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Bilinear Strichartz estimates and applications to the cubic nonlinear Schrödinger equation in two space dimensions*

Hideo Takaoka

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Abstract. The initial value problem for the defocusing cubic nonlinear Schrödinger equation on $\mathbb{R}^2$ is locally well-posed in H^s for $s \geq 0$. The L^2-space norm is invariant under rescaling to the equation, then the critical regularity is $s = 0$. In this note, we prove the global well-posedness in H^s for all $s > 1/2$. The proof uses the almost conservation approach by adding additional (non-resonant) correction terms to the original almost conserved energy.

Key words: Strichartz estimate, nonlinear Schrödinger equation, global well-posedness.

1. Introduction

This note concerns with the initial value problem (IVP) for the cubic nonlinear Schrödinger equation in $\mathbb{R}^{1+2}$

$$\begin{cases} i\partial_t u(t,\,x) + \Delta u(t,\,x) = |u(t,\,x)|^2 u(t,\,x), & (t,\,x) \in \mathbb{R}^{1+2} \\ u(0,\,x) = u_0(x) \in H^s(\mathbb{R}^2), \end{cases} \tag{1.1}$$

where $H^s(\mathbb{R}^2)$ denotes the inhomogeneous Sobolev space. In general, the conservation laws of L^2-mass and $\dot{H}^1$-energy can be used to obtain the global well-posedness results in L^2 and H^1 spaces. We will be interested in the global-in-time well-posedness of (1.1) for low-regularity s below the energy regularity.

The equation (1.1) has the L^2-mass conservation law

$$\int_{\mathbb{R}^2} |u(t,\,x)|^2 dx = \int_{\mathbb{R}^2} |u_0(x)|^2 dx, \tag{1.2}$$

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*This is joint work with James Colliander, Markus Keel, Gigliola Staffilani and Terence Tao [8]. This note summarizes the result the author presented at the Nonlinear Wave Equations at Hokkaido University.

and the (total) energy conservation law

$$E[u(t)] := \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u(t, x)|^2 + \frac{1}{4}|u(t, x)|^4 dx = E[u_0]. \tag{1.3}$$

It is known that the IVP (1.1) is locally well-posed when $s \geq 0$, and the time-interval of existence of solution can be obtained in the term of H^s norm of the data when $s > 0$ (cf. [4, 14])[1]. Moreover, the solution-map $u_0 \mapsto u(t)$ is continuous[2] for $s \geq 0$, and not uniformly continuous for $s < 0$.

The L^2-space is the critical space for (1.1) with respect to the scale invariant space under the scaling symmetry

$$u(t, x) \mapsto \frac{1}{\lambda} u\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big), \quad \lambda > 0 \tag{1.4}$$

or the Galilean invariant space under the Galilean symmetry

$$u(t, x) \mapsto e^{ix\cdot v + it(|v|^2/2)} u(t, x - vt), \quad v \in \mathbb{R}^2.$$

From the conservation laws (1.2)–(1.3) (and Sobolev inequality), we obtain the time-global a priori estimate for solutions in H^s for $s = 0$ or $s = 1$ of the form

$$\sup_{|t| \leq T} \|u(t)\|_{H^s} \leq C(s, \|u_0\|_{H^s}, T) \tag{1.5}$$

for all $T > 0$. This form of the a priori bound in conjunction with the proof of the local existence theory (in particular the time interval to guarantee the existence of solution depends on the H^s-norm of data) can be used to prove the global well-posedness in H^s for $s \geq 1$. The a priori estimate (1.5) holds for $s = 0$, but the lack of the L^2-local well-posedness theorem can not immediately prove the global well-posedness result in L^2 (If data are small, the global well-posedness holds in L^2 including scattering result).

The first breakthrough to establish the H^s-global well-posedness for fractional exponent s of H^s has been developed by the *Fourier truncation method* of J. Bourgain [2] who obtained for $s > 3/5$. This result was improved by the *Almost conserved quantities* [7], which obtained the estimate (1.5) for $s > 4/7$. These two methods use a low-frequency/high-frequency

[1] When $s = 0$, the time-interval of existence of solution depends upon the profile of data.

[2] When $s > 0$, the H^s well-posedness is shown by the contraction mapping theorem. Then the solution-map is analytic.

decomposition approach, but estimate the nonlinear interactions (low-high energy cascade) in different ways.

Fourier truncation method. With a cut-off frequency $|\xi| = N$, we assess the low-frequency component in $|\xi| \leq N$ and the high-frequency component in $|\xi| > N$, respectively. Roughly, if the solution is decomposed into three components: low frequencies nonlinearly (via original equation (1.1)), high frequencies nonlinearly (via original equation (1.1)) and the low-high frequencies interaction nonlinearly (coupled equation of low-high frequencies). The low frequencies solution conserves the H^1-energy (1.3), but this is large. One observes that the high frequencies solution an be approximated to evolve linearly. One prove that the low-high (high-high also) frequency interactions are small error in H^1, under certain smoothing property, compared to the low-low frequency interactions, provided choosing N. On the other hand, the almost conserved quantities proceed slightly different with the method.

Almost conserved quantities. This method uses the modified energy $E_N[u(t)] = E[I_N u(t)]$, where $I = I_N$ is a Fourier multiplier operator mapping from H^s to H^1 (defined in Section 2.1). More precisely, $I = identy$ for low frequency, while $I = N^{1-s}|\nabla|^{s-1}$ for high frequency. If N large, the modified energy $E_N[(u(t)]$ is qualifiedly equal to the original energy $E[u(t)]$. The low frequency interaction is estimated in H^1 energy space. The low-high (high-high also) frequency interaction is estimated with approximately conserved energy E_N in $I_N H^s$. Thus the multiplier operator I_N has an advantage of improving the estimate developed in [7] for the low-high frequency nonlinear interactions.

In this note we improve the result of [7] and present the following theorem.

Theorem 1 *Let $s > 1/2$. Then the initial value problem* (1.1) *is globally well-posed in H^s. More specifically, for any $u_0 \in H^s$, there exists a unique global solution $u(t)$ to* (1.1) *in $C_t(\mathbb{R}\colon H^s_x)$. rthermore, the a priori bound* (1.5) *holds.*

The proof relies on the modification of the almost conserved quantities used in [7] by adding *resonant correction terms*.

Remark 1 Theorem 1 holds for the focusing cubic nonlinear Schrödinger equation (replacing the sign of nonlinearity), assuming the smallness of the L^2-norm of the initial data $\|u_0\|_{L^2} < \|Q\|_{L^2}$, where Q is the positive solution of $\Delta Q - Q = -Q^3$ (grand state solution for the focusing nonlinear Schrödinger equation).

Remark 2 Fang and Grillakis obtained the global well-posedness at $s = 1/2$ by using the interaction Morawetz estimate [11], and Colliander, Grillakis and Tzirakis [5] improved this for $s > 2/5$ by combining the Morawetz estimate with the almost conserved quantities. In more recently, Kllip, Tao and Visan [12] obtained global well-posedness and scattering for all $s \geq 0$, though radial data. But Theorem 1 (in particular Theorem 2) seems interesting, because the angularly constrained Strichartz estimate (Corollary 1) in conjunction with [5, 11] may improve the global well-posedness for $s > 4/13$ without radial condition on the initial data.

Open problem It is conjectured that (1.1) is globally well-posed and scatters to free solution for all data in L^2. This conjecture still remains open.

2. Sketch of the proof of Theorem 1

The strategy of the *almost conserved quantities* is as follows: First fix an arbitrary time interval $[0, T]$. Let $E_N[u(t)]$ be a new energy for solutions in H^s depending on a parameter $N \gg 1$ and take the rescaling. We prove again the local well-posedness result in the space associated to $E_N[(u(t)]$ on time intervals of length $\delta \sim 1$. Finally, we perform the iteration on the time interval $[0, T]$ to derive the a priori estimate of solutions with rescaling. How is it that our argument is successful? The variant of the energy E_N is very slowly in t. In particular, the energy E_N is *almost conserved* to evolve of (1.1). For Theorem 1, we use a slight variant $\widetilde{E}[u(t)] = \widetilde{E_N}[u(t)]$ of $E_N[u(t)]$.

2.1. Almost conserved quantity

An almost conserved quantity is defined as follows: Let $N \gg 1$ and

$$\widehat{Iu}(\xi) = \widehat{I_N u}(\xi) = m(\xi)\widehat{u}(\xi),$$

where $m(\xi)$ is an even C^∞-monotone function which equals to 1 for $|\xi| < N$ and equals to $(|\xi|/N)^{s-1}$ for $|\xi| > 2N$. We define

$$E_N[u(t)] = E[Iu(t)]. \tag{2.1}$$

This quantity is almost conserved to evolve the solution of (1.1) in the following sense:

$$\begin{aligned}\frac{d}{dt}E_N[u(t)] &= -2\,\mathrm{Re}\int_{\mathbb{R}^2} \overline{I\partial_t u}(I(|u|^2u) - |Iu|^2 Iu)dx \\ &= O(N^{-\alpha}),\end{aligned} \tag{2.2}$$

for some $\alpha > 0$ ($\alpha = 3/2 - \varepsilon$ is obtained in [7]). With E_N, we obtain the a priori estimate (1.5) for $s > 4/7$.

2.2. Resonant correction terms

Improving the error term $N^{-\alpha}$ in (2.2), we try to remove the biquadratic term in (2.2). At the present, we use the following modified energy functional $\widetilde{E}[u(t)]$:

$$\widetilde{E}[u(t)] = \Lambda_2(\sigma_2; u) + \Lambda_4(\widetilde{\sigma_4}; u)$$

where

$$\begin{aligned}\Lambda_k(\sigma; u) &= \int_{\xi_1+\cdots+\xi_k=0} \sigma(\xi_1, \ldots, \xi_k)\widehat{u}(\xi_1)\cdots\widehat{\overline{u}}(\xi_k), \\ \sigma_2 &= -\frac{1}{2}\xi_1 m_1 \xi_2 m_2, \\ \widetilde{\sigma_4} &= \frac{|\xi_1|^2 m_1^2 - |\xi_2|^2 m_2^2 + |\xi_3|^2 m_3^2 - |\xi_4|^2 m_4^2}{4(|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2)} 1_{\Omega_{nr}}, \\ \Omega_{nr} &= \{(\xi_1, \ldots, \xi_4)\colon\ \max|\xi_k| \le N \text{ or } |\cos\angle(\xi_{12}, \xi_{14})| \ge \theta > 0\},\end{aligned}$$

($\xi_{ij} = \xi_i + \xi_j$ etc) $1_{\Omega_{nr}}$ is the characteristic function on Ω_{nr}, and $m_k = m(\xi_k)$. $\theta = \theta(N) > 0$ is defined later depending on N (Section 2.4).

Remark 3 With the above functions, we can write the first generation of the modified energy $E_N[u(t)]$ as follows:

$$E_N[u(t)] = \Lambda_2(\sigma_2; u) + \Lambda_4(\sigma_4; u),$$

where

$$\sigma_4 = \frac{1}{4}m_1 m_2 m_3 m_4.$$

Remark 4 The resonant sets

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0,$$

$$0 = |\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2 = 2\xi_{12} \cdot \xi_{14},$$

(ξ_{12} and ξ_{14} are almost orthogonal) are almost canceled from the biquadratic form of $\Lambda_4(\sigma_4; u)$ in the following sense

$$\frac{d}{dt}\widetilde{E}[u(t)] = \Lambda_4(\widetilde{\sigma_4}; u) + \Lambda_6(\widetilde{\sigma_6}; u) \tag{2.3}$$

where

$$\widetilde{\sigma_4} = (|\xi_1|^2 m_1^2 - |\xi_2|^2 m_2^2 + |\xi_3|^2 m_3^2 - |\xi_4|^2 m_4^2) 1_{\Omega_r}$$
$$\Omega_r = \{(\xi_1, \ldots, \xi_4)\colon \max|\xi_k| > N \text{ and } |\cos\angle(\xi_{12}, \xi_{14})| < \theta\}$$

(we skip the details for the symbol $\widetilde{\sigma_6}$ in this note, because the biquadratic term is leading). Exploiting the presence of resonance condition Ω_r is our improvement of the previous work in [7].

The interest of the resonance condition lies in the following estimate.

Theorem 2 Angularly refined bilinear Strichartz estimate *Let* $0 < N_1 < N_2$ *and* $\theta \in (0, 1/100)$. *For* ϕ_1, $\phi_2 \in L^2$ *with Fourier frequencies* N_1, N_2, *respectively, we have*

$$\|F\|_{L^2_{t,x}} \lesssim \min\Big\{\theta, \frac{N_1}{N_2}\Big\}^{1/2} \|\phi_1\|_{L^2} \|\phi_2\|_{L^2},$$

where

$$F(t, x) = \int_{\xi_1+\xi_2=0} e^{ix(\xi_1+\xi_2)} 1_{|\cos\angle(\xi_1,\xi_2)| \le \theta} \widehat{e^{it\Delta}\phi_1}(\xi_1) \widehat{e^{it\Delta}\phi_2}(\xi_2).$$

The above estimate without angularly constrained was already obtained in [2]. The proof of Theorem 2 follows an argument in [2] under the additional restriction on the angle between interacting frequencies.

We recall the Fourier restriction norm space $X_{s,b}[0, T]$ with the following norm [1]:

$$\|f\|_{X_{s,b}[0,T]} = \inf\{\|g\|_{X_{s,b}} \mid f = g \text{ on } (t, x) \in [0, T] \times \mathbb{R}^2\}$$

where

$$\|g\|^2_{X_{s,b}} = \int_{\mathbb{R}^3} (1 + |\tau + |\xi|^2|)^{2b} (1 + |\xi|)^{2s} |\widehat{g}(\tau, \xi)|^2 d\xi d\tau.$$

The following corollary immediately follows from Theorem 2.

Corollary 1 *Let* $0 < N_1 < N_2$ *and* $\theta \in (0,\, 1/100)$. *For* $u_1,\, u_2 \in X_{0,1/2+\varepsilon}$ *such that*

$$\operatorname{supp}\widehat{u_1}(t,\,\xi) = \{|\xi| \sim N_1\},\ \operatorname{supp}\widehat{u_2}(t,\,\xi) = \{|\xi| \sim N_2\},$$

and $|\cos\angle(\xi_1,\,\xi_2)| \le \theta$ *for* $\xi_1 \in \operatorname{supp}\widehat{u_1}(t,\,\xi)$ *and* $\xi_2 \in \operatorname{supp}\widehat{u_2}(t,\,\xi)$, *we have*

$$\|u_1u_2\|_{L^2(\mathbb{R}^{1+2})} \lesssim \min\Big\{\theta,\, \frac{N_1}{N_2}\Big\}^{1/2} \|u_1\|_{X_{0,1/2+\varepsilon}} \|u_2\|_{X_{0,1/2+\varepsilon}}. \tag{2.4}$$

2.3. Local well-posedness in IH^s-space

In this section we prove the local well-posedness of the initial value problem obtained by acting on (1.1) with the operator I

$$\begin{cases} iIu_t(t,\,x) + \Delta Iu(t,\,x) = I(|u(t,\,x)|^2u(t,\,x)), \\ Iu(0,\,x) = Iu_0(x). \end{cases} \tag{2.5}$$

We still have the following local well-posedness theorem (cf. [4, 14, 7]).

Lemma 1 (Modified Local well-posedness) *Let* $s > 0$. *The Cauchy problem* (2.5) *is locally well-posed on* $[0,\,T_0]$, $T_0 = T_0(\|Iu_0\|_{H^1})$ *with solution* $u(t)$ *such that*

$$Iu \in C([0,\,T]\colon H^1), \quad \|Iu\|_{X_{1,1/2+\varepsilon}[0,T_0]} \lesssim (\|Iu_0\|_{H^1}).$$

Next we give that $\widetilde{E}[u(t)]$ controls data size as follows:

Lemma 2 *Let* $u(t) \in H^s$ $(s > 1/2)$ *be a solution to* (1.1). *Then*

$$\|Iu(t)\|^2_{\dot{H}^1} \le \widetilde{E}[u(t)] + \frac{c}{\theta N^{2-\varepsilon}} \|Iu(t)\|^2_{\dot{H}^1} \|Iu(t)\|^2_{H^1},$$

where θ *is given by* (2.3).

The proof of Lemma 2 is essentially similar to [7]. Therefore we omit details.

2.4. $\widetilde{E}[u(t)]$ obeys the almost conservation law

Lemma 3 (Almost conservation) *Let* $u(t) \in H^s$ $(s > 1/2)$ *be a solution to* (1.1). *For* $t \ge 0$, *we have*

$$\widetilde{E}[u(t)] \le \widetilde{E}[u(0)] + \Big(\frac{1}{N^{2-\varepsilon}} + \frac{\theta^{1/2}}{N^{3/2-\varepsilon}} + \frac{1}{\theta N^{3-\varepsilon}}\Big) C(\|Iu\|_{X_{1,1/2+\varepsilon}[0,t]}).$$

Note that the choice $\theta = 1/N$ produces the pre-factor $cN^{-2+\varepsilon}$.

A brief outline of the proof of Lemma 3. By (2.3), we have

$$\begin{aligned}\widetilde{E}[u(t)]-\widetilde{E}[u(0)] &= \int_0^t \Lambda_4(\widetilde{\sigma_4};u)+\Lambda_6(\widetilde{\sigma_6};u)ds \\ &= I_1+I_2.\end{aligned}$$

We aim to show

$$I_1+I_2 \le \Big(\frac{1}{N^{2-\varepsilon}}+\frac{\theta^{1/2}}{N^{3/2-\varepsilon}}+\frac{1}{\theta N^{3-\varepsilon}}\Big)C(\|Iu\|_{X_{1,1/2+\varepsilon}[0,t]}). \tag{2.6}$$

We use the Littlewood-Paley decomposition and break u into $u=\sum_N u_N$ where $\operatorname{supp}\widehat{u}(\xi)=\{|\xi|\sim N\}$.

We consider (2.6) for the term I_1 and provide proofs for some special cases: $N_1\sim N_2\ge N$, $N_3\gg N_4$ under the resonant condition $|\cos\angle(\xi_{12},\,\xi_{14})|\le\theta$. We need the following calculations

$$\begin{aligned}&|\widetilde{\sigma_4}|\le c(m(N_1)^2N_1N_3\theta+m(N_3)^2N_3^2),\\ &|\cos\angle(\xi_1,\,\xi_3)|=|\cos\angle(\xi_{14},\,\xi_{34})|+O\Big(\frac{N_4}{N_3}\Big)\le\theta+O\Big(\frac{N_4}{N_3}\Big).\end{aligned}$$

Then taking $u_{N_2}u_{N_4}$ in L^2, and $u_{N_1}u_{N_3}$ in L^2, respectively, and using Corollary 1, we can estimate

$$\le c\big(m(N_1)^2N_1N_3\theta+m(N_3)^2N_3^2\big)\Big(\frac{N_4}{N_2}\Big)^{1/2}\Big(\theta+\frac{N_4}{N_3}\Big)^{1/2} \times\prod_{j=1}^4\|u_{N_j}\|_{X_{0,1/2+\varepsilon}},$$

this is bounded by

$$\le c\frac{m(N_1)^2N_1N_3\theta+m(N_3)^2N_3^2}{m(N_1)^2N_1^2N_3m(N_3)N_4}\Big(\frac{N_4}{N_2}\Big)^{1/2}\Big(\theta+\frac{N_4}{N_3}\Big)^{1/2} \times\prod_{j=1}^4\|Iu_{N_j}\|_{X_{1,1/2+\varepsilon}}.$$

Splitting into two cases; $N_3\ge N_4/\theta$ and $N_3<N_4/\theta$ and summing over $N_1\sim N_2\ge N$, $N_3\gg N_4$, we have the bound (2.6). □

2.5. Induction energy implies the a priori estimate (1.5)

Finally, we give a sketch of the induction argument that Lemmas 1, 2 and 3 imply Theorem 1.

Let $u(t)$ be a smooth solution of (1.1). By (1.4), consider the rescaled solution u_λ

$$u_\lambda(t, x) = \frac{1}{\lambda} u\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big), \quad \lambda > 0.$$

Fix an arbitrary time interval $[0, T]$. We prove Theorem 1, if we construct a rescaled solution u_λ on $[0, \lambda^2 T]$. As in [6, 7, 8] the idea is to reach time $t = \lambda^2 T$ inductively using the energy estimates in Lemmas 2 and 3 and the local theory in Lemma 1. An easy computation shows that

$$\|Iu_0\|_{\dot H^1} \le c\lambda^{-s} N^{1-s} \|u_0\|_{H^s} \ll 1$$

provided we choose $\lambda \gg \|u_0\|_{H^s}^{1/s} N^{(1-s)/s}$. From Lemmas 2 and 3, $u(t)$ has the a priori estimate

$$\begin{aligned} \|Iu(t)\|_{\dot H^1} &\le \widetilde{E}[u(t)] \\ &\le \widetilde{E}[u(0)] + cN^{-2+\varepsilon} C(\|Iu\|_{X_{1,1/2+\varepsilon}[0,t]}). \end{aligned} \tag{2.7}$$

We will show that by Lemma 1, $\|Iu\|_{X_{1,1/2+\varepsilon}[0,T_0]} \le C$ and $T_0 = 1$ whenever $\|Iu_0\|_{\dot H^1} \ll 1$. Thus there is an increment in the energy of size at most $N^{-2+\varepsilon}$, if $\|Iu\|_{X_{1,1/2+\varepsilon}} \le C$ in (2.7). Hence we want to ensure

$$N^{2-\varepsilon} \ge c\lambda^2 T = N^{2(1-s)/s} T C(\|u_0\|_{H^s}),$$

which is achieved for $s > 1/2$ by letting $N = N(T)$ sufficiently large. Notice that

$$\frac{1}{2}\|u(t)\|_{H^s}^2 \le \widetilde{E}[u(t)] + c\|u(t)\|_{L^2}^2.$$

This proves Theorem 1, using the L^2-conservation law (1.2). □

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Department of Mathematics
Kobe University