gal(\ell_{\mathbf{C}})$ to $(\overline{a},\ a')$.

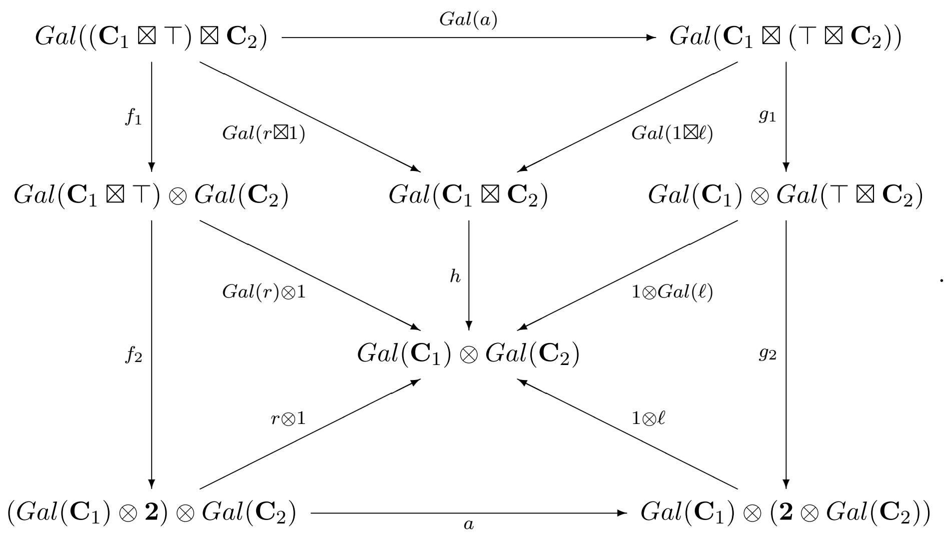
In the same way, it can be shown that the right triangle commutes. □

Proposition 97 *The coherence conditions hold.*

Proof. The coherence conditions of the $*$-autonomous category $\mathcal{S}lat$ and Propositions 96, 90, 85, 92 implies the corresponding coherence conditions of $ChuCors$. For example, the coherence condition of commutativity of



follows from the following commutative diagram.



In fact

$$
\begin{aligned}
& h \circ Gal(1 \boxtimes \ell) \circ Gal(a) \\
& \qquad = (1 \otimes Gal(\ell)) \circ g_1 \circ Gal(a) \quad \text{by naturality of } \kappa \\
& \qquad = (1 \otimes \ell) \circ g_2 \circ g_1 \circ Gal(a) \quad \text{by definition of } \ell \\
& \qquad = (1 \otimes Gal(\ell)) \circ a \circ f_2 \circ f_1 \quad \text{by definition of } a \\
& \qquad = (r \otimes 1) \circ f_2 \circ f_1 \\
& \qquad = (Gal(r) \otimes 1) \circ f_1 \\
& \qquad = h \circ (Gal(r \boxtimes 1)).
\end{aligned}
$$

□

In short, we have proved

Theorem 98 *The category ChuCors has a structure of *-autonomous category with the unit $\top$, the tensor $(-)\boxtimes(-)$, the internalhom $(-) \multimap (-)$, the the dualizing object $\bot$, and the natural isomorphisms r, ℓ, a, s, δ, which makes the Galois functor a $*$-autonomous functor.*

Concluding Remarks

Chu maps and Chu correspondences The concept of formal contexts appear in quite a few contexts under various terminologies such as Chu space [14], classification [3], etc. besides the formal concept analysis [9]. When the framework of category theory is used, the Chu maps are usually adopted as arrows. However there can generally be few Chu maps between two formal contexts, which seems to make the category theory rather uninteresting in some field of research.

In contrast, there are abundant Chu correspondences between two formal contexts and give justification of the usage of the category theoretical machinery in studying formal concepts.

The concpet of bonds The $*$-autonomous category structure of *ChuCors* is described combinatorially and is defined more straightforward way than that of the category *Slat* owing especially to the beautiful concept of *bonds* introduced in [9]. In fact, we could as well have developed the category of bonds, which is isomorphic to our *ChuCors*.

Heyting valued contexts We can define the concept of Chu correspondence when $\{0, 1\}$ is replaced by a Heyting algebra as in [13]. Although the Galois functor does not seem full and faithful in general, most of our results seem to hold. In particular the $*$-autonomous category structure of *ChuCors* is defined similarly.

Chu construction It seems that the procedure of constructing $*$-autonomous category from a complete closed category given by Chu [1], when applied to the category *Rel* of correspondences between sets, gives us most of the structures of *ChuCors* described in this paper. However, since *Rel* is not complete, the verification of Chu [1] does not apply to our category and operators directly.

Application Since our theory allow us to introduce "natural" correspondence in the topics where objects have dualism description mentioned in the introduction, we expect that our theory has theoretical applications.

As an example, we explain a usage of Chu correspondence in model theory in Section A of the appendix.

A. Application to model theory

For basic terminologies, see [12].

Let $T = (L, \Phi)$ be a first order theory, where L is a first order language and Φ is its axiom which is an arbitrary set of L-sentences. Let F_x^L be the set of L-formulas with one free variable x. A subset of F_x^L is called a 1-type. Let M be a model of T. Then we have the following formal context

$$\mathbf{C}(M) := (M,\, F_x^L,\, I),$$

with

$$mI\varphi \overset{\text{def}}{\iff} M \models \varphi|_{x=m},$$

for $m \in M$ and $\varphi \in F_x^L$. The polar of $m \in M$, denoted by $F_x(m)$ is the 1-type defined by m. The polar of $\varphi \in F_x^L$ is the set $M(\varphi)$ of $m \in M$ satisfying φ, namely, $M \models \varphi|_{x=m}$. A subset of M is closed if there is a 1-type $N \subset F_x^L$ with $N' = M$. A 1-type $Q \subset F_x^L$ is closed if there is a subset $P \subset M$ whose polar is Q, namely, Q is the set of formulas $\varphi \in F_x^L$ which are satisfied by all the elements of N.

The closure operator on F_x^L is the semantic implication in the model M. Namely, for $Q \subset F_x^L$ and $\varphi \in F_x^L$, $\varphi \in \overline{Q}$ if, for every $m \in M$, the condition $mI\psi$ for all $\psi \in Q$ implies $mI\varphi$.

Suppose M_1 and M_2 are two models of T. Define a correspondence

$$L\colon M_1 \to M_2$$

by

$$m_2 \in Lm_1 \overset{\text{def}}{\iff} m_2 I F_x(m_1).$$

Proposition 99 (L, id) *is a Chu correspondence from* $\mathbf{C}(M_1)$ *to* $\mathbf{C}(M_2)$ *in the weak sense, whence defines a Chu correspondence*

$$\gamma(M_1, M_2)\colon \mathbf{C}(M_1) \to \mathbf{C}(M_2).$$

Proof. For $\varphi \in F_x^L$,

$$Lm_1 I\varphi \Leftrightarrow m_1 I\varphi.$$

In fact, by definition, $m_1 I\varphi$ implies $Lm_1 I\varphi$. Suppose $Lm_1 I\varphi$. If $m_1 I\varphi$ does not hold, then $m_1 I\neg\varphi$. Hence $Lm_1 I\neg\varphi$, which contradicts the assumption. Hence we have $m_1 I\varphi$. □

Two models M_i ($i = 1, 2$) are called elementarily equivalent if $M_1 \models \phi$ iff $M_2 \models \phi$ for all L-sentences ϕ.

Proposition 100 *Models M_1 and M_2 are elementarily equivalent if and only if*

$$\overline{L_{\gamma(M_1,M_2)*} M_1} = M_2. \tag{10}$$

Proof. Note that for $\varphi \in F_x^L$ and a model M, the condition $M \models \forall x\varphi$ means $MI\varphi$, i.e., $\varphi \in M'$.

Suppose M_1 and M_2 are elementarily equivalent. Then $M_1' = M_2^* \subset F_x^L$, where $(-)^*$ denotes the polar operator of $\mathbf{C}(M_2)$, whence (10) holds.

Conversely suppose (10) holds. Let ϕ be an L-sentence. Then there is a $\varphi \in F_x^L$, possibly without x, such that ϕ is logically equivalent to either $\forall x\varphi$ or $\exists x\varphi$.

Suppose $\phi \equiv \forall x\varphi$. Then

$$\begin{aligned} M_2 \models \phi &\Leftrightarrow M_2 I\varphi \\ &\Leftrightarrow \overline{L_* M_1} I\varphi \\ &\Leftrightarrow M_1 I\varphi \\ &\Leftrightarrow M_1 \models \phi. \end{aligned}$$

If $\phi \equiv \exists x\varphi$, then $\neg\phi \equiv \forall x\neg\varphi$, whence

$$M_2 \models \neg\phi \Leftrightarrow M_1 \models \neg\phi$$

and we conclude

$$M_2 \models \phi \Leftrightarrow M_1 \models \phi.$$

□

These samples seem to suggest that the Chu correspondences between the formal contexts $\mathbf{C}(M)$ might be useful tool as well as significant objects to study in the theory of model theories.

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