Hokkaido Mathematical Journal Vol. 37 (2008) p. 561–610

Flux of simple ends of maximal surfaces in $\mathbf{R}^{2,1}$

Taishi IMAIZUMI and Shin KATO

(Received June 11, 2007; Revised November 5, 2007)

Abstract. Simple ends of maximal surfaces in $\mathbf{R}^{2,1}$ naturally correspond to catenoidal or planar ends of minimal surfaces in $\mathbf{R}^3$. We study some properties of flux of simple ends, which are different from those of catenoidal or planar ends. We also give a classification of maximal surfaces of genus zero with 3 simple ends.

Key words: maximal surface, flux formula, simple end.

1. Introduction

Let M be a Riemann surface, and $\mathbf{R}^{2,1} = (\mathbf{R}^3, g_{\mathbf{R}^{2,1}})$ be the Lorentzian 3-space. In this paper, we call a map $X\colon M \to \mathbf{R}^{2,1}$ a *maximal map*, and $X(M)$ a *maximal surface*, if, roughly speaking, $X|_{M\setminus\Sigma}$ is a conformal space-like maximal immersion, where Σ is the set of points where the extended normal vectors are null. (We give a precise definition in § 2.)

Let $\overline{M}$ be a compact Riemann surface, and consider the case when the domain M of a maximal map X is $\overline{M}$ punctured by n points, say $M = \overline{M} \setminus \{q_1, \ldots, q_n\}$. As in the case of minimal surfaces in the Euclidean 3-space $\mathbf{R}^3 = (\mathbf{R}^3, g_{\mathbf{R}^3})$, we can define the *flux vector* by $\varphi(\gamma) := \int_\gamma \vec{n} ds$ for any loop γ in M which intersects the singular set Σ at a discrete set, where $\vec{n}$ is a conormal such that $(\gamma, \vec{n})$ is positively oriented, and ds is the line element of $X(M)$.

In particular, the flux vector at the end q_j is defined by $\varphi_j := \varphi(\gamma)$, where γ is a loop surrounding q_j from the left. In general, $\varphi(\gamma)$ can be described by the residue of a certain $\mathbf{C}^3$-valued holomorphic 1-form on M, and is independent of the choice of γ in each homology class. Hence φ_j is well-defined. By the residue theorem, we get the flux formula $\sum_{j=1}^{n} \varphi_j = 0$. *Simple ends*, which we study in this paper, are ends of order at most 2. They correspond to catenoidal or planar ends in $\mathbf{R}^3$, which always have order 2, although the simple ends considered here can have order either 2 or 1. In [5], the first author classified the asymptotic behaviors of simple

---

2000 Mathematics Subject Classification : Primary 53C42, Secondary 58E12.

ends. We call X (or $X(M)$) an *n-noid* if all of the ends of X are simple ends of order 2.

In the case of $\mathbf{R}^3$, the flux vector of any catenoidal end must be parallel to the limit normal at the end. Also in the case of $\mathbf{R}^{2,1}$, the flux vector of a simple end has the same property if its limit normal is not null, i.e. the end is arranged to be on $M \setminus \Sigma$. However, a simple end arranged to be on Σ does not have such a property in general. We show, in § 2, that the flux vector of such a simple end is a vector in the null plane including the limit normal. Hence we must prepare a certain new formulation to study n-noids with prescribed flux (see § 3–4).

Other than the property above, we can find various phenomena in the case of $\mathbf{R}^{2,1}$, which do not occur in the case of $\mathbf{R}^3$. For instance, there exists a nontrivial $\mathbf{Z}^2$-action on the space of maximal surfaces whose flux vectors span a timelike plane. This action switches timelike flux and spacelike flux, and preserves branch points. We also give a characterization of the symmetry of $X(M)$ with respect to a point, or that of the property that the image doubles up, by using the metric $X^*(g_{\mathbf{R}^3})$ which degenerates on Σ (see § 5).

In § 6, we give some obstructions for the existence of n-noids with prescribed flux. Some of them apply only to the case of $\mathbf{R}^{2,1}$. In § 7, we classify all of the 3-noids of genus zero. The space of 3-noids of genus zero is more complicated than that of 3-end catenoids in $\mathbf{R}^3$. In § 8, we show a general existence result for 4-noids with prescribed flux, that corresponds to [9, Theorem 3.6].

The authors thank Professor Kotaro Yamada for helpful advice. They also thank the referees for useful comments.

2. Flux of simple ends

Let $\mathbf{R}^{2,1}$ be the Lorentzian 3-space. Set $\mathbf{H}^2_\pm := \{{}^t(x_1, x_2, x_3) \in \mathbf{R}^{2,1} \mid {x_1}^2 + {x_2}^2 - {x_3}^2 = -1, \pm x_3 > 0\}$, $\mathbf{H}^2 := \mathbf{H}^2_+ \cup \mathbf{H}^2_-$, and $\mathbf{S}^{1,1} := \{{}^t(x_1, x_2, x_3) \in \mathbf{R}^{2,1} \mid {x_1}^2 + {x_2}^2 - {x_3}^2 = 1\}$, where ^{t} means the transposition of rows and columns. Let $\hat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$, $\mathbf{S}^1 := \{p \in \mathbf{C} \mid |p| = 1\}$, $\Delta := \{(p, p') \in \hat{\mathbf{C}}^2 \mid p = p'\}$, and $T^2 := \mathbf{S}^1 \times \mathbf{S}^1$. Define the map $v\colon (\Delta \setminus T^2) \cup (T^2 \setminus \Delta) \to \mathbf{C}^3$ by

$$v(p, p') := \frac{1}{\overline{p'}p - 1} \begin{pmatrix} -(p + \overline{p'}) \\ \sqrt{-1}(p - \overline{p'}) \\ \overline{p'}p + 1 \end{pmatrix} \ (p, p' \neq \infty), \ v(\infty, \infty) := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and $\check{v}\colon \mathbf{S}^1 \to \mathbf{C}^3$ by

$$\check{v}(p) := \begin{pmatrix} \operatorname{Re} p \\ \operatorname{Im} p \\ -1 \end{pmatrix}.$$

Let $\sigma\colon \mathbf{H}^2 \to \hat{\mathbf{C}} \setminus \mathbf{S}^1$ be the stereographic projection from the north pole $\mathbf{e}_3 := {}^t(0, 0, 1)$. Then the inverse of this map is given by the following:

$$\sigma^{-1}(p) = v(p, p) = \frac{1}{|p|^2 - 1} \begin{pmatrix} -2 \operatorname{Re} p \\ -2 \operatorname{Im} p \\ |p|^2 + 1 \end{pmatrix}.$$

On the other hand, it holds that

$$\sqrt{-1} v(p, p') = \frac{1}{\operatorname{Im} \xi} \begin{pmatrix} \operatorname{Re}(p\xi) \\ \operatorname{Im}(p\xi) \\ -\operatorname{Re} \xi \end{pmatrix}$$

for any $(p, p') \in T^2 \setminus \Delta$, where ξ is a unit number such that $p' = p\xi^2$. For any $p \in \mathbf{S}^1$, denote the null plane including $\check{v}(p)$ by $NP(p)$. Set

$$NP_+(p) := \{v \in NP(p) \mid \det(\mathbf{e}_3, \check{v}(p), v) > 0\},$$
$$NP_-(p) := \{v \in NP(p) \mid \det(\mathbf{e}_3, \check{v}(p), v) < 0\}.$$

Then we have $\sqrt{-1} v(p, p') \in NP_+(p) \cap NP_-(p') \cap \mathbf{S}^{1,1}$.

Define the map $v^*\colon (\Delta \setminus (T^2 \cup \{(\infty, \infty)\})) \cup (T^2 \setminus \Delta) \to \mathbf{C}^3$ by

$$v^*(p, p') := \frac{1}{\overline{p'}p - 1} \begin{pmatrix} -(p^2 + 1) \\ \sqrt{-1}(p^2 - 1) \\ 2p \end{pmatrix},$$

$\check{v}^*\colon \mathbf{S}^1 \to \mathbf{C}^3$ by

$$\check{v}^*(p) := \frac{1}{2} \begin{pmatrix} -(p^2 - 1) \\ \sqrt{-1}(p^2 + 1) \\ 0 \end{pmatrix},$$

and $\widetilde{v}\colon \mathbf{S}^1 \to \mathbf{C}^3$ by

$$\widetilde{v}(p) := \begin{pmatrix} -(p^2 + 1) \\ \sqrt{-1}(p^2 - 1) \\ 2p \end{pmatrix}.$$

Then it holds that $v^*(p,\,p') = (\overline{p'}p-1)^{-1}\widetilde{v}(p)$ and $\check{v}(p) = -(2p)^{-1}\widetilde{v}(p)$. We use these maps to describe dependence between the equations which we will derive in § 2 and § 4.

Let M be a Riemann surface. Then, by the *Weierstrass representation formula* (cf. [12]), any conformal spacelike maximal immersion $X\colon M \to \mathbf{R}^{2,1}$ is given by

$$ {}^tX(z) = \operatorname{Re}\int_{z_0}^{z}(1+g^2,\,\sqrt{-1}(1-g^2),\,-2g)\eta, \tag{2.1}$$

where g is a meromorphic function on M, and η is a holomorphic 1-form on M such that the 1-forms $g\eta$ and $g^2\eta$ are also holomorphic on M. We call $(g,\,\eta)$ the *Weierstrass data* of X. g is the stereographic image of the Gauss map $G\colon M \to \mathbf{R}^{2,1}$ of X, i.e. $g := \sigma \circ G$. The induced metrics on M are given by $X^*(g_{\mathbf{R}^{2,1}}) = (1-|g|^2)^2|\eta|^2$ and $X^*(g_{\mathbf{R}^3}) = (1-|g|^2)^2|\eta|^2 + 8(\operatorname{Re} g\eta)^2$.

Conversely, for any Riemann surface M, any meromorphic function g on M, and any holomorphic 1-form η on M, the map X given by (2.1) is a (branched) conformal spacelike maximal immersion on $\widetilde{M\setminus\Sigma}$, where $\Sigma := \{z \in M \mid |g(z)| = 1\}$. When the map X is well-defined on M, we call $X\colon M \to \mathbf{R}^{2,1}$ a *maximal map*, and $X(M)$ a *maximal surface*. We call $q \in M$ a *branch point* of X if $X^*(g_{\mathbf{R}^3}) = 0$ at q. We say X is *nonbranched* if X has no branch point. The rank of the metric $X^*(g_{\mathbf{R}^3})$ is 1 on $\Sigma \setminus \{\text{branch points}\}$. For any $z \in \Sigma$, we regard $G(z) = \sigma^{-1}(g(z)) = \infty \times \check{v}(g(z))$ as a kind of null vector, since $G(z)$ cannot be extended to Σ, but $\mathbf{R}G(z)$ and $g(z)$ can be naturally extended.

The map X given by (2.1) is well-defined on M if and only if

$$\operatorname{Re}\int_{\gamma}(1+g^2,\,\sqrt{-1}(1-g^2),\,-2g)\eta = 0 \tag{2.2}$$

holds for any loop γ in M. Set

$$R_i := \operatorname{Res}_{\gamma} g^i\eta = \frac{1}{2\pi\sqrt{-1}}\int_{\gamma} g^i\eta \quad (i = 0,\,1,\,2).$$

Then the condition (2.2) is rewritten as

$$R_0 + R_2 \in \mathbf{R}, \quad R_0 - R_2 \in \sqrt{-1}\mathbf{R}, \quad R_1 \in \mathbf{R},$$

and it is equivalent to

$$R_0 = \overline{R_2}, \quad R_1 = \overline{R_1}. \tag{2.3}$$

Note here that, for any curve $z = z(s)$ in $M \setminus \Sigma$, its conormal is given by

$${}^t\vec{n} = -\operatorname{Im}(1 + g^2, \sqrt{-1}(1 - g^2), -2g)\eta(z'(s)).$$

Hence, if a loop γ in M intersects Σ at a discrete set, then the integral $\varphi = \varphi(\gamma) = \int_\gamma \vec{n}ds$ makes sense, and satisfies

$${}^t\varphi = -\operatorname{Im}\int_\gamma (1 + g^2, \sqrt{-1}(1 - g^2), -2g)\eta.$$

Therefore, we can define $\varphi = \varphi(\gamma)$ even when the intersection of γ and Σ is not discrete. From the equality above and (2.2), it follows that

$$\begin{aligned} {}^t\varphi &= -2\pi \operatorname{Res}_q(1 + g^2, \sqrt{-1}(1 - g^2), -2g)\eta \\ &= -2\pi(R_0 + R_2, \sqrt{-1}(R_0 - R_2), -2R_1). \end{aligned} \tag{2.4}$$

(cf. [4] for the case when φ is timelike.) We call φ the *flux vector* of the loop γ. It depends only on the homology class of γ.

Let $\hat{M}$ be a Riemann surface, q an inner point of $\hat{M}$, and set $M := \hat{M} \setminus \{q\}$. Consider a maximal map $X\colon M \to \mathbf{R}^{2,1}$ which cannot be extended to q. We call the image of a neighborhood of q the *end* q, and the end q is called a *simple end* if its Weierstrass data (g, η) can be meromorphically extended to $\hat{M}$, and η, $g\eta$ and $g^2\eta$ have poles of order at most 2 at q.

Denote the residues R_0, R_1, R_2 for a loop γ surrounding q once from the left also by R_0, R_1, R_2 respectively. If a maximal map X given by (2.1) has a simple end at q and $g(q) = p \neq \infty$, then $(g - p)^2\eta$ does not have a pole at q. Hence we have

$$0 = \operatorname{Res}_q(g - p)^2\eta = R_2 - 2pR_1 + p^2R_0. \tag{2.5}$$

For the loop γ as above, we call $\varphi = \varphi(\gamma)$ the flux vector of the end q, as we have already mentioned in the introduction. We denote it by $\varphi = \varphi(q)$.

Let ξ be a unit complex number such that $\overline{p}R_2 \in \xi\mathbf{R}$. (When $R_2 = 0$, we may choose an arbitrary ξ.) Set $p' := p\xi^2$. Then we have the following:

Theorem 2.1 *Let X be a maximal map from the universal cover of $M = \hat{M} \setminus \{q\}$ to $\mathbf{R}^{2,1}$ given by* (2.1)*. If X has a simple end at q, then at least one of $p = p'$ or $|p| = 1$ holds. Furthermore, X is then well-defined on a neighborhood of q in M itself if and only if it holds that* (2.5) *and*

$$\begin{cases} w := pR_0 - R_1 \in \begin{cases} \mathbf{R} & \textit{if} \;\; p = p' \\ \sqrt{-1}\mathbf{R} & \textit{if} \;\; |p| = 1 \end{cases} \\ \\ w^* := \dfrac{1}{2}(\overline{p'}p + 1)R_0 - \overline{p'}R_1 = 0. \end{cases} \tag{2.6}$$

(*resp.*

$$\begin{cases} \breve{w} := -R_1 \in \mathbf{R} \\ w^* := R_0 - \overline{p}R_1 = 0.) \end{cases} \tag{2.7}$$

when the flux vector $\varphi(q)$ of the end q is not null (*resp. is null*)*. If the limit normal $G(q)$ is not null, then $\varphi(q)$ is proportional to $G(q)$. On the other hand, if $G(q)$ is null, then $\varphi(q)$ is not necessarily proportional to $G(q)$, but arranged in the null plane including $G(q)$.*

Proof. If the end q is well-defined and simple, then, by (2.3) and (2.5), we have

$$\begin{aligned} 0 &= 2|p|^2(R_1 - \overline{R_1}) = \overline{p} \cdot 2pR_1 - p \cdot 2\overline{p}\overline{R_1} \\ &= \overline{p}(R_2 + p^2R_0) - p(\overline{R_2} + \overline{p}^2\overline{R_0}) = \overline{p}(R_2 + p^2R_0) - p(R_0 + \overline{p}^2R_2) \\ &= (1 - |p|^2)(\overline{p}R_2 - pR_0). \end{aligned}$$

Hence, if $|p| \neq 1$, i.e. the limit normal vector at q is not null, then we have $\overline{p}R_2 = pR_0 = p\overline{R_2}$, namely $\overline{p}R_2 \in \mathbf{R}$. Therefore, if $R_2 \neq 0$, then $\xi^2 = 1$ and $p = p'$.

Since $\overline{\xi}\overline{p}R_2 \in \mathbf{R}$, we have $\overline{\xi}\overline{p}R_2 = \xi p\overline{R_2} = \xi pR_0$, and hence

$$pR_0 \in \overline{\xi}\mathbf{R}, \tag{2.8}$$

and

$$2pR_1 = p^2R_0 + R_2 = p \cdot \overline{\xi}^2\overline{p}R_2 + R_2 = (\overline{p'}p + 1)R_2. \tag{2.9}$$

If $p = p'$, i.e. $\xi^2 = 1$, then, by (2.3) and (2.8), $w = pR_0 - R_1 \in \mathbf{R}$. If

$|p| = 1$, then, by (2.3) and (2.5),

$$\begin{aligned} w &= \frac{1}{2p}(2p^2R_0 - 2pR_1) = \frac{\overline{p}}{2}(2p^2R_0 - R_2 - p^2R_0) \\ &= \frac{\overline{p}}{2}(p^2R_0 - R_2) = \frac{\overline{p}}{2}(p^2R_0 - \overline{R_0}) \\ &= \frac{1}{2}(pR_0 - \overline{p}\overline{R_0}) \in \sqrt{-1}\mathbf{R}. \end{aligned}$$

In particular, if $|p| = 1$ and $p = p'$, then $w = 0$ and $\check{w} = -R_1 = w - pR_0 = -pR_0 \in \mathbf{R}$.

On the other hand, by (2.3) and (2.9), we also have

$$\begin{aligned} w^* &= \frac{1}{2}(\overline{p'}p + 1)R_0 - \overline{p'}R_1 = \frac{1}{2}(\overline{\xi}^2|p|^2 + 1)R_0 - \overline{\xi}^2\overline{p}\overline{R_1} \\ &= \frac{1}{2}(\overline{\xi}^2|p|^2 + 1)\overline{R_2} - \overline{\xi}^2 \cdot \frac{1}{2}(\xi^2|p|^2 + 1)\overline{R_2} \\ &= \frac{1}{2}(\overline{\xi}^2 - 1)(|p|^2 - 1)\overline{R_2} = 0. \end{aligned}$$

Now, we get (2.6) in any case, and (2.7) in the case when $|p| = 1$ and $p = p'$.

Conversely, in the case when $\overline{p'}p \neq 1$, i.e. $|p| \neq 1$ or $p \neq p'$, if we assume (2.5) and (2.6), then, by (2.6), we have

$$-\frac{1}{2}(\overline{p'}p + 1)(R_1 + w) + \overline{p'}pR_1 = 0,$$

and hence

$$R_1 = w\frac{\overline{p'}p + 1}{\overline{p'}p - 1},$$

and

$$R_0 = \frac{1}{p}(R_1 + w) = w\frac{2\overline{p'}}{\overline{p'}p - 1}.$$

Even when $p = 0$, this is true since $R_0 = 0$. Moreover, by (2.5), we have

$$R_2 = -p^2R_0 + 2pR_1 = w\frac{2p}{\overline{p'}p - 1}.$$

Now, we have

$$R_1 = w\frac{\overline{p'}p+1}{\overline{p'}p-1} \in \mathbf{R},$$

and

$$\overline{R_0} = w\overline{\frac{2\overline{p'}}{\overline{p'}p-1}} = w\frac{2p}{\overline{p'}p-1} = R_2,$$

and we get (2.3).

On the other hand, in the case when $\overline{p'}p = 1$, i.e. $|p| = 1$ and $p = p'$, the two equalities in (2.6) are equivalent to each other, and (2.6) is not equivalent to (2.3) even if we assume (2.5). However, if we assume (2.5) and (2.7), then we have $R_2 = -p^2R_0 + 2pR_1 = pR_1 = p\overline{R_1} = \overline{R_0}$, and we get (2.3).

Now, let us calculate the flux vector $\varphi(q)$ of the end q.

When $\overline{p'}p \neq 1$, we have, by (2.4),

$$\varphi = \frac{4\pi w}{\overline{p'}p-1}\begin{pmatrix} -(p+\overline{p'}) \\ \sqrt{-1}(p-\overline{p'}) \\ \overline{p'}p+1 \end{pmatrix} = 4\pi w v(p,\, p'). \tag{2.10}$$

Hence, if $|p| \neq 1$, φ is parallel to the limit normal $G(q) = v(p,\, p)$. On the other hand, if $|p| = 1$ and $p \neq p'$, then $\varphi \in NP_+(p) \cap NP_-(p')$.

When $\overline{p'}p = 1$, i.e. $|p| = 1$ and $p = p'$, since $R_0 = -\check{w}\overline{p}$, $R_1 = -\check{w}$, $R_2 = -\check{w}p$, we have $\varphi = 4\pi\check{w}\check{v}(p)$. □

Even when X satisfies $g(q) = p = \infty$, if we choose a congruent transformation F of $\mathbf{R}^{2,1}$ such that $F(\mathbf{e}_3) \neq \mathbf{e}_3$, then the maximal map $F \circ X$ satisfies $\tilde{g}(q) \neq \infty$, where we set $\tilde{g} := \sigma \circ F \circ G$. Hence the assumption $g(q) = p \neq \infty$ is not restrictive.

In keeping with the notation of Kobayashi [12], we call the end q satisfying $|p| \neq 1$ (resp. $|p| = 1$ and $p \neq p'$) a simple end of *the first* (resp. *second*) *kind*, and the value $w = w(q)$ the *weight* of the end q. $w(q)$ is invariant under the action of conformal coordinate transformations of $\hat{M}$ and the orientation preserving congruent transformations of $\mathbf{R}^{2,1}$ (cf. § 3).

On the other hand, we call the end q satisfying $|p| = 1$ and $p = p'$, a simple end of *the third kind*. When the end q is of the third kind, $w(q)$ automatically vanishes. $\check{w} = \check{w}(q)$ is not invariant under the action of the orientation preserving congruent transformations. When $\check{w}$ also vanishes, we may regard such an end as being of the second kind with zero flux vector. Any result in this paper holds independently of this choice.

For later use, we note here that w, $\check{w}$ and w^* satisfy the following equalities, given so that the assumption (2.5) holds.

The case when $\overline{p'}p \neq 1$:

$$\begin{cases} \frac{1}{2}(R_0+R_2) &= -\frac{-(p+\overline{p'})}{\overline{p'}p-1}w + \frac{-(p^2+1)}{\overline{p'}p-1}w^* \\ \frac{\sqrt{-1}}{2}(R_0-R_2) &= -\frac{\sqrt{-1}(p-\overline{p'})}{\overline{p'}p-1}w + \frac{\sqrt{-1}(p^2-1)}{\overline{p'}p-1}w^* \\ -R_1 &= -\frac{\overline{p'}p+1}{\overline{p'}p-1}w + \frac{2p}{\overline{p'}p-1}w^*. \end{cases} \tag{2.11}$$

The case when $\overline{p'}p = 1$:

$$\begin{cases} \frac{1}{2}(R_0+R_2) &= -\operatorname{Re} p\cdot\check{w} + \frac{-(p^2-1)}{2}w^* \\ \frac{\sqrt{-1}}{2}(R_0-R_2) &= -\operatorname{Im} p\cdot\check{w} + \frac{\sqrt{-1}(p^2+1)}{2}w^* \\ -R_1 &= 1\cdot\check{w} + 0\cdot w^*. \end{cases} \tag{2.12}$$

For simple ends of order 1, we have the following:

Proposition 2.2 *Let* $X\colon M = \hat{M}\setminus\{q\} \to \mathbf{R}^{2,1}$ *be a maximal map given by* (2.1)*, and* q *a well-defined simple end of* X*. If* q *is of order* 1*, then* q *is of the third kind. In particular, the flux vector of* q *is a nonzero vector.*

Proof. Since $(g-p)\eta$ and $(g-p)g\eta$ do not have a pole at q, we have

$$\begin{cases} 0 = \operatorname{Res}_q(g-p)\eta = R_1 - pR_0, \\ 0 = \operatorname{Res}_q(g-p)g\eta = R_2 - pR_1. \end{cases} \tag{2.13}$$

Since the end q is well-defined, by (2.3) and (2.13), we have

$$R_0 = \overline{R_2} = \overline{pR_1} = \overline{p}R_1 = \overline{p}pR_0 = |p|^2R_0.$$

Now, if $R_0 = 0$, then $R_1 = R_2 = 0$. This contradicts the assumption that q is a simple end of order 1. Hence $R_0 \neq 0$ and we have $|p| = 1$. Since $w = pR_0 - R_1 = 0$ and $\check{w} = -R_1 \neq 0$, the end q is of the third kind, and its flux vector is a nonzero vector. □

3. Standard examples, flux formula, and general existence of maximal herissons

The maximal map given by the data $(g, \eta) = (-z, z^{-2}dz)$ is called a *catenoid of the first kind*. This surface has two simple ends of the first kind. The maximal map given by the data

$$(g, \eta) = \Big(\frac{z-1}{z+1}, \frac{-\sqrt{-1}(z+1)^2}{2z^2}dz\Big)$$

is called a *helicoid of the second kind*. This surface has two simple ends of the second kind (cf. [12]). The maximal map given by the data

$$(g, \eta) = \Big(1, \frac{-2}{z^2-1}dz\Big)$$

has two simple ends of the third kind and of order 1. The image of this map is not a surface but a null line. As we shall see in § 4, there exists no maximal map of genus zero with only two simple ends of the third kind and of order 2.

Let $X\colon M = \hat{M} \setminus \{q\} \to \mathbf{R}^{2,1}$ be a maximal map which has a simple end at q, as in § 2. When $w \in \mathbf{R} \setminus \{0\}$ (resp. $\sqrt{-1}\mathbf{R} \setminus \{0\}$), the simple end q is asymptotically an end of a catenoid of the first kind (resp. helicoid of the second kind) (cf. [5]). By $\varphi(q) = 4\pi w v(p, p')$, we see that w (resp. $w/\sqrt{-1}$) is the ratio of the size of the flux vector φ of the end q to that of the standard catenoid of the first kind if $|p| \neq 1$ (resp. helicoid of the second kind if $|p| = 1$ and $p \neq p'$).

Let $\overline{M}$ be a compact Riemann surface, $q_1, \ldots, q_n$ distinct points on $\overline{M}$, and set $M := \overline{M} \setminus \{q_1, \ldots, q_n\}$. Then, for any maximal map $X\colon M \to \mathbf{R}^{2,1}$, by (2.4) and the residue theorem, we have the balancing formula, also called the *flux formula*, $\sum_{j=1}^{n} \varphi_j = \mathbf{0}$, where $\varphi_j := \varphi(q_j)$. In particular, in the case when all the ends are simple, we have

$$\sum_{\overline{p'_j}p_j \neq 1} w_j v(p_j, p'_j) + \sum_{\overline{p'_j}p_j = 1} \check{w}_j \check{v}(p_j) = \mathbf{0}, \tag{3.1}$$

where $w_j := w(q_j)$, $p_j := g(q_j)$ and p'_j is p' for q_j.

In this paper, we consider the following:

Problem 3.1 Let p_j, p'_j be complex numbers or ∞ such that $|p_j| = |p'_j|$, and that $p_j = p'_j$ if $|p_j| \neq 1$ $(j = 1, \ldots, n)$. For any j such that $|p_j| \neq 1$ (resp. $|p_j| = 1$ and $p_j \neq p'_j$), let a_j be a real (resp. purely imaginary) number. For any j such that $|p_j| = 1$ and $p_j = p'_j$, let $\check{a}_j$ be a real number. Suppose that these numbers satisfy

$$\sum_{\overline{p'_j}p_j \neq 1} a_j v(p_j, p'_j) + \sum_{\overline{p'_j}p_j = 1} \check{a}_j \check{v}(p_j) = \mathbf{0}. \tag{3.2}$$

Does there exists a maximal map $X\colon M := \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^{2,1}$ satisfying the following condition?

$$\begin{cases} g(q_j) = p_j,\ w(q_j) = a_j,\ \varphi(q_j) = a_j v(p_j, p'_j) & \text{if } \overline{p'_j}p_j \neq 1 \\ g(q_j) = p_j,\ \check{w}(q_j) = \check{a}_j,\ \varphi(q_j) = \check{a}_j \check{v}(p_j) & \text{if } \overline{p'_j}p_j = 1 \end{cases} \tag{3.3}$$
$$(j = 1, \dots, n).$$

In this section, as an example of an application of the formula we derived in § 2, we show an existence result for maximal herissons with simple ends realizing prescribed flux, where a *maximal herisson* means a maximal map whose Gauss map is of degree 1.

Theorem 3.2 *For any flux data satisfying* (3.2), *there exists a unique maximal herisson* $X\colon \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^{2,1}$ *satisfying* (3.3) *if all the* p_j*'s are different from each other, and if* a_j *or* $\check{a}_j$ *is nonzero for any* j*.*

Proof. Let $X\colon M := \overline{M} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^{2,1}$ be a maximal herisson with simple ends. Since the degree of g is 1, we may set $\overline{M} := \hat{\mathbf{C}}$. By a suitable congruent transformation of $\mathbf{R}^{2,1}$, we may assume that the limit normal of any end is not ${}^t(0, 0, 1) = \sigma^{-1}(\infty)$ without loss of generality. Moreover, by a suitable coordinate transformation, we may set $g(z) = z$.

By this normalization, the positions of the ends satisfy $q_j = g(q_j) = p_j$ ($j = 1, \dots, n$). Hence, if X satisfies the condition (3.3), then η must be of the following form:

$$\eta := \Big[\sum_{j=1}^{n} \Big\{ \frac{b_j}{(z - p_j)^2} + \frac{c_j}{z - p_j} \Big\} + f(z) \Big] dz,$$

where b_j and c_j are complex numbers at least one of which does not vanish for each $j = 1, \dots, n$, and $f(z)$ is a holomorphic function on $\mathbf{C}$. Hence

$$g\eta = z\eta = \Big[\sum_{j=1}^{n} \Big\{ \frac{b_j p_j}{(z - p_j)^2} + \frac{b_j + c_j p_j}{z - p_j} + c_j \Big\} + z f(z) \Big] dz,$$
$$g^2\eta = z^2\eta$$

$$= \Big[\sum_{j=1}^{n}\Big\{\frac{b_j {p_j}^2}{(z-p_j)^2} + \frac{2b_j p_j + c_j {p_j}^2}{z-p_j} + b_j + c_j p_j + c_j z\Big\} + z^2 f(z)\Big] dz.$$

Denote the residues R_0, R_1 for γ_j surrounding $q_j = p_j$ once from the left by R_{0j}, R_{1j}, respectively. Then it holds that

$$R_{0j} = c_j, \quad R_{1j} = b_j + c_j q_j.$$

Now, for each end $q_j = p_j$ of the first or second kind, i.e. $\overline{p'_j} p_j \neq 1$, by (2.6), we have

$$\begin{cases} w(q_j) \equiv p_j R_{0j} - R_{1j} = p_j c_j - (b_j + c_j p_j) = -b_j = a_j \\ w^*(q_j) \equiv \dfrac{1}{2}(\overline{p'_j} p_j + 1) R_{0j} - \overline{p'_j} R_{1j} \\ \qquad\quad = \dfrac{1}{2}(\overline{p'_j} p_j + 1) c_j - \overline{p'_j}(b_j + c_j p_j) = -\overline{p'_j} b_j - \dfrac{1}{2}(\overline{p'_j} p_j - 1) c_j \\ \qquad\quad = 0. \end{cases}$$

Solving this as an algebraic equation with respect to b_j, c_j, we get the following solution.

$$b_j = -a_j, \quad c_j = \frac{2a_j \overline{p'_j}}{\overline{p'_j} p_j - 1}.$$

On the other hand, for each end $q_j = p_j$ of the third kind, i.e. $\overline{p'_j} p_j = 1$, by (2.7), we have

$$\begin{cases} \check{w}(q_j) \equiv -R_{1j} = -(b_j + c_j p_j) = \check{a}_j \\ w^*(q_j) \equiv R_{0j} - \overline{p_j} R_{1j} = c_j - \overline{p_j}(b_j + c_j p_j) = -\overline{p_j} b_j = 0. \end{cases}$$

Solving this equation, we get the following solution.

$$b_j = 0, \quad c_j = -\check{a}_j \overline{p_j}.$$

Now, by the assumption (3.2), we have

$$\begin{aligned} \sum_{j=1}^{n} c_j &= \sum{}' a_j \frac{2\overline{p_j}}{\overline{p'_j} p_j - 1} + \sum{}'' \check{a}_j(-\overline{p_j}) \\ &= \sum{}' a_j(-v_1(p_j, p'_j) + \sqrt{-1} v_2(p_j, p'_j)) \\ &\quad + \sum{}'' \check{a}_j(-\check{v}_1(p_j) + \sqrt{-1} \check{v}_2(p_j)) = 0, \end{aligned}$$

$$\begin{aligned}\sum_{j=1}^{n}(b_j + c_j p_j) &= \sum\nolimits' a_j \frac{\overline{p'_j} p_j + 1}{\overline{p'_j} p_j - 1} + \sum\nolimits'' \check{a}_j(-1) \\ &= \sum\nolimits' a_j v_3(p_j, p'_j) + \sum\nolimits'' \check{a}_j \check{v}_3(p_j) = 0, \\ \sum_{j=1}^{n}(2b_j p_j + c_j {p_j}^2) &= \sum\nolimits' a_j \frac{2p_j}{\overline{p'_j} p_j - 1} + \sum\nolimits'' \check{a}_j(-p_j) \\ &= \sum\nolimits' a_j(-v_1(p_j, p'_j) - \sqrt{-1} v_2(p_j, p'_j)) \\ &\quad + \sum\nolimits'' \check{a}_j(-\check{v}_1(p_j) - \sqrt{-1}\check{v}_2(p_j)) = 0,\end{aligned}$$

where $\sum' := \sum_{\overline{p'_j} p_j \neq 1}$, $\sum'' := \sum_{\overline{p'_j} p_j = 1}$, and v_i (resp. $\check{v}_i$) is the i-th component of v (resp. $\check{v}$).

Hence X does not have an end at ∞ if and only if $f(z) \equiv 0$. Now, we get the following Weierstrass data for X satisfying (3.3):

$$\begin{aligned}g(z) &= z, \\ \eta &= \Big[\sum\nolimits' \Big\{ \frac{-a_j}{(z - p_j)^2} + \frac{2a_j \overline{p'_j}/(\overline{p'_j} p_j - 1)}{z - p_j} \Big\} + \sum\nolimits'' \frac{-\check{a}_j \overline{p_j}}{z - p_j} \Big] dz.\end{aligned}$$

□

Theorem 3.2 is a natural analogue of [17, Theorem 2.5]. Regrettably, generic maximal herissons with more than 2 simple ends have branch points. To find nonbranched examples, we must consider the Gauss map of higher order.

4. Algebraic equation for n-noids of genus zero in $\mathbf{R}^{2,1}$, and relative weights

Let $\overline{M}$ be a compact Riemann surface, $q_1, \ldots, q_n$ distinct points on $\overline{M}$, and set $M := \overline{M} \setminus \{q_1, \ldots, q_n\}$. Let $X\colon M \to \mathbf{R}^{2,1}$ be a nonbranched maximal map all of whose ends are simple ends, and (g, η) its Weierstrass data.

Consider the case when $\overline{M} = \hat{\mathbf{C}}(= \mathbf{C} \cup \{\infty\})$. Assume $q_j \neq \infty$ and $G(q_j) \neq \sigma^{-1}(\infty) = {}^t(0, 0, 1)$, i.e. $p_j = g(q_j) \neq \infty$, for any $j = 1, \ldots, n$. Suppose that the ends $q_1, \ldots, q_m$ are of order 1, and that the ends $q_{m+1}, \ldots, q_n$ are of order 2. Then the divisor of η is given by $\prod_{j=1}^{m}(z - q_j) \prod_{k=m+1}^{n}(z - q_k)^2$. Since ∞ is not an end of X, η does not have a pole at ∞. Hence the sum of orders of poles of η is $m + 2(n - m) = 2n - m$, and the sum of orders of zeroes of η is $2n - m - 2$. On the other hand, since X has no branch point, η and $g^2\eta$ have no common zero. Hence the zeroes of η must coincide with the poles of g, and the order of η at any zero is the double of the order of g at the same point as a pole. Now, we see that the degree of g must be equal to $(2n - m - 2)/2 = n - 1 - m/2$. In particular, the number of the ends of order 1 must be an even number.

As an easy application of this observation, we give here a proof of the fact which we mentioned in § 3. If a maximal map X of genus 0 has only two simple ends of the third kind, then the number of the ends of order 1 is 2 or 0. When the number is 2, X is a map whose image is a null line. We have already presented its Weierstrass data in § 3. On the other hand, when both the ends are of order 2, by the consideration above, the degree of g must be equal to $2 - 1 - 0/2 = 1$. Hence the two null vectors φ_1 and φ_2 must be linearly independent. They cannot satisfy the flux formula, and hence we see that *there exists no two-ended maximal map of genus* 0 *whose ends are simple of the third kind and of order* 2.

We call a nonbranched maximal map $X\colon M \to \mathbf{R}^{2,1}$ an *n-noid*, if all the ends are simple ends of order 2. It is a natural analogue of a minimal surface we call an n-end catenoid. By the consideration above, the Weierstrass data (g, η) of any n-noid X satisfying $q_j \neq \infty$, $p_j \neq \infty$ $(j = 1, \ldots, n)$ is given by

$$g(z) = \frac{P(z)}{Q(z)}, \quad \eta = -Q(z)^2 dz, \tag{4.1}$$

where

$$P(z) = \sum_{j=1}^{n} \frac{p_j b_j}{z - q_j}, \quad Q(z) = \sum_{j=1}^{n} \frac{b_j}{z - q_j},$$

$b_j \in \mathbf{C} \setminus \{0\}$. (cf. See [9] for the case of n-end catenoids in $\mathbf{R}^3$.) This data automatically satisfies the condition (2.5).

Now, for each end q_j, denote the corresponding R_0, R_1, R_2, w, w^* by R_{0j}, R_{1j}, R_{2j}, w_j, w_j^* respectively. Then we have

$$\begin{aligned} R_{0j} &= -\operatorname*{Res}_{z=q_j} Q(z)^2 dz \\ &= -\operatorname*{Res}_{z=q_j} \Big\{ \frac{{b_j}^2}{(z-q_j)^2} - \sum_{k=1;k\neq j}^{n} \frac{2b_j b_k}{q_k - q_j} \frac{1}{z-q_j} + \cdots \Big\} dz \end{aligned}$$

$$
\begin{aligned}
&= b_j \sum_{k=1;k\neq j}^{n} b_k \frac{2}{q_k - q_j}, \\
R_{1j} &= -\operatorname*{Res}_{z=q_j} P(z)Q(z)dz \\
&= -\operatorname*{Res}_{z=q_j} \Big\{ \frac{p_j {b_j}^2}{(z-q_j)^2} - \sum_{k=1;k\neq j}^{n} \frac{(p_k+p_j)b_j b_k}{q_k - q_j} \frac{1}{z-q_j} + \cdots \Big\} dz \\
&= b_j \sum_{k=1;k\neq j}^{n} b_k \frac{p_k + p_j}{q_k - q_j}, \\
R_{2j} &= -\operatorname*{Res}_{z=q_j} P(z)^2 dz \\
&= -\operatorname*{Res}_{z=q_j} \Big\{ \frac{{p_j}^2 {b_j}^2}{(z-q_j)^2} - \sum_{k=1;k\neq j}^{n} \frac{2p_j p_k b_j b_k}{q_k - q_j} \frac{1}{z-q_j} + \cdots \Big\} dz \\
&= b_j \sum_{k=1;k\neq j}^{n} b_k \frac{2p_j p_k}{q_k - q_j},
\end{aligned}
$$

and hence

$$
\begin{aligned}
w_j = p_j R_{0j} - R_{1j} &= b_j \sum_{k=1;k\neq j}^{n} b_k \frac{2p_j - (p_k + p_j)}{q_k - q_j} \\
&= -b_j \sum_{k=1;k\neq j}^{n} b_k \frac{p_k - p_j}{q_k - q_j}, \\
w_j^* = \frac{1}{2}(\overline{p'_j} p_j + 1) R_{0j} - \overline{p'_j} R_{1j} &= b_j \sum_{k=1;k\neq j}^{n} b_k \frac{(\overline{p'_j} p_j + 1) - \overline{p'_j}(p_k + p_j)}{q_k - q_j} \\
&= -b_j \sum_{k=1;k\neq j}^{n} b_k \frac{\overline{p'_j} p_k - 1}{q_k - q_j}.
\end{aligned}
$$

By using these equalities, we can rewrite the conditions (2.7) and (2.11), and get the following fact.

Theorem 4.1 *There exists an n-noid* $X\colon M = \hat{\mathbf{C}} \setminus \{q_1, \ldots, q_n\} \to \mathbf{R}^{2,1}$ *satisfying* (3.3) *if and only if there exist nonzero complex numbers* $b_1, \ldots, b_n$ *satisfying*

$$\begin{cases} w_j \equiv -b_j \displaystyle\sum_{k=1;k\neq j}^{n} b_k \frac{p_k - p_j}{q_k - q_j} = a_j & \textit{if}\ \ \overline{p'_j} p_j \neq 1, \\ \check{w}_j \equiv -b_j \displaystyle\sum_{k=1;k\neq j}^{n} b_k \frac{p_k + p_j}{q_k - q_j} = \check{a}_j & \textit{if}\ \ \overline{p'_j} p_j = 1, \\ w_j^* \equiv -b_j \displaystyle\sum_{k=1;k\neq j}^{n} b_k \frac{\overline{p'_j} p_k - 1}{q_k - q_j} = 0, & \end{cases} \tag{4.2}$$

$$(j = 1, \ldots, n),$$

and the degree of g *given by* (4.1) *is* $n - 1$.

In the case when $q_1 = \infty \neq p_j$ $(j = 1, \ldots, n)$, $P(z)$ and $Q(z)$ are given by

$$P(z) = -p_1 b_1 + \sum_{j=2}^{n} \frac{p_j b_j}{z - q_j}, \quad Q(z) = -b_1 + \sum_{j=2}^{n} \frac{b_j}{z - q_j},$$

and hence, we must replace w_j, w_j^* and $\check{w}_j$ by the following.

$$w_1 = b_1 \sum_{k=2}^{n} b_k (p_k - p_1),$$

$$w_j = -b_j \Big\{ b_1 (p_1 - p_j) + \sum_{k=2;k\neq j}^{n} b_k \frac{p_k - p_j}{q_k - q_j} \Big\}$$

$$\check{w}_1 = b_1 \sum_{k=2}^{n} b_k (p_k + p_1),$$

$$\check{w}_j = -b_j \Big\{ b_1 (p_1 + p_j) + \sum_{k=2;k\neq j}^{n} b_k \frac{p_k + p_j}{q_k - q_j} \Big\}$$

$$w_1^* = b_1 \sum_{k=2}^{n} b_k (\overline{p'_1} p_k - 1),$$

$$w_j^* = -b_j \Big\{ b_1 (\overline{p'_j} p_1 - 1) + \sum_{k=2;k\neq j}^{n} b_k \frac{\overline{p'_j} p_k - 1}{q_k - q_j} \Big\}$$

$$(j = 2, \ldots, n).$$

By the flux formula, each given data p_j, p'_j, a_j (or $\check{a}_j$) $(j = 1, \ldots, n)$ must satisfy (3.2). To find an n-noid satisfying (3.3), we have only to solve (4.2) as an algebraic equation with respect to $q_1, \ldots, q_n$, $b_1, \ldots, b_n$, and show that $P(z)^2dz$ and $Q(z)^2dz$ have no common zeroes.

In the case when $q_j \neq \infty$, $p_j \neq \infty$ $(j = 1, \ldots, n)$, define the square matrix $A = (a_{jk})_{j,k=1,\ldots,n}$ by

$$a_{jj} := 0 \quad (j = 1, \ldots, n), \quad a_{jk} := \frac{\overline{p'_j} p_k - 1}{q_k - q_j} \quad (j \neq k).$$

In the case when $q_1 = \infty \neq p_j$ $(j = 1, \ldots, n)$, replace the definitions of a_{j1} and a_{1k} by

$$a_{j1} := \overline{p'_j} p_1 - 1 \quad (j = 2, \ldots, n),$$
$$a_{1k} := -(\overline{p'_1} p_k - 1) \quad (k = 2, \ldots, n).$$

Set $\mathbf{q} := {}^t(q_1, \ldots, q_n)$, $\mathbf{b} := {}^t(b_1, \ldots, b_n)$, $\mathbf{w} := {}^t(w_1, \ldots, w_n)$, and $\mathbf{w}^* := {}^t(w_1^*, \ldots, w_n^*)$. Then it holds that $\mathbf{w}^* = -\operatorname{diag}[b_1, b_2, b_3, b_4] A\mathbf{b}$. Since $b_j \neq 0$ $(j = 1, \ldots, n)$, the third equalities in (4.2) are equivalent to $A\mathbf{b} = \mathbf{0}$. Hence, if there exists a maximal map satisfying (4.2), then $\mathbf{q}$ satisfies $\det A = 0$, and $\mathbf{b} \in \operatorname{Ker} A$.

The following lemma is also useful for solving the equation (4.2).

Lemma 4.2 *When* $\overline{p'_j} p_j \neq 1$ *for any* $j = 1, \ldots, n$, *set* $V := (v(p_1, p'_1), \ldots, v(p_n, p'_n))$ *and* $V^* := (v^*(p_1, p'_1), \ldots, v^*(p_n, p'_n))$. *When* $\overline{p'_j} p_j = 1$ *for some* $j = 1, \ldots, n$, *replace the* j*-th column* $v(p_j, p'_j)$ *of* V *by* $\check{v}(p_j)$, *the* j*-th column* $v^*(p_j, p'_j)$ *of* V^* *by* $\check{v}^*(p_j)$, *and the* j*-th component* w_j *of* $\mathbf{w}$ *by* $\check{w}_j$, *respectively. Then it holds that*

$$V\mathbf{w} = V^*\mathbf{w}^*. \tag{4.3}$$

Proof. By the residue theorem, it holds that $\sum_{j=1}^n R_{0j} = \sum_{j=1}^n R_{1j} = \sum_{j=1}^n R_{2j} = 0$. Hence by taking the sum of (2.11) (or (2.12)) for $q = q_j$ $(j = 1, \ldots, n)$, we have (4.3). □

In general, it is difficult to judge whether $P(z)^2dz$ and $Q(z)^2dz$ have a common zero or not. We give here two criteria, that are some special cases.

Lemma 4.3 $P(z)^2dz$ *and* $Q(z)^2dz$ *have no common zero if one of the following conditions holds*:

(1) $q_j = p_j$ $(j = 1, \ldots, n)$ *and* $\sum_{j=1}^n b_j \neq 0$,

(2) $p_2 = p_3 = \cdots = p_n \neq p_1$.

Proof. (1) When $q_j = p_j$ for all $j = 1, \ldots, n$,

$$Q(z) = \sum_{j=1}^{n} \frac{b_j}{z - p_j}, \quad P(z) = zQ(z) - \sum_{j=1}^{n} b_j.$$

Now, our assertion is clear.
(2) When $p_2 = p_3 = \cdots = p_n \neq p_1$,

$$Q(z) = \sum_{j=1}^{n} \frac{b_j}{z - q_j}, \quad P(z) = \frac{(p_1 - p_2)b_1}{z - q_1} + p_2 Q(z).$$

Now, our assertion is clear since $(p_1 - p_2)b_1 \neq 0$. □

Let X be an n-noid given by (4.1) and (4.2). For any data (g, η) as in (4.1), set

$$w_{jk} := -b_j b_k \frac{p_k - p_j}{q_k - q_j}, \quad w^*_{jk} := -b_j b_k \frac{\overline{p'_j} p_k - 1}{q_k - q_j}$$
$$(j,\, k = 1, \ldots, n; j \neq k).$$

When (g, η) satisfies (4.2), we call w_{jk} the *relative weight* of the end-pair (q_j, q_k) $(j, k = 1, \ldots, n; j \neq k)$ of an n-noid X given by (g, η). In the case when $q_1 = \infty \neq p_j$ $(j = 1, \ldots, n)$, replace the definitions of w_{j1}, w^*_{j1}, w_{1k}, w^*_{1k} by the following:

$$w_{j1} := -b_j b_1 (p_1 - p_j), \quad w^*_{j1} := -b_j b_1 (\overline{p'_j} p_1 - 1) \quad (j = 2, \ldots, n),$$
$$w_{1k} := b_1 b_k (p_k - p_1), \quad w^*_{1k} := b_1 b_k (\overline{p'_1} p_k - 1) \quad (k = 2, \ldots, n).$$

The values of w_{jk} are independent of the parameterizations. More precisely, *w_{jk} are invariant under the conformal transformations of $\hat{\mathbf{C}}$ and the orientation preserving congruent transformations of $\mathbf{R}^{2,1}$*. We can prove this fact in the same way as [8, Proposition 2.3].

By direct computation, we can show the following relationship between the arrangement of the ends in the domain and the relative weights (cf. [7]).

$$\frac{w_{jm} w_{k\ell} - w_{j\ell} w_{km}}{w_{jm} w_{k\ell} - w_{jk} w_{\ell m}} = \begin{cases} \dfrac{(q_j - q_k)(q_\ell - q_m)}{(q_j - q_\ell)(q_k - q_m)} & \text{for } q_j, q_k, q_\ell, q_m \neq \infty \\ \dfrac{q_\ell - q_m}{q_k - q_m} & \text{for } q_j = \infty, \end{cases} \tag{4.4}$$

from which it also follows that

$$\frac{(w_{jm}w_{k\ell} - w_{j\ell}w_{km})w_{jk}w_{\ell m}}{(w_{jm}w_{k\ell} - w_{jk}w_{\ell m})w_{j\ell}w_{km}} = \frac{(p_j - p_k)(p_\ell - p_m)}{(p_j - p_\ell)(p_k - p_m)} \quad \text{for } p_j \neq p_\ell \quad \text{and} \quad p_k \neq p_m. \tag{4.5}$$

In the case when all the ends are not of the third kind, we can rewrite the equation (4.2) by using the relative weights:

$$\begin{cases} \displaystyle\sum_{k=1;k\neq j}^{n} w_{jk} = a_j \\ \displaystyle\sum_{k=1;k\neq j}^{n} w_{jk}^* \equiv \sum_{k=1;k\neq j}^{n} w_{jk}\frac{\overline{p_j'}p_k - 1}{p_k - p_j} = 0 \end{cases} \qquad (j = 1, \ldots, n). \tag{4.6}$$

5. Symmetry of maximal surfaces

We observe here some natural correspondences between the Weierstrass data of maximal surfaces in $\mathbf{R}^{2,1}$ and that of minimal surfaces in $\mathbf{R}^3$. By using these correspondences, we can find various examples. In this section, we are free from the assumption in the previous sections that the genus of a surface is zero and all the ends are simple ends.

Set $\mathbf{e}_1 := {}^t(1, 0, 0)$, $\mathbf{e}_2 := {}^t(0, 1, 0)$ and $\mathbf{e}_3 := {}^t(0, 0, 1)$.

At first, we note here that, for any conformal minimal immersion $X\colon M \to \mathbf{R}^3$ given by

$${}^tX(z) = \operatorname{Re}\int_{z_0}^{z}(1 - g^2, \sqrt{-1}(1 + g^2), 2g)\eta,$$

the condition corresponding to (2.3) is as follows.

$$R_0 = -\overline{R_2}, \quad R_1 = \overline{R_1}. \tag{5.1}$$

It is easy to see that (g, η) represents a minimal surface in $\mathbf{R}^3$ satisfying

$$\varphi(\gamma) \in \mathbf{R}\mathbf{e}_3 \quad (\forall\gamma\colon \text{ a loop in } M) \tag{5.2}$$

if and only if the condition

$$R_0 = R_2 = 0, \quad R_1 = \overline{R_1} \tag{5.3}$$

holds for any γ. On the other hand, (g, η) also represents a maximal surface in $\mathbf{R}^{2,1}$ satisfying (5.2) if and only if (5.3) holds for any γ. Therefore the

Weierstrass data $(g,\,\eta)$ of a minimal surface in $\mathbf{R}^3$ with (5.2) can be regarded as that of a maximal surface in $\mathbf{R}^{2,1}$ also with (5.2). This correspondence is discussed in [18].

It is also easy to see that $(g,\,\eta)$ represents a minimal surface in $\mathbf{R}^3$ satisfying

$$\varphi(\gamma) \in \mathbf{R}\mathbf{e}_1 + \mathbf{R}\mathbf{e}_2 \quad (\forall\gamma\colon\ \text{a loop in } M) \tag{5.4}$$

if and only if the condition

$$R_0 = -\overline{R_2}, \quad R_1 = 0 \tag{5.5}$$

holds for any γ. On the other hand, $(g,\,\eta)$ also represents a maximal surface in $\mathbf{R}^{2,1}$ satisfying (5.4) if and only if the condition

$$R_0 = \overline{R_2}, \quad R_1 = 0 \tag{5.6}$$

holds for any γ. Therefore $(g,\,\eta)$ is the Weierstrass data of a minimal surface in $\mathbf{R}^3$ with (5.4) if and only if $(g_1,\,\eta_1) = (g,\,\sqrt{-1}\eta)$ is that of a maximal surface in $\mathbf{R}^{2,1}$ also with (5.4). By this, we can find a large family of maximal correspondents to the family of minimal surfaces, which are called of TYPE II in [9], [10], [11].

We also see that $(g,\,\eta)$ represents a maximal surface in $\mathbf{R}^{2,1}$ satisfying

$$\varphi(\gamma) \in \mathbf{R}\mathbf{e}_1 + \mathbf{R}\mathbf{e}_3 \quad (\forall\gamma\colon\ \text{a loop in } M) \tag{5.7}$$

if and only if the condition

$$R_0 = \overline{R_0} = R_2 = \overline{R_2}, \quad R_1 = \overline{R_1} \tag{5.8}$$

holds for any γ, and that $(g,\,\eta)$ is the Weierstrass data of a maximal surface in $\mathbf{R}^{2,1}$ with (5.7) if and only if

$$(g_2,\,\eta_2) = \Big(\sqrt{-1}\frac{g-\sqrt{-1}}{g+\sqrt{-1}},\ \frac{\sqrt{-1}}{2}(g+\sqrt{-1})^2\eta\Big)$$

is also. In particular, any branch point of the latter surface, if one exists, coincides with that of the former surface. The most typical example of this correspondence can be found between a catenoid of the first kind and a helicoid of the second kind. In general, this correspondence gives a transformation between two n-noids which trades a simple end of the first kind with that of the second kind.

	Target space	(a) type of symmetry	(b) necessary and sufficient condition
(5.9)	$\mathbf{R}^3$	w.r.t. the x_1x_2-plane	$g \circ I = 1/\overline{g}$, $I^*\eta = -\overline{g^2\eta}$
(5.10)	$\mathbf{R}^{2,1}$	w.r.t. a point	$g \circ I = 1/\overline{g}$, $I^*\eta = -\overline{g^2\eta}$
(5.11)	$\mathbf{R}^{2,1}$	double	$g \circ I = 1/\overline{g}$, $I^*\eta = \overline{g^2\eta}$
(5.12)	$\mathbf{R}^{2,1}$	w.r.t. the x_1x_3-plane	$g \circ I = \overline{g}$, $I^*\eta = \overline{\eta}$

Table 5.1.

We can show the following facts about symmetry of minimal or maximal surfaces, in the same way as the condition for a minimal surface to be a double cover of a nonorientable minimal surface. Here we call a maximal surface $X(M) \subset \mathbf{R}^{2,1}$ a *double surface* if there exists an antiholomorphic involution $I\colon M \to M$ satisfying $X(I(z)) = X(z)$ $(z \in M)$ and $G(I(z)) = -G(z)$ $(z \in M \setminus \Sigma)$.

Proposition 5.1 *Let X be a conformal minimal immersion into $\mathbf{R}^3$, or a maximal map into $\mathbf{R}^{2,1}$, defined on a Riemann surface M with the Weierstrass data (g, η). Assume that any nontrivial holomorphic covering transformation group does not act on X. Then, with* (a) *and* (b) *as in* Table 5.1*, X has the symmetry of type* (a) *(up to parallel transformation) if and only if (g, η) satisfies the condition* (b) *for some antiholomorphic involution $I\colon M \to M$, which satisfies $I^2(z) = z$ and $I_z = 0$.*

Each correspondence we considered in the top of this section preserves some symmetry in Proposition 5.1.

Since the condition (b) is the same for (5.9) and (5.10), we see that, if a minimal surface with parallel flux is symmetric with respect to the x_1x_2-plane (or a spacelike plane), then it has a natural correspondent maximal surface which is symmetric with respect to a point.

On the other hand, (g, η) satisfies (5.9) if and only if $(g_1, \eta_1) = (g, \sqrt{-1}\eta)$ satisfies (5.11). (g, η) satisfies (5.12) if and only if

$$(g_2, \eta_2) = \left(\sqrt{-1}\frac{g - \sqrt{-1}}{g + \sqrt{-1}},\ \frac{\sqrt{-1}}{2}(g + \sqrt{-1})^2\eta\right)$$

satisfies (5.11). Namely, any minimal (resp. maximal) surface whose flux vectors are arranged on the common (resp. common timelike) plane has a natural correspondent maximal double surface whose flux vectors are arranged on the common spacelike (resp. timelike) plane.

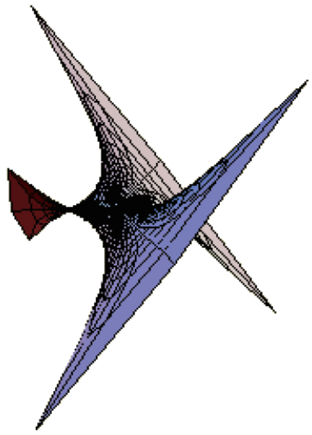

Fig. 5.1.

The helicoid of the second kind is a double surface, which satisfies (5.11) (cf. [12]). Any maximal correspondent to the Cosín-Ros' family of strongly symmetric Alexandrov embedded n-end catenoids (cf. [1]) is also a double surface (see Fig. 5.1).

Any maximal herisson all of whose ends are of the second or third kind is also a double surface. As we shall see in § 7, any 3-noid all of whose ends are of the second or third kind is a double surface (cf. Proposition 7.7). However, this cannot be expected in general when $n \geq 4$, even if the condition (5.7) holds. For instance, consider the 4-noid given by the following Weierstrass data:

$$(g,\, \eta) = \Big(\frac{z(z^2+z+1)+\sqrt{-1}(z^2-z+1)}{(z^2+z+1)+\sqrt{-1}z(z^2-z+1)},\\ \frac{6(3+2\sqrt{-1})}{13}\Big(\frac{1}{z^2-z+1}+\frac{\sqrt{-1}z}{z^2+z+1}\Big)^2 dz\Big).$$

Though this 4-noid has ends of the second kind only, and satisfies the condition (5.7), it is not a double surface. In any double surface, two single sheets are bounded by common null curves, which are subsets of the singular set $X(\Sigma)$.

Another typical singularity which appears on a maximal surface is a conelike singularity, which is a special case of degenerate null curves. The

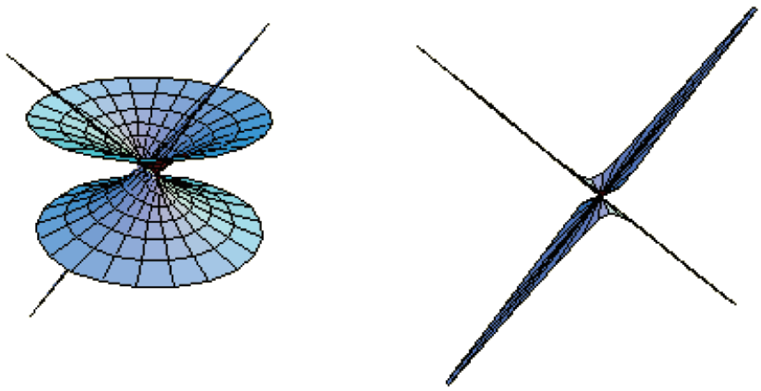

Fig. 5.2.

catenoid of the first kind has a conelike singularity with symmetry of type (5.10). If a maximal surface includes a degenerate null curve, then it is symmetric with respect to the point where the null curve degenerates (cf. Theorem 5.4). By applying the correspondence "(g,η) with (5.9)"$\mapsto$ "(g,η) with (5.10)" to a special case of [6, Example 2.6], we see that the 4-noid given by

$$(g,\,\eta) = \Big(\frac{z^3+3z}{3z^2+1},\, -\Big\{\frac{3z^2+1}{z(z^2-1)}\Big\}^2 dz\Big)$$

also includes a degenerate null curve. In this surface, two ends with vanishing flux are arranged on the degenerate null curve, and the singularity is not conelike (see Fig. 5.2). It seems that maximal surfaces with conelike singularities are less abundant than the double surfaces (cf. [13], [4], [15], [2]).

Here we present some facts for symmetric maximal surfaces.

Proposition 5.2 *Let* $X\colon M \to \mathbf{R}^{2,1}$ *be a nonbranched maximal map with the Weierstrass data* $(g,\,\eta)$*. If* $X(M)$ *is a double surface as in* Proposition 5.1 (5.11)*, then any connected component of the fixed point set* Σ_0 *of* I *is not a loop.*

Proof. Suppose that Σ_1 is a connected component of Σ_0 and that Σ_1 is a loop. Let $z(t)$ $(0 \le t \le \ell)$ be a regular parameterization of Σ_1. Then it holds that

$$\frac{d}{dt}X_3(z(t)) = \mathrm{Re}\,\frac{-2g\eta_{z(t)}}{dt}.$$

By (5.11), we have $|g| \equiv 1$ on Σ_0. Moreover, by (5.11) again, we also get

$$\frac{g\eta_{z(t)}}{dt} = \frac{g(z(t))I^*\eta_{z(t)}}{dt} = \frac{g(z(t))\overline{g^2\eta_{z(t)}}}{dt} = \frac{\overline{g\eta_{z(t)}}}{dt},$$

and hence

$$\frac{d}{dt}X_3(z(t)) = \frac{-2g\eta_{z(t)}}{dt}.$$

Since X has no branch point and $|g| \equiv 1$ on Σ_1, $-2g\eta$ has no zero on Σ_1, and hence $dX_3(z(t))/dt$ does not change sign. Hence we get

$$X_3(z(\ell)) - X(z(0)) = \int_0^\ell \frac{d}{dt}X_3(z(t))dt \neq 0,$$

namely X is not a well-defined map. □

Proposition 5.3 *Let $X\colon M = \hat{M}\setminus\{q\} \to \mathbf{R}^{2,1}$ be a nonbranched maximal map with the Weierstrass data (g, η) which has a simple end at q. If $X(M)$ is symmetric with respect to a point as in* Proposition 5.1 (5.10)*, and if q is included in the fixed point set Σ_0 of $I\colon \hat{M} \to \hat{M}$, then the flux of the end q vanishes.*

Proof. Set $R_i := R_i(q)$ $(i = 0, 1, 2)$. By (5.10), we have

$$\overline{\eta} = -\frac{1}{\overline{g}^2}I^*\eta = -(g\circ I)^2 I^*\eta = -I^*(g^2\eta),$$
$$\overline{g\eta} = -\frac{1}{\overline{g}}I^*\eta = -g\circ I\cdot I^*\eta = -I^*(g\eta),$$

from which it holds that

$$R_0 = \frac{1}{2\pi\sqrt{-1}}\int_\gamma \eta = \frac{1}{2\pi\sqrt{-1}}\int_\gamma \overline{-I^*(g^2\eta)} = \overline{\frac{1}{2\pi\sqrt{-1}}\int_{I(\gamma)} g^2\eta} = -\overline{R_2},$$
$$R_1 = \frac{1}{2\pi\sqrt{-1}}\int_\gamma g\eta = \frac{1}{2\pi\sqrt{-1}}\int_\gamma \overline{-I^*(g\eta)} = \overline{\frac{1}{2\pi\sqrt{-1}}\int_{I(\gamma)} g\eta} = -\overline{R_1},$$

where γ is a loop surrounding q once from the left. By these equalities and (2.3), we have $R_0 = R_1 = R_2 = 0$, which implies $w = w(q) = 0$ (and $\check{w} = \check{w}(q) = 0$). □

In the case when $M = \overline{M} \setminus \{q_1, \ldots, q_n\}$ for some compact Riemann surface $\overline{M}$, the conclusion of Proposition 5.2 (resp. 5.3) means that X has at least one end on each connected component of $\overline{\Sigma_0}$ (resp. X has no simple end with nonzero flux on $\overline{\Sigma_0}$).

Theorem 5.4 *Let $X\colon M \to \mathbf{R}^{2,1}$ be a nonbranched maximal map with the Weierstrass data (g, η).*

(1) *If $X(M)$ is a double surface as in* Proposition 5.1 (5.11) (*resp. symmetric with respect to a point as in* Proposition 5.1 (5.10))*, then the tangent space $T_q\Sigma_0$ of the fixed point set Σ_0 of $I\colon M \to M$ is the nonzero-eigenspace* (*resp. zero-eigenspace*) *with respect to $X^*(g_{\mathbf{R}^3})$ for any $q \in \Sigma_0$.*

(2) *If there exists an open submanifold Σ_1 of the singular set $\Sigma = \{q \in M \mid |g(q)| = 1\}$ such that the tangent space $T_q\Sigma_1$ is the nonzero-eigenspace* (*resp. zero-eigenspace*) *with respect to $X^*(g_{\mathbf{R}^3})$ for any $q \in \Sigma_1$, then $X(U)$ is a double surface* (*resp. symmetric with respect to a point*) *for some open subset U of M.*

Proof. For any coordinate function z on a domain in M, set $f(z) := \eta/dz$. Then the eigenvalues of $X^*(g_{\mathbf{R}^3}) = (1-|g|^2)^2|\eta|^2 + 8(\operatorname{Re} g\eta)^2$ are $(1-|g|^2)^2|f|^2$ and $(1-|g|^2)^2|f|^2 + 8|gf|^2$, and the associated eigenspaces are spanned by ${}^t(\operatorname{Im} gf, \operatorname{Re} gf)$ and ${}^t(\operatorname{Re} gf, -\operatorname{Im} gf)$ respectively. In particular, for any point on Σ, the eigenvalues are 0 and $8|gf|^2$. Note here that $8|gf|^2 \neq 0$ holds at any nonbranched point.

Let $z(t)$ be a regular curve along the fixed point set Σ_0 of an anti-holomorphic involution $I\colon M \to M$. Then, since $dz/dt = I_{\overline{z}} \cdot d\overline{z}/dt$, the tangent space $T_q\Sigma_0$ of Σ_0 at any nonbranched point $q \in \Sigma_0$ is spanned by ${}^t(1 + \operatorname{Re} I_{\overline{z}}, \operatorname{Im} I_{\overline{z}})$ or ${}^t(\operatorname{Im} I_{\overline{z}}, 1 - \operatorname{Re} I_{\overline{z}})$. Note here that these two vectors are linear independent, and that

$$\begin{vmatrix} \operatorname{Re} gf & 1 + \operatorname{Re}\dfrac{\partial I}{\partial \overline{z}} \\ -\operatorname{Im} gf & \operatorname{Im}\dfrac{\partial I}{\partial \overline{z}} \end{vmatrix} = \operatorname{Im}\Big(gf + gf\frac{\partial I}{\partial \overline{z}}\Big),$$

$$\begin{vmatrix} \operatorname{Re} gf & \operatorname{Im}\dfrac{\partial I}{\partial \overline{z}} \\ -\operatorname{Im} gf & 1 - \operatorname{Re}\dfrac{\partial I}{\partial \overline{z}} \end{vmatrix} = \operatorname{Re}\Big(gf - gf\frac{\partial I}{\partial \overline{z}}\Big) \qquad (5.13)$$

$$(\text{resp.}\ \begin{vmatrix} \operatorname{Im} gf & 1+\operatorname{Re}\dfrac{\partial I}{\partial \overline{z}} \\ \operatorname{Re} gf & \operatorname{Im}\dfrac{\partial I}{\partial \overline{z}} \end{vmatrix} = -\operatorname{Re}\Big(gf + gf\frac{\partial I}{\partial \overline{z}}\Big),$$

$$\begin{vmatrix} \operatorname{Im} gf & \operatorname{Im}\dfrac{\partial I}{\partial \overline{z}} \\ \operatorname{Re} gf & 1-\operatorname{Re}\dfrac{\partial I}{\partial \overline{z}} \end{vmatrix} = \operatorname{Im}\Big(gf - gf\frac{\partial I}{\partial \overline{z}}\Big)).$$

(1) If $X(M)$ is a double surface (resp. symmetric with respect to a point), then, by Proposition 5.1 (5.11) (resp. (5.10)), there exists an antiholomorphic involution $I\colon M \to M$ such that $g \circ I = 1/\overline{g}$, and

$$(gf)\circ I \cdot \frac{\partial I}{\partial \overline{z}} = \overline{gf} \quad (\text{resp.}\ -\overline{gf}) \tag{5.14}$$

holds for any coordinate function z. In particular, $|g| = 1$ holds on Σ_0, i.e. $\Sigma_0 \subset \Sigma$.

Now, by (5.14) and $I|_{\Sigma_0} = \mathrm{id}_{\Sigma_0}$, the right-hand-sides of the equalities in (5.13) vanish on Σ_0. Hence any tangent vectors of Σ_0 at q are parallel to the nonzero-eigenvector ${}^t(\operatorname{Re} gf, -\operatorname{Im} gf)$ (resp. the zero-eigenvector ${}^t(\operatorname{Im} gf, \operatorname{Re} gf)$).

(2) Define $I_1\colon \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ by $I_1(\zeta) := 1/\overline{\zeta}$.

Set $M_1 := \{q \in M \mid dg(q) \neq 0\}$. Then $g|_{M_1}\colon M_1 \to \hat{\mathbf{C}}$ is a local diffeomorphism. Choose and fix $q_1 \in \Sigma_1 \cap M_1$ such that $g(q_1) \neq \infty$. Then there exists a neighborhood U_1 of q_1 such that $U_1 \subset M_1$, $\infty \notin g(U_1)$, and $g|_{U_1}$ is injective. Set $V_1 := I_1(g(U_1)) \cap g(U_1)$. Since $\Sigma_1 \subset \Sigma$, we have $|g(q_1)| = 1$, and hence $V_1 \cap \mathbf{S}^1 \neq \emptyset$. Let V be a connected component of V_1 including $g(q_1)$, and set $U := g^{-1}(V) \cap U_1$. Then U is also a neighborhood of q_1, and we can define $I\colon U \to U$ by $I := g^{-1} \circ I_1 \circ g$. I is an antiholomorphic involution on U, and $I|_{\Sigma_1 \cap U} = \mathrm{id}_{\Sigma_1 \cap U}$.

Set $z := g|_U$. Then we can use z as a coordinate function on U. By the definition of I, we have

$$g \circ I(z) = \frac{1}{\overline{g(z)}} \tag{5.15}$$

on U.

Suppose that the tangent space $T_q\Sigma_1$ is the nonzero-eigenspace (resp. zero-eigenspace) with respect to $X^*(g_{\mathbf{R}^3})$ for any $q \in \Sigma_1$. Then the left-hand-sides of the equalities in (5.13) vanish on $\Sigma_1 \cap U$, and hence $gf \cdot I_{\overline{z}} =$

$\overline{gf}$ (resp. $-\overline{gf}$) on $\Sigma_1 \cap U$. Now, by $I|_{\Sigma_1\cap U} = \mathrm{id}_{\Sigma_1\cap U}$, we have (5.14) on $\Sigma_1 \cap U$.

Since both $(gf) \circ I \cdot I_{\overline{z}}$ and $\overline{gf}$ are antiholomorphic on U, and U is connected, (5.14) holds also on U. By (5.14) and (5.15) on U, we get (5.11) (resp. (5.10)) (b) on U. □

In (2), if $I : U \to U$ can be extended to M, then the assertion holds also for $X(M)$.

We will use the following criterion in § 7.

Proposition 5.5 *Let X be an n-noid of genus 0 defined by (4.1). Let A be as in § 4. If $|p_j| = |p'_j| = |q_j| = 1$ ($j = 1, \ldots, n$), a_j (or $\breve{a}_j$) $\neq 0$ for some j, and $\operatorname{rank} A = n - 1$, then $X(M)$ is a double surface.*

Proof. Let $(\mathbf{q}, \mathbf{b}) = ({}^t(q_1, \ldots, q_n), {}^t(b_1, \ldots, b_n))$ be a solution of the equation (4.2) realizing the given n-noid X. Then we have

$$\begin{aligned}
a_j = -\overline{a_j} &= \overline{b_j} \sum_{k=1;k\neq j}^{n} \overline{b_k}\frac{\overline{p_k} - \overline{p_j}}{\overline{q_k} - \overline{q_j}} \\
&= -\frac{\sqrt{-1}\,\overline{p_j b_j}}{\overline{q_j}} \sum_{k=1;k\neq j}^{n} \frac{\sqrt{-1}\,\overline{p_k b_k}}{\overline{q_k}} \frac{p_k - p_j}{q_k - q_j}, \\
\Big(\breve{a}_j = \overline{\breve{a}_j} &= -\overline{b_j} \sum_{k=1;k\neq j}^{n} \overline{b_k}\frac{\overline{p_k} + \overline{p_j}}{\overline{q_k} - \overline{q_j}} \\
&= -\frac{\sqrt{-1}\,\overline{p_j b_j}}{\overline{q_j}} \sum_{k=1;k\neq j}^{n} \frac{\sqrt{-1}\,\overline{p_k b_k}}{\overline{q_k}} \frac{p_k + p_j}{q_k - q_j}, \Big) \\
0 = \overline{0} &= -\sum_{k=1;k\neq j}^{n} \overline{b_k}\frac{p'_j\overline{p_k} - 1}{\overline{q_k} - \overline{q_j}} \\
&= \frac{\sqrt{-1} p'_j}{\overline{q_j}} \sum_{k=1;k\neq j}^{n} \frac{\sqrt{-1}\,\overline{p_k b_k}}{\overline{q_k}} \frac{\overline{p'_j} p_k - 1}{q_k - q_j}.
\end{aligned}$$

Hence, if we define $\widetilde{\mathbf{b}} = {}^t(\widetilde{b}_1, \ldots, \widetilde{b}_n)$ by $\widetilde{b}_j := \sqrt{-1}\,\overline{p_j b_j}/\overline{q_j}$ ($j = 1, \ldots, n$), then $(\mathbf{q}, \widetilde{\mathbf{b}})$ is also a solution of (4.2). If $\operatorname{rank} A = n-1$, then it holds that $\widetilde{\mathbf{b}} = t\mathbf{b}$ for some $t \in \mathbf{C}$, since $\mathbf{b}, \widetilde{\mathbf{b}} \in \operatorname{Ker} A$. By the equalities above, we have $t = \pm 1$, and we get (b) in Proposition 5.1 (5.11). □

For the behaviors of singularities in a general situation, see [18], [3].

6. Obstructions for the existence of n-noids of genus zero with prescribed flux

Denote

$$v_j := \begin{cases} v(p_j, p_j') & \text{if } |p_j| \neq 1 & \text{(the first kind)} \\ \sqrt{-1}v(p_j, p_j') & \text{if } |p_j| = 1 \text{ and } p_j \neq p_j' & \text{(the second kind)} \\ \check{v}(p_j) & \text{if } |p_j| = 1 \text{ and } p_j = p_j' & \text{(the third kind).} \end{cases}$$

In the same way as for the Euclidean case, we can show the following fact.

Theorem 6.1 *There exists no n-noid $X : \hat{\mathbf{C}} \setminus \{q_1, \ldots, q_n\} \to \mathbf{R}^{2,1}$ satisfying* (3.3)*, if one of the following conditions holds*:

(6.1) $p_1 = p_2 = \cdots = p_n$,

(6.2) *all v_j are not null,* $-v_1 = \cdots = -v_N = v_{N+1} = \cdots = v_n$, *and* $\sum_{j=1}^{N} {a_j}^2 \neq \sum_{k=N+1}^{n} {a_k}^2$,

(6.3) *all v_j are not null,* $-v_1 = -v_2 = v_3 = \cdots = v_n$,

(6.4) $n = 4$, *v_1 and v_2 are not null,* $-v_1 = v_2$, $v_3 = v_4 \neq \pm v_1$.

Proof. The condition (6.1) implies that the degree of the Gauss map is at least n. This contradicts the fact that the degree of the Gauss map must be $n - 1$.

When all of the flux vectors are parallel and not null, by a suitable rotation in $\mathbf{R}^{2,1}$, we may assume that the given flux data satisfies (5.2) or (5.4) without loss of generality. In each case, there exists a maximal surface with the given flux data if and only if there exists a corresponding minimal surface as in § 5. Since, by [16, Theorem 1], the condition (6.2) is an obstruction for the existence of such minimal surfaces, it is also an obstruction for maximal surfaces.

In the case when the condition (6.3) holds, one of the following holds:

(i) $|p_j| \neq 1$, $p_j' = p_j$ $(j = 1, \ldots, n)$, $p_1 = p_2$, $p_3 = \cdots = p_n = 1/\overline{p_1}$,

(ii) $|p_j| = 1$, $p_j' \neq p_j$ $(j = 1, \ldots, n)$, $p_1 = p_2 = p_3' = \cdots = p_n'$, $p_1' = p_2' = p_3 = \cdots = p_n$.

In each case, it holds that $w_{12}^* \neq 0$ and $w_{1k}^* = 0$ $(k = 3, \ldots, n)$. Hence $w_1^* = w_{12}^* \neq 0$, i.e. the condition (4.6) cannot be satisfied.

In the case when the condition (6.4) holds, one of the following holds:

(i) $|p_1| \neq 1$, $p_1' = p_1$, $p_2 = 1/\overline{p_1}$, $p_2' = p_2$, $p_3 = p_4 \neq p_1, p_2$, and $p_3' = p_4'$,

(ii) $|p_1| = 1$, $p_1' \neq p_1$, $p_2 = p_1'$, $p_2' = p_1$, $p_3 = p_4$, $p_3' = p_4'$,

and one of the following holds:

$$p_3 \neq p_1,\, p_2; \quad p_3 = p_1,\, p_3' \neq p_1'; \quad p_3 = p_2,\, p_3' \neq p_2',$$

In each case, it holds that $w_{34} = w_{43} = 0$ and $w_{12}^* = 0 = w_{21}^*$. By (4.6), we have

$$w_{13}^* + w_{14}^* = 0 = w_{23}^* + w_{24}^*. \tag{6.5}$$

On the other hand, by (3.2) and the assumption that v_1 and v_3 are linearly independent, it holds that

$$a_1 = a_2, \quad a_3 + a_4 = 0.$$

By (4.6) again, we also have

$$w_{12} + w_{13} + w_{14} = w_{21} + w_{23} + w_{24}, \quad w_{31} + w_{32} + w_{41} + w_{42} = 0,$$

which implies that

$$w_{13} + w_{14} = 0 = w_{23} + w_{24}. \tag{6.6}$$

Since $p_3 \neq p_1,\, p_2$, or $p_3 \neq p_1 = p_2'$, or $p_3 \neq p_2 = p_1'$ holds, we have, by (6.5), (6.6) and $b_1,\, b_2 \neq 0$,

$$\frac{1}{q_3 - q_1} b_3 + \frac{1}{q_4 - q_1} b_4 = 0, \quad \frac{1}{q_3 - q_2} b_3 + \frac{1}{q_4 - q_2} b_4 = 0.$$

Since $b_3,\, b_4 \neq 0$, it must hold that

$$\begin{aligned} 0 &= \frac{1}{q_3 - q_1}\frac{1}{q_4 - q_2} - \frac{1}{q_4 - q_1}\frac{1}{q_3 - q_2} \\ &= \frac{(q_1 - q_2)(q_4 - q_3)}{(q_3 - q_1)(q_4 - q_1)(q_3 - q_2)(q_4 - q_2)}. \end{aligned}$$

This contradicts the assumption that $q_1,\, q_2,\, q_3,\, q_4$ are different from each other. □

We can also show the following fact, which has no analogue in the Euclidean case.

Theorem 6.2 *There exists no* n*-noid* $X\colon \hat{\mathbf{C}} \setminus \{q_1, \ldots, q_n\} \to \mathbf{R}^{2,1}$ *satisfying* (3.3)*, if one of the following conditions holds*:
(6.7) $|p_2| = 1$ *and* $p_1 \neq p_2' = p_3 = \cdots = p_n$,
(6.8) *all* v_j *are null,* $v_{N+1} = \cdots = v_n \neq v_1, \ldots, v_N$ $(N \leq n/2)$.

Proof. In the case when the condition (6.7) holds, it holds that $w^*_{2k} = 0$ $(k = 3, \ldots, n)$. By (4.6) with $j = 2$, we have $w^*_{21} = 0$. Since we assume $p_1 \neq p'_2$, we have $b_2 b_1 = 0$.

In the case when the condition (6.8) holds, it holds that

$$p_{N+1} = \cdots = p_n \neq p_1, \ldots, p_N \qquad \text{and} \quad |p_j| = 1,\, p'_j = p_j \ (j = 1, \ldots, n),$$

and hence $w^*_{jk} = 0$ $(j, k = N+1, \ldots, n)$. By (4.6) with $j = N+1, \ldots, n$, we have $\sum_{k=1}^{N} w^*_{jk} = 0$ $(j = N+1, \ldots, n)$. Since $p_j = p_{N+1} \neq 0$ $(j = N+1, \ldots, n)$, we have

$$\sum_{k=1}^{N} \frac{1}{q_k - q_j} b_k(p_k - p_{N+1}) = 0 \quad (j = N+1, \ldots, n).$$

Since $b_k(p_k - p_{N+1}) \neq 0$ $(k = 1, \ldots, N)$, it must hold that

$$\begin{aligned} 0 &= \det\left(\frac{1}{q_k - q_j}\right)_{j=N+1,\ldots,2N;k=1,\ldots,N} \\ &= (-1)^{N(N-1)/2} \frac{\prod_{j,j'=N+1;j<j'}^{2N}(q_j - q_{j'}) \prod_{k,k'=1;k<k'}^{N}(q_k - q_{k'})}{\prod_{j=N+1}^{2N}\prod_{k=1}^{N}(q_k - q_j)}. \end{aligned}$$

This contradicts the assumption that $q_1, \ldots, q_n$ are different from each other. □

7. Classification of 3-noids of genus zero

In this section, we solve the equation (4.2) with $n = 3$, and classify 3-noids of genus zero.

In the case of $\mathbf{R}^3$, the space of 3-noids of genus zero essentially consists of only two 1-parameter families. One is the family of 3-end catenoids which includes the Jorge-Meeks' 3-noid. The other is the family of surfaces called Lopez-Ros' surfaces, each of which has 2 catenoidal ends and 1 planar end (cf. [14]).

On the other hand, the space of 3-noids in $\mathbf{R}^{2,1}$ is more complicated. Let p_j, p'_j, a_j (or $\check{a}_j$) $(j = 1, 2, 3)$ be as in Problem 3.1 with the condition (3.2). Assume that $p_j \neq \infty$ $(j = 1, 2, 3)$. Let v_j be as in § 6 $(j = 1, 2, 3)$, and denote $\langle v_1, v_2, v_3\rangle := \mathbf{R}v_1 + \mathbf{R}v_2 + \mathbf{R}v_3$. Set $a_j := 0$ if $|p_j| = 1$ and $p_j = p'_j$. Recall here that, if $|p_j| = 1$ and $p_j \neq p'_j$, then $v_j \in NP_+(p_j) \cap$

$NP_-(p'_j) \cap \mathbf{S}^{1,1}$. In the same way as in (4.6), the equation

$$w_j \equiv -\sum_{k=1;k\neq j}^{3} b_j b_k \frac{p_k - p_j}{q_k - q_j} = a_j \quad (j = 1, 2, 3) \tag{7.1}$$

is rewritten as $\sum_{k=1;k\neq j}^{3} w_{jk} = a_j$ $(j = 1, 2, 3)$. This is also equivalent to

$$w_{kl} = \frac{1}{2}(a_k + a_l - a_m) \qquad \text{for} \quad (k, l, m) = (1, 2, 3), (2, 3, 1), (3, 1, 2). \tag{7.2}$$

By this equality, we get the following:

Lemma 7.1 *There exists no* 3*-noid* $X : \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3\} \to \mathbf{R}^{2,1}$ *satisfying* $w(q_j) = a_j$ $(j = 1, 2, 3)$*, if the following condition holds*:
(7.3) $a_k = 0$*,* $a_\ell = a_m$ *for* $(k, l, m) = (1, 2, 3)$ *or* $(2, 3, 1)$ *or* $(3, 1, 2)$.

Proof. By (7.2) and (7.3), we have $w_{k\ell} = w_{km} = 0$, from which it must hold that $p_\ell = p_k = p_m$, i.e. $p_1 = p_2 = p_3$. However, by Theorem 6.1 (6.1), there exists no n-noid satisfying such a condition. □

In particular, there exists no 3-noid all of whose ends have weight 0. Note here that, if an end is of the third kind, i.e. its flux vector is null, then the weight of the end is 0 even when the flux vector does not vanish. By Theorems 6.1, 6.2 and Lemma 7.1, we have the following:

Theorem 7.2 *There exists no* 3*-noid* $X : \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3\} \to \mathbf{R}^{2,1}$ *satisfying* (3.3)*, if one of the following conditions hold*:
(7.4) $v_1 = v_2 = v_3$,
(7.5) $-v_1 = v_2 = v_3$,
(7.6) $-v_1 = v_2 \neq \pm v_3$,
(7.7) $\pm v_1 \neq v_2 = v_3$*, and* v_2*,* v_3 *are null,*
(7.8) v_1 *is null, and* $v_2, v_3 \in NP_+(p_1) \cup NP_-(p_1)$.
(7.9) $v_1, v_2, v_3 \in NP_+(p_1)$,
(7.10) $v_1 \in NP_+(p_1)$*, and* $v_2, v_3 \in NP_-(p_1)$,
(7.11) $\dim\langle v_1, v_2, v_3\rangle = 3$.

Proof. If (7.4) or (7.9) (resp. (7.5)) holds, then our assertion follows from Theorem 6.1 (6.1) (resp. (6.3)) with $n = 3$.

If (7.6) holds, then, by the assumption (3.2), we have $a_1 = a_2$, $a_3 = 0$. If (7.7) holds, then, by the assumption (3.2), we have $a_1 = 0$. Since v_2

and v_3 are null, we also have $a_2 = a_3 = 0$. If (7.11) holds, then, by the assumption (3.2), we have $a_1 = a_2 = a_3 = 0$. Hence our assertion follows from Lemma 7.1 in each of these cases.

If (7.8) holds, then, since v_1 is null, we have $a_1 = 0$. Since $\langle v_1, v_2, v_3\rangle$ is a null plane, we also have $a_2 = a_3$ or $a_2 + a_3 = 0$. If $a_2 = a_3$, then, by Lemma 7.1, there exists no 3-noid satisfying (3.3). On the other hand, if $a_2 + a_3 = 0$, then we have $p_2 = p_3 = p_1$ or $p'_2 = p'_3 = p_1 (\neq p_3)$. Now our assertion follows from Theorem 6.1 (6.1) or Theorem 6.2 (6.7).

If (7.10) holds, then $|p_2| = 1$, $p_3 \neq p'_2 = p_1$, and our assertion follows from Theorem 6.2 with (6.7). □

By Theorem 7.2 with (7.4), (7.5), (7.11), we see that, for any 3-noid of genus zero, the v_j's must span a 2-dimmensional vector space, i.e. $\dim\langle v_1, v_2, v_3\rangle = 2$. We call this space the *flux plane* of X.

By Theorem 7.2 with (7.7), we see that *if* $p_1 \neq p_2 = p_3$ *and* $\overline{p'_3}p_2 = \overline{p'_3}p_3 = 1$*, then there exists no* 3*-noid* $X \colon \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3\} \to \mathbf{R}^{2,1}$ *satisfying* (3.3). Hence, in all other cases, one of the following conditions holds if we make a suitable change of the indices of the ends:

(7.12) $p_1 \neq p_2 = p_3$ and $\overline{p'_3}p_3 \neq 1$,

(7.13) $p_1 \neq p_2 \neq p_3 \neq p_1$.

To solve the equation (4.2) with $n = 3$, we prepare the following:

Lemma 7.3 *Set* $(\mathbf{q}, \mathbf{b}) = ({}^t(q_1, q_2, q_3), {}^t(b_1, b_2, b_3))$*.*

(1) *In the case when* (7.12) *holds, if* $(\mathbf{q}, \mathbf{b})$ *satisfies* (7.1) *and* $w_3^* = 0$*, then* $(\mathbf{q}, \mathbf{b})$ *is a solution of the equation* (4.2)*.*

(2) *In the case when* (7.13) *holds, if* $(\mathbf{q}, \mathbf{b})$ *satisfies* (7.1)*, then* $(\mathbf{q}, \mathbf{b})$ *is a solution of the equation* (4.2)*.*

Proof. (1) The 2×2 minor determinants of the matrix $(\widetilde{v}(p_1), \widetilde{v}(p_2))$ are $2\sqrt{-1}(p_1 - p_2)(p_1p_2 + 1)$, $-2(p_1 - p_2)(p_1p_2 - 1)$ and $2\sqrt{-1}(p_1 - p_2)(p_1 + p_2)$. By the assumption $p_1 \neq p_2$, at least one of them does not vanish, and hence $\widetilde{v}(p_1)$ and $\widetilde{v}(p_2)$ are linearly independent.

(i) *The case when* $\overline{p'_j}p_j \neq 1$ $(j = 1, 2, 3)$.

By the assumption $w_3^* = 0$ and (4.3), we have

$$\sum_{j=1}^{2} w_j^* v^*(p_j, p'_j) = \sum_{j=1}^{3} w_j^* v^*(p_j, p'_j) = V^*\mathbf{w}^* = V\mathbf{w} = V\mathbf{a} = \mathbf{0}.$$

Since $v^*(p_1, p'_1) = (\overline{p'_1}p_1 - 1)^{-1}\widetilde{v}(p_1)$ and $v^*(p_2, p'_2) = (\overline{p'_2}p_2 - 1)^{-1}\widetilde{v}(p_2)$ are

linearly independent, we get $w_1^* = w_2^* = 0$. Hence $(\mathbf{q}, \mathbf{b})$ is a solution of the equation (4.2).
(ii) *The case when* $\overline{p_1'}p_1 = 1$ *and* $\overline{p_j'}p_j \neq 1$ $(j = 2, 3)$.
Since $\overline{p_1'}p_1 = 1$, we have $w_1^* = \overline{p_1}w_1 = \overline{p_1}a_1 = 0$. By the assumption $w_3^* = 0$ and (4.3), we have

$$\begin{aligned}
&(\breve{w}_1 - \breve{a}_1)(-\breve{v}(p_1)) + w_2^* v^*(p_2, p_2') \\
&= (\breve{w}_1 - \breve{a}_1)(-\breve{v}(p_1)) + \sum_{j=2}^{3} w_j^* v^*(p_j, p_j') \\
&= (\breve{w}_1 - \breve{a}_1)(-\breve{v}(p_1)) + \breve{w}_1\breve{v}(p_1) + \sum_{j=2}^{3} w_j v(p_j, p_j') - w_1^* \breve{v}^*(p_1) \\
&= \breve{a}_1\breve{v}(p_1) + \sum_{j=2}^{3} a_j v(p_j, p_j') - 0 \cdot \breve{v}^*(p_1) = \mathbf{0}.
\end{aligned}$$

Since $-\breve{v}(p_1) = (2p_1)^{-1}\widetilde{v}(p_1)$ and $v^*(p_2, p_2') = (\overline{p_2'}p_2 - 1)^{-1}\widetilde{v}(p_2)$ are linearly independent, we get $\breve{w}_1 - \breve{a}_1 = w_2^* = 0$. Hence $(\mathbf{q}, \mathbf{b})$ is a solution of the equation (4.2).
(iii) *The case when* $\overline{p_2'}p_2 = 1$ *and* $\overline{p_j'}p_j \neq 1$ $(j = 1, 3)$.
(iv) *The case when* $\overline{p_j'}p_j = 1$ $(j = 1, 2)$ *and* $\overline{p_3'}p_3 \neq 1$.
We can prove our assertion in the same way as in the second case.
(2) Set $\widetilde{V} = (\widetilde{v}(p_1), \widetilde{v}(p_2), \widetilde{v}(p_3))$. Then we have $\det \widetilde{V} = 4\sqrt{-1}(p_1 - p_2)(p_2 - p_3)(p_3 - p_1) \neq 0$.
(i) *The case when* $\overline{p_j'}p_j \neq 1$ $(j = 1, 2, 3)$.
By (4.3), we have $V^*\mathbf{w}^* = V\mathbf{w} = V\mathbf{a} = \mathbf{0}$. Since $\det V^* = \prod_{j=1}^{3}(\overline{p_j'}p_j - 1)^{-1} \det \widetilde{V} \neq 0$, we get $\mathbf{w}^* = (V^*)^{-1}\mathbf{0} = \mathbf{0}$. Hence $(\mathbf{q}, \mathbf{b})$ is a solution of the equation (4.2).
(ii) *The case when* $\overline{p_j'}p_j \neq 1$ $(j = 1, 2)$ *and* $\overline{p_3'}p_3 = 1$.
Since $\overline{p_3'}p_3 = 1$ we have $w_3^* = \overline{p_3}w_3 = \overline{p_3}a_3 = 0$. Set

$$V_{(1)}^* := (v^*(p_1, p_1'), v^*(p_2, p_2'), -\breve{v}(p_3)).$$

By (4.3), we have

$$V_{(1)}^* \begin{pmatrix} w_1^* \\ w_2^* \\ \breve{w}_3 - \breve{a}_3 \end{pmatrix} = \sum_{j=1}^{2} w_j^* v^*(p_j, p_j') + (\breve{w}_3 - \breve{a}_3)(-\breve{v}(p_3))$$

$$= \sum_{j=1}^{2} w_j v(p_j, p_j') + \check{w}_3 \check{v}(p_3) - w_3^* \check{v}^*(p_3) - (\check{w}_3 - \check{a}_3)\check{v}(p_3)$$

$$= \sum_{j=1}^{2} a_j v(p_j, p_j') - 0 \cdot \check{v}^*(p_3) + \check{a}_3 \check{v}(p_3) = \mathbf{0}.$$

Since $\det V_{(1)}^* = \prod_{j=1}^{2}(\overline{p_j'}p_j - 1)^{-1}(2p_3)^{-1} \det \widetilde{V} \neq 0$, we get ${}^t(w_1^*, w_2^*, \check{w}_3 - \check{a}_3) = (V_{(1)}^*)^{-1}\mathbf{0} = \mathbf{0}$. Hence $(\mathbf{q}, \mathbf{b})$ is a solution of the equation (4.2).

(iii) *The case when* $\overline{p_1'}p_1 \neq 1$ *and* $\overline{p_j'}p_j = 1$ $(j = 2, 3)$.

(iv) *The case when* $\overline{p_j'}p_j = 1$ $(j = 1, 2, 3)$.

We can prove our assertion in the same way as in the second case. □

Now, let us solve the equation (4.2) with $n = 3$. In the case when (7.12) holds, we have the following:

Lemma 7.4 *For any flux data satisfying* (3.2) *and* (7.12)*, there exists a unique* 3*-noid* $X : \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3\} \to \mathbf{R}^{2,1}$ *satisfying* (3.3) *if and only if the following condition holds*:

$$a_2 \neq 0, \quad a_3 \neq 0, \quad \textit{and} \quad \overline{p_3'}p_1 \neq 1. \tag{7.14}$$

Proof. In this case, by Lemma 7.3 (1), we have only to consider data (g, η) satisfying (7.1) and $w_3^* = 0$. We may set $q_j = p_j$ $(j = 1, 2)$ and $q_3 = (p_1 + p_2)/2$. Under this setting, it holds that $w_1 = -b_1b_2 - 2b_1b_3$, $w_2 = -b_1b_2$, $w_3 = -2b_1b_3$, and

$$w_3^* = \frac{2b_3}{p_2 - p_1}\{(\overline{p_3'}p_1 - 1)b_1 - (\overline{p_3'}p_3 - 1)b_2\}.$$

Note here that, by (7.12) and the assumption (3.2), it holds that $a_1 = a_2 + a_3$. Then the equation "(7.1) and $w_3^* = 0$" is rewritten as

$$b_1b_2 = -a_2, \quad b_1b_3 = -\frac{a_3}{2}, \quad (\overline{p_3'}p_1 - 1)b_1 = (\overline{p_3'}p_3 - 1)b_2.$$

By solving this, we get

$${b_1}^2 = -a_2\frac{\overline{p_3'}p_3 - 1}{\overline{p_3'}p_1 - 1}, \quad b_2 = \frac{\overline{p_3'}p_1 - 1}{\overline{p_3'}p_3 - 1}b_1, \quad b_3 = \frac{a_3}{2a_2}b_2.$$

This solution makes sense if and only if (7.14) holds. By the criterion in Lemma 4.3 (2), the given surface has no branch points. □

In the case when (7.13) holds, we have the following:

Lemma 7.5 *For any flux data satisfying* (3.2) *and* (7.13)*, there exists a unique* 3*-noid* $X: \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3\} \to \mathbf{R}^{2,1}$ *satisfying* (3.3) *if and only if both of the following conditions hold*:

$$a_k + a_\ell \neq a_m \quad \textit{for} \quad (k, \ell, m) = (1, 2, 3), (2, 3, 1), (3, 1, 2), \tag{7.15}$$
$$D := {a_1}^2 + {a_2}^2 + {a_3}^2 - 2(a_2a_3 + a_3a_1 + a_1a_2) \neq 0. \tag{7.16}$$

Proof. In this case, by Lemma 7.3 (2), we have only to consider data (g, η) satisfying (7.2). We may set $q_j = p_j$ $(j = 1, 2, 3)$. Under this setting, it holds that $w_{jk} = -b_jb_k$, and the equation (7.2) is rewritten as

$$b_kb_\ell = -\frac{1}{2}(a_k + a_\ell - a_m) \qquad \text{for} \quad (k, \ell, m) = (1, 2, 3), (2, 3, 1), (3, 1, 2).$$

By solving this, we get

$$\begin{aligned} {b_1}^2 &= -\frac{(a_1 + a_2 - a_3)(a_3 + a_1 - a_2)}{2(a_2 + a_3 - a_1)}, \\ b_2 &= \frac{a_2 + a_3 - a_1}{a_3 + a_1 - a_2} b_1, \quad b_3 = \frac{a_2 + a_3 - a_1}{a_1 + a_2 - a_3} b_1. \end{aligned}$$

This solution makes sense if and only if $a_k + a_\ell \neq a_m$ for $(k, \ell, m) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$. By the criterion in Lemma 4.3 (1), the given surface have no branch point if and only if

$$\begin{aligned} D &= -4b_1b_2b_3(b_1 + b_2 + b_3) \\ &= {a_1}^2 + {a_2}^2 + {a_3}^2 - 2(a_2a_3 + a_3a_1 + a_1a_2) \neq 0. \end{aligned}$$

□

Describing the conditions (7.14), (7.15) and (7.16) by using the layout of the v_j's, we have the following:

Theorem 7.6 *For any flux data satisfying* (3.2)*, there exists a unique* 3*-noid* $X: \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3\} \to \mathbf{R}^{2,1}$ *satisfying* (3.3) *if* $\dim\langle v_1, v_2, v_3\rangle = 2$*, the conditions* (7.6), (7.7), (7.8) *do not hold,* $a_j \neq 0$ *for some* j*, and if one of the following conditions holds*:

(7.17) $\langle v_1, v_2, v_3\rangle$ *is timelike, and, for any Lorentzian transformation* F, $\{F(v_1), F(v_2), F(v_3)\}$ *does not coincide with any of the following* *sets*:

$$\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix},\begin{pmatrix}\sinh\theta\\0\\\cosh\theta\end{pmatrix},\begin{pmatrix}-\sinh\theta'\\0\\\cosh\theta'\end{pmatrix}\right\},\qquad \left\{\begin{pmatrix}-1\\0\\0\end{pmatrix},\begin{pmatrix}\cosh\theta\\0\\\sinh\theta\end{pmatrix},\begin{pmatrix}\cosh\theta'\\0\\-\sinh\theta'\end{pmatrix}\right\}$$

where θ *and* θ' *are positive numbers satisfying* $\cosh(\theta+\theta') = \cosh\theta + \cosh\theta' + 3$,

$$\left\{\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}\sqrt{2}\\0\\1\end{pmatrix},\begin{pmatrix}\sqrt{2}\\0\\-1\end{pmatrix}\right\},\quad \left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}-1\\0\\\sqrt{2}\end{pmatrix},\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix}\right\},$$

(7.18) $\langle v_1, v_2, v_3\rangle$ *is spacelike,*

(7.19) $\langle v_1, v_2, v_3\rangle$ *is null, and* v_k, $v_\ell \in NP_+(p_k)$, $v_m \in NP_-(p_k)$ *for* $(k, \ell, m) = (1, 2, 3)$ *or* $(2, 3, 1)$ *or* $(3, 1, 2)$,

(7.20) $\langle v_1, v_2, v_3\rangle$ *is null, and* $v_1, v_2, v_3 \in NP_-(p)$ *for some* $p \neq p_1, p_2, p_3$.

Proof. Note here that, if the assumption of this theorem is satisfied, then all of the conditions (7.i) ($i = 4, \ldots, 11$) do not hold, and either (7.12) or (7.13) must hold. In the former part of this proof, we show that, if (7.12) (resp. (7.13)) holds, and if (7.14) (resp. (7.15)) does not hold, then one of the conditions (7.i) ($i = 4, \ldots, 11$) holds. Since this contradicts our assumption, we see that (7.14) or (7.15) must hold, and we can apply Lemma 7.4 or 7.5. Thereafter, we rewrite the condition (7.16) to a more concrete form.

First, we consider the case when (7.12) holds. By $p_1 \neq p_2 = p_3$, it holds that $v_1 \neq v_2, v_3$ and "$v_2 = v_3$ or $v_2, v_3 \in NP_+(p_2) \cup \{\tilde{v}(p_2)\}$". By $\overline{p_3'}p_3 \neq 1$, v_3 is not null.

If $a_2 = 0$ and v_2 is null, then, since $p_3 = p_2$ and v_3 is not null, we have $v_3 \in NP_+(p_2)$. Moreover, since $p_1 \neq p_2$ and $v_2 \neq v_3$, we have $v_1 \in NP_-(p_2)$. This is the case of (7.8).

If $a_2 = 0$ and v_2 is not null, then, since $a_1v_1 + a_3v_3 = \mathbf{0}$, a_1 or $a_3 \neq 0$, and $p_1 \neq p_3$, we have $-v_1 = v_3$. If $a_3 = 0$, then, in the same way as above, we have $-v_1 = v_2$. If $\overline{p_3'}p_1 = 1$ and $|p_3'| \neq 1$, then, since $p_3' = p_3$ and $\overline{p_3}p_1 = 1$, we have $-v_1 = v_3$. These are the case of (7.6).

If $\overline{p_3'}p_1 = 1$ and $|p_3'| = 1$, then $p_3' = p_1$ implies $v_3 \in NP_-(p_1)$. Since $p_3 = p_2$ and v_3 is not null, we also have $v_3 \in NP_+(p_2)$. If $v_2 = v_3$, then $v_2 = v_3 \in NP_-(p_1)$. This is the case of (7.8) or (7.10). If $v_2 \neq v_3$, then $v_1 \in NP(p_2)$. Since $p_1 \neq p_2$, we have $v_1 \in NP_-(p_2) \cap NP_+(p_1)$, and hence $-v_1 = v_3$. This is the case of (7.6).

Hence, if (7.12) and the assumption of this theorem hold, then (7.14) holds.

Secondly, we consider the case when (7.13) holds. Since the p_j's take distinct values, the v_j's are also different from each other. Moreover, since we assume that (7.6) does not hold, all the $\pm v_j$'s are different from each other,

Suppose that $a_1 = a_2 + a_3$. Note here that, if $a_1 \in \mathbf{R} \setminus \{0\}$ (resp. $\sqrt{-1}\mathbf{R} \setminus \{0\}$), then $a_2, a_3 \in \mathbf{R}$ (resp. $\sqrt{-1}\mathbf{R}$). Denote the Lorentzian inner product by $(\,\cdot\,,\,\cdot\,)$, and its corresponding norm by $||\,\cdot\,||$.

If v_1 is timelike (resp. null), v_2 and v_3 are timelike, and $a_2a_3 \neq 0$, then, by

$$\begin{aligned} -{a_2}^2 - {a_3}^2 + 2a_2a_3(v_2, v_3) &= ||a_2v_2 + a_3v_3||^2 \\ &= ||-a_1v_1||^2 = -(a_2+a_3)^2 \\ (\text{resp. } &= ||-\check{a}_1v_1||^2 = 0), \end{aligned}$$

we have $(v_2, v_3) = -1$, from which it follows that $v_2 = v_3$. This is not our case.

If v_1 is spacelike (resp. null), v_2 and v_3 are spacelike, and $a_2a_3 \neq 0$, then, by

$$\begin{aligned} -{a_2}^2 - {a_3}^2 - 2a_2a_3(v_2, v_3) &= ||-\sqrt{-1}a_2v_2 - \sqrt{-1}a_3v_3||^2 \\ &= ||\sqrt{-1}a_1v_1||^2 = -(a_2+a_3)^2 \\ (\text{resp. } &= ||-\check{a}_1v_1||^2 = 0), \end{aligned}$$

we have $(v_2, v_3) = 1$ from which it follows that $v_2, v_3 \in NP_+(p)$ or $v_2, v_3 \in NP_-(p)$ for some $p \in \mathbf{S}^1$. Since $p_2 \neq p_3$, we have $v_2, v_3 \in NP_-(p)$ for $p \neq p_2, p_3$. Since $v_2 \neq v_3$ and $a_1 = a_2 + a_3$, we have $v_1 \in NP_+(p)$ (resp. $v_1 = \check{v}(p)$) and $p = p_1$. This is the case of (7.10) (resp. (7.8)).

If $a_2 = 0$ (resp. $a_3 = 0$), then $a_1 = a_3 \neq 0$ (resp. $a_1 = a_2 \neq 0$), and v_1 and v_3 (resp. v_1 and v_2) are not null. Hence we have $-v_1 = v_3$ (resp. $-v_1 = v_2$). This is the case of (7.6).

Hence, if (7.13) and the assumption of this theorem hold, then (7.15) holds.

Next, let us rewrite the condition (7.16), or its negation $D = 0$, under the assumption that the condition (7.15) holds. If (7.15) holds and $a_k = 0$ for some k, then $D = (a_\ell - a_m)^2 \neq 0$, where $\{k, \ell, m\} = \{1, 2, 3\}$. Hence we have only to consider the case when $a_j \neq 0$ for any j.

(i) *The case when v_j is timelike* ($j = 1, 2, 3$).

Let σ_j be 1 or -1 such that $\sigma_j v_j \in \mathbf{H}^2_+$ ($j = 1, 2, 3$), and θ_{jk} the hyperbolic distance between $\sigma_j v_j$ and $\sigma_k v_k$. Set $c_{jk} := \cosh\theta_{jk}$ and $s_{jk} := \sinh\theta_{jk}$. Then the Lorentzian inner product of v_j and v_k is given by $(v_j, v_k) = -\sigma_j\sigma_k c_{jk}$. Since $\sum_{j=1}^3 a_j v_j = \mathbf{0}$, $\mathbf{a} = {}^t(a_1, a_2, a_3)$ is an element of $\mathrm{Ker}((v_j, v_k))_{j,k=1,2,3} = \mathrm{Ker}(-\sigma_j\sigma_k c_{jk})_{j,k=1,2,3}$. By changing indices of the v_j's if necessary, we may assume that the v_j's are arranged so that $\theta_{12}+\theta_{23} = \theta_{13}$ holds. Under this assumption, it holds that $c_{12} = c_{13}c_{23} - s_{13}s_{23}$, $c_{23} = c_{12}c_{13} - s_{12}s_{13}$, $c_{13} = c_{12}c_{23} + s_{12}s_{23}$ and $s_{13} = s_{12}c_{23} + c_{12}s_{23}$. By using these equalities, we have $\mathrm{Ker}((v_j, v_k))_{j,k=1,2,3} = \mathbf{R}\,{}^t(\sigma_2\sigma_3 s_{23}, -\sigma_1\sigma_3 s_{13}, \sigma_1\sigma_2 s_{12})$. Hence $D = 0$ holds if and only if

$$\begin{aligned} 0 &= {s_{12}}^2 + {s_{23}}^2 + {s_{13}}^2 - 2(-\sigma_1\sigma_2 s_{13}s_{23} - \sigma_2\sigma_3 s_{12}s_{13} + \sigma_1\sigma_3 s_{12}s_{23}) \\ &= D_I(D_I - 4\sigma_1\sigma_3), \end{aligned}$$

where we set

$$\begin{aligned} D_I &:= \sigma_1\sigma_3(\sigma_1\sigma_2 c_{12} + \sigma_2\sigma_3 c_{23} + \sigma_1\sigma_3 c_{13} + 1) \\ &= (c_{12} + \sigma_1\sigma_2)(c_{23} + \sigma_2\sigma_3) + s_{12}s_{23}. \end{aligned}$$

Note here that $c_{jk} > 1$ and $s_{jk} > 0$ holds for any $j \neq k$ under our assumption. Then we see that, if $\sigma_1 = \sigma_2 = \sigma_3$, then $D_I > 4 = 4\sigma_1\sigma_3 > 0$. We also see that, if $-\sigma_1 = \sigma_2 = \sigma_3$ or $\sigma_1 = \sigma_2 = -\sigma_3$, then $D_I > 0 > -4 = 4\sigma_1\sigma_3$. On the other hand, we see that, if $\sigma_1 = -\sigma_2 = \sigma_3$, then $D_I > 0$, but $4\sigma_1\sigma_3 = 4 > 0$. Hence $D = 0$ holds if and only if $\sigma_1 = -\sigma_2 = \sigma_3$ and $D_I = 4$, i.e. $c_{13} = c_{12} + c_{23} + 3$. $\sum_{j=1}^3 a_j v_j = \mathbf{0}$ is satisfied only if $F(v_1) = {}^t(s_{12}, 0, c_{12})$, $F(v_2) = {}^t(0, 0, -1)$, $F(v_3) = {}^t(-s_{23}, 0, c_{23})$, hold for some Lorentzian transformation F.

(ii) *The case when v_j is spacelike* ($j = 1, 2, 3$), *and $\langle v_1, v_2, v_3\rangle$ is a timelike plane.*

In this case, we also get the quite similar conclusion by the quite similar calculation as in (i). This fact is shown also by using the correspondence $(g, \eta) \leftrightarrow (g_2, \eta_2)$ given in § 5.

(iii) *The case when* v_j *is spacelike* $(j = 1, 2, 3)$, *and* $\langle v_1, v_2, v_3\rangle$ *is a spacelike plane.*
Let θ_{jk} be the angle between v_j and v_k, chosen to satisfy $\theta_{12} + \theta_{23} = \theta_{13}$. Set $c_{jk} := \cos\theta_{jk}$ and $s_{jk} := \sin\theta_{jk}$. Then the Lorentzian inner product of v_j and v_k is given by $(v_j, v_k) = c_{jk}$. By the choice above, we have $c_{12} = c_{13}c_{23} + s_{13}s_{23}$, $c_{23} = c_{12}c_{13} + s_{12}s_{13}$, $c_{13} = c_{12}c_{23} - s_{12}s_{23}$ and $s_{13} = s_{12}c_{23} + c_{12}s_{23}$. By the same reason as in the case (i), $\mathbf{a} = {}^t(a_1, a_2, a_3) \in \mathrm{Ker}((v_j, v_k))_{j,k=1,2,3} = \mathrm{Ker}(c_{jk})_{j,k=1,2,3}$. In this case, we have $\mathrm{Ker}((v_j, v_k))_{j,k=1,2,3} = \mathbf{R}\,{}^t(s_{23}, -s_{13}, s_{12})$. Hence $D = 0$ holds if and only if

$$\begin{aligned} 0 &= {s_{12}}^2 + {s_{23}}^2 + {s_{13}}^2 - 2(-s_{13}s_{23} - s_{12}s_{13} + s_{12}s_{23}) \\ &= -D_{II}(D_{II} - 4), \end{aligned}$$

where we set

$$D_{II} := c_{12} + c_{23} + c_{13} + 1 = 4\cos\frac{\theta_{12}}{2}\cos\frac{\theta_{23}}{2}\cos\frac{\theta_{13}}{2}.$$

However, since $-1 < \cos(\theta_{jk}/2) < 1$ and $\cos(\theta_{jk}/2) \neq 0$ for any $j \neq k$ under our assumption, we have $D_{II} < 4$ and $D_{II} \neq 0$. Hence we get $D \neq 0$.
(iv) *The case when* v_j *is spacelike* $(j = 1, 2, 3)$, *and* $\langle v_1, v_2, v_3\rangle$ *is a null plane.*
Since we assume that the p_j's take distinct values and that (7.15) holds, we have only to consider the case of (7.20). In this case, since $a_1 + a_2 + a_3 = 0$, we have

$$\begin{aligned} D &= {a_1}^2 + {a_2}^2 + (-a_1 - a_2)^2 \\ &\quad -2\{a_1a_2 + a_2(-a_1 - a_2) + (-a_1 - a_2)a_1\} \\ &= 4({a_1}^2 + a_1a_2 + {a_2}^2) \neq 0. \end{aligned}$$

(v) *The case when* v_1 *is timelike, and* v_2, v_3 *are spacelike.*
Note here that $D = \{{a_1}^2 + (a_2 - a_3)^2\} - 2a_1(a_2 + a_3)$. Since ${a_1}^2 + (a_2 - a_3)^2$ is real and $a_1(a_2 + a_3)$ is purely imaginary, $D = 0$ holds if and only if ${a_1}^2 + (a_2 - a_3)^2 = 0$ and $a_1(a_2 + a_3) = 0$. Since $a_1 \neq 0$, $D = 0$ is equivalent to

$$a_1 : a_2 : a_3 = \pm 2\sqrt{-1} : 1 : (-1) = \pm 2 : (-\sqrt{-1}) : \sqrt{-1}.$$

$\sum_{j=1}^{3} a_j v_j = \mathbf{0}$ is satisfied only if $F(v_1) = {}^t(0, 0, \pm 1)$, $F(v_2) = {}^t(\sqrt{2}, 0, 1)$, $F(v_3) = {}^t(\sqrt{2}, 0, -1)$ hold for some Lorentzian transformation F. Now, if

d	directions of flux	flux plane	additional conditions	Does X exist?	cf.
1	$v_k = v_\ell = v_m$			No	(7.4)
	$-v_k = v_\ell = v_m$			No	(7.5)
	$-v_k = v_\ell \neq \pm v_m$			No	(7.6)
	$\pm v_k \neq v_\ell = v_m$:null			No	(7.7)
			$c_{km} = c_{k\ell} + c_{\ell m} + 3$	No(branched)	
		timelike	$a_k : a_\ell : a_m = 2\sqrt{-1} : 1 : (-1)$	No(branched)	(7.17)
			otherwise	Yes	
2		spacelike		Yes	(7.18)
	otherwise		v_j:null ($\exists j$)	No	(7.8)
			$v_k,\, v_\ell,\, v_m \in NP_+(p_k)$	No	(7.9)
		null	$v_k,\, v_\ell \in NP_+(p_k)$, $v_m \in NP_-(p_k)$	Yes	(7.19)
			$v_k \in NP_+(p_k)$, $v_\ell,\, v_m \in NP_-(p_k)$	No	(7.10)
			$v_k,\, v_\ell,\, v_m \in NP_-(p)$, $p \neq p_k,\, p_\ell,\, p_m$	Yes	(7.20)
3				No	(7.11)

Table 7.1.

we transpose v_2 and v_3, then $\pm$ are interchanged. Hence these two obstructions are essentially the same as each other.

(vi) *The case when v_1 is spacelike, and v_2, v_3 are timelike.*
In this case, we also get the quite similar conclusion by the quite similar calculation as in (v). This fact is shown also by using the correspondence $(g,\, \eta) \leftrightarrow (g_2,\, \eta_2)$ given in § 5. $\square$

Theorems 7.2 and 7.6 complete the classification of 3-noids of genus zero in $\mathbf{R}^{2,1}$.

We can show the following fact about the symmetry of 3-noids:

Proposition 7.7 *Let $X\colon M = \hat{\mathbf{C}} \setminus \{q_1,\, q_2,\, q_3\} \to \mathbf{R}^{2,1}$ be a 3-noid all of whose ends are of the second or third kind. Then $X(M)$ is a double surface.*

Proof. Let A be the matrix as in § 4. Since all the v_j's are spacelike or null, $|p_j| = |p'_j| = 1$ for any j. In the case when (7.12) holds, by $p_2 = p_3$ and $\overline{p'_3}p_3 \neq 1$, we have $\overline{p'_3}p_2 \neq 1$. Moreover, by $p_1 \neq p_3$, we also have $\overline{p'_2}p_1 \neq 1$ or $\overline{p'_2}p_3 \neq 1$. Hence we get $\operatorname{rank} A = 2 = 3 - 1$. On the other hand, in the case when (7.13) holds, by $p_k \neq p_\ell$, we have $\overline{p'_m}p_k \neq 1$ or $\overline{p'_m}p_\ell \neq 1$ for $(k,\, \ell,\, m) = (1,\, 2,\, 3),\, (2,\, 3,\, 1),\, (3,\, 1,\, 2)$. Hence we get $\operatorname{rank} A = 2 = 3 - 1$ also in this case. Since $n = 3$, we may choose q_j's so that $|q_j| = 1$ for any j. Therefore, by Proposition 5.5, we get our assertion. $\square$

In the remainder of this section, let us remark on the case when some end is of order 1. Let $X\colon \hat{\mathbf{C}} \setminus \{q_1,\, q_2,\, q_3\} \to \mathbf{R}^{2,1}$ be a maximal map all of whose ends are simple ends. By the consideration in § 4, the number

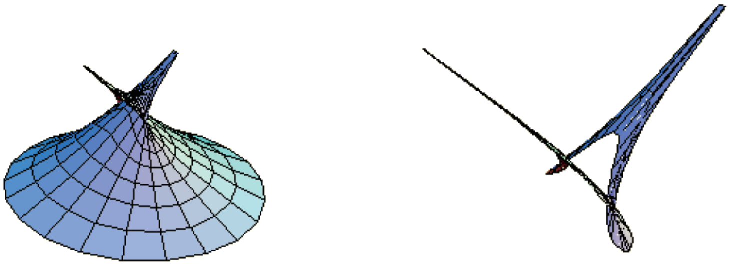

Fig. 7.1.

of the ends of order 1 must be even, if X has no branch point. Hence, if $n = 3$, then all the ends are of order 2 or two ends are of order 1. Here we consider the latter case. In this case, the degree of the Gauss map must be $3 - 1 - 2/2 = 1$, and hence X must be a maximal herrison, which we have already considered in § 3. Let q_1 and q_2 be of order 1, and q_3 of order 2. Then, by Proposition 2.2, v_1 and v_2 is null and both $\breve{a}_1$ and $\breve{a}_2$ do not vanish. Moreover, since the degree of the Gauss map is 1, $p_1 \neq p_2$ and hence $v_1 \neq v_2$. Therefore v_3 cannot be null, by the flux formula.

Conversely, if the given data satisfies these conditions, then, by Theorem 3.2, we have the following:

Proposition 7.8 *Given limiting normals* v_j *and values* a_j *(or* $\breve{a}_j$*), there exists a corresponding nonbranched maximal herisson* $X\colon \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3\} \to \mathbf{R}^{2,1}$ *satisfying* (3.3)*, if* $\dim\langle v_1, v_2, v_3\rangle = 2$*,* $a_3 \neq 0$*, and if the following condition holds*:

(7.21) $\langle v_1, v_2, v_3\rangle$ *is timelike,* v_1, v_2 *are null, and* v_3 *is not null.*

We present here figures of two 3-noids with the same flux data. One has two ends of order 1 (Fig. 7.1). The other has ends of order 2 only (Fig. 7.2).

8. General existence of 4-noids of genus zero with prescribed flux

In this section, we show that there exists a nonbranched 4-noid for a generic flux data. Let p_j, p'_j, a_j (or $\breve{a}_j$) ($j = 1, 2, 3, 4$) be as in Problem 3.1 with the condition (3.2), and V, V^*, $\mathbf{w}$, $\mathbf{w}^*$ as in § 4. When $\overline{p'_j}p_j \neq 1$ for

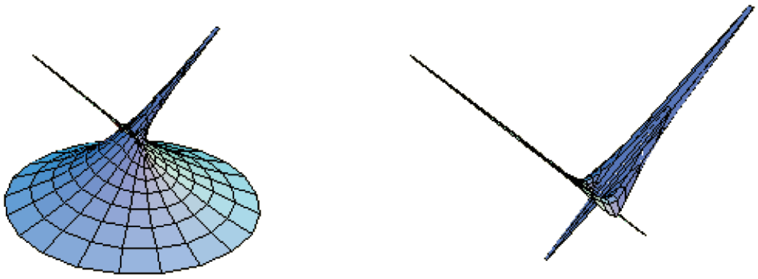

Fig. 7.2.

$j = 1, 2, 3, 4$, set $\mathbf{a} := {}^t(a_1, a_2, a_3, a_4)$. When $\overline{p'_j} p_j = 1$ for some j, replace the j-th component a_j of $\mathbf{a}$ by $\check{a}_j$. Let A be the matrix as in § 4 with $n = 4$ and $q_1 = \infty \neq p_j$ $(j = 1, 2, 3, 4)$, and $\tilde{A}$ the cofactor matrix of A. Let v_j be as in § 6 $(j = 1, 2, 3, 4)$. We fix $q_1 = \infty$, $q_2, q_3 \in \mathbf{C}$ $(q_2 \neq q_3)$, and regard each component of $\mathbf{w}$, $\mathbf{w}^*$, A and $\tilde{A}$ as a function with respect to q_4. We keep these assumptions throughout this section unless otherwise stated.

To solve the equation (4.2) with $n = 4$, we prepare the following:

Lemma 8.1 *Assume that* $\dim\langle v_k, v_\ell, v_m\rangle = 3$ *for some distinct* $k, \ell, m \in \{1, 2, 3, 4\}$, $q_4 \neq q_j$ $(j = 1, 2, 3)$, *and* $\mathbf{a} \neq \mathbf{0}$. *If* $(\mathbf{q}, \mathbf{b}) = ({}^t(q_1, q_2, q_3, q_4), {}^t(b_1, b_2, b_3, b_4))$ *satisfies* $\mathbf{w}^* = \mathbf{0}$ *and* $\mathbf{w} \neq \mathbf{0}$, *then there exists a nonzero complex number* t *satisfying* $t\mathbf{w} = \mathbf{a}$.

Proof. By (3.2), we have $V\mathbf{a} = \mathbf{0}$, namely $\mathbf{a} \in \operatorname{Ker} V$. On the other hand, by our assumption and (4.3), we also have $V\mathbf{w} = V^*\mathbf{w}^* = V^*\mathbf{0} = \mathbf{0}$, namely $\mathbf{w} \in \operatorname{Ker} V$. Since $\operatorname{rank} V = 3$, we have $\dim \operatorname{Ker} V = 4 - 3 = 1$. Therefore, if $\mathbf{w} \neq \mathbf{0}$, then $\mathbf{w}$ spans $\operatorname{Ker} V$, from which we get our assertion. □

In general, the following holds:

Lemma 8.2 *Let* B *be an* $n \times n$ *matrix. If* $\operatorname{rank} B = 1$, *and if* $b_{j\sigma(j)} \neq 0$ $(j = 1, \ldots, n)$ *for some* $\sigma \in S_n$, *then* $b_{jk} \neq 0$ *for any* $j, k \in \{1, \ldots, n\}$.

Proof. Since $\operatorname{rank} B = 1$, any two columns of B are proportional to each other. Hence, for any j, k, it holds that $b_{jk} b_{\sigma^{-1}(k)\sigma(j)} = b_{j\sigma(j)} b_{\sigma^{-1}(k)k} = b_{j\sigma(j)} b_{\sigma^{-1}(k)\sigma(\sigma^{-1}(k))} \neq 0$. □

By using Lemma 8.2, we get the following:

Lemma 8.3 *Assume*

$$\dim\langle v_k, v_\ell, v_m\rangle = 3 \quad \textit{for any distinct} \quad k, \ell, m \in \{1, 2, 3, 4\}, \tag{8.1}$$

and $q_4 \neq q_j$ $(j = 1, 2, 3)$. *If* $\det A = 0$, *then* $\operatorname{rank} A = 3$, $\operatorname{rank} \tilde{A} = 1$, *and* $\tilde{a}_{jk} \neq 0$ *holds for any* $j, k \in \{1, 2, 3, 4\}$.

Proof. $\dim\langle v_2, v_3, v_4\rangle = 3$ is equivalent to

$$\begin{aligned}
0 \neq &\begin{vmatrix} -(p_2+\overline{p_2'}) & -(p_3+\overline{p_3'}) & -(p_4+\overline{p_4'}) \\ \sqrt{-1}(p_2-\overline{p_2'}) & \sqrt{-1}(p_3-\overline{p_3'}) & \sqrt{-1}(p_4-\overline{p_4'}) \\ \overline{p_2'}p_2+1 & \overline{p_3'}p_3+1 & \overline{p_4'}p_4+1 \end{vmatrix} \\
&= -2\sqrt{-1}\{(\overline{p_2'p_3'}p_4 + \overline{p_4'})(p_2-p_3) \\
&\qquad + (\overline{p_3'p_4'}p_2 + \overline{p_2'})(p_3-p_4) + (\overline{p_4'p_2'}p_3 + \overline{p_3'})(p_4-p_2)\} \\
&= 2\sqrt{-1}(q_2-q_3)(q_3-q_4)(q_4-q_2)\tilde{a}_{11}.
\end{aligned}$$

Hence we have $\tilde{a}_{11} \neq 0$, $\operatorname{rank}\tilde{A} \geq 1$, and $\operatorname{rank} A \geq 3$. In the same way, we can show $\tilde{a}_{jj} \neq 0$ also for $j = 2, 3, 4$.

Since we assume $\det A = 0$, we have $\operatorname{rank} A = 3$, and hence $\dim \operatorname{Ker} A = 4 - 1 = 3$. Since $A\tilde{A} = \det A \cdot E = 0 \cdot E = O$, each column vector of $\tilde{A}$ is an element of $\operatorname{Ker} A$. Hence $\operatorname{rank}\tilde{A} = 1$. Now, by Lemma 8.2, we have $\tilde{a}_{jk} \neq 0$ $(\forall j, k)$. □

We also prepare the following:

Lemma 8.4 *Assume that the condition* (8.1) *holds, and* $q_4 \neq q_j$ $(j = 1, 2, 3)$. *If* $\mathbf{w}^* = \mathbf{0}$, $\mathbf{w} = \mathbf{0}$, *and* $b_j \neq 0$ $(j = 1, 2, 3, 4)$, *then it holds that*

$$\frac{(p_1-p_2)(p_3-p_4)}{(p_1-p_3)(p_2-p_4)} = \zeta_6 \quad \textit{or} \quad \overline{\zeta_6} \tag{8.2}$$

and

$$\left(\frac{(q_1-q_2)(q_3-q_4)}{(q_1-q_3)(q_2-q_4)} = \right)\frac{q_3-q_4}{q_2-q_4} = \overline{\zeta_6} \quad \textit{or} \quad \zeta_6, \tag{8.3}$$

where $\zeta_6 := e^{2\pi\sqrt{-1}/6}$.

Proof. Recall here that $\mathbf{w}$ is defined by replacing w_j by $\check{w}_j$ when $\overline{p_j'}p_j = 1$. By the assumption, it holds that $w_j \equiv \sum_{k=1;k\neq j}^4 w_{kj} = 0$ $(j = 1, 2, 3, 4)$, from which it follows that

$$w_{12} = w_{34}, \quad w_{13} = w_{24}, \quad w_{14} = w_{23}. \tag{8.4}$$

Suppose that $p_j = p_k$ for some distinct $j, k \in \{1, 2, 3, 4\}$. For instance, if $p_1 = p_2$, then, by (8.4), we have $w_{34} = w_{12} = 0$, which implies $p_3 = p_4$. When $\overline{p_1'}p_1 \neq 1$ (resp. $\overline{p_1'}p_1 = 1$), by $w_1 = 0$ (resp. $\breve{w}_1 = 0$) and $w_1^* = 0$, we have

$$\begin{cases} (p_3 - p_1)b_1(b_3 + b_4) = 0 \\ \qquad\qquad\qquad (\text{resp. } 2p_1b_1b_2 + (p_3 + p_1)b_1(b_3 + b_4) = 0) \\ (\overline{p_1'}p_1 - 1)b_1b_2 + (\overline{p_1'}p_3 - 1)b_1(b_3 + b_4) = 0. \end{cases}$$

This and $b_1b_2 \neq 0$ imply $p_3 = p_1$, and we get $p_1 = p_2 = p_3 = p_4$. However this contradicts (8.1). Hence the p_j's must be different from each other, i.e.

$$p_{1234} := \frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)} \neq 0,\ 1,\ \infty$$

under our assumption.

In the case when $\overline{p_1'}p_1 \neq 1$, set

$$F_{1k} := \frac{\overline{p_1'}p_k - 1}{p_k - p_1}.$$

Then $w_{1k}^* = F_{1k}w_{1k}$ ($k = 2, 3, 4$). Note here that

$$F_{1k} - F_{1\ell} = \frac{-(\overline{p_1'}p_1 - 1)(p_k - p_\ell)}{(p_k - p_1)(p_\ell - p_1)} \neq 0 \quad (k,\ \ell = 2,\ 3,\ 4; k \neq \ell)$$

holds now. By $w_1 \equiv w_{12} + w_{13} + w_{14} = 0$ and $w_1^* \equiv F_{12}w_{12} + F_{13}w_{13} + F_{14}w_{14} = 0$, we have

$$\frac{w_{12}}{w_{13}} = \frac{F_{13} - F_{14}}{F_{14} - F_{12}} = -p_{1234}.$$

By this equality, (4.5), (8.4), and $w_{14} = -w_{12} - w_{13}$, we have

$$\begin{aligned} p_{1234} &= \frac{(w_{14}w_{23} - w_{13}w_{24})w_{12}w_{34}}{(w_{14}w_{23} - w_{12}w_{34})w_{13}w_{24}} = \frac{({w_{14}}^2 - {w_{13}}^2){w_{12}}^2}{({w_{14}}^2 - {w_{12}}^2){w_{13}}^2} \\ &= \left(\frac{w_{12}}{w_{13}}\right)^3 \frac{(w_{12}/w_{13}) + 2}{2(w_{12}/w_{13}) + 1} = (-p_{1234})^3 \frac{-p_{1234} + 2}{-2p_{1234} + 1}, \end{aligned}$$

from which we get the equation

$$p_{1234}(p_{1234}-1)({p_{1234}}^2-p_{1234}+1)=0.$$

Since $p_{1234}\neq 0,\ 1$, we get (8.2). Now, by

$$\Big(\frac{(q_1-q_2)(q_3-q_4)}{(q_1-q_3)(q_2-q_4)}=\Big)\frac{q_3-q_4}{q_2-q_4}=p_{1234}\frac{w_{13}w_{24}}{w_{12}w_{34}} = p_{1234}\frac{{w_{13}}^2}{{w_{12}}^2}=\frac{1}{p_{1234}},$$

we also get (8.3).

In the case when $\overline{p_1'}p_1=1$, set

$$\check{F}_{1k}:=\frac{p_k+p_1}{p_k-p_1}\quad (k=2,\ 3,\ 4).$$

Then, by

$$\check{F}_{1k}-\check{F}_{1\ell}=\frac{-2(p_k-p_\ell)}{(p_k-p_1)(p_\ell-p_1)}\neq 0\qquad (k,\ell=2,3,4\ ;\ k\neq\ell),$$

$w_1\equiv w_{12}+w_{13}+w_{14}=0$ and $\check{w}_1\equiv \check{F}_{12}w_{12}+\check{F}_{13}w_{13}+\check{F}_{14}w_{14}=0$, we can derive the same conclusion (8.2) and (8.3). □

The condition (8.2) means that the arrangement of the p_j's is "conformal" with that of the vertices of a regular tetrahedron.

Now, for any nonnegative integer n_1, n_2 such that $n_1+n_2=4$, set

$$\begin{aligned}\mathcal{F}_{n_1,n_2}:=\{&\mathbf{z}=(z_1,\ z_1',\ z_2,\ z_2',\ z_3,\ z_3',\ z_4,\ z_4')\in\mathbf{C}^8\mid\\ &z_j=\overline{z_j'}\ (j=1,\ \ldots,\ n_1),\ |z_k|=|z_k'|=1\ (k=n_1+1,\ \ldots,\ 4)\}\\ &(\cong \mathbf{C}^{n_1}\times(\mathbf{S}^1)^{2n_2}).\end{aligned}$$

We regard $\mathcal{F}_{n_1,n_2}$ as a set of parameters

$$\mathbf{p}=(p_1,\ \overline{p_1'},\ p_2,\ \overline{p_2'},\ p_3,\ \overline{p_3'},\ p_4,\ \overline{p_4'}),$$

a collection of the directions of flux of 4-noids with n_1 (resp. n_2) ends of the first (resp. second) or third kind. Note here that the ratio of components of $\mathbf{a}$ is uniquely determined by $\mathbf{p}$ when $\dim\langle v_1,\ v_2,\ v_3,\ v_4\rangle=3$. Hence $\mathcal{F}_{n_1,n_2}$ can be regarded also as a parameter space of flux data of 4-noids with $\dim\langle v_1,\ v_2,\ v_3,\ v_4\rangle=3$.

Recall here that we fix $q_1 = \infty$, q_2, $q_3 \in \mathbf{C}$ ($q_2 \neq q_3$). Set $q := q_4$, and

$$F_{\mathbf{p}}(q) := \det A \cdot (q_2 - q_3)^2 (q - q_2)^2 (q - q_3)^2.$$

Then $F_{\mathbf{p}}(q)$ is a polynomial with respect to q whose degree is at most 4, and whose coefficients are polynomials with respect to $\mathbf{p}$ (and fixed q_2, q_3).

Lemma 8.5 *Assume*

$$-v_j \neq v_k,\ v_j \notin NP_-(p_k) \quad (j,\, k = 1,\, 2,\, 3,\, 4; j \neq k). \tag{8.5}$$

Then the degree of $F_{\mathbf{p}}(q)$ *is* 4*, and* $F_{\mathbf{p}}(q_j) \neq 0$ ($j = 2, 3$).

Proof. By direct computation, we have

$$\begin{aligned} &\text{the top term of } F_{\mathbf{p}}(q) = (\overline{p_1'}p_4 - 1)(\overline{p_4'}p_1 - 1)(\overline{p_2'}p_3 - 1)(\overline{p_3'}p_2 - 1)q^4, \\ &F_{\mathbf{p}}(q_2) = (\overline{p_1'}p_3 - 1)(\overline{p_3'}p_1 - 1)(\overline{p_2'}p_4 - 1)(\overline{p_4'}p_2 - 1)(q_2 - q_3)^2, \\ &F_{\mathbf{p}}(q_3) = (\overline{p_1'}p_2 - 1)(\overline{p_2'}p_1 - 1)(\overline{p_3'}p_4 - 1)(\overline{p_4'}p_3 - 1)(q_2 - q_3)^2. \end{aligned}$$

Hence we get our assertion. □

Set

$$\begin{aligned} F_1(\mathbf{p}) &:= \prod_{j=1}^{4} \Big\{ \sum\nolimits_{\{k,\ell,m\} = \{1,2,3,4\} \setminus \{j\}} (\overline{p_k' p_\ell' p_m + p_m'})(p_k - p_\ell) \Big\}, \\ F_2(\mathbf{p}) &:= (p_1 - p_2)^2 (p_3 - p_4)^2 - (p_1 - p_2)(p_3 - p_4)(p_1 - p_3)(p_2 - p_4) \\ &\quad + (p_1 - p_3)^2 (p_2 - p_4)^2, \\ F_3(\mathbf{p}) &:= \prod_{j \neq k} (\overline{p_j'} p_k - 1). \end{aligned}$$

$F_1(\mathbf{p}) \neq 0$ (resp. $F_2(\mathbf{p}) = 0$, $F_3(\mathbf{p}) \neq 0$) is satisfied if and only if the condition (8.1) (resp. (8.2), (8.5)) holds.

Let $\mathbf{p}$ be an element of $\mathcal{F}_{n_1,n_2}$ satisfying $F_i(\mathbf{p}) \neq 0$ ($i = 1, 2, 3$).

Since $F_3(\mathbf{p}) \neq 0$, by Lemma 8.5, $F_{\mathbf{p}}(q) = 0$ is an equation of degree 4. Let q be a solution of $F_{\mathbf{p}}(q) = 0$. By Lemma 8.5 again, we see that $q \neq q_2$, q_3. Hence we have $\det A(q) = 0$.

Let $\mathbf{b}(q) = {}^t(b_1(q), b_2(q), b_3(q), b_4(q))$ be the first column vector of $\tilde{A}(q)$ multiplied by $(q_2 - q_3)^2 (q - q_2)^2 (q - q_3)^2$. Then all the $b_j(q)$'s are polynomials with respect to $\mathbf{p}$ and q (and q_2, q_3). Since $F_1(\mathbf{p}) \neq 0$, all components of $\tilde{A}(q)$ do not vanish, and hence $b_j(q) \neq 0$ ($j = 1, 2, 3, 4$). Note here that $A(q)\mathbf{b}(q) = \mathbf{0}$ implies $\mathbf{w}^*(q) = -\operatorname{diag}[b_1(q), b_2(q), b_3(q), b_4(q)] A(q)\mathbf{b}(q) =$

$\mathbf{0}$.

Since $F_2(\mathbf{p}) \neq 0$, by Lemma 8.4, we have $\mathbf{w}(q) \neq \mathbf{0}$. Hence, by Lemma 8.1, there exists a nonzero complex number $t(q)$ satisfying $t(q)\mathbf{w}(q) = \mathbf{a}$, and $(\mathbf{q}, \mathbf{b}) = (\mathbf{q}, \sqrt{t(q)}\mathbf{b}(q))$ satisfies (4.2).

Define $P(z)$ and $Q(z)$ as in § 4 with this $(\mathbf{q}, \mathbf{b})$. Let $G_{\mathbf{p}}(q)$ be the resultant of $P(z)$ and $Q(z)$, $H_{\mathbf{p}}(q)$ the remainder of the integral quotient of $G_{\mathbf{p}}(q)^{12}$, the twelfth power of $G_{\mathbf{p}}(q)$, by $F_{\mathbf{p}}(q)$. Then $H_{\mathbf{p}}(q)$ is a polynomial with respect to q whose degree is at most 3. Let $H_d(\mathbf{p})$ be its coefficients of the term of degree d ($d = 0, 1, 2, 3$). $H_d(\mathbf{p})$'s are rational functions with respect to $\mathbf{p}$ (and fixed q_2, q_3).

If $H_d(\mathbf{p}) \neq 0$ for some d, then, for at least one solution q of $F_{\mathbf{p}}(q)$, the degree of $g(z) = P(z)/Q(z)$ is 3, i.e. the maximal map given by (4.1) has no branch points, and hence we get a desired solution of Problem 3.1.

In the same way as in [9], we have the following lemma.

Lemma 8.6 *If there exists a nonbranched* 4*-noid whose flux data is given by one particular element* $\mathbf{p} \in \mathcal{F}_{n_1,n_2}$*, then for almost all other elements* $\mathbf{p} \in \mathcal{F}_{n_1,n_2}$*, there exists a nonbranched* 4*-noid realizing* $\mathbf{p}$ *as its flux data.*

Proof. If there exists a nonbranched 4-noid whose flux data is given by an element $\mathbf{p} \in \mathcal{F}_{n_1,n_2}$, then $H_d \not\equiv 0$ on $\mathbf{C}^8$ for some $d = 0, 1, 2, 3$. Since $\mathcal{F}_{n_1,n_2} \cong \mathbf{C}^{n_1} \times (\mathbf{S}^1)^{2n_2}$, it also holds on $\mathcal{F}_{n_1,n_2}$. Indeed, $H_d \circ \varphi^{-1}$ is a nonconstant meromorphic function on $\mathbf{C}^{n_1} \times \mathbf{R}^{2n_2}$, where φ is a local coordinate function on $\mathcal{F}_{n_1,n_2}$ given by

$$\varphi(\mathbf{z}) := (z_1, \ldots, z_{n_1}, \varphi_0(z_{n_1+1}), \varphi_0(z'_{n_1+1}), \ldots, \varphi_0(z_4), \varphi_0(z'_4)),$$
$$\varphi_0(z) := \sqrt{-1}\frac{1-z}{1+z}, \quad {\varphi_0}^{-1}(x) := \frac{\sqrt{-1}-x}{\sqrt{-1}+x}.$$

Hence the subset

$$\mathcal{F}'_{n_1,n_2} := \{\mathbf{p} \in \mathcal{F}_{n_1,n_2} \mid F_i(\mathbf{p}) \neq 0 \quad \text{for any} \quad i = 1, 2, 3, \quad H_d(\mathbf{p}) \neq 0 \quad \text{for some} \quad d = 0, 1, 2, 3.\}$$

is open and dense in $\mathcal{F}_{n_1,n_2}$. By the consideration before this lemma, there exists at least one 4-noid which is a solution of Problem 3.1 for any $\mathbf{p} \in \mathcal{F}'_{n_1,n_2}$. □

Now, we present examples of nonbranched 4-noids, i.e. sampling points for $H_d(\mathbf{p}) \neq 0$ for some d, for each pair (n_1, n_2).

Example 8.7 Let φ_1, φ_2, φ_3, φ_4 be nonzero null vectors in $\mathbf{R}^{2,1}$ such that $\sum_{j=1}^4 \varphi_j = \mathbf{0}$. By a suitable Lorentzian transformation, they can be transformed to $\check{a}\check{v}(1)$, $\check{a}\check{v}(-1)$, $-\check{a}\check{v}(\zeta)$, $-\check{a}\check{v}(-\zeta)$ for some $\zeta \in \mathbf{S}^1$. If this collection is realized as the flux vectors of some 4-noid, then, by Theorem 6.2 (6.8), $\zeta \neq \pm 1$.

Now, solving the equation (4.2) with

$$\begin{cases} p_1 = p_1' = 1,\ p_2 = p_2' = -1,\ p_3 = p_3' = \zeta,\ p_4 = p_4' = -\zeta, \\ a_1 = a_2 = \check{a},\ a_3 = a_4 = -\check{a},\ \check{a} \neq 0, \end{cases} \tag{8.6}$$

we get the following solution

$$\begin{cases} q_1 = 1,\ q_2 = -1,\ q_3 = q,\ q_4 = -q, \\ b_1 = b_2 = \sqrt{\dfrac{\check{a}(q-\overline{\zeta})}{q+\overline{\zeta}}},\ b_3 = b_4 = -\dfrac{b_1(q^2-1)}{2\zeta(q-\overline{\zeta})}, \end{cases} \tag{8.7}$$

where q is an arbitrary solution of the equation

$$F(q) := (q^2-1)^2 - 4\zeta q(q-\overline{\zeta})^2 = 0. \tag{8.8}$$

By $\det A(q) = 0$, we also have another equation $(q^2-1)^2 - 4\overline{\zeta} q(q-\zeta)^2 = 0$. However, for any solution q of this equation, q^{-1} is a solution of (8.8), and $\hat{\mathbf{C}} \setminus \{1, -1, q^{-1}, -q^{-1}\}$ is conformally equivalent to $\hat{\mathbf{C}} \setminus \{1, -1, q, -q\}$. Hence we have only to consider the equation (8.8) to classify the solutions of (4.2) with (8.6).

For any solution q of (8.8), it is clear that $q \neq 0$. Moreover, by $\zeta \neq \pm 1$, we see that $q \neq \pm 1, \overline{\zeta}$, and hence $b_1, b_3 \neq 0$.

Now, $g(z)$ is given by

$$g(z) = \frac{(b_1 + b_3\zeta q)z^2 - (b_1 q + b_3 \zeta) q}{z\{(b_1+b_3)z^2 - (b_1 q^2 + b_3)\}}.$$

Since $q \neq 0$ and

$$\begin{aligned} &(b_1 + b_3\zeta q)(b_1 q^2 + b_3) - (b_1 q + b_3\zeta) q (b_1 + b_3) \\ &\qquad = \zeta(q-\overline{\zeta})(q^2-1) b_1 b_3 \neq 0, \end{aligned}$$

$\deg g \leq 2$ holds only if $b_1 + b_3 = 0$ or $b_1 q + b_3 \zeta = 0$. However, both of these equalities do not hold in our case. Indeed, by (8.8), we have

$$\frac{b_3}{b_1} = -\frac{q^2-1}{2\zeta(q-\overline{\zeta})} = -\frac{2q(q-\overline{\zeta})}{q^2-1},$$

from which it follows that

$$b_1 + b_3 = b_1\Big(1 + \frac{b_3}{b_1}\Big)$$
$$= \begin{cases} b_1\Big\{1 - \dfrac{q^2-1}{2\zeta(q-\bar{\zeta})}\Big\} = \dfrac{-b_1}{2\zeta(q-\bar{\zeta})}(q^2 - 2\zeta q + 1), \\ b_1\Big\{1 - \dfrac{2q(q-\bar{\zeta})}{q^2-1}\Big\} = \dfrac{-b_1}{q^2-1}(q^2 - 2\bar{\zeta} q + 1). \end{cases}$$

Hence, if $b_1 + b_3 = 0$ holds, then we have $q^2 - 2\zeta q + 1 = 0 = q^2 - 2\bar{\zeta} q + 1$, which implies $2\bar{\zeta}(\zeta^2 - 1)q = 0$. This is not our case. In the same way, we can show that $b_1 q + b_3 \zeta = 0$ also does not hold.

Hence, for any $\zeta \in \mathbf{S}^1 \setminus \{\pm 1\}$, the solution (8.7) gives the Weierstrass data of nonbranched 4-noids satisfying (3.3) with (8.6).

Since

$$F(q) = \frac{1}{4}(q-\zeta)F'(q) - 3(\zeta^2 - 1)q(q - \bar{\zeta}),$$

the equation (8.8) has no multiple root, and hence we get 4 distinct solutions for any $\zeta \in \mathbf{S}^1 \setminus \{\pm 1\}$. These solutions are essentially different from each other, since $w_{12} = -\check{a}(q-\bar{\zeta})/(q+\bar{\zeta})$ takes different values. This completes the classification of 4-noids all of whose ends are of the third kind.

Since each data $\mathbf{p}$ given by (8.6) is an element of $\mathcal{F}_{n_1,n_2}$ for any n_1, n_2 such that $n_1 + n_2 = 4$, by Lemma 8.6, we conclude the following:

Theorem 8.8 *For generic flux data satisfying* (3.2) *with* $n = 4$, *there exists a corresponding nonbranched* 4*-noid* $X\colon \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3, q_4\} \to \mathbf{R}^{2,1}$ *satisfying* (3.3).

References

[1] Cosín C. and Ros A., *A Plateau problem at infinity for properly immersed minimal surfaces with finite total curvature*. Indiana Univ. Math. J. **50** (2001), 847–879.

[2] Fernández I., López F. and Souam R., *The space of complete embedded maximal surfaces with isolated singularities in the* 3*-dimensional Lorentz-Minkowski space*. Math. Ann. **332** (2005), 605–643.

[3] Fujimori S., Saji K., Umehara M. and Yamada K., *Cuspidal cross caps and singularities of maximal surfaces*. `arXiv:math. DG/0510366`.

[4] Imaizumi T., *Maximal surfaces with conelike singularities of finite type*. Kobe J. Math. **18** (2001), 51–60.

[5] Imaizumi T., *Maximal surfaces with simple ends.* Kyushu J. Math. **58** (2004), 59–70.
[6] Kato S., *Construction of n-end catenoids with prescribed flux.* Kodai Math. J. **18** (1995), 86–98.
[7] Kato S., *On the weights of end-pairs in n-end catenoids of genus zero* II. Kyushu J. Math. **61** (2007), 275–319.
[8] Kato S. and Nomura K., *On the weights of end-pairs in n-end catenoids of genus zero.* Osaka J. Math. **41** (2004), 507–532.
[9] Kato S., Umehara M. and Yamada K., *An inverse problem of the flux for minimal surfaces.* Indiana Univ. Math. J. **46** (1997), 529–559.
[10] Kato S., Umehara M. and Yamada K., *General existence of minimal surfaces of genus zero with catenoidal ends and prescribed flux.* Comm. Anal. Geom. **8** (2000), 83–114.
[11] Kato S., Umehara M. and Yamada K., *General existence of minimal surfaces with prescribed flux.* II. Topics in complex analysis, differential geometry and mathematical physics (St. Konstantin, 1996), World Sci. Publishing, River Edge, NJ, 1997, pp. 116–135.
[12] Kobayashi O., *Maximal surfaces in the 3-dimensional Minkowski space L^3.* Tokyo J. Math. **6** (1983), 297–309.
[13] Kobayashi O., *Maximal surfaces with conelike singularities.* J. Math. Soc. Japan **36** (1984), 609–617.
[14] López F.J., *The classification of complete minimal surfaces with total curvature greater than -12π.* Trans. Amer. Math. Soc. **334** (1992), 49–74.
[15] López F.J., López R. and Souam R., *Maximal surfaces of Riemann type in Lorentz-Minkowski space L^3.* Michigan J. Math. **47** (2000), 469–497.
[16] Pérez J., *Riemann bilinear relations on minimal surfaces.* Math. Ann. **310** (1998), 307–332.
[17] Rosenberg H. and Toubiana É., *Complete minimal surfaces and minimal herissons.* J. Differential Geom. **28** (1988), 115–132.
[18] Umehara M. and Yamada K., *Maximal surfaces with singularities in Minkowski space.* Hokkaido Math. J. **35** (2006), 13–40.

T. Imaizumi
TIERRA. COM
Kobe Harbor Land Center Bldg. 19F
1-3-3 Higashi-Kawasaki-cho, Chuo-ku
Kobe, 650-0044, Japan

S. Kato
Department of Mathematics
Osaka City University
3-3-138 Sugimoto, Sumiyoshi-ku
Osaka, 558-8585, Japan
E-mail: shinkato@sci.osaka-cu.ac.jp