Hokkaido Mathematical Journal Vol. 37 (2008) p. 507–530

Uniqueness of meromorphic functions sharing two or three sets

(The authors dedicate the paper to their teacher respected Prof. I. Lahiri.)

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(Received April 25, 2007)

Abstract. In this paper, we employ the notion of weighted sharing to consider the problem of uniqueness of meromorphic functions when they share two or three sets. Our results not only improve the results of Qiu and Fang [21], Lin and Yi [27], Lin and Yi [28], Fang and Lahiri [6] but also supplement the result of Lahiri [13] in a new direction and consequently provide an answer to the question of Gross [7].

Key words: meromorphic functions, uniqueness, weighted sharing, shared set.

1. Introduction, Definitions and Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside a possible exceptional set of finite linear measure. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z\colon f(z) -a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S) = \bigcup_{a\in S}\{z\colon f(z)-a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

In 1976 F. Gross [7] posed the following question:

Question A ([7]) *Can one find two finite sets S_j $(j = 1, 2)$ such that any two nonconstant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

Now it is natural to ask the following question.

2000 Mathematics Subject Classification : 30D35.

Question B ([19]) *Can one find two finite sets S_j $(j = 1, 2)$ such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

Also for meromorphic functions in [22] the following question was asked.

Question C ([22]) *Can one find three finite sets S_j $(j = 1, 2, 3)$ such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical ?*

Perhaps to the knowledge of the authors during the last several years the possible answer of *Question B* {cf. [2], [3], [4], [6], [9], [13], [17], [19], [22], [25], [28], [29]} is studied by more authors than that of *Question C* {cf. [1], [5], [14], [18], [21], [22], [23], [27]} and continuous efforts is being put in to relax the hypothesis of the results.

In 1998 in the direction of *Question B* improving all the previous results Lahiri [9] proved the following result.

Theorem A ([9]) *Let $S_1 = \{z\colon z^n + az^{n-1} + b = 0\}$, and $S_2 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 8) is an integer. If f and g are two nonconstant meromorphic functions having no simple poles such that $E_f(S_i) = E_g(S_i)$ for $i = 1, 2$ then $f \equiv g$.*

Afterwards Fang and Lahiri improved *Theorem A* as follows.

Theorem B ([6]) *Let S_1, and S_2, be defined as in* Theorem A *and n (≥ 7) is an integer. If f and g are two nonconstant meromorphic functions having no simple poles such that $E_f(S_i) = E_g(S_i)$ for $i = 1, 2$ then $f \equiv g$.*

It should be noted that if two meromorphic functions f and g have no simple pole then clearly $\Theta(\infty, f) \geq 1/2$ and $\Theta(\infty, g) \geq 1/2$.

In 2001 an idea of gradation of sharing of values and sets known as weighted sharing was introduced in [11, 12] which measure how close a shared value is to being shared IM or to being shared CM. We now give the definition.

Definition 1.1 ([11, 12]) Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2 ([11]) Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $E_f(S, k) = \bigcup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

With the notion of weighted sharing Lahiri improved *Theorem B* as follows.

Theorem C ([13]) *Let $S_1 = \{z\colon z^n + az^{n-1} + b = 0\}$, and $S_2 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 7) is an integer. If f and g are two nonconstant meromorphic functions such that $E_f(S_1, 2) = E_g(S_1, 2)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $\Theta(\infty; f) + \Theta(\infty; g) > 1$ then $f \equiv g$.*

Definition 1.3 ([27, 28]) We put

$$\delta_{1)}(\infty; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N(r, \infty; f \mid= 1)}{T(r, f)}$$

Clearly $0 \leq 1/2\delta_{1)}(\infty; f) \leq \Theta(\infty; f) \leq \delta_{1)}(\infty; f)$.

Recently Yi and Lin [28] have improved *Theorem B* and obtained the following result.

Theorem D ([28]) *Let S_1, and S_2, be defined as in* Theorem C. *If f and g are two nonconstant meromorphic functions such that $E_f(S_1, \infty) = E_g(S_1, \infty)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $\delta_{1)}(\infty; f) > 9/14$ then $f \equiv g$.*

In the direction of *Question C* Fang and Xu [5] proved the following result.

Theorem E ([5]) *Let $S_1 = \{z\colon z^3 - z^2 - 1 = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$. Suppose that f and g are two nonconstant meromorphic functions such that $\Theta(\infty; f) > 1/2$ and $\Theta(\infty; g) > 1/2$. If $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2, 3$ then $f \equiv g$.*

Dealing with the question of Gross, Qiu and Fang [21] proved the following theorem.

Theorem F ([21]) *Let $n \geq 3$ be a positive integer $S_1 = \{z\colon z^n - z^{n-1} - 1 = 0\}$, $S_2 = \{0\}$, and $S_3 = \{\infty\}$. Let f and g be two nonconstant meromorphic functions whose poles are of multiplicities at least 2. If $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2, 3$ then $f \equiv g$.*

They also gave example to show that the condition that the poles of $f(z)$ and $g(z)$ are of multiplicities at least 2 can not be removed in *Theorem F*. In 2004 Yi and Lin [27] proved the following theorems.

Theorem G ([27]) *Let $S_1 = \{z\colon z^n + az^{n-1} + b = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 3) is an integer. If for two nonconstant meromorphic functions f and g $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2, 3$ and $\delta_{1)}(\infty; f) > 5/6$ then $f \equiv g$.*

In the same paper Yi and Lin [27] also proved the following result.

Theorem H ([27]) *Let $S_1 = \{z\colon z^n + az^{n-1} + b = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 4) is an integer. If for two nonconstant meromorphic functions f and g, $E_f(S_i) = E_g(S_i)$ for $i = 1, 2, 3$ and $\Theta(\infty; f) > 0$ then $f \equiv g$.*

Yi and Lin [27] remarked that the assumption $E_f(S_2, \infty) = E_g(S_2, \infty)$ in *Theorems G* and *H* can be relaxed to $E_f(S_2, 0) = E_g(S_2, 0)$ and also the assumption $\Theta(\infty; f) > 0$ in *Theorem H* can be replaced by $\delta_{1)}(\infty; f) > 0$.

It is to be noted that to deal with *Question B* and *Question C* none of the previous authors considered the situation of further relaxation of the nature of sharing the set $\{\infty\}$ in the aforesaid theorems. So it will be interesting to consider the following problem. *Is it possible to further relax the nature of sharing the set $\{\infty\}$ without increasing the cardinalities of the other range sets ?*

In the paper we deal with this problem.

We now state the following seven theorems which are the main results of the paper.

Theorem 1.1 *Let* S_1*, and* S_2*, be defined as in* Theorem C*. If* f *and* g *are two nonconstant meromorphic functions having no simple pole such that* $E_f(S_1, 2) = E_g(S_1, 2)$*,* $E_f(S_2, 2) = E_g(S_2, 2)$ *then* $f \equiv g$*.*

Theorem 1.2 *Let* S_1*, and* S_2*, be defined as in* Theorem C*. If* f *and* g *are two nonconstant meromorphic functions such that* $E_f(S_1, 3) = E_g(S_1, 3)$*,* $E_f(S_2, \infty) = E_g(S_2, \infty)$ *and* $\delta_{1)}(\infty; f) > 14/3n$ *then* $f \equiv g$*.*

Remark 1.1 *Theorem 1.1* and *Theorem 1.2* are respectively the improvements of *Theorem B* and *Theorem D*.

Theorem 1.3 *Let* S_1*, and* S_2*, be defined as in* Theorem C*. If* f *and* g *are two nonconstant meromorphic functions such that* $E_f(S_1, 3) = E_g(S_1, 3)$*,* $E_f(S_2, \infty) = E_g(S_2, \infty)$ *and* $\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) > 8/(n-1)$ *then* $f \equiv g$*.*

Theorem 1.4 *Let* S_1*, and* S_2*, be defined as in* Theorem C*. If* f *and* g *are two nonconstant meromorphic functions such that* $E_f(S_1, 2) = E_g(S_1, 2)$*,* $E_f(S_2, \infty) = E_g(S_2, \infty)$ *and* $\delta_{1)}(\infty; f) > 31/6n$ *then* $f \equiv g$*.*

Theorem 1.5 *Let* S_1*,* S_2 *and* S_3 *be defined as in* Theorem G*. If for two nonconstant meromorphic functions* f *and* g *having no simple pole* $E_f(S_1, 5) = E_g(S_1, 5)$*,* $E_f(S_2, 1) = E_g(S_2, 1)$ *and* $E_f(S_3, \infty) = E_g(S_3, \infty)$ *then* $f \equiv g$*.*

Remark 1.2 *Theorem 1.5* improves *Theorem F*.

Theorem 1.6 *Let* S_1*,* S_2 *and* S_3 *be defined as in* Theorem G*. If* f *and* g *are two nonconstant meromorphic functions such that* $E_f(S_1, 6) = E_g(S_1, 6)$*,* $E_f(S_2, 0) = E_g(S_2, 0)$ *and* $E_f(S_3, \infty) = E_g(S_3, \infty)$ *and* $\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) > 5/n$ *then* $f \equiv g$*.*

Theorem 1.7 *Let* S_1*,* S_2 *and* S_3 *be defined as in* Theorem H*. If* f *and* g *are two nonconstant meromorphic functions such that* $E_f(S_1, 4) = E_g(S_1, 4)$*,* $E_f(S_2, 0) = E_g(S_2, 0)$ *and* $E_f(S_3, 6) = E_g(S_3, 6)$ *and* $\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) > 0$ *then* $f \equiv g$*.*

Remark 1.3 *Theorem 1.6* and *Theorem 1.7* are respectively the improvements of *Theorem G* and *Theorem H*.

Following example shows that the condition $\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) > 0$

is sharp in *Theorem 1.7*.

Example 1.1 Let

$$g=-a\frac{e^{(n-1)z}-1}{e^{nz}-1}, \quad f(z)=e^z g(z)$$

and S_i's be as in *Theorem 1.7*. Then $E_f(S_i,\infty)=E_g(S_i,\infty)$ for $i=1,2,3$ because $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$ and $f\equiv e^z g$. Also $\delta_{1)}(\infty;f)+\delta_{1)}(\infty;g)=0$ and $f\not\equiv g$.

Though for the standard definitions and notations of the value distribution theory we refer to [8], we now explain some notations which are used in the paper.

Definition 1.4 ([10]) For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $N(r,a;f\mid=1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r,a;f\mid\leq m)$ $(N(r,a;f\mid\geq m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m where each a point is counted according to its multiplicity.

$\overline{N}(r,a;f\mid\leq m)$ $(\overline{N}(r,a;f\mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f\mid<m)$, $N(r,a;f\mid>m)$, $\overline{N}(r,a;f\mid<m)$ and $\overline{N}(r,a;f\mid>m)$ are defined analogously.

Definition 1.5 We denote by $\overline{N}(r,a;f\mid=k)$ the reduced counting function of those a-points of f whose multiplicities is exactly k, where $k\geq 2$ is an integer.

Definition 1.6 Let f and g be two nonconstant meromorphic functions such that f and g share (a,k) where $a\in\mathbb{C}\cup\{\infty\}$. Let z_0 be a a-point of f with multiplicity p, a a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p>q$, by $\overline{N}_E^{(k+1}(r,a;f)$ the counting function of those a-points of f and g where $p=q\geq k+1$; each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,a;g)$ and $\overline{N}_E^{(k+1}(r,a;g)$. Clearly $\overline{N}_E^{(k+1}(r,a;f)=\overline{N}_E^{(k+1}(r,a;g)$.

Definition 1.7 ([12]) We denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f)$ $+\overline{N}(r,a;f\mid\geq 2)$.

Definition 1.8 ([11, 12]) Let f, g share a value a IM. We denote by $\overline{N}_*(r,\, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,\, a; f,\, g) \equiv \overline{N}_*(r,\, a; g,\, f)$ and $\overline{N}_*(r,\, a; f,\, g) = \overline{N}_L(r,\, a; f)$ $+\overline{N}_L(r,\, a; g)$.

Definition 1.9 ([15]) Let $a,\, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,\, a; f \mid g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.10 ([15]) Let $a,\, b_1,\, b_2,\, \ldots,\, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,\, a; f \mid g \neq b_1,\, b_2,\, \ldots,\, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1,\, 2,\, \ldots,\, q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two nonconstant meromorphic functions defined as follows.

$$F = \frac{f^{n-1}(f+a)}{-b}, \quad G = \frac{g^{n-1}(g+a)}{-b}. \tag{2.1}$$

Henceforth we shall denote by H, Φ and V the following three functions

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big),$$
$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}$$

and

$$V = \Big(\frac{F'}{F-1} - \frac{F'}{F}\Big) - \Big(\frac{G'}{G-1} - \frac{G'}{G}\Big) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Lemma 2.1 ([12], Lemma 1) *Let F, G share $(1,1)$ and $H \not\equiv 0$. Then*

$$N(r,1; F \mid= 1) = N(r,1; G \mid= 1) \le N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2 *Let $S_1 = \{z\colon z^n + az^{n-1} + b = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has* *no repeated root,* n (≥ 3) *is an integer and* F*,* G *be given by* (2.1). *If for two nonconstant meromorphic functions* f *and* g $E_f(S_1, 0) = E_g(S_1, 0)$*,* $E_f(S_2, 0) = E_g(S_2, 0)$*,* $E_f(S_3, 0) = E_g(S_3, 0)$ *and* $H \not\equiv 0$ *then*

$$\begin{aligned} N(r, H) \leq & \overline{N}_*(r, 0, f, g) + \overline{N}(r, 0; f + a \mid\geq 2) \\ & + \overline{N}(r, 0; g + a \mid\geq 2) + \overline{N}_*(r, 1; F, G) \\ & + \overline{N}_*(r, \infty; f, g) + \overline{N}_0(r, 0; F^{'}) + \overline{N}_0(r, 0; G^{'}), \end{aligned}$$

where $\overline{N}_0(r, 0; F^{'})$ *is the reduced counting function of those zeros of* $F^{'}$ *which are not the zeros of* $F(F-1)$ *and* $\overline{N}_0(r, 0; G^{'})$ *is similarly defined.*

Proof. Since $E_f(S_1, 0) = E_g(S_1, 0)$ it follows that F and G share $(1, 0)$. We can easily verify that possible poles of H occur at (i) those zeros of f and g whose multiplicities are distinct from the multiplicities of the corresponding zeros of g and f respectively, (ii) multiple zeros of $f + a$ and $g + a$, (iii) those poles of f and g whose multiplicities are distinct from the multiplicities of the corresponding poles of g and f respectively, (iv) those 1-points of F and G with different multiplicities, (v) zeros of $F^{'}$ which are not the zeros of $F(F-1)$, (v) zeros of $G^{'}$ which are not zeros of $G(G-1)$. Since H has only simple poles, the lemma follows from above. This proves the lemma. $\square$

Lemma 2.3 ([15], Lemma 4) *If two nonconstant meromorphic functions* F *and* G *share* $(1, 0)$*,* $(\infty, 0)$ *and* $H \not\equiv 0$ *then*

$$\begin{aligned} N(r, H) \leq & \overline{N}(r, 0; F \mid\geq 2) + \overline{N}(r, 0; G \mid\geq 2) \\ & + \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) \\ & + \overline{N}_0(r, 0; F^{'}) + \overline{N}_0(r, 0; G^{'}), \end{aligned}$$

where $\overline{N}_0(r, 0; F^{'})$ *is the reduced counting function of those zeros of* $F^{'}$ *which are not the zeros of* $F(F-1)$ *and* $\overline{N}_0(r, 0; G^{'})$ *is similarly defined.*

Lemma 2.4 ([20]) *Let* f *be a nonconstant meromorphic function and let*

$$R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f *with constant coefficients* $\{a_k\}$ *and* $\{b_j\}$ *where* $a_n \neq 0$ *and* $b_m \neq 0$ *Then*

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 2.5 *Let* F *and* G *be given by* (2.1). *If* f, g *share* $(0, 0)$ *and* 0 *is not a Picard exceptional value of* f *and* g. *Then* $\Phi \equiv 0$ *implies* $F \equiv G$.

Proof. Suppose

$$\Phi \equiv 0.$$

Then by integration we obtain

$$F - 1 \equiv C(G - 1).$$

It is clear that if z_0 is a zero of f then it is a zero of g. So from (2.1) it follows that $F(z_0) = 0$ and $G(z_0) = 0$. So $C = 1$ and hence $F \equiv G$. □

Lemma 2.6 *Let* F *and* G *be given by* (2.1), $n \geq 3$ *an integer and* $\Phi \not\equiv 0$. *If* F, G *share* $(1, m)$ *and* f, g *share* $(0, p)$, (∞, k), *where* $0 \leq m \leq \infty$, $0 \leq p < \infty$ *and* $0 \leq k \leq \infty$ *then*

$$\begin{aligned}&[(n-1)p + n - 2]\overline{N}(r, 0; f \mid\geq p+1)\\ &\qquad\qquad \leq \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) + S(r, f) + S(r, g).\end{aligned}$$

Proof. Suppose 0 is an e.v.P. (exceptional value Picard) of f and g then the lemma follows immediately.
Next suppose 0 is not an e.v.P. of f and g. Let z_0 be a zero of f with multiplicity q and a zero of g with multiplicity r. From (2.1) we know that z_0 is a zero of F with multiplicity $(n-1)q$ and a zero of G with multiplicity $(n-1)r$. We note that F and G have no zero of multiplicity t where $(n-1)p < t < (n-1)(p+1)$. So from the definition of Φ it is clear that z_0 is a zero of Φ with multiplicity at least $(n-1)(p+1) - 1$. So we have

$$\begin{aligned}&[(n-1)p + n - 2]\overline{N}(r, 0; f \mid\geq p+1)\\ &= [(n-1)p + n - 2]\overline{N}(r, 0; g \mid\geq p+1)\\ &= [(n-1)p + n - 2]\overline{N}\left(r, 0; F \mid\geq n(p+1)\right)\\ &\leq N(r, 0; \Phi)\\ &\leq N(r, \infty; \Phi) + S(r, f) + S(r, g)\\ &\leq \overline{N}_*(r, \infty; F, G) + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).\end{aligned}$$

□

Lemma 2.7 *Let F and G be given by* (2.1) *and f, g share* $(\infty,\, 0)$ *and ∞ is not a Picard exceptional value of f and g. Then $V \equiv 0$ implies $F \equiv G$*

Proof. Suppose

$$V \equiv 0.$$

Then by integration we obtain

$$1-\frac{1}{F} \equiv A\Big(1-\frac{1}{G}\Big).$$

It is clear that if z_0 is a pole of f then it is a pole of g. Hence from the definition of F and G we have $1/F(z_0)=0$ and $1/G(z_0)=0$. So $A=1$ and hence $F \equiv G$. □

Lemma 2.8 *Let F, G be given by* (2.1) *and $V \not\equiv 0$. If f, g share* $(0,\, 0)$, $(\infty,\, k)$, *where $0 \le k < \infty$ and F, G share* $(1,\, m)$ *then the poles of F and G are the zeros of V and*

(i)
$$\begin{aligned}&(nk+n-1)\overline{N}(r,\,\infty;f)\\ \le\,& nkN(r,\,\infty;f\mid=1)+n(k-1)\overline{N}(r,\,\infty;f\mid=2)\\ &+\cdots+n\overline{N}(r,\,\infty;f\mid=k)\\ &+\overline{N}_*(r,\,0;f,\,g)+\overline{N}(r,\,0;f+a)+\overline{N}(r,\,0;g+a)\\ &+\overline{N}_L(r,\,1;F)+\overline{N}_L(r,\,1;G)+S(r,\,f)+S(r,\,g).\end{aligned}$$

(ii)
$$\begin{aligned}&(2n-1)\overline{N}(r,\,\infty;f)\\ \le\,&\overline{N}_*(r,\,0;f,\,g)+\overline{N}(r,\,0;f+a)+\overline{N}(r,\,0;g+a)+\overline{N}_L(r,\,1;F)\\ &+\overline{N}_L(r,\,1;G)+nN(r,\,\infty;f\mid=1)+S(r,\,f)+S(r,\,g).\end{aligned}$$

Similar expressions hold for g.

Proof. Suppose ∞ is an e.v.P. of f and g then the lemma follows immediately.

Next suppose ∞ is not an e.v.P. of f and g. Since f, g share $(\infty;k)$, it follows that F, G share $(\infty;nk)$ and so a pole of F with multiplicity p $(\ge nk+1)$ is a pole of G with multiplicity r $(\ge nk+1)$ and vice versa. We note that F and G have no pole of multiplicity q where $nk<q<nk+n$. Also any common pole of F and G of multiplicity $p \le nk$ is a zero of V of multiplicity $p-1$. Since

$$\begin{aligned}&(n-1)N(r,\,\infty;f\mid=1)+(2n-1)\overline{N}(r,\,\infty;f\mid=2)+\dots\\ &+(nk-1)\overline{N}(r,\,\infty;f\mid=k)+(nk+n-1)\overline{N}(r,\,\infty;f\mid\ge k+1)\\ =\,&(nk+n-1)\overline{N}(r,\,\infty;f)-nkN(r,\,\infty;f\mid=1)\end{aligned}$$

$$- n(k-1)\overline{N}(r, \infty; f \mid= 2) - \cdots - n\overline{N}(r, \infty; f \mid= k)$$

using *Lemma 2.4* we get from the definition of V

$$\begin{aligned}
&(n-1)\overline{N}(r, \infty; f)\\
&\leq (2n-1)\overline{N}(r, \infty; f) - n\overline{N}(r, \infty; f \mid= 1)\\
&\leq (n-1)N(r, \infty; f \mid= 1) + (2n-1)\overline{N}(r, \infty; f \mid= 2) + \cdots\\
&\quad + (nk-1)\overline{N}(r, \infty; f \mid= k) + (nk+n-1)\overline{N}(r, \infty; f \mid\geq k+1)\\
&\leq N(r, 0; V)\\
&\leq N(r, \infty; V) + S(r, f) + S(r, g)\\
&\leq \overline{N}_*(r, 0; f, g) + \overline{N}(r, 0; f+a) + \overline{N}(r, 0; g+a)\\
&\quad + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).
\end{aligned}$$

Now (i) and (ii) follows from the above discussion. □

Lemma 2.9 *Let F, G be given by* (2.1) *and $V \not\equiv 0$. If f, g share (∞, k), where $0 \leq k < \infty$ and F, G share $(1, m)$ then the poles of F and G are the zeros of V and*

(i) $$\begin{aligned}
&(nk+n-1)\overline{N}(r, \infty; f)\\
&\leq nkN(r, \infty; f \mid= 1) + n(k-1)\overline{N}(r, \infty; f \mid= 2)\\
&\quad + \cdots + n\overline{N}(r, \infty; f \mid= k)\\
&\quad + \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, 0; f+a)\\
&\quad + \overline{N}(r, 0; g+a) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + S(r, f) + S(r, g).
\end{aligned}$$

(ii) $$\begin{aligned}
&(2n-1)\overline{N}(r, \infty; f)\\
&\leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, 0; f+a) + \overline{N}(r, 0; g+a)\\
&\quad + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + nN(r, \infty; f \mid= 1) + S(r, f) + S(r, g).
\end{aligned}$$

Similar expressions hold for g.

Proof. We omit the proof since the proof of the lemma can be carried out in the line of proof of *Lemma 2.8*. □

Lemma 2.10 ([1], Lemma 3) *Let f and g be two nonconstant meromorphic functions sharing $(1, m)$, where $2 \leq m < \infty$. Then*

$$\begin{aligned}
&\overline{N}(r, 1; f \mid= 2) + 2\overline{N}(r, 1; f \mid= 3) + \cdots + (m-1)\overline{N}(r, 1; f \mid= m)\\
&+ m\overline{N}_L(r, 1; f) + (m+1)\overline{N}_L(r, 1; g) + m\overline{N}_E^{(m+1}(r, 1; f)\\
&\leq N(r, 1; g) - \overline{N}(r, 1; g).
\end{aligned}$$

Lemma 2.11 *Let F, G be given by* (2.1) *and they share* (1, m). *If f, g share* (0, p), (∞, k) *where* $2 \leq m < \infty$ *and* $H \not\equiv 0$. *Then*

$$\begin{aligned}T(r,\,F) \leq\, & \overline{N}(r,\,0;f) + \overline{N}(r,\,0;g) + \overline{N}_*(r,\,0;f,\,g) + N_2(r,\,0;f+a)\\ & + N_2(r,\,0;g+a) + \overline{N}(r,\,\infty;f) + \overline{N}(r,\,\infty;g)\\ & + \overline{N}_*(r,\,\infty;f,\,g) - m(r,\,1;G) - \overline{N}(r,\,1;F\mid= 3)\\ & - \cdots - (m-2)\overline{N}(r,1;F\mid= m) - (m-2)\overline{N}_L(r,\,1;F)\\ & -(m-1)\overline{N}_L(r,\,1;G) - (m-1)\overline{N}_E^{(m+1}(r,\,1;F)\\ & + S(r,\,F) + S(r,\,G)\end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned}T(r,\,F) + T(r,\,G) \leq\, & \overline{N}(r,\,0;F) + \overline{N}(r,\,\infty;F) + \overline{N}(r,\,0;G)\\ & + \overline{N}(r,\,\infty;G) + \overline{N}(r,\,1;F) + \overline{N}(r,\,1;G)\\ & - N_0(r,\,0;F^{'}) - N_0(r,\,0;G^{'})\\ & + S(r,\,F) + S(r,\,G).\end{aligned} \tag{2.2}$$

In view of *Definition 1.8*, using *Lemmas 2.1*, *2.2* and *2.10* we see that

$$\begin{aligned}& \overline{N}(r,\,1;F) + \overline{N}(r,\,1;G)\\ & \leq N(r,\,1;F\mid= 1) + \overline{N}(r,\,1;F\mid= 2) + \overline{N}(r,\,1;F\mid= 3)\\ & \quad + \cdots + \overline{N}(r,\,1;F\mid= m) + \overline{N}_E^{(m+1}(r,\,1;F)\\ & \quad + \overline{N}_L(r,\,1;F) + \overline{N}_L(r,\,1;G) + \overline{N}(r,\,1;G)\\ & \leq \overline{N}_*(r,\,0;f,\,g) + \overline{N}(r,\,0;f+a\mid\geq 2) + \overline{N}(r,\,0;g+a\mid\geq 2)\\ & \quad + \overline{N}_*(r,\,\infty;f,\,g) + \overline{N}_L(r,\,1;F) + \overline{N}_L(r,\,1;G)\\ & \quad + \overline{N}(r,\,1;F\mid= 2) + \cdots + \overline{N}(r,\,1;F\mid= m)\\ & \quad + \overline{N}_E^{(m+1}(r,\,1;F) + \overline{N}_L(r,\,1;F) + \overline{N}_L(r,\,1;G)\\ & \quad + T(r,\,G) - m(r,\,1;G) + O(1) - \overline{N}(r,\,1;F\mid= 2)\\ & \quad - 2\overline{N}(r,\,1;F\mid= 3) - (m-1)\overline{N}(r,\,1;F\mid= m) - \cdots\\ & \quad - m\overline{N}_E^{(m+1}(r,\,1;F) - m\overline{N}_L(r,\,1;F) - (m+1)\overline{N}_L(r,\,1;G)\\ & \quad + \overline{N}_0(r,\,0;F^{'}) + \overline{N}_0(r,\,0;G^{'}) + S(r,\,F) + S(r,\,G)\\ & \leq \overline{N}_*(r,\,0;f,\,g) + \overline{N}(r,\,0;f+a\mid\geq 2) + \overline{N}(r,\,0;g+a\mid\geq 2)\\ & \quad + \overline{N}_*(r,\,\infty;f,\,g) + T(r,\,G) - m(r,\,1;G) - \overline{N}(r,\,1;F\mid= 3)\\ & \quad - 2\overline{N}(r,\,1;F\mid= 4) - \cdots - (m-2)\overline{N}(r,\,1;F\mid= m)\end{aligned} \tag{2.3}$$

$$
\begin{aligned}
&- (m-2)\overline{N}_L(r,\,1;F) - (m-1)\overline{N}_L(r,\,1;G) \\
&- (m-1)\overline{N}_E^{(m+1}(r,\,1;F) + \overline{N}_0(r,\,0;F^{'}) + \overline{N}_0(r,\,0;G^{'}) \\
&+ S(r,\,F) + S(r,\,G)
\end{aligned}
$$

From (2.2) and (2.3) in view of *Definition 1.7* the lemma follows. □

Lemma 2.12 *Let* $F,\,G$ *be given by* (2.1) *and they share* $(1,\,m)$*. If* $f,\,g$ *share* $(\infty,\,k)$ *where* $2 \le m < \infty$ *and* $H \not\equiv 0$*. Then*

$$
\begin{aligned}
T(r,\,F) \le\, & 2\overline{N}(r,\,0;f) + 2\overline{N}(r,\,0;g) + N_2(r,\,0;f+a) \\
& + N_2(r,\,0;g+a) + \overline{N}(r,\,\infty;f) + \overline{N}(r,\,\infty;g) \\
& + \overline{N}_*(r,\,\infty;f,\,g) - m(r,\,1;G) - \overline{N}(r,\,1;F\,|{=}\,3) \\
& - \cdots - (m-2)\overline{N}(r,\,1;F\,|{=}\,m) - (m-2)\overline{N}_L(r,\,1;F) \\
& - (m-1)\overline{N}_L(r,\,1;G) - (m-1)\overline{N}_E^{(m+1}(r,\,1;F) \\
& + S(r,\,F) + S(r,\,G)
\end{aligned}
$$

Proof. We omit the proof since using *Lemmas 2.1*, *2.3* and *2.10* the proof of the lemma can be carried out in the line of proof of *Lemma 2.11.* □

Lemma 2.13 ([14], Lemma 3) *Let* $f,\,g$ *be two nonconstant meromorphic functions sharing* $(0,\,\infty)$*,* $(\infty,\,\infty)$ *and* $\Theta(\infty;f) + \Theta(\infty;g) > 0$*. Then* $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ *implies* $f \equiv g$*, where* n (≥ 2) *is an integer and* a *is a nonzero finite constant.*

Lemma 2.14 ([13], Lemma 5) *If two nonconstant meromorphic functions* $f,\,g$ *share* $(\infty,\,0)$ *then for* $n \ge 2$

$$f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2,$$

where $a,\,b$ *are finite nonzero constants.*

Lemma 2.15 ([26], Lemma 6) *If* $H \equiv 0$*, then* $F,\,G$ *share* $(1,\,\infty)$*. If further* $F,\,G$ *share* $(\infty,\,0)$ *then* $F,\,G$ *share* $(\infty,\,\infty)$*.*

Lemma 2.16 ([16]) *If* $N(r,\,0;f^{(k)}\mid f \ne 0)$ *denotes the counting function of those zeros of* $f^{(k)}$ *which are not the zeros of* f*, where a zero of* $f^{(k)}$ *is counted according to its multiplicity then*

$$
\begin{aligned}
N(r,\,0;f^{(k)}\mid f \ne 0) \le k\overline{N}(r,\,\infty;f) + N(r,\,0;f\,|{<}\,k) \\
+ k\overline{N}(r,\,0;f\,|{\ge}\,k) + S(r,\,f).
\end{aligned}
$$

Lemma 2.17 *Let F, G be given by* (2.1) *and they share* $(1,\, m)$. *Also let $\omega_1,\, \omega_2 \ldots \omega_n$ are the members of the set $S_1 = \{z\colon z^n + az^{n-1} + b = 0\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 3) is an integer. Then*

$$\begin{aligned}\overline{N}_L(r,\, 1; F) \leq & \frac{1}{m+1}\big[\overline{N}(r,\, 0; f) + \overline{N}(r,\, \infty; f) - N_{\otimes}(r,\, 0; f^{'})\big]\\ & + S(r,\, f),\end{aligned}$$

where $N_{\otimes}(r,\, 0; f^{'}) = N(r,\, 0; f^{'} \mid f \neq 0,\, \omega_1,\, \omega_2,\, \ldots,\, \omega_n)$.

Proof. Using *Lemma 2.4* and *Lemma 2.16* we see that

$$\begin{aligned}\overline{N}_L(r,\, 1; F) &\leq \overline{N}(r,\, 1; F \mid\geq m+2)\\ &\leq \frac{1}{m+1}\big(N(r,\, 1; F) - \overline{N}(r,\, 1; F)\big)\\ &\leq \frac{1}{m+1}\Big[\sum_{j=1}^{n}\big(N(r,\, \omega_j; f) - \overline{N}(r,\, \omega_j; f)\big)\Big]\\ &\leq \frac{1}{m+1}\big(N(r,\, 0; f^{'} \mid f \neq 0) - N_{\otimes}(r,\, 0; f^{'})\big)\\ &\leq \frac{1}{m+1}\big[\overline{N}(r,\, 0; f) + \overline{N}(r,\, \infty; f) - N_{\otimes}(r,\, 0; f^{'})\big]\\ &\quad + S(r,\, f).\end{aligned}$$

This proves the lemma. □

Lemma 2.18 *Under the condition of* Lemma 2.17

$$\begin{aligned}\overline{N}_*(r,\, 1; F,\, G) \leq & \frac{1}{m}\big[\overline{N}(r,\, 0; f) + \overline{N}(r,\, \infty; f) - N_{\otimes}(r,\, 0; f^{'})\big]\\ & + S(r,\, f).\end{aligned}$$

Proof. Since

$$\begin{aligned}\overline{N}_*(r,\, 1; F,\, G) &\leq \overline{N}(r,\, 1; F \mid\geq m+1)\\ &\leq \frac{1}{m}\big(N(r,\, 1; F) - \overline{N}(r,\, 1; F)\big),\end{aligned}$$

the proof of the lemma can be carried out in the line of proof of *Lemma 2.17*. □

Lemma 2.19 ([24]) *Let F, G be two nonconstant meromorphic functions sharing* $(1, \infty)$ *and* (∞, ∞). *If*

$$N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) < \lambda T_1(r) + S_1(r),$$

where $\lambda < 1$ *and* $T_1(r) = \max\{T(r, F), T(r, G)\}$ *and* $S_1(r) = o(T_1(r))$, $r \longrightarrow \infty$, *outside a possible exceptional set of finite linear measure, then* $F \equiv G$ *or* $FG \equiv 1$.

Lemma 2.20 *Let F, G be given by* (2.1) $n \geq 3$ *and F, G share* $(1, m)$. *If f, g share* $(0, 0)$, (∞, k), $\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) > 5/n$ *and* $H \equiv 0$. *Then* $f \equiv g$.

Proof. Since $H \equiv 0$ we get from *Lemma 2.15* F and G share $(1, \infty)$ and (∞, ∞). If possible let us suppose $F \not\equiv G$. Then from *Lemma 2.5* and *Lemma 2.6* we have

$$\overline{N}(r, 0; f) = \overline{N}(r, 0; g) = S(r).$$

Again from *Lemma 2.7* and *Lemma 2.8* we have for $\varepsilon > 0$

$$\begin{aligned}
&\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) \\
&\leq \frac{4}{2n-1}T(r) + \frac{n}{2n-1}N(r, \infty; f \mid= 1) + \frac{n}{2n-1}N(r, \infty; g \mid= 1) \\
&\leq \left[\frac{2n+4}{2n-1} - \frac{n}{2n-1}\big(\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) - 2\varepsilon\big)\right]T(r)
\end{aligned}$$

Therefore we see that

$$\begin{aligned}
&N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \\
&\leq 2\overline{N}(r, 0; f) + 2\overline{N}(r, 0; g) + N_2(r, 0; f + a) + N_2(r, 0; g + a) \\
&\quad + 2\overline{N}(r, \infty; f) \\
&\leq \left[2 + \frac{2n+4}{2n-1} - \frac{n}{2n-1}\big(\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) - 2\varepsilon\big)\right]T(r) + S(r).
\end{aligned} \tag{2.4}$$

Using *Lemma 2.4* we obtain

$$T_1(r) = n\max\{T(r, f), T(r, g)\} + O(1) = nT(r) + O(1). \tag{2.5}$$

So again using *Lemma 2.4* we get from (2.4) and (2.5)

$$\begin{aligned}
&N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \\
&\leq \frac{\big[3 + 5/(2n-1) - \{n/(2n-1)\}\big(\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) - 2\varepsilon\big)\big]}{n}
\end{aligned}$$

$$\times T_1(r) + S(r).$$

Since $\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) > 5/n$ and $\varepsilon > 0$ is arbitrary for $n \geq 3$ we have by *Lemma 2.19* $FG \equiv 1$, which is impossible by *Lemma 2.14*. Hence $F \equiv G$ i.e. $f^{n-1}(f + a) \equiv g^{n-1}(g + a)$. This together with the assumption that f and g share $(0, 0)$ implies that f and g share $(0, \infty)$. Now the lemma follows from *Definition 1.3* and *Lemma 2.13*. □

Lemma 2.21 ([28], Lemma 2) *Suppose F and G be defined as in* (2.1) *and $\Theta(\infty; f) > 2/(n-1)$. Then $F \equiv G$ implies $f \equiv g$, where n (≥ 7) is an integer and a is a nonzero finite constant.*

Lemma 2.22 *Let F, G be given by* (2.1) *$n \geq 7$ and F, G share* $(1, m)$. *If f, g share (∞, k), $\delta_{1)}(\infty; f) > 14/3n$ and $H \equiv 0$. Then $f \equiv g$.*

Proof. Since $H \equiv 0$ we get from *Lemma 2.15* F and G share $(1, \infty)$ and (∞, ∞). If possible let us suppose $F \not\equiv G$. From *Lemmas 2.7* and *2.9* we have for $\varepsilon > 0$ that

$$\begin{aligned}
&N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \\
&\leq 2\overline{N}(r, 0; f) + 2\overline{N}(r, 0; g) + N_2(r, 0; f + a) + N_2(r, 0; g + a) \\
&\quad + 2\overline{N}(r, \infty; f) \\
&\leq \left[7 + \frac{9}{2n-1} - \frac{2n}{2n-1}\big(\delta_{1)}(\infty; f) - \varepsilon\big)\right]T(r) + S(r),
\end{aligned}$$

So using *Lemmas 2.19*, *2.14* we can deduce a contradiction. Hence $F \equiv G$. Noting that $(1/2)\delta_{1)}(\infty; f) > 7/3n \geq 2/(n-1)$ for $n \geq 7$, in view of *Definition 1.3* and *Lemma 2.21* we can prove $f \equiv g$. □

Lemma 2.23 *Let F, G be given by* (2.1) *$n \geq 4$ and F, G share* $(1, m)$. *If f, g share* $(0, 0)$, (∞, k), *$\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) > 0$ and $H \equiv 0$. Then $f \equiv g$.*

Proof. We omit the proof since the proof can be carried out in the line of proof of *Lemma 2.20*. □

Lemma 2.24 ([14], Lemma 9) *Let f and g be two nonconstant meromorphic functions such that $\Theta(\infty; f) + \Theta(\infty; g) > 4/(n-1)$, where n (≥ 4) is an integer. Then $f^{n-1}(f + a) \equiv g^{n-1}(g + a)$ implies $f \equiv g$, a is a finite nonzero constant.*

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1). Then F and G share $(1, 2)$, $(\infty; 2n)$. We consider the following cases.

Case 1: Let $H \not\equiv 0$. Then $F \not\equiv G$. Suppose ∞ is not an e.v.P. of f and g. Then by *Lemma 2.7* we get $V \not\equiv 0$. Hence from *Lemmas 2.4*, *2.9*, *2.12* and *2.17* we obtain

$$\begin{aligned} nT(r,f) \leq & 2\overline{N}(r,\,0;f) + 2\overline{N}(r,\,0;g) + N_2(r,\,0;f+a) \\ & + N_2(r,\,0;g+a) + \overline{N}(r,\,\infty;f\mid\geq 2) + \overline{N}(r,\,\infty;g\mid\geq 2) \\ & + \overline{N}(r,\,\infty;f\mid\geq 3) - \overline{N}_L(r,\,1;G) + S(r,\,f) + S(r,\,g) \\ \leq & 3T(r,\,f) + 3T(r,\,g) + \frac{2}{2n-1}\Big\{2T(r,\,f) + 2T(r,\,g) \\ & + \frac{1}{3}(\overline{N}(r,\,0;f) + \overline{N}(r,\,\infty;f))\Big\} + \frac{1}{3n-1}\Big\{2T(r,\,f) \\ & + 2T(r,\,g) + \frac{1}{3}(\overline{N}(r,\,0;f) + \overline{N}(r,\,\infty;f))\Big\} \\ & + S(r,\,f) + S(r,\,g) \\ \leq & \Big[6 + \frac{28}{3(2n-1)} + \frac{14}{3(3n-1)}\Big]T(r) + S(r). \end{aligned} \tag{3.1}$$

If ∞ is an e.v.P. of f and g then (3.1) automatically holds.
In the same manner we can obtain

$$nT(r,\,g) \leq \Big[6 + \frac{28}{3(2n-1)} + \frac{14}{3(3n-1)}\Big]T(r) + S(r). \tag{3.2}$$

Combining (3.1) and (3.2) we see that

$$\Big[n - 6 - \frac{28}{3(2n-1)} - \frac{14}{3(3n-1)}\Big]T(r) \leq S(r),$$

which leads to a contradiction for $n \geq 7$.

Case 2: Let $H \equiv 0$. Then noting that f and g have no simple poles implies $\delta_{1)}(\infty;f) = 1 > 14/3n$ the theorem follows from *Lemma 2.22*. □

Proof of Theorem 1.2. Let F, G be given by (2.1). Then F and G share $(1, 3)$, $(\infty;\infty)$. We consider the following cases.

Case 1: Let $H \not\equiv 0$. Then $F \not\equiv G$. Suppose ∞ is not an e.v.P. of f and g. Then by *Lemma 2.7* we get $V \not\equiv 0$. Hence from *Lemmas 2.4*, *2.9* and *2.12* we obtain for $\varepsilon > 0$

$$\begin{aligned} nT(r,\, f) \leq\, & 2\overline{N}(r,\, 0; f) + 2\overline{N}(r,\, 0; g) + N_2(r,\, 0; f+a) \qquad (3.3)\\ & + N_2(r,\, 0; g+a) + \overline{N}(r,\, \infty; f) + \overline{N}(r,\, \infty; g)\\ & - \overline{N}_*(r,\, 1; F,\, G) - \overline{N}_L(r,\, 1; G) + S(r,\, f) + S(r,\, g)\\ \leq\, & 3T(r,\, f) + 3T(r,\, g)\\ & + \frac{2}{2n-1}\{2T(r,\, f) + 2T(r,\, g) + \overline{N}_*(r,\, 1; F,\, G)\}\\ & + \frac{2n}{2n-1} N(r,\, \infty; f \mid= 1) - \overline{N}_*(r,\, 1; F,\, G)\\ & + S(r,\, f) + S(r,\, g)\\ \leq\, & \left[6 + \frac{8}{(2n-1)}\right]T(r) + \frac{2n}{2n-1}\{1 - \delta_{1)}(\infty; f) + \varepsilon\}T(r)\\ & + S(r)\\ \leq\, & \left[7 + \frac{9}{(2n-1)} - \frac{2n}{2n-1}\{\delta_{1)}(\infty; f) - \varepsilon\}\right]T(r) + S(r). \end{aligned}$$

If ∞ is an e.v.P. of f and g then (3.3) automatically holds.
In the same way we can obtain

$$\begin{aligned} nT(r,\, g) \leq\, & \left[7 + \frac{9}{(2n-1)} - \frac{2n}{2n-1}\{\delta_{1)}(\infty; f) - \varepsilon\}\right]T(r) \qquad (3.4)\\ & + S(r). \end{aligned}$$

Combining (3.3) and (3.4) we see that

$$\left[n - 7 - \frac{9}{(2n-1)} + \frac{2n}{2n-1}\{\delta_{1)}(\infty; f) - \varepsilon\}\right]T(r) \leq S(r).$$

Since $\delta_{1)}(\infty; f) > 14/3n$ there exist a $\rho > 0$ such that $\delta_{1)}(\infty; f) = 14/3n + \rho$. We choose $0 < \varepsilon < \rho$ then we get a contradiction.

Case 2: Let $H \equiv 0$. Now the theorem follows from *Lemma 2.22*. □

Proof of Theorem 1.3. Let F, G be given by (2.1). Then F and G share $(1,\, 3)$, $(\infty; \infty)$. We consider the following cases.

Case 1: Let $H \not\equiv 0$. Then $F \not\equiv G$. Suppose ∞ is not an e.v.P. of f and g. Then by *Lemma 2.7* we get $V \not\equiv 0$. Hence from *Lemmas 2.4*, *2.9* and *2.12* we obtain for $\varepsilon > 0$

$$
\begin{aligned}
& nT(r, f) && (3.5)\\
& \leq 3T(r, f) + 3T(r, g) + \frac{2}{2n-1}\{2T(r, f) + 2T(r, g) + \overline{N}_*(r, 1; F, G)\}\\
& \quad + \frac{n}{2n-1}N(r, \infty; f\,|{=}1) + \frac{n}{2n-1}N(r, \infty; g\,|{=}1) - \overline{N}_*(r, 1; F, G)\\
& \quad + S(r, f) + S(r, g)\\
& \leq \left[6 + \frac{8}{(2n-1)}\right]T(r) + \frac{n}{2n-1}\{2 - \delta_{1)}(\infty; f) - \delta_{1)}(\infty; g) + 2\varepsilon\}T(r)\\
& \quad + S(r)\\
& \leq \left[7 + \frac{9}{(2n-1)} - \frac{n}{2n-1}\{\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) - 2\varepsilon\}\right]T(r) + S(r).
\end{aligned}
$$

If ∞ is an e.v.P. of f and g then (3.5) automatically holds.
In the same way we can obtain

$$
\begin{aligned}
nT(r, g) \leq & \left[7 + \frac{9}{(2n-1)} - \frac{n}{2n-1}\{\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) - 2\varepsilon\}\right]\\
& \times T(r) + S(r). \qquad (3.6)
\end{aligned}
$$

Combining (3.5) and (3.6) we see that

$$
\begin{aligned}
\left[n - 7 - \frac{9}{(2n-1)} + \frac{n}{2n-1}\{\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) - 2\varepsilon\}\right]T(r)&\\
\leq S(r).&
\end{aligned}
$$

Noting that $\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) > 8/(n-1)$ and $\varepsilon > 0$ is arbitrary we get a contradiction.

Case 2: Let $H \equiv 0$. Since $(1/2)\big(\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g)\big) > 4/(n-1)$, proceeding in the same way as done in the proof of *Lemma 2.22* the theorem follows from *Definition 1.3* and *Lemma 2.24*. □

Proof of Theorem 1.4. We omit the proof since using *Lemmas 2.4*, *2.9*, *2.12* and *2.17* and proceeding in the same way as done in Theorem 1.2 the proof of the theorem can be carried out. □

Proof of Theorem 1.5. Let F, G be given by (2.1). Then F and G share $(1, 5)$, $(\infty; \infty)$. We consider the following cases.

Case 1: Let $H \not\equiv 0$. Then $F \not\equiv G$. Suppose 0, ∞ are not exceptional values Picard of f and g. Then by *Lemma 2.5* and *Lemma 2.7* we get $\Phi \not\equiv 0$ and $V \not\equiv 0$. Hence from *Lemmas 2.4*, *2.6*, *2.8* and *2.11* we obtain

$$\begin{aligned} nT(r,\, f) \leq\, & 2\overline{N}(r,\, 0; f) + \overline{N}(r,\, 0; f \mid\geq 2) + N_2(r,\, 0; f+a) \\ & + N_2(r,\, 0; g+a) + \overline{N}(r,\, \infty; f) + \overline{N}(r,\, \infty; g) \\ & - 3\overline{N}_*(r,\, 1; F,\, G) - \overline{N}_L(r,\, 1; G) + S(r,\, f) + S(r,\, g) \\ \leq\, & \Big(\frac{2}{n-2} + \frac{2}{2n-1}\Big)\overline{N}_*(r,\, 1; F,\, G) \\ & + \Big(1 + \frac{2}{2n-1}\Big)\overline{N}(r,\, 0; f \mid\geq 2) + \Big(2 + \frac{4}{2n-1}\Big)T(r) \\ & - 3\overline{N}_*(r,\, 1; F,\, G) - \overline{N}_L(r,\, 1; G) + S(r) \\ \leq\, & \Big(2 + \frac{4}{2n-1}\Big)T(r) + \frac{2n+1}{(2n-1)(2n-3)}\overline{N}_L(r,\, 1; F) \\ & + S(r) \\ \leq\, & \Big(2 + \frac{4}{2n-1} + \frac{2n+1}{3(2n-1)(2n-3)}\Big)T(r) + S(r). \end{aligned} \tag{3.7}$$

If 0, ∞ are e.v.P. of f and g then (3.7) automatically holds.
In the same manner we can obtain

$$nT(r,\, g) \leq \Big(2 + \frac{4}{2n-1} + \frac{2n+1}{3(2n-1)(2n-3)}\Big)T(r) + S(r). \tag{3.8}$$

Combining (3.7) and (3.8) we see that

$$\Big(n - 2 - \frac{4}{(2n-1)} - \frac{2n+1}{3(2n-1)(2n-3)}\Big)T(r) \leq S(r),$$

which is a contradiction for $n \geq 3$.

Case 2: Let $H \equiv 0$. Noting that f and g have no simple poles implies $\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) = 2 > 5/n$ the theorem follows from *Lemma 2.20.* □

Proof of Theorem 1.6. Let F, G be given by (2.1). Then F and G share $(1,\, 6)$, $(\infty; \infty)$. We consider the following cases.

Case 1: Let $H \not\equiv 0$. Then $F \not\equiv G$. Suppose 0, ∞ are not exceptional values Picard of f and g. Then by *Lemma 2.5* and *Lemma 2.7* we get $\Phi \not\equiv 0$

and $V \not\equiv 0$. Hence from *Lemmas 2.4, 2.6, 2.8* and *2.11* we obtain for $\varepsilon > 0$

$$\begin{aligned}
nT(r,\, f) \le\, & 3\overline{N}(r,\, 0; f) + N_2(r,\, 0; f+a) + N_2(r,\, 0; g+a) \qquad (3.9)\\
& +\overline{N}(r,\, \infty; f) + \overline{N}(r,\, \infty; g) - 4\overline{N}_*(r,\, 1; F,\, G)\\
& -\overline{N}_L(r,\, 1; G) + S(r,\, f) + S(r,\, g)\\
\le\, & \Big[3 + \frac{2}{2n-1}\Big]\overline{N}(r,\, 0; f) + \Big(2 + \frac{4}{2n-1}\Big)T(r)\\
& +\frac{2}{2n-1}\overline{N}_*(r,\, 1; F,\, G) + \frac{n}{2n-1}N(r,\, \infty; f \mid= 1)\\
& +\frac{n}{2n-1}N(r,\, \infty; g \mid= 1) - 4\overline{N}_*(r,\, 1; F,\, G)\\
& +S(r,\, f) + S(r,\, g)\\
\le\, & \Big[\frac{3}{n-2} + \frac{2}{(n-2)(2n-1)} + \frac{2}{2n-1}\Big]\overline{N}_*(r,\, 1; F,\, G)\\
& +\Big(2 + \frac{4}{2n-1}\Big)T(r)\\
& + \frac{n}{2n-1}\{2 - \delta_{1)}(\infty; f) - \delta_{1)}(\infty; g) + 2\varepsilon\}T(r)\\
& -4\overline{N}_*(r,\, 1; F,\, G) + S(r)\\
\le\, & \Big[3 + \frac{5}{(2n-1)} - \frac{n}{2n-1}\{\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) - 2\varepsilon\}\Big]\\
& \times T(r) + S(r).
\end{aligned}$$

If 0, ∞ are e.v.P. of f and g then (3.9) automatically holds.
In the same way we can obtain

$$\begin{aligned}
nT(r,\, g) \le\, & \Big[3 + \frac{5}{(2n-1)} - \frac{n}{2n-1}\{\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) - 2\varepsilon\}\Big]\\
& \times T(r) + S(r). \qquad (3.10)
\end{aligned}$$

Combining (3.9) and (3.10) we see that

$$\Big[n - 3 - \frac{5}{(2n-1)} + \frac{n}{2n-1}\{\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) - 2\varepsilon\}\Big] \times T(r) \le S(r).$$

Since $\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) > 5/n$ there exist a $\rho > 0$ such that $\delta_{1)}(\infty; f) + \delta_{1)}(\infty; g) = 5/n + \rho$. We choose $0 < \varepsilon < \rho/2$ then we get a contradiction.

Case 2: Let $H \equiv 0$. Now the theorem follows from *Lemma 2.20.* □

Proof of Theorem 1.7. Let F, G be given by (2.1). Then F and G share $(1,\, 4)$, $(\infty; 6n)$. We consider the following cases.

Case 1: Let $H \not\equiv 0$. Then $F \not\equiv G$. Suppose 0, ∞ are not exceptional values Picard of f and g. Then by *Lemma 2.5* and *Lemma 2.7* we get $\Phi \not\equiv 0$ and $V \not\equiv 0$. Noting that f, g share $(0,\, 0)$ and $(\infty,\, 6)$ implies $\overline{N}_*(r,\, 0; f,\, g) \leq \overline{N}(r,\, 0; f) = \overline{N}(r,\, 0; g)$ and $\overline{N}_*(r,\, \infty; f, g) \leq \overline{N}(r,\, \infty; f \mid\geq 7) = \overline{N}(r,\, \infty; g \mid\geq 7)$ from *Lemmas 2.4*, *2.6* and *2.11* we obtain

$$\begin{aligned}
&nT(r,\, f) + nT(r,\, g) &(3.11)\\
&\leq 6\overline{N}(r,\, 0; f) + 2T(r,\, f) + 2T(r,\, g) + 4\overline{N}(r,\, \infty; f)\\
&\quad + 2\overline{N}(r,\, \infty; f \mid\geq 7) - 5\overline{N}_*(r,\, 1; F,\, G) + S(r,\, f) + S(r,\, g)\\
&\leq 2T(r,\, f) + 2T(r,\, g) + \left\{2 + \frac{6}{n-2}\right\}\overline{N}(r,\, \infty; f \mid\geq 7)\\
&\quad + \frac{6}{n-2}\overline{N}_*(r,\, 1; F,\, G) + 4\overline{N}(r,\, \infty; f) - 5\overline{N}_*(r,\, 1; F,\, G)\\
&\quad + S(r,\, f) + S(r,\, g)
\end{aligned}$$

So respectively using *Lemma 2.8* for $k = 6$ and $k = 0$, *Lemma 2.6* for $p = 0$ and *Lemma 2.18* we get from (3.11) that

$$\begin{aligned}
&nT(r,\, f) + nT(r,\, g) &(3.12)\\
&\leq \left(2 + \frac{3(n+1)}{(n-2)(7n-1)}\right)\{T(r,\, f) + T(r,\, g)\}\\
&\quad + \left(\frac{6}{n-2} + \frac{2(n+1)}{(n-2)(7n-1)}\right)\overline{N}_*(r,\, 1; F,\, G)\\
&\quad + \frac{4}{n-1}[T(r,\, f) + T(r,\, g) + \overline{N}(r,\, 0; f) + \overline{N}_*(r,\, 1; F,\, G)]\\
&\quad - 5\overline{N}_*(r,\, 1; F,\, G) + S(r,\, f) + S(r,\, g)\\
&\leq \left(2 + \frac{4}{n-1} + \frac{3(n+1)}{(n-2)(7n-1)}\right)\{T(r,\, f) + T(r,\, g)\}\\
&\quad + \left(\frac{6}{n-2} + \frac{4}{n-1} + \frac{2(n+1)}{(n-2)(7n-1)}\right)\overline{N}_*(r,\, 1; F,\, G)\\
&\quad + \frac{4}{(n-1)(n-2)}[\overline{N}(r,\, \infty; f \mid\geq 7) + \overline{N}_*(r,\, 1; F,\, G)]\\
&\quad - 5\overline{N}_*(r,\, 1; F,\, G) + S(r,\, f) + S(r,\, g)\\
&\leq \left(2 + \frac{4n-6}{(n-1)(n-2)} + \frac{3(n+1)}{(n-2)(7n-1)}\right)\{T(r,\, f) + T(r,\, g)\}
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{10}{n-2}+\frac{2(n+1)}{(n-2)(7n-1)}-5\Big)\overline{N}_*(r,\,1;F,\,G)\\
&+S(r,\,f)+S(r,\,g)\\
\le\;&\Big(2+\frac{(4n-6)}{(n-1)(n-2)}+\frac{7(n+1)}{2(n-2)(7n-1)}\Big)\{T(r,\,f)+T(r,\,g)\}\\
&+S(r,\,f)+S(r,\,g).
\end{aligned}
$$

From (3.12) we get a contradiction for $n \geq 4$.
If 0, ∞ are e.v.P. of f and g then (3.12) automatically holds.

Case 2: Let $H \equiv 0$. Now the theorem follows from *Lemma 2.23*. □

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