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An inclusion between sets of orbits and surjectivity of the restriction map of rings of invariants

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Abstract. Let V be a finite dimensional vector space over the complex number field $\mathbb{C}$. Suppose that, by the adjoint action, a reductive subgroup $\tilde{G}$ of $GL(V)$ acts on a subspace $\tilde{L}$ of $\mathrm{End}(V)$ and a closed subgroup G of $\tilde{G}$ acts on a subspace L of $\tilde{L}$. In this paper, we give a sufficient condition on the inclusion $(G, L) \hookrightarrow (\tilde{G}, \tilde{L})$ for which the orbits correspondence $L/G \to \tilde{L}/\tilde{G}$ ($\mathcal{O} \mapsto \tilde{\mathcal{O}} := \mathrm{Ad}(\tilde{G}) \cdot \mathcal{O}$) is injective. Moreover we show that the ring $\mathbb{C}[L]^G$ of G-invariants on L is the integral closure of $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$ in its quotient field. Then, if the ring $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$ is normal, the restriction map rest: $\mathbb{C}[\tilde{L}]^{\tilde{G}} \to \mathbb{C}[L]^G$ ($f \mapsto f|_L$) is surjective. By using this, we give some examples for which $L/G \to \tilde{L}/\tilde{G}$ is injective and rest: $\mathbb{C}[\tilde{L}]^{\tilde{G}} \to \mathbb{C}[L]^G$ is surjective.

Key words: inclusion theorem between sets of orbits, the restriction map of rings of invariants.

0. Introduction

Let V be a finite dimensional vector space over the complex number field $\mathbb{C}$ and $\sigma\colon \mathrm{End}(V) \to \mathrm{End}(V)$ a $\mathbb{C}$-linear anti-automorphism of the associative algebra. Let $\tilde{G}$ be a subgroup of $GL(V)$ such that $\sigma(\tilde{G}) = \tilde{G}$ and $\sigma^2|_{\tilde{G}} = \mathrm{id}_{\tilde{G}}$. Suppose that $\tilde{G}$ acts on a σ-stable subspace $\tilde{L}$ of $\mathrm{End}(V)$ by the adjoint action. Define a subgroup G of $\tilde{G}$ and a subspace L of $\tilde{L}$ by

$$G := \{g \in \tilde{G} \mid \sigma(g) = g^{-1}\}, \quad L := \{X \in \tilde{L} \mid \sigma(X) = \alpha X\},$$

where $\alpha \in \mathbb{C}^\times$. Then the group G also acts on L by the adjoint action. The following are examples of such situation $(G, L) \hookrightarrow (\tilde{G}, \tilde{L})$.

(1) Put $\tilde{G} = GL(V)$, $\tilde{L} = \mathfrak{gl}(V)$, $\sigma(X) = {}^tX$ and $\alpha = -1$. Then $G = O(V)$ and $L = \mathfrak{o}(V)$.

(2) Put $V = \mathbb{C}^{m+n}$, $\tilde{G} = \{\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right) \mid g_1 \in GL(m, \mathbb{C}),\, g_2 \in GL(n, \mathbb{C})\}$, $\tilde{L} = \{\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) \mid A \in \mathrm{Mat}_{m\times n}(\mathbb{C}), B \in \mathrm{Mat}_{n\times m}(\mathbb{C})\}$, $\sigma(X) = {}^tX$ and $\alpha = -1$.

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Then $G = O(m, \mathbb{C}) \times O(n, \mathbb{C})$ and $L = \{X \in \tilde{L} \mid \sigma(X) = -X\}$ (L is the -1-eigenspace of the symmetric pair $(O(m+n, \mathbb{C}), O(m, \mathbb{C}) \times O(n, \mathbb{C}))$).

(3) Put $V = \mathbb{C}^{m+n}$, $\tilde{G} = \{\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \mid g_1 \in GL(m, \mathbb{C}), g_2 \in GL(n, \mathbb{C})\}$,
$\tilde{L} = \{\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \mid A \in \mathrm{Mat}_{m\times n}(\mathbb{C}), B \in \mathrm{Mat}_{n\times m}(\mathbb{C})\}$,
$J = \begin{pmatrix} 1_m & 0 & 0 \\ 0 & 0 & 1_{n/2} \\ 0 & -1_{n/2} & 0 \end{pmatrix}$,
$\sigma(X) = J^{-1\,t}XJ$ and $\alpha = -\sqrt{-1}$. Then
$G = O(m, \mathbb{C}) \times Sp(n, \mathbb{C})$ and $L = \{X \in \tilde{L} \mid \sigma(X) = -\sqrt{-1}X\}$
(L is the $\sqrt{-1}$-eigenspace of the $\mathbb{Z}_4$-graded Lie algebra defined by $(\mathfrak{gl}(V), -\sigma)$).

These examples $(G, L) \hookrightarrow (\tilde{G}, \tilde{L})$ can be seen as inclusions of $\mathbb{Z}_m$-graded Lie algebras. For these examples, it is known that the inclusions $\mathcal{N}(L)/G \hookrightarrow \mathcal{N}(\tilde{L})/\tilde{G}$ $(\mathcal{O} \mapsto \mathrm{Ad}(\tilde{G}) \cdot \mathcal{O})$ of nilpotent orbits holds, and as a consequence, nilpotent G-orbits in L are classified by Young diagrams or ab-diagrams (see for example [He], [O2], [KP]). We can show that the inclusions of orbits hold not only for nilpotent orbits, but also for general orbits (i.e., $L/G \hookrightarrow \tilde{L}/\tilde{G}$). In this paper, we show that the inclusion $L/G \hookrightarrow \tilde{L}/\tilde{G}$ hold for more general situations (Theorem 1).

We are going to explain the contents of this paper briefly. In §1, we give a sufficient condition for which the inclusion $L/G \hookrightarrow \tilde{L}/\tilde{G}$ holds. In §2, we study the relationship between the rings $\mathbb{C}[L]^G$ and $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$ (restrictions of $\tilde{G}$-invariants on $\tilde{L}$ to L). Suppose that the inclusion $L/G \hookrightarrow \tilde{L}/\tilde{G}$ holds and that closed G-orbits are mapped to closed $\tilde{G}$-orbits by this correspondence. Then the correspondence of closed orbits $L^{G\text{-cl}}/G \hookrightarrow \tilde{L}^{\tilde{G}\text{-cl}}/\tilde{G}$ is identified with the morphism $\mathrm{Spec}(\mathbb{C}[\tilde{L}]^{\tilde{G}}) \to \mathrm{Spec}(\mathbb{C}[L]^G)$ defined by the restriction map rest: $\mathbb{C}[\tilde{L}]^{\tilde{G}} \to \mathbb{C}[L]^G$ $(f \mapsto f|_L)$. Since $L^{G\text{-cl}}/G$ is a subset of $\tilde{L}^{\tilde{G}\text{-cl}}/\tilde{G}$, it is wishful that functions on $L^{G\text{-cl}}/G$ extend to those on $\tilde{L}^{\tilde{G}\text{-cl}}/\tilde{G}$ and the restriction map rest: $\mathbb{C}[\tilde{L}]^{\tilde{G}} \to \mathbb{C}[L]^G$ becomes surjective. We show that *rest* is surjective under the assumption of Theorem 1 and the condition that the ring $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$ is normal (Theorem 12).

In §3, §4 and §5, we give some examples of inclusions $(G, L) \hookrightarrow (\tilde{G}, \tilde{L})$ for which $L/G \hookrightarrow \tilde{L}/\tilde{G}$ and $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L = \mathbb{C}[L]^G$ hold. In particular, in §4, we show that FFT for $O(n, \mathbb{C})$ and $Sp(n, \mathbb{C})$ can be proved by using FFT for $GL(n, \mathbb{C})$ and Theorem 12, under some restriction on the size of matrices.

We mention that the results of this paper can be applied to the classical

$\mathbb{Z}_m$-graded Lie algebras to obtain classification of orbits and determination of rings of invariants. These applications will be treated in the forthcoming paper [O3].

1. Inclusion theorem between sets of orbits

The following theorem is a generalization of [O1, Proposition 4].

Theorem 1 *Let V be a finite dimensional vector space over the complex number field $\mathbb{C}$ and $\sigma\colon \mathrm{End}(V) \to \mathrm{End}(V)$ a $\mathbb{C}$-linear anti-automorphism of the associative algebra. Let $\tilde{G}$ be a subgroup of $GL(V)$ such that*
(a) $\langle \tilde{G} \rangle_{\mathbb{C}} \cap GL(V) = \tilde{G}$, *where $\langle \tilde{G} \rangle_{\mathbb{C}}$ denotes the subspace of* $\mathrm{End}(V)$ *spanned by $\tilde{G}$.*
(b) $\sigma(\tilde{G}) = \tilde{G}$ *and* $\sigma^2|_{\tilde{G}} = \mathrm{id}_{\tilde{G}}$.
Let $\tilde{L}$ be an $\mathrm{Ad}(\tilde{G})$-stable and σ-stable subspace of $\mathrm{End}(V)$, and α an element of $GL(\tilde{L})$ such that $\alpha(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\alpha(X)$ for any $g \in \tilde{G}$ and $X \in \tilde{L}$ (i.e., $\alpha \in Z_{GL(\tilde{L})}(\mathrm{Ad}_{\tilde{L}}(\tilde{G}))$). Define the subgroup $G := \{g \in \tilde{G} \mid \sigma(g) = g^{-1}\}$ of $\tilde{G}$ and the subspace $L := \{X \in \tilde{L} \mid \sigma(X) = \alpha(X)\}$. Then the correspondence $L/G \to \tilde{L}/\tilde{G}$, $\mathcal{O} \mapsto \tilde{\mathcal{O}} := \mathrm{Ad}(\tilde{G}) \cdot \mathcal{O}$ of adjoint orbits is injective.

Proof. Suppose that two elements X, $Y \in L$ are conjugate by an element $g \in \tilde{G}$; $Y = gXg^{-1}$. It is sufficient to show that X and Y are conjugate under G.

Since

$$\begin{aligned} gXg^{-1} &= Y = \alpha^{-1}(\sigma(Y)) \\ &= \alpha^{-1}(\sigma(gXg^{-1})) = \alpha^{-1}(\sigma(g)^{-1}\sigma(X)\sigma(g)) \\ &= \alpha^{-1}(\sigma(g)^{-1}\alpha(X)\sigma(g)) = \sigma(g)^{-1}X\sigma(g), \end{aligned}$$

we have $\sigma(g)gX = X\sigma(g)g$ and hence $h := g^{-1}\sigma(g)^{-1} = (\sigma(g)g)^{-1} \in Z_{\tilde{G}}(X)$. Since h is invertible, there exsits a polynomial $f(T) \in \mathbb{C}[T]$ of a variable T such that $h = f(h)^2$ by Lemma 2 below. Then we see

$$\begin{aligned} &\sigma(h) = \sigma(g^{-1}\sigma(g)^{-1}) = (\sigma^2(g))^{-1}\sigma(g)^{-1} = g^{-1}\sigma(g)^{-1} = h, \\ &\sigma(f(h)) = f(h), \quad g^{-1}\sigma(g)^{-1} = h = f(h)^2 = f(h)\sigma(f(h)), \end{aligned}$$

and

$$1 = g(g^{-1}\sigma(g)^{-1})\sigma(g) = gf(h)\sigma(f(h))\sigma(g) = gf(h)\sigma(gf(h)).$$

Hence we have $\sigma(gf(h)) = (gf(h))^{-1}$. Since $f(h) \in \tilde{G}$ by condition (a), we have $gf(h) \in G$. Since $h \in Z_{\tilde{G}}(X)$, we also have $f(h) \in Z_{\tilde{G}}(X)$. Then by

$$Y = gXg^{-1} = gf(h)Xf(h)^{-1}g^{-1} = gf(h)X(gf(h))^{-1},$$

X and Y are conjugate under $gf(h) \in G$. □

The next lemma easily follows from the Chinese remainder theorem.

Lemma 2 *For any invertible element $h \in \mathrm{End}(V)$, there exsits a polynomial $f(T) \in \mathbb{C}[T]$ such that $h = f(h)^2$.*

Remark 3 (1) Let $\langle\ ,\ \rangle$ be a non-degenerate bilinear form on V and $\sigma(X) = X^*$ the adjoint of an element $X \in \mathrm{End}(V)$. Then clearly $\sigma\colon \mathrm{End}(V) \to \mathrm{End}(V)$ is a $\mathbb{C}$-linear anti-automorphism of the associative algebra.

Conversely, if $\sigma\colon \mathrm{End}(V) \to \mathrm{End}(V)$ is a $\mathbb{C}$-linear anti-automorphism of the associative algebra, we can easily show that there exsists a non-degenerate bilinear form $\langle\ ,\ \rangle$ on V for which the adjoint with respect to $\langle\ ,\ \rangle$ coinsides with σ.

(2) In Theorem 1, of course $\alpha \in GL(\tilde{L})$ can be choosen as a non-zero scalar multiplication; $\alpha\colon \tilde{L} \to \tilde{L}$, $\alpha(X) = \alpha X$ $(\alpha \in \mathbb{C}^\times)$. The author cannot find meaningful example for which α is not a scalar multiplication. The examples in §4 and §5 are all the cases where α are non-zero scalar multiplications.

By Theorem 1, $L/G \hookrightarrow \tilde{L}/\tilde{G}$ holds for the three examples in Introduction.

2. Invariant theory related to the inclusion theorem

(2.1) Preliminaries from invariant theory

Suppose a reductive group G acts on an affine variety X. We denote by $\mathbb{C}[X]^G$ the subring of the coordinate ring $\mathbb{C}[X]$ consisting of G-invariant functions and call $\mathbb{C}[X]^G$ the ring of G-invariants. Since $\mathbb{C}[X]^G$ is finitely generated by Hilbert's theorem, we can consider the affine variety $X//G := \mathrm{Spec}(\mathbb{C}[X]^G)$. It is known that $X//G$ is the categorical quotient of X under the action of G. The morphism $\pi_{(G,X)}\colon X \to X//G$ defined by the inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$ is called the affine quotient map under G. Clearly $\pi_{(G,X)}$ maps any G-orbit of X to a point of $X//G$.

Theorem 4 (See [PV, Thorem 4.6 and Corollary to Theorem 4.7] for example) *$\pi_{(G,X)}\colon X \to X//G$ is surjective and any fibre of $\pi_{(G,X)}$ contains exactly one closed G-orbit.*

For a G-stable subset Y of X, we denote by Y/G the set-theoretical quotient, that is, the set of G-orbits in Y. We denote by $X^{G\text{-cl}}$ the set of points $x \in X$ for which the orbit $G \cdot x$ is closed in X. The map $\pi_{(G,X)}$ defines a map $\overline{\pi}_{(G,X)} \colon X/G \to X//G$ and the restriction $\overline{\pi}_{(G,X)}|_{X^{G\text{-cl}}/G} \colon X^{G\text{-cl}}/G \to X//G$ is bijective by Theorem 4. Hence we can identify $X//G$ with the set $X^{G\text{-cl}}/G$ of closed G-orbits in X.

Next, we consider the following situation. Suppose a reductive group $\tilde{G}$ acts on an affine variety $\tilde{X}$ and a reductive closed subgroup G of $\tilde{G}$ acts on a closed subvariety X of $\tilde{X}$. We denote such a situation by $(G, X) \hookrightarrow (\tilde{G}, \tilde{X})$. For an orbit $\mathcal{O} \in X/G$, we denote by $\tilde{\mathcal{O}} := \tilde{G} \cdot \mathcal{O} \in \tilde{X}/\tilde{G}$ the $\tilde{G}$-orbit generated by $\mathcal{O}$. We also denote by $\tilde{\mathcal{O}}^{\tilde{G}\text{-cl}}$ the unique closed $\tilde{G}$-orbit in the closure $\overline{\tilde{\mathcal{O}}}$. Thus we obtain a map

$$X^{G\text{-cl}}/G \to \tilde{X}^{\tilde{G}\text{-cl}}/\tilde{G}, \ \mathcal{O} \mapsto \tilde{\mathcal{O}}^{\tilde{G}\text{-cl}}.$$

Proposition 5 *Let $r \colon X//G \to \tilde{X}//\tilde{G}$ be the morphism defined by the restriction map* rest: $\mathbb{C}[\tilde{X}]^{\tilde{G}} \to \mathbb{C}[X]^G$, *$f \mapsto f|_X$. Then by the above identification $X//G = X^{G\text{-cl}}/G$ and $\tilde{X}//\tilde{G} = \tilde{X}^{\tilde{G}\text{-cl}}/\tilde{G}$, the morphism r coincides with the map $\mathcal{O} \mapsto \tilde{\mathcal{O}}^{\tilde{G}\text{-cl}}$.*

Remark 6 Let us consider the correspondence

$$X/G \to \tilde{X}/\tilde{G}, \ \mathcal{O} \mapsto \tilde{\mathcal{O}} := \tilde{G} \cdot \mathcal{O}.$$

Suppose that any closed G-orbit in X is mapped, by this correspondence, to a closed $\tilde{G}$-orbit in $\tilde{X}$. Then the morphism $r \colon X//G \to \tilde{X}//\tilde{G}$ coincides with the natural correspondence

$$X^{G\text{-cl}}/G \to \tilde{X}^{\tilde{G}\text{-cl}}/\tilde{G}, \ \mathcal{O} \mapsto \tilde{\mathcal{O}} := \tilde{G} \cdot \mathcal{O}.$$

In particular, if this correspondence is injective, so is r.

Let us give a geometric interpretation of the ring $\mathbb{C}[\tilde{X}]^{\tilde{G}}|_X$ (the image of rest: $\mathbb{C}[\tilde{X}]^{\tilde{G}} \to \mathbb{C}[X]^G$).

Proposition 7 *Suppose that a reductive algebraic group $\tilde{G}$ acts on an affine variety $\tilde{X}$ and that X is a closed subvariety of $\tilde{X}$.*

(i) *Let us consider the $\tilde{G}$-stable subvariety $N := \overline{\tilde{G} \cdot X}$ of $\tilde{X}$. Then the restriction map* rest: $\mathbb{C}[\tilde{X}]^{\tilde{G}} \to \mathbb{C}[N]^{\tilde{G}}$, *$f \mapsto f|_N$ is surjective.*

(ii) *From* (i), *we obtain a ring homomorphism $\mathbb{C}[N]^{\tilde{G}} \to \mathbb{C}[\tilde{X}]^{\tilde{G}}|_X$, $f \mapsto$*

$f|_X$. This is an isomorphism. In paticular, we obtain $\mathrm{Spec}(\mathbb{C}[\tilde{X}]^{\tilde{G}}|_X) \simeq N//\tilde{G}$.

(iii) *Let* $\pi = \pi_{(\tilde{G},\tilde{X})}\colon \tilde{X} \to \tilde{X}//\tilde{G}$ *be the affine quotient map under* $\tilde{G}$. *Then the closure* $\overline{\pi(X)}$ *of the image* $\pi(X)$ *is isomorphic* $N//\tilde{G}$*:*

$$\overline{\pi(X)} \simeq N//\tilde{G} \simeq \mathrm{Spec}(\mathbb{C}[\tilde{X}]^{\tilde{G}}|_X).$$

Proof. (i) Since $\tilde{G}$ is reductive and $\mathbb{C}[\tilde{X}] \to \mathbb{C}[N]$, $f \mapsto f|_N$ is a surjective $\tilde{G}$-module homomorphism of the locally finite $\tilde{G}$-modules, the sum of trivial representations in $\mathbb{C}[\tilde{X}]$ is mapped by this homomorphism onto that in $\mathbb{C}[N]$. This means $\mathbb{C}[\tilde{X}]^{\tilde{G}} \to \mathbb{C}[N]^{\tilde{G}}$ is surjective.

(ii) Since $X \subset N$ and $\mathbb{C}[\tilde{X}]^{\tilde{G}} \to \mathbb{C}[N]^{\tilde{G}}$ is surjective, we obtain a surjective homomorphism $\mathbb{C}[N]^{\tilde{G}} \to \mathbb{C}[\tilde{X}]^{\tilde{G}}|_X$. Since $\tilde{G} \cdot X$ is dense in N, this homomorphism is injective.

(iii) Let us consider the commutative diagram

$$\begin{array}{ccc} \mathbb{C}[N] & \leftarrow & \mathbb{C}[\tilde{X}] \\ \uparrow & & \uparrow \\ \mathbb{C}[N]^{\tilde{G}} & \leftarrow & \mathbb{C}[\tilde{X}]^{\tilde{G}} \end{array}$$

and the corresponding diagram

$$\begin{array}{ccc} N & \hookrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ N//\tilde{G} & \hookrightarrow & \tilde{X}//\tilde{G} \end{array}.$$

Since the vertical arrows in the first diagram are surjectuve, those in the second diagram are closed immersions. Hence we have $N//\tilde{G} = \pi(N)$. Since π is continuous and $\pi(N)$ is a closed subset of $\tilde{X}//\tilde{G}$, we easily see that $\overline{\pi(X)} = \overline{\pi(\tilde{G} \cdot X)} = \pi(N)$. □

(2.2) An application of Luna's criterion

As an application of Luna's criterion, let us give a condition on $(G, L) \hookrightarrow (\tilde{G}, \tilde{L})$ for which the correspondence $L/G \to \tilde{L}/\tilde{G}$ maps a closed orbit to a closed orbit and the ring extension $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L \subset \mathbb{C}[L]^G$ is integral.

Theorem 8 *Let* $\mathcal{G}$ *be a reductive algebraic group over* $\mathbb{C}$ *and* $\theta\colon \mathcal{G} \to \mathcal{G}$ *an automorphism of* $\mathcal{G}$. *We denote by* $\theta\colon \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{G})$ *the corresponding automorphism of the Lie algebra of* $\mathcal{G}$. *Let* $\tilde{G}$ *be a* θ*-stable reductive subgroup of* $\mathcal{G}$ *and* $\tilde{L}$ *a* θ*-stable,* $\mathrm{Ad}(\tilde{G})$*-stable subspace of* $\mathrm{Lie}(\mathcal{G})$. *Define a closed* *subgroup* G' *of* $\tilde{G}$ *by* $G' = \{g \in \tilde{G} \mid \mathrm{Ad}_{\tilde{L}}(g) = \mathrm{Ad}_{\tilde{L}}(\theta(g))\}$. *Let* α *be an element of* $GL(\tilde{L})$ *such that* $\alpha(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\alpha(X)$ *for any* $g \in \tilde{G}$ *and* $X \in \tilde{L}$. *Define an element* $\varphi \in GL(\tilde{L})$ *by* $\varphi(X) = \alpha^{-1}(\theta(X))$ $(X \in \tilde{L})$. *Put* $L := \{X \in \tilde{L} \mid \varphi(X) = X \ (\Leftrightarrow \ \theta(X) = \alpha(X))\}$. *Suppose that* φ *has finite order. Then* $\mathrm{Ad}_{\tilde{L}}(G')$ *is reductive and we have the following*:

(i) *For the correspondence*

$$L/G' \to \tilde{L}/\tilde{G}, \ \mathcal{O} \mapsto \tilde{\mathcal{O}} := \mathrm{Ad}(\tilde{G}) \cdot \mathcal{O},$$

$\tilde{\mathcal{O}}$ *is closed in* $\tilde{L}$ *if and only if* $\mathcal{O}$ *is closed in* L.

(ii) *The morphism* $L//G' \to \tilde{L}//\tilde{G}$ *corresponding to the restriction map* $\mathrm{rest}\colon \mathbb{C}[\tilde{L}]^{\tilde{G}} \to \mathbb{C}[L]^{G'}$ *is finite, that is,* $\mathbb{C}[L]^{G'}$ *is integral over the image* $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$.

(iii) *Suppose that the morphism* $L//G' \to \tilde{L}//\tilde{G}$ *of* (ii) *is injective. Then the morphism* $L//G' = \mathrm{Spec}(\mathbb{C}[L]^{G'}) \to \mathrm{Spec}(\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L)$ *corresponding to* $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L \hookrightarrow \mathbb{C}[L]^{G'}$ *is bijective and birational* (*i.e., the quotient fields of* $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$ *and* $\mathbb{C}[L]^{G'}$ *coinside*). *In particular, since* $\mathbb{C}[L]^{G'}$ *is normal* (*i.e., integrally closed in its quotient field*), $\mathbb{C}[L]^{G'}$ *is the integral closure of* $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$ *in its quotient field.*

By Proposition 7, we have the following:

Corollary to Theorem 8 *In the setting of* Theorem 8, (iii), *if* $\overline{\pi_{(\tilde{G},\tilde{L})}(L)}$ *is a normal variety, we have* $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L = \mathbb{C}[L]^{G'}$, *that is the restriction map* $\mathrm{rest}\colon \mathbb{C}[\tilde{L}]^{\tilde{G}} \to \mathbb{C}[L]^{G'}$ *is surjective.*

We begin the proof of Theorem 8 with showing the following lemma.

Lemma 9 *In the setting of* Theorem 8, *let* $\tilde{H}$ *be the subgroup of* $GL(\tilde{L})$ *generated by* $\mathrm{Ad}_{\tilde{L}}(\tilde{G})$ *and* φ; $\tilde{H} := \langle \mathrm{Ad}_{\tilde{L}}(\tilde{G}) \cup \{\varphi\}\rangle$.

(i) *For* $g \in \tilde{G}$, *we have* $\varphi \circ \mathrm{Ad}_{\tilde{L}}(g) \circ \varphi^{-1} = \mathrm{Ad}_{\tilde{L}}(\theta(g))$. *Therefore* $\mathrm{Ad}_{\tilde{L}}(\tilde{G})$ *is a normal subgroup of* $\tilde{H}$ *and the identity component of* $\mathrm{Ad}_{\tilde{L}}(\tilde{G})$ *coinsides with that of* $\tilde{H}$. *In particular* $\tilde{H}$ *is a reductive subgroup of* $GL(\tilde{L})$.

(ii) *Let* $H := \langle\varphi\rangle$ *be the finite subgroup of* $\tilde{H}$ *generated by* φ. *Then the fixed points set* $\tilde{L}^H := \{X \in \tilde{L} \mid h \cdot X = X \text{ for any } h \in H\}$ *of* $\tilde{L}$ *under the action of* H *coinsides with* L.

(iii) *We have* $Z_{\tilde{H}}(H) = \langle \mathrm{Ad}_{\tilde{L}}(G') \cup \{\varphi\}\rangle$. *Moreover* $\mathrm{Ad}_{\tilde{L}}(G')$ *is reductive.*

Proof. For $g \in \tilde{G}$ and $X \in \tilde{L}$, since $\theta(g) \in \tilde{G}$ and α commutes with $\mathrm{Ad}_{\tilde{L}}(\theta(g))$, we compute

$$\begin{aligned}\varphi \circ \mathrm{Ad}_{\tilde{L}}(g) \circ \varphi^{-1}(X) &= \alpha^{-1}(\theta(\mathrm{Ad}_{\tilde{L}}(g)\theta^{-1}(\alpha(X)))) \\ &= \alpha^{-1}(\mathrm{Ad}_{\tilde{L}}(\theta(g))\alpha(X)) = \mathrm{Ad}_{\tilde{L}}(\theta(g))X.\end{aligned}$$

Hence (i) follows.

(ii) is obvious.

By (i), any $\tilde{g} \in \tilde{H}$ can be written as $\tilde{g} = \mathrm{Ad}_{\tilde{L}}(g) \circ \varphi^k$ for some $g \in \tilde{G}$ and an integer $k \geq 0$. Again by (i), we have

$$\begin{aligned}\varphi \circ \tilde{g} \circ \varphi^{-1} &= \varphi \circ \{\mathrm{Ad}_{\tilde{L}}(g) \circ \varphi^k\} \circ \varphi^{-1} \\ &= \varphi \circ \{\mathrm{Ad}_{\tilde{L}}(g) \circ \varphi^{-1}\} \circ \varphi^k = \mathrm{Ad}_{\tilde{L}}(\theta(g)) \circ \varphi^k.\end{aligned}$$

Therefore we see

$$\begin{aligned}\tilde{g} \in Z_{\tilde{H}}(H) &\Leftrightarrow \varphi \circ \tilde{g} \circ \varphi^{-1} = \tilde{g} \Leftrightarrow \mathrm{Ad}_{\tilde{L}}(\theta(g)) \circ \varphi^k = \mathrm{Ad}_{\tilde{L}}(g) \circ \varphi^k \\ &\Leftrightarrow \mathrm{Ad}_{\tilde{L}}(\theta(g)) = \mathrm{Ad}_{\tilde{L}}(g).\end{aligned}$$

Hence $Z_{\tilde{H}}(H) = \langle \mathrm{Ad}_{\tilde{L}}(G') \cup \{\varphi\} \rangle$. Since $\tilde{H}$ is reductive and H is a finite subgroup of $\tilde{H}$, $Z_{\tilde{H}}(H)$ is reductive by [LR, Lemma 1.1]. It is clear that the identity component of $\mathrm{Ad}_{\tilde{L}}(G')$ coincides with that of $Z_{\tilde{H}}(H)$. Hence $\mathrm{Ad}_{\tilde{L}}(G')$ is also reductive. □

In the setting of Lemma 9, we notice that $\mathbb{C}[\tilde{L}]^{\tilde{H}} = (\mathbb{C}[\tilde{L}]^{\tilde{G}})^{\langle\varphi\rangle} \hookrightarrow \mathbb{C}[\tilde{L}]^{\tilde{G}}$, $\mathbb{C}[L]^{Z_{\tilde{H}}(H)} = \mathbb{C}[L]^{G'}$, and $\tilde{H} \cdot \mathcal{O} = \langle\varphi\rangle \cdot (\tilde{G} \cdot \mathcal{O})$ for $\mathcal{O} \in L/G' = L/Z_{\tilde{H}}(H)$. Then Theorem 8, (i) and (ii) follow from the next theorem due to Luna.

Theorem 10 ([L], see also [PV, Theorem 6.16 and Theorem 6.17]) *Suppose that a reductive group $\tilde{H}$ acts on an affine variety $\tilde{X}$ and that H is a reductive subgroup of $\tilde{H}$. Let $X = \tilde{X}^H := \{x \in \tilde{X} \mid h \cdot x = x$ for any $h \in H\}$ be the fixed points set of $\tilde{X}$ under the action of H. Then we have the following.*

(i) *The morphism $X//Z_{\tilde{H}}(H) \to \tilde{X}//\tilde{H}$ defined by the resrtriction map* $\mathrm{rest}\colon \mathbb{C}[\tilde{X}]^{\tilde{H}} \to \mathbb{C}[X]^{Z_{\tilde{H}}(H)}$ *is finite (i.e., $\mathbb{C}[X]^{Z_{\tilde{H}}(H)}$ is integral over $\mathbb{C}[\tilde{X}]^{\tilde{H}}|_X$).*

(ii) *For a point $x \in X$, the orbit $\tilde{H} \cdot x$ is closed in $\tilde{X}$ if and only if $Z_{\tilde{H}}(H) \cdot x$ is closed in X.*

Let us give a proof of Theorem 8, (iii). Since the restriction map

$\mathbb{C}[\tilde{L}]^{\tilde{G}} \to \mathbb{C}[L]^{G'}$ is decomposed as

$$\mathbb{C}[\tilde{L}]^{\tilde{G}} \to \mathbb{C}[\tilde{L}]^{\tilde{G}}|_L \hookrightarrow \mathbb{C}[L]^{G'},$$

the morphism $L//G' \to \tilde{L}//\tilde{G}$ is also decopmosed as

$$L//G' \xrightarrow{\pi} \mathrm{Spec}(\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L) \to \tilde{L}//\tilde{G}.$$

Since π is finite (closed map) and dominant, π is surjective. On the other hand, since $L//G' \to \tilde{L}//\tilde{G}$ is injective, so is π. Then the birationality of π follows from the next theorem.

Theorem 11 ([Hu, Theorem 4.6]) *Let $\pi\colon X \to Y$ be a dominant, injective morphism of irreducible varieties over an algebraically closed field K. Then via π, the function field $K(X)$ is a finite, purely inseparable extension of $K(Y)$.*

(2.3) Inclusion theorem and rings of invariants

Theorem 12 *In the setting of* Theorem 1*, we assume the following in addition to* (a), (b) *of* Theorem 1.

(c) *The element $\varphi \in GL(\tilde{L})$, defined by $\varphi(X) = \alpha^{-1}(\sigma(X))$ $(X \in \tilde{L})$, has finite order.*

Then we have the following:

(i) *For the correspondence*

$$L/G \to \tilde{L}/\tilde{G},\ \mathcal{O} \mapsto \tilde{\mathcal{O}} := \mathrm{Ad}(\tilde{G}) \cdot \mathcal{O},$$

$\tilde{\mathcal{O}}$ is closed in $\tilde{L}$ if and only if $\mathcal{O}$ is closed in L.

(ii) *The morphism $L//G \to \mathrm{Spec}(\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L)$, defined by $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L \hookrightarrow \mathbb{C}[L]^G$, is bijective and gives a normalization of the variety* $\mathrm{Spec}(\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L)$ *(i.e., $L//G$ is normal and the morphism is finite, birational). In particular, if the ring $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$ is normal (it is equivalent that $\overline{\pi_{(\tilde{G},\tilde{L})}(L)}$ is normal by* Proposition 7*), then $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L = \mathbb{C}[L]^G$ and the restriction map* $\mathrm{rest}\colon \mathbb{C}[\tilde{L}]^{\tilde{G}} \to \mathbb{C}[L]^G$ *is surjective.*

Proof. Let us consider the automorphism $\theta\colon GL(V) \to GL(V)$ defined by $\theta(g) = \sigma(g)^{-1}$ $(g \in GL(V))$. Then the corresponding Lie algebra automorphism $\theta\colon \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ is given by $\theta(X) = -\sigma(X)$ $(X \in \mathfrak{gl}(V))$. Moreover the group G and the subspace L can be written as

$$G = \{g \in \tilde{G} \mid \theta(g) = g\}, \quad L = \{X \in \tilde{L} \mid \theta(X) = -\alpha(X)\}$$

We also consider the subgroup $G' = \{g \in \tilde{G} \mid \mathrm{Ad}_{\tilde{L}}(g) = \mathrm{Ad}_{\tilde{L}}(\theta(g))\}$ of $\tilde{G}$ which contains G. Since the correspondence $L/G \to \tilde{L}/\tilde{G}$ decomposed as

$$L/G \to L/G' \to \tilde{L}/\tilde{G}$$

and $L/G \to \tilde{L}/\tilde{G}$ is injective by Theorem 1, the correspondence

$$L/G \to L/G' \ (\mathcal{O} \mapsto \mathrm{Ad}(G') \cdot \mathcal{O})$$

is bijective. This means that, for any point $x \in L$, two orbits $\mathrm{Ad}(G)x$ and $\mathrm{Ad}(G')x$ coinside. In particular, we have $\mathbb{C}[L]^G = \mathbb{C}[L]^{G'}$. Therefor we can apply Theorem 8 by taking G instead of G' and obtain Theorem 12. □

3. Examples

Let us give some examples for which Theorem 1, Theorem 8 and Theorem 12 can be applied.

(3.1) $(O(n, \mathbb{C}), \mathfrak{o}(n, \mathbb{C})) \hookrightarrow (GL(n, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C}))$

Put $\tilde{G} = GL(n, \mathbb{C})$, $\tilde{L} = \mathrm{Mat}_{n\times n}(\mathbb{C})$ (the set of $n \times n$-matrices) and consider the anti-involution

$$\sigma\colon \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Mat}_{n\times n}(\mathbb{C}),\ X \mapsto {}^tX.$$

We take

$$G := \{g \in \tilde{G} \mid \sigma(g) = g^{-1}\} = O(n, \mathbb{C}) \quad \text{and } L := \{X \in \tilde{L} \mid \sigma(X) = -X\}.$$

By Theorem 1 and Theorem 12, we have

(1) $L/G \to \tilde{L}/\tilde{G}$ is injective.

(2) The quotient fields of $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$ and $\mathbb{C}[L]^G$ coinside and $\mathbb{C}[L]^G$ is the integral closure of $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$ in its quotient field.

Let us show that $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L = \mathbb{C}[L]^G$. Define functions $P_j \in \mathbb{C}[\tilde{L}]$ by

$$\det(T1_n - X) = T^n + P_1(X)T^{n-1} + \cdots + P_n(X),\ (X \in \tilde{L}).$$

It is well known that $P_1, \ldots, P_n$ are algbraically independent and $\mathbb{C}[\tilde{L}]^{\tilde{G}} = \mathbb{C}[P_1, \ldots, P_n]$.

For $X \in L$, it is clear that $P_j(X) = 0$ for odd j. Hence

$$\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L = \mathbb{C}[P_2|_L, P_4|_L, \ldots, P_{2[n/2]}|_L].$$

Put

$$A = \begin{pmatrix} 0 & a_1 & & & & & & \\ -a_1 & 0 & & & & \mathbf{0} & & \\ & & 0 & a_2 & & & & \\ & & -a_2 & 0 & & & & \\ & & & & \ddots & & & \\ & \mathbf{0} & & & & 0 & a_{[n/2]} \\ & & & & & -a_{[n/2]} & 0 \end{pmatrix}$$

and consider an element $X = A \in L$ (n is even) or $X = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) \in L$ (n is odd). Then we see

$$\det(T1_n - X) = (T^2 + a_1^2) \dots (T^2 + a_{[n/2]}^2) T^{n-2[n/2]}.$$

From this, we find that $P_2|_L$, $P_4|_L$, $\dots$, $P_{[n/2]}|_L$ are algebraically independent. Hence $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$ is isomorphic to a polynomial ring. By (2) above, we obtain

(3) $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L = \mathbb{C}[L]^G$.

(3.2) Symmetric pairs $(\mathfrak{sp}(2m, \mathbb{C}), \mathfrak{gl}(m, \mathbb{C})) \hookrightarrow (\mathfrak{gl}(2m, \mathbb{C}), \mathfrak{gl}(m, \mathbb{C}) + \mathfrak{gl}(m, \mathbb{C}))$

Let us consider a vector space $V = \mathbb{C}^{2m}$, a matrix $S = \left(\begin{smallmatrix} 1_m & 0 \\ 0 & -1_m \end{smallmatrix}\right)$ and an automorphism $\theta\colon GL(V) \to GL(V)$, $\theta(g) = SgS^{-1}$. Let us take subgroups

$$\begin{aligned} \tilde{G} &= \{g \in GL(V) \mid \theta(g) = g\} = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \,\middle|\, g_1,\, g_2 \in GL(m, \mathbb{C}) \right\}, \\ GL(V)' &= \{g \in GL(V) \mid \mathrm{Ad}(\theta(g)) = \mathrm{Ad}(g)\} \\ &= \left\langle \tilde{G} \cup \left\{ \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix} \right\} \right\rangle \end{aligned}$$

of $GL(V)$ and a subspace

$$\begin{aligned} \tilde{\mathfrak{s}} &= \{X \in \mathfrak{gl}(V) \mid \theta(X) = -X\} \\ &= \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \,\middle|\, B,\, C \in \mathrm{Mat}_{m\times m}(\mathbb{C}) \right\} \end{aligned}$$

of $\mathfrak{gl}(V)$. Apply Theorem 8, (i) to the inclusion $(GL(V)', \tilde{\mathfrak{s}}) \hookrightarrow (GL(V), \mathfrak{gl}(V))$. Then, for an orbit $\mathcal{O}' \in \tilde{\mathfrak{s}}/GL(V)'$, $\mathcal{O}'$ is closed in $\tilde{\mathfrak{s}}$ if and only if $\mathrm{Ad}(GL(V)) \cdot \mathcal{O}'$ is a semisimple orbit. Since ${}^\sharp(GL(V)'/\tilde{G}) < \infty$, we obtain the following well-known fact due to [KR].

(0) For an orbit $\mathcal{O} \in \tilde{\mathfrak{s}}/\tilde{G}$, $\mathcal{O}$ is closed in $\tilde{\mathfrak{s}}$ if and only if $\mathrm{Ad}(GL(V))\mathcal{O}$ is a semisimple orbit.

By [O3], we have the following:

(1) The eigenvalues of an element of $\tilde{\mathfrak{s}}$ are of the form $\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \ldots, \alpha_m, -\alpha_m$ ($\alpha_j \in \mathbb{C}$). Moreover, for given $\alpha_j \in \mathbb{C}$ ($1 \leq j \leq m$), there exsist an element of $\tilde{\mathfrak{s}}$ with eigenvalues $\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \ldots, \alpha_m, -\alpha_m$.

(2) For two semisimple elements $X, Y \in \tilde{\mathfrak{s}}$, X and Y are conjugate under $\tilde{G}$ if and only if the eigenvalues (with multiplicities) of X and Y coinside.

The statement (2) implies

(3) The morphism $\tilde{\mathfrak{s}}//\tilde{G} \to \mathfrak{gl}(V)//GL(V)$ defined by rest: $\mathbb{C}[\mathfrak{gl}(V)]^{GL(V)} \to \mathbb{C}[\tilde{\mathfrak{s}}]^{\tilde{G}}$ is injective.

By Theorem 8, (iii), we have

(4) The quotient fields of $\mathbb{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\tilde{\mathfrak{s}}}$ and $\mathbb{C}[\tilde{\mathfrak{s}}]^{\tilde{G}}$ coinside and $\mathbb{C}[\tilde{\mathfrak{s}}]^{\tilde{G}}$ is the integral closure of $\mathbb{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\tilde{\mathfrak{s}}}$ in its quotient field.

Define functions $P_1, P_2, \ldots, P_{2m} \in \mathbb{C}[\mathfrak{gl}(V)]$ by

$$\det(T1_{2m} - X) = T^{2m} + P_1(X)T^{2m-1} + \cdots + P_{2m}(X), \quad (X \in \mathfrak{gl}(V)).$$

For $X \in \tilde{\mathfrak{s}}$, since $SXS^{-1} = -X$, $P_j(X) = 0$ for odd j and hence

$$\mathbb{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\tilde{\mathfrak{s}}} = \mathbb{C}[P_2|_{\tilde{\mathfrak{s}}}, P_4|_{\tilde{\mathfrak{s}}}, \ldots, P_{2m}|_{\tilde{\mathfrak{s}}}].$$

Suppose that the eigenvalues of an element $X \in \tilde{\mathfrak{s}}$ are $\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \ldots, \alpha_m, -\alpha_m$. Then we have

$$\det(T1_{2m} - X) = (T^2 - \alpha_1^2)(T^2 - \alpha_2^2) \cdots (T^2 - \alpha_m^2).$$

From this, we know that $P_2|_{\tilde{\mathfrak{s}}}, P_4|_{\tilde{\mathfrak{s}}}, \ldots, P_{2m}|_{\tilde{\mathfrak{s}}}$ are algebraically independent. Hence $\mathbb{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\tilde{\mathfrak{s}}}$ is isomorphic to a polynomial ring. By (4) above, we obtain

(5) $\mathbb{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\tilde{\mathfrak{s}}} = \mathbb{C}[\tilde{\mathfrak{s}}]^{\tilde{G}}$.

Next we consider an anti-automorphism $\sigma\colon \mathrm{End}(V) \to \mathrm{End}(V)$ defined by $\sigma(X) = J^{-1}{}^tXJ$ ($X \in \mathrm{End}(V)$), where we put $J := \left(\begin{smallmatrix} 0 & 1_m \\ -1_m & 0 \end{smallmatrix}\right)$. We also consider the subgroup $G = \{g \in \tilde{G} \mid \sigma(g) = g^{-1}\}$ of $\tilde{G}$ and a subspace

$$\mathfrak{s} = \{X \in \tilde{\mathfrak{s}} \mid \sigma(X) = -X\} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \middle| B, C \in \mathrm{Sym}_m(\mathbb{C}) \right\}$$

of $\tilde{\mathfrak{s}}$. Then by Theorem 1, the orbits correspondence $\mathfrak{s}/G \to \tilde{\mathfrak{s}}/\tilde{G}$ is injective. For

$$X = \begin{pmatrix} & & b_1 & \\ & \mathbf{0} & & \ddots \\ & & & & b_m \\ c_1 & & & & \\ & \ddots & & \mathbf{0} & \\ & & c_m & & \end{pmatrix} \in \mathfrak{s},$$

we find $\det(T1_{2m}-X) = (T^2-b_1c_1)(T^2-b_2c_2)\cdots(T^2-b_mc_m)$. From this, we find that $P_2|_{\mathfrak{s}}, P_4|_{\mathfrak{s}}, \ldots, P_{2m}|_{\mathfrak{s}}$ are algebraically independent and $\mathbb{C}[\tilde{\mathfrak{s}}]^{\tilde{G}}|_{\mathfrak{s}} = \mathbb{C}[P_2|_{\mathfrak{s}}, P_4|_{\mathfrak{s}}, \ldots, P_{2m}|_{\mathfrak{s}}]$ is isomorphic to a polynomial ring. Therefore by Theorem 12, we obtain

(6) $\mathbb{C}[\mathfrak{s}]^G = \mathbb{C}[\tilde{\mathfrak{s}}]^{\tilde{G}}|_{\mathfrak{s}} = \mathbb{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\mathfrak{s}}$.

$(G, \mathfrak{s})$ is an example of classical graded Lie algebras. Generalization of these results for general classical graded Lie algebras will be given in [O3].

4. FFT for GL_n and that for O_n, Sp_n

Let us consider a vector space $V = \mathbb{C}^{n+m}$ and a matrix $J = \begin{pmatrix} K & 0 \\ 0 & 1_m \end{pmatrix}$, where we put

$$K = \begin{cases} 1_n & (\varepsilon = 1) \\ \begin{pmatrix} 0 & 1_{n/2} \\ -1_{n/2} & 0 \end{pmatrix} & (\varepsilon = -1,\ n\colon \text{even}) \end{cases}.$$

Define an anti-automorphism $\sigma\colon \operatorname{End}(V) \to \operatorname{End}(V)$ by $\sigma(X) = J^{-1t}XJ$ $(X \in \operatorname{End}(V))$. We consider the following subgroups of $GL(V)$:

$$\begin{aligned} \tilde{G} &:= \left\{ \begin{pmatrix} g & 0 \\ 0 & c1_m \end{pmatrix} \,\middle|\, g \in GL(n, \mathbb{C}),\ c \in \mathbb{C}^\times \right\} \simeq GL(n, \mathbb{C}) \times \mathbb{C}^\times, \\ G &= \{x \in \tilde{G} \mid \sigma(x) = x^{-1}\} \\ &= \left\{ \begin{pmatrix} g & 0 \\ 0 & c1_m \end{pmatrix} \,\middle|\, J^{-1t}gJ = g^{-1},\ c \in \{\pm 1\} \right\} \\ &\simeq \begin{cases} O(n, \mathbb{C}) \times \{\pm 1_m\} & (\varepsilon = 1) \\ Sp(n, \mathbb{C}) \times \{\pm 1_m\} & (\varepsilon = -1) \end{cases}. \end{aligned}$$

We also consider the following subspaces of $\mathrm{End}(V)$:

$$\tilde{L} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \middle| \; B \in \mathrm{Mat}_{n\times m}(\mathbb{C}),\ C \in \mathrm{Mat}_{m\times n}(\mathbb{C}) \right\},$$
$$L = \{X \in \tilde{L} \mid \sigma(X) = X\} = \left\{ \begin{pmatrix} 0 & B \\ {}^tBK & 0 \end{pmatrix} \middle| \; B \in \mathrm{Mat}_{n\times m}(\mathbb{C}) \right\}.$$

Then we can easily verify that the assumptions of Theorem 1 and Theorem 12 hold in this situation (with $\alpha = 1$). Therefore we have the following:

(1) The correspondence $L/G \to \tilde{L}/\tilde{G},\ \mathcal{O} \mapsto \tilde{\mathcal{O}} = \mathrm{Ad}(\tilde{G}) \cdot \mathcal{O}$, is injective.
(2) $\mathcal{O} \in L/G$ is closed in L if and only if $\tilde{\mathcal{O}} \in \tilde{L}/\tilde{G}$ is closed in $\tilde{L}$.
(3) The quotient fields of $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$ and $\mathbb{C}[L]^G$ coinside and $\mathbb{C}[L]^G$ is the integral closure of $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L$ in its quotient field.

We easily see that

$$\mathrm{Ad}_{\tilde{L}}(\tilde{G}) = \mathrm{Ad}_{\tilde{L}}(GL(n, \mathbb{C}) \times \{1_m\}) \text{ and}$$
$$\mathrm{Ad}_{\tilde{L}}(G) = \begin{cases} \mathrm{Ad}_{\tilde{L}}(O(n, \mathbb{C}) \times \{1_m\}) & (\varepsilon = 1) \\ \mathrm{Ad}_{\tilde{L}}(Sp(n, \mathbb{C}) \times \{1_m\}) & (\varepsilon = -1) \end{cases}.$$

In such way, we can consider $\tilde{G}$, G, $\tilde{L}$, L as

$$G = \begin{cases} O(n, \mathbb{C}) & (\varepsilon = 1) \\ Sp(n, \mathbb{C}) & (\varepsilon = -1) \end{cases} \hookrightarrow \tilde{G} = GL(n, \mathbb{C}),$$
$$L = \mathrm{Mat}_{n\times m}(\mathbb{C}) \hookrightarrow$$
$$\tilde{L} = \mathrm{Mat}_{m\times n}(\mathbb{C}) \times \mathrm{Mat}_{n\times m}(\mathbb{C}) \ (B \mapsto ({}^tBK,\, B)),$$

where the action of $\tilde{G}$ on $\tilde{L}$ is given by $g \cdot (C, B) = (Cg^{-1}, gB)$ ($g \in \tilde{G}$, $(C, B) \in \tilde{L}$) and that of G on L is the left action. Notice that the inclusion $L \hookrightarrow \tilde{L}$ is G-equivariant.

For $x = (C, B) \in \tilde{L}$, we put $\pi(x) = CB \in \mathrm{Mat}_{m\times m}(\mathbb{C})$ and obtain a map $\pi\colon \tilde{L} \to \mathrm{Mat}_{m\times m}(\mathbb{C})$. Denote by $\pi_{i,j}(x)$ ($1 \le i, j \le m$) the (i, j)-entry of $\pi(x)$. Clearly $\pi_{i,j} \in \mathbb{C}[\tilde{L}]^{\tilde{G}}$. First fundamental theorem (FFT) for invariant theory for GL_n says that

FFT for $\boldsymbol{GL_n}$ $\mathbb{C}[\tilde{L}]^{\tilde{G}} = \mathbb{C}[\pi_{i,j}]_{1\le i,j\le m}$.

This implies that $\pi\colon \tilde{L} \to \pi(\tilde{L})$ is the affine quotient map under $\tilde{G}$; $\pi(\tilde{L}) \simeq \tilde{L}/\!/\tilde{G}$. Then, if we can show that $\overline{\pi(L)}$ is normal, we obtain $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L = \mathbb{C}[L]^G$ by Theorem 12, (ii).

Suppose $n \geq m$. Then it is easy to see that

$$\pi(L) = \begin{cases} \mathrm{Sym}_m(\mathbb{C}) & (\varepsilon = 1) \\ \mathrm{Alt}_m(\mathbb{C}) & (\varepsilon = -1) \end{cases} \text{ and}$$

$$\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L = \begin{cases} \mathbb{C}[\pi_{i,j}|_L]_{1\leq i\leq j\leq m} & (\varepsilon = 1) \\ \mathbb{C}[\pi_{i,j}|_L]_{1\leq i<j\leq m} & (\varepsilon = -1) \end{cases}.$$

Hence $\overline{\pi(L)} = \pi(L)$ is normal and we obtain

(4) $\mathbb{C}[\tilde{L}]^{\tilde{G}}|_L = \mathbb{C}[L]^G$, $\mathbb{C}[L]^G = \begin{cases} \mathbb{C}[\pi_{i,j}|_L]_{1\leq i\leq j\leq m} & (\varepsilon = 1) \\ \mathbb{C}[\pi_{i,j}|_L]_{1\leq i<j\leq m} & (\varepsilon = -1) \end{cases}$ and the functions $\pi_{i,j}|_L$ ($1 \leq i \leq j \leq m$ in case $\varepsilon = 1$ and $1 \leq i < j \leq m$ in case $\varepsilon = -1$) are algebraically independent generaters of $\mathbb{C}[L]^G$.

These are FFT for $O(n,\mathbb{C})$ and $Sp(n,\mathbb{C})$ in case $m \leq n$. In such way, we can prove FFT for $O(n,\mathbb{C})$ and $Sp(n,\mathbb{C})$ by using FFT for $GL(n,\mathbb{C})$ and Theorem 12.

5. Embedding of the action of Doković, Sekiguchi and Zhao

Let us consider a vector space $V = \mathbb{C}^{4n}$ and a matrix $J = \begin{pmatrix} 0 & 0 & 0 & 1_n \\ 0 & 0 & 1_n & 0 \\ 0 & 1_n & 0 & 0 \\ 1_n & 0 & 0 & 0 \end{pmatrix}$.
Define an anti-involution $\sigma\colon \mathrm{End}(V) \to \mathrm{End}(V)$ by $\sigma(X) = J^{-1}{}^tXJ$ ($X \in \mathrm{End}(V)$). We consider the following subgroups of $GL(V)$:

$$\tilde{G} := \left\{ \begin{pmatrix} g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & h \end{pmatrix} \middle| g,\, h \in GL(n,\mathbb{C}) \right\},$$

$$G = \{x \in \tilde{G} \mid \sigma(x) = x^{-1}\}$$
$$= \left\{ \begin{pmatrix} g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & {}^tg^{-1} & 0 \\ 0 & 0 & 0 & {}^tg^{-1} \end{pmatrix} \middle| g \in GL(n,\mathbb{C}) \right\}.$$

We also consider the following subspaces of $\mathrm{End}(V)$:

$$\tilde{L} = \left\{ \begin{pmatrix} 0 & 0 & X & 0 \\ 0 & 0 & 0 & Y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| X,\, Y \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \right\},$$

$$L = \{A \in \tilde{L} \mid \sigma(A) = A\} = \left\{ \begin{pmatrix} 0 & 0 & X & 0 \\ 0 & 0 & 0 & {}^tX \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, X \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \right\}.$$

It is easy to see that these satisfy the assumption of Theorem 1. Hence
(1) The correspondence $L/G \to \tilde{L}/\tilde{G},\ \mathcal{O} \mapsto \tilde{\mathcal{O}} = \mathrm{Ad}(\tilde{G}) \cdot \mathcal{O}$ is injective.
By the natural identifications, we can consider $\tilde{G}$, G, $\tilde{L}$, L as

$$\begin{aligned} &G = GL(n,\, \mathbb{C}) \hookrightarrow \tilde{G} = GL(n,\, \mathbb{C}) \times GL(n,\, \mathbb{C}),\ g \mapsto (g,\, {}^tg^{-1}), \\ &L = \mathrm{Mat}_{n\times n}(\mathbb{C}) \hookrightarrow \tilde{L} = \mathrm{Mat}_{n\times n}(\mathbb{C}) \times \mathrm{Mat}_{n\times n}(\mathbb{C}),\ X \mapsto (X,\, {}^tX), \end{aligned}$$

where the action of $\tilde{G}$ on $\tilde{L}$ is given by $(g,\, h) \cdot (X,\, Y) = (gXh^{-1},\, gYh^{-1})$ $((g,\, h) \in \tilde{G},\ (X,\, Y) \in \tilde{L})$ and that of G on L is given by $g \cdot X = gX{}^tg$ $(g \in G,\ X \in L)$. Notice that the inclusion $L \hookrightarrow \tilde{L}$ is G-equivariant. The action $G \times L \to L$ is that considered in [DSZ].

For these actions, we easily see that $\mathbb{C}[\tilde{L}]^{\tilde{G}} = \mathbb{C}[L]^G = \mathbb{C}$. Let us determine the fields of rational invariants $\mathbb{C}(\tilde{L})^{\tilde{G}}$ and $\mathbb{C}(L)^G$, and show that generaters of $\mathbb{C}(L)^G$ are obtained by restrictions of some elements of $\mathbb{C}(\tilde{L})^{\tilde{G}}$.

Define functions $P_0,\, P_1,\, \ldots,\, P_n \in \mathbb{C}[\tilde{L}]$ by

$$\begin{aligned} \det(TX + Y) = P_0(X,\, Y)T^n &+ P_1(X,\, Y)T^{n-1} \\ &+ \cdots + P_n(X,\, Y) \quad ((X,\, Y) \in \tilde{L}). \end{aligned}$$

Notice that $P_0(X,\, Y) = \det(X)$ and $P_n(X,\, Y) = \det(Y)$. Since

$$P_j\big((g,\, h) \cdot (X,\, Y)\big) = \det(g)\det(h)^{-1} P_j\big((X,\, Y)\big) \quad ((g,\, h) \in \tilde{G}),$$

rational functions $f_j := P_j/P_0$ $(1 \le j \le n)$ are elements of $\mathbb{C}(\tilde{L})^{\tilde{G}}$. Define dense open subset $\tilde{L}_0$ of $\tilde{L}$ by

$$\begin{aligned} \tilde{L}_0 := \{(X,\, Y) \in \tilde{L} \mid \det(X) \neq 0 \neq \det(Y) &\text{ and} \\ X^{-1}Y &\text{ has distinct eigenvalues}\}. \end{aligned}$$

For $(X_1,\, Y_1),\, (X_2,\, Y_2) \in \tilde{L}_0$, suppose that $f_j(X_1,\, Y_1) = f_j(X_2,\, Y_2)$ for any $1 \le j \le n$. Then we have

$$\begin{aligned} \det(T1_n + X_2^{-1}Y_2) &= \det(X_2)^{-1}\det(TX_2 + Y_2) \\ &= \det(X_1)^{-1}\det(TX_1 + Y_1) \\ &= \det(T1_n + X_1^{-1}Y_1). \end{aligned}$$

Since $X_1^{-1}Y_1$ and $X_2^{-1}Y_2$ have distinct eigenvalues, there exists $h \in GL(n, \mathbb{C})$ such that $X_2^{-1}Y_2 = h(X_1^{-1}Y_1)h^{-1}$. If we put $g := X_2hX_1^{-1}$, we have $gX_1h^{-1} = X_2$ and $Y_2 = (X_2hX_1^{-1})Y_1h^{-1} = gY_1h^{-1}$. Hence (X_1, Y_1) and (X_2, Y_2) are conjugate under $\tilde{G}$. Therefore the rational invariants $f_1, f_2, \ldots, f_n \in \mathbb{C}(\tilde{L})^{\tilde{G}}$ separate $\tilde{G}$-orbits in $\tilde{L}_0$. By [PV, Lemma 2.1], we have

(2) $\mathbb{C}(\tilde{L})^{\tilde{G}} = \mathbb{C}(f_1, f_2, \ldots, f_n)$.

We easily see that

(3) $P_j(X, Y) = P_{n-j}(Y, X)$ $(0 \le j \le n)$. In particular, $P_j(X, {}^tX) = P_j({}^tX, X) = P_{n-j}(X, {}^tX)$ $(0 \le j \le n)$ for any $(X, {}^tX) \in L$.

Thus we see $f_j|_L = f_{n-j}|_L$ $(1 \le j \le n-1)$ and obtain rational invariants $f_1|_L, f_2|_L, \ldots, f_{[n/2]}|_L \in \mathbb{C}(L)^G$.

Let us show that

(4) $\mathbb{C}(L)^G = \mathbb{C}(f_1|_L, f_2|_L, \ldots, f_{[n/2]}|_L)$.

For this purpose we first show that $\tilde{L}_0 \cap L \neq \emptyset$. Suppose that $n = 2m+1$ is odd. Take a skew-symmetric matrix A with distinct eigenvalues $a_1, a_2, \ldots, a_m, 0, -a_1, -a_2, \ldots, -a_m$ and a scalar $c \in \mathbb{C}^\times$ such that $c \neq \pm a_j$ $(1 \le j \le m)$. Put $X = c1_n + A$. Then $X^{-1\,t}X = (c1_n + A)^{-1}(c1_n - A)$ has distinct eigenvalues

$$\frac{c-a_1}{c+a_1}, \ldots, \frac{c-a_m}{c+a_m}, 1, \frac{c+a_1}{c-a_1}, \ldots, \frac{c+a_m}{c-a_m}$$

Hence $(X, {}^tX) \in \tilde{L}_0 \cap L$ and $\tilde{L}_0 \cap L \neq \emptyset$. Similarly we can show that $\tilde{L}_0 \cap L \neq \emptyset$ for even n.

For $(X_1, {}^tX_1), (X_2, {}^tX_2) \in \tilde{L}_0 \cap L$, suppose that $f_j(X_1, {}^tX_1) = f_j(X_2, {}^tX_2)$ for any $1 \le j \le [n/2]$. Then the same equations hold for any $1 \le j \le n$. Since $f_1, f_2, \ldots, f_n \in \mathbb{C}(\tilde{L})^{\tilde{G}}$ separate $\tilde{G}$-orbits in $\tilde{L}_0$, $(X_1, {}^tX_1)$ and $(X_2, {}^tX_2)$ are conjugate under $\tilde{G}$. By the fact (1) above, $(X_1, {}^tX_1)$ and $(X_2, {}^tX_2)$ are conjugate under G. Therefor the rational invariants $f_1|_L, f_2|_L, \ldots, f_{[n/2]}|_L \in \mathbb{C}(L)^G$ separate G-orbits in $\tilde{L}_0 \cap L$. Again by [PV, Lemma 2.1], we obtain the fact (4).

Therefore, for the inclusion $(G, L) \hookrightarrow (\tilde{G}, \tilde{L})$, $\mathbb{C}(L)^G$ is generated by the restrictions of elements of $\mathbb{C}(\tilde{L})^{\tilde{G}}$.

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