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The lifespan of solutions to nonlinear Schrödinger and Klein-Gordon equations

(Dedicated to Professor Rentaro Agemi on the occasion of his seventieth birthday)

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Abstract. Precise information on the lifespan sometimes tells us how the nonlinearity affects large time behavior of solutions to nonlinear evolution equations. As pointed out by John and Hörmander in 1987, there are surprising connections between the lifespan and the null condition in the wave equation case. In this paper we give a review of analogous lifespan estimates for nonlinear Schrödinger and Klein-Gordon equations. We also discuss how this viewpoint could give a unified understanding of many previous results.

Key words: lifespan, nonlinear Schrödinger equations, nonlinear Klein-Gordon equations.

1. Introduction

First of all, let us recall some facts on nonlinear wave equations which will be our point of departure. For simplicity we restrict our attentions to the following model equation:

$$\begin{cases} \partial_t^2 u - \Delta_x u = \displaystyle\sum_{j,k,l=0}^{3} \eta_{jkl} \partial_j u \partial_k \partial_l u, & t > 0,\ x \in \mathbb{R}^3, \\ u(0,\,x) = \varepsilon\varphi(x),\ \partial_t u(0,\,x) = \varepsilon\psi(x) & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where η_{jkl} is a real constant, φ, ψ are smooth functions with compact support, $\partial_0 = \partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ ($j = 1, 2, 3$), and Δ_x denotes the Laplacian in $\mathbb{R}^3$. It is well known that the solution to (1.1) can develop singularities in finite time even if the initial data are sufficiently small and smooth. For instance, it was proved by John [22] that the classical solution for (1.1) with the nonlinearity $u_t u_{tt}$ blows up in finite time with the estimate $T_\varepsilon \le \exp(C/\varepsilon)$, where T_ε denotes the lifespan, i.e., the supremum of all $T > 0$ such that a unique classical solution for (1.1) exists for $0 \le t \le T$. On the other hand, in 1986 Christdoulou [8] and Klainerman [29] introduced a

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structural condition on the nonlinearities, called the null condition, under which the small amplitude solution remains smooth for all time. In 1987 John [23] and Hörmander [20] obtained an explicit lower bound estimate for T_ε:

$$\liminf_{\varepsilon\to+0} \varepsilon \log T_\varepsilon \geq H := \Big[\sup_{(\rho,\omega)\in\mathbb{R}\times\mathbb{S}^2} \Big(\frac{-1}{2}\partial_\rho^2\Phi(\rho,\,\omega)N(\omega)\Big)\Big]^{-1} \tag{1.2}$$

(with the convention $1/0 = +\infty$), where $\Phi(\rho,\,\omega)$ is a smooth function of $(\rho,\,\omega)\in\mathbb{R}\times\mathbb{S}^2$, vanishing when ρ is large, which can be written explicitly in terms of φ and ψ by using the Radon transform, and $N(\omega)$ is given by

$$N(\omega) = \sum_{j,k,l=0}^{3} \eta_{jkl}\omega_j\omega_k\omega_l$$

for $\omega = (\omega_1,\,\omega_2,\,\omega_3)\in\mathbb{S}^2$, $\omega_0 = -1$. What is important in (1.2) is that $N(\omega)$ vanishes identically if and only if the null condition is satisfied (i.e., the nonlinear term is written as a linear combination of the so-called null forms). Note also that $\Phi(\rho,\,\omega)$ naturally appears when we consider large time asymptotics of the solution $u_0(t,\,x)$ of the free wave equation. Roughly saying, it holds that

$$u_0(t,\,x) \sim \frac{\varepsilon}{|x|}\Phi\Big(|x|-t,\,\frac{x}{|x|}\Big)$$

when $||x|-t|\leq C$ and $t\to+\infty$. The proof of (1.2) relies on the fact that the solution $u(t,\,x)$ of (1.1) is well approximated by

$$\frac{\varepsilon}{|x|}A\Big(\varepsilon\log t,\,|x|-t,\,\frac{x}{|x|}\Big)$$

in the large time, where $A(s,\,\rho,\,\omega)$ solves

$$\partial_s(\partial_\rho A) + \frac{N(\omega)}{4}\partial_\rho\big((\partial_\rho A)^2\big) = 0; \quad A(0,\,\rho,\,\omega) = \Phi(\rho,\,\omega).$$

Observe that $s = H$ is characterized by the "blow-up time" of $A(s,\,\rho,\,\omega)$, which is a reason why H appears in the right hand side of (1.2) (see e.g., [20], [21], [23], [24] for the details). It should be remarkable that similar reduced equations play an important role in the study of wave equations with more general long-range nonlinearities. Recent development in this direction can be found in [6], [7], [26], [30], [31], [32], [33], etc.

Now we turn to our problem. We raise the question whether analogous conclusions are valid for other nonlinear evolution equations. To be more specific, consider the initial value problem in the form $Lu = N(u)$, $u(0) = \varepsilon\varphi$ and address the following questions:

- *Are there analogous asymptotic lower bound estimates* (*as* $\varepsilon \to +0$) *on the lifespan, which can be explicitly computed from* N *and* φ*?*
- *Can we find some structural conditions, like the null condition for the wave equation case, from the quantities appearing in these estimates?*

The purpose of this paper is to give a review of recent results on this issue in the case of Schrödinger and Klein-Gordon equations.

2. Nonlinear Schrödinger equations

This section is devoted to the Schrödinger equation case. Our model equation here will be the cubic derivative nonlinear Schrödinger equation in one space dimension:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\partial_x^2 u = N(u,\, \partial_x u), & t > 0,\ x \in \mathbb{R}, \\ u(0,\, x) = \varepsilon\varphi(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where φ belongs to the Schwartz class $\mathcal{S}$, and the nonlinear term N is supposed to be a cubic homogeneous polynomial in $(u,\, \overline{u},\, u_x,\, \overline{u_x})$ and gauge invaiant, i.e.,

$$N(e^{i\theta}v,\, e^{i\theta}q) = e^{i\theta}N(v,\, q) \quad \text{for all } v,\, q \in \mathbb{C} \text{ and } \theta \in \mathbb{R}. \tag{2.2}$$

We denote the lifespan by T_ε, which is the supremum of all $T > 0$ such that there exists a unique solution $u \in C([0,\, T]; \mathcal{S})$ of (2.1). In [46], the following estimate is obtained.

Theorem 2.1 *We have*

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \geq H := \big[\sup_{\xi \in \mathbb{R}} \big(2|\hat{\varphi}(\xi)|^2 \operatorname{Im} N(1,\, i\xi)\big)\big]^{-1} \tag{2.3}$$

($1/0$ *is interpreted as* $+\infty$), *where* $\hat{\varphi}$ *denotes the Fourier transform, i.e.,*

$$\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iy\xi}\varphi(y)dy.$$

Of our interest is what the quantity H tells us. For this purpose, we first observe that $H = +\infty$ implies either φ vanishes identically or

$$\operatorname{Im} N(1, i\xi) \le 0 \quad \text{for all } \xi \in \mathbb{R}. \tag{2.4}$$

In the recent paper [17], small data global existence is proved under the condition (2.4) and large time asymptotics of the solution is also obtained:

Theorem 2.2 *Suppose that N satisfies* (2.4). *Let $\varphi \in H^{2,1} \cap H^{3,0}$, where $H^{s,k}$ denotes the weighted Sobolev space. Then* (2.1) *admits a unique global solution $u \in C([0, \infty); H^{2,1} \cap H^{3,0})$ if ε is small enough. Moreover, the following asymptotic expression is valid as $t \to \infty$ uniformly in $x \in \mathbb{R}$:*

$$u(t, x) = \frac{1}{\sqrt{it}} A\Big(\log t, \frac{x}{t}\Big) e^{ix^2/2t} + O(t^{-3/4+\mu}), \tag{2.5}$$

where $\mu > 0$ is an arbitrarily small constant, and $A(s, \xi)$ solves

$$i\partial_s A = N(1, i\xi)|A|^2 A. \tag{2.6}$$

Next we shall explain how previous results can be unified by the above theorem. We focus on three typical cases:
(i) $\sup_{\xi\in\mathbb{R}} \operatorname{Im} N(1, i\xi) < 0$,
(ii) $\operatorname{Im} N(1, i\xi) \equiv 0$,
(iii) $N(1, i\xi) \equiv 0$.

- The condition (i) implies that the solution $A(s, \xi)$ of (2.6) decays like $O(s^{-1/2})$ in L^∞_ξ as $s \to +\infty$ because (2.6) leads to

  $$\partial_s\big(|A(s, \xi)|^2\big) = 2 \operatorname{Im} N(1, i\xi)|A(s, \xi)|^4,$$

  so

  $$|A(s, \xi)|^2 = \frac{|A(0, \xi)|^2}{1 - 2 \operatorname{Im} N(1, i\xi)|A(0, \xi)|^2 s}.$$

  As a consequence, the solution $u(t, x)$ of (2.1) gains an additional logarithmic time decay, i.e., $u(t, x) = O((t \log t)^{-1/2})$ in L^∞_x as $t \to +\infty$ (this should be compared with the fact that $O(t^{-1/2})$ is the best decay rate if $N \equiv 0$ and $\varphi \ne 0$). Note that this is a generalization of the previous result by Shimomura [44], in which the dissipative nonlinearity ($N = \lambda|u|^2 u$ with $\operatorname{Im}\lambda < 0$) was considered.
- Under the condition (ii), the ODE (2.6) can be solved explicitly as

  $$A(s, \xi) = \Phi_+(\xi) e^{-i \operatorname{Re} N(1,i\xi)|\Phi_+(\xi)|^2 s}$$

with a suitable $\Phi_+(\xi)$ (depending on ε), which yields the asymptotic formura

$$u(t, x) = \frac{1}{\sqrt{it}}\Phi_+\Big(\frac{x}{t}\Big)e^{-i\,\mathrm{Re}\,N(1,ix/t)|\Phi_+(x/t)|^2\log t}e^{ix^2/2t} + O(t^{-3/4+\mu}) \quad (t \to \infty).$$

Main feature of this formula is that the solution has an additional logarithmic factor in the phase, which is typical in 1D cubic NLS. In particular, this covers the previous result due to Hayashi-Naumkin-Uchida [18] (see also [12], [13], [19], [36], [40], [41], etc.)

- The condition (iii) is nothing but a special case of (ii) where the real part of $N(1, i\xi)$ is also identically zero. However, it is worth noting that there is an interesting similarity between (iii) and the null condition. To be more precise, let us introduce $J = x + it\partial_x$. (Note that $[i\partial_t + (1/2)\partial_x^2, J] = 0$ and $[\partial_x, J] = 1$, where $[\,\cdot\,,\,\cdot\,]$ denotes the commutator. Because of these commutation relations, one may expect that Ju decays like u or ∂u.) The key observation is as follows: Since

$$N\Big(u,\, u_x - \frac{1}{it}Ju\Big) = N\Big(1,\, i\frac{x}{t}\Big)|u|^2u = 0,$$

we have

$$\begin{aligned} N(u, u_x) &= N(u, u_x) - N\Big(u,\, u_x - \frac{1}{it}Ju\Big) \\ &= \frac{1}{it}Ju\int_0^1 \frac{\partial N}{\partial u_x}\Big(u,\, u_x - \frac{\theta}{it}Ju\Big)d\theta \\ &\quad - \frac{1}{it}\overline{Ju}\int_0^1 \frac{\partial N}{\partial \overline{u_x}}\Big(u,\, u_x - \frac{\theta}{it}Ju\Big)d\theta, \end{aligned}$$

which implies a gain of extra time-decay in the worst contribution of the nonlinear term satisfying (iii). Using this gain, we can directly show that the small amplitude solution is asymptotically free. (This fact was first pointed out by Y. Tsutsumi [48]. See also [27], [28].)

We shall add a few more remarks on the Schrödinger case. The first one is related to the upper bound on the lifespan in the case of $H < \infty$. As far as the author knows, there are no previous results on small data blow-up nor global existence in this case. However, in view of the corresponding results on the wave equation case ([1]–[5], [24], etc.), it would be quite reasonable to conjecture that the small data solution can develop singularities in finite time with the upper estimate

$$\limsup_{\varepsilon\to+0} \varepsilon^2 \log T_\varepsilon \le H.$$

Note also that we can find blow-up examples in the periodic case. For instance, consider the evolution governed by

$$iu_t + \frac{1}{2}u_{xx} = \lambda_2|u|^2 u_x$$

on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. For the solution $u(t, x)$, we put

$$M(t) = -\operatorname{sgn}(\operatorname{Re}\lambda_2)\cdot \operatorname{Im}\int_{\mathbb{T}} u\overline{u_x}dx.$$

Then, differentiating $M(t)$ and integrating by part, we see that

$$\frac{dM}{dt}(t) = 2|\operatorname{Re}\lambda_2|\int_{\mathbb{T}} |u|^2|u_x|^2 dx.$$

On the other hand, the Schwarz inequality implies

$$M(t)^2 \le 2\pi\int_{\mathbb{T}} |u|^2|u_x|^2 dx,$$

whence we have

$$\frac{dM}{dt}(t) \ge \frac{|\operatorname{Re}\lambda_2|}{\pi}M(t)^2.$$

This implies $M(t)$ must blow up in finite time if $M(0) > 0$ and $\operatorname{Re}\lambda_2 \ne 0$. (Remark that $\operatorname{Im} N(1, i\xi) = (\operatorname{Re}\lambda_2)\xi$ if $N(u, u_x) = \lambda_2|u|^2u_x$.) Unfortunately, however, this argument fails in the case of $\mathbb{R}$ because $\int_{\mathbb{R}} 1dx = \infty$. Similar examples were considered in [42].

Another remark concerns the gauge invariance (2.2). It is possible to relax (2.2) but impossible to remove it completely. In fact, when (2.2) is replaced by

$$N(e^{i\theta}, 0) = e^{i\theta}N(1, 0) \quad \text{for all } \theta\in\mathbb{R}, \tag{2.7}$$

we can show that the above assertions remain valid if we replace $N(1, i\xi)$ in the statement by

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta}N(e^{i\theta}, i\xi e^{i\theta})d\theta.$$

Note that (2.7) is just what excludes u^3, $\overline{u}^3$, $u\overline{u}^2$ from all possible cubic nonlinear terms, but it is not a technical restriction because for these three nonlinearities we can find a class of initial data for which the solution has another kind of asymptotic profile than that Theorem 2.2 could give (see [14] and [15] for the details).

3. Nonlinear Schrödinger equations, continued

In this section we give a sketch of the proof of Theorems 2.1 and 2.2. First let us denote by u_0 the solution of the linearized problem near zero:

$$\Big(i\partial_t + \frac{1}{2}\partial_x^2\Big)u_0 = 0, \quad u_0(0,\, x) = \varepsilon\varphi(x).$$

Then we know that

$$u_0(t,\, x) = \frac{\varepsilon}{\sqrt{it}}\hat{\varphi}\Big(\frac{x}{t}\Big)e^{ix^2/2t} + \cdots$$

in the large time. Regarding u_0 as a rough approximation of the solution u for (2.1), we observe as in the wave equation case ([1], [20], etc) that $s = \varepsilon^2 \log t$ should be the slow time correponding to 1D cubic NLS. This indicates that u could be better approximated by

$$u_1(t,\, x) = \frac{\varepsilon}{\sqrt{t}}A\Big(\varepsilon^2 \log t,\, \frac{x}{t}\Big)e^{ix^2/2t}$$

with a smooth function $A(s,\, \xi)$ satisfying $A(0,\, \xi) = e^{-i\pi/4}\hat{\varphi}(\xi)$. Substituting this expression into (2.1), keeping only the leading terms and dividing by $\varepsilon^3 t^{-3/2} e^{ix^2/2t}$ leads to the ODE (2.6). We can verify that u_1 does give a better approximation when t is fairly large, but it does not have correct Cauchy data. So the next task is to piece it together with u_0. We choose a smooth cut-off function χ, equal to 1 in $(-\infty,\, 1)$ and equal to 0 in $(2,\, \infty)$, and we set

$$u_a(t,\, x) = \chi(\varepsilon t)u_0(t,\, x) + (1 - \chi(\varepsilon t))u_1(t,\, x).$$

Then we can see that $u_a(t,\, x)$ is a smooth function defined on $[0,\, e^{H/\varepsilon^2}) \times \mathbb{R}$ which satisfies

$$\begin{aligned} &u_a(0,\, x) = \varepsilon\varphi(x), \\ &\sup_{t\in[0, e^{K/\varepsilon^2}]} \|\partial_x^j J^k u_a(t,\, \cdot\,)\|_{L^2} = O(\varepsilon), \end{aligned}$$

$$\int_0^{e^{K/\varepsilon^2}} \|\partial_x^j J^k R(t,\,\cdot\,)\|_{L^2} dt = O(\varepsilon^2)$$

for any j, $k = 0, 1, 2, \dots$ and $K < H$, where

$$R(t,\,x) = \Big(i\partial_t + \frac{1}{2}\partial_x^2\Big)u_a - N(u_a,\,\partial_x u_a).$$

Once such an approximate solution is obtained, suitable energy estimate involving J enables us to prove that u cannot differ too much from u_a as long as the latter remains smooth, which implies Theorem 2.1 (see [46] for more details).

To prove Theorem 2.2, the above approximation is insufficient (even though $H = +\infty$). So we appeal to a different approach. Let $\mathcal{U} = \mathcal{U}(t)$ be the free Schrödinger evolution group, i.e., $u_0(t,\,x) = (\mathcal{U}(t)\varepsilon\varphi)(x)$. It is well known that $\mathcal{U}$ is decomposed into $\mathcal{MDFM}$, where $(\mathcal{M}\varphi)(x) = e^{ix^2/2t}\varphi(x)$, $(\mathcal{D}\varphi)(x) = (it)^{-1/2}\varphi(x/t)$ and $\mathcal{F}$ denotes the Fourier transform with respect to x. Because of the estimate

$$\|u(t) - \mathcal{MDFU}^{-1}u(t)\|_{L^\infty} \le Ct^{-3/4}(\|u(t)\|_{L^2} + \|Ju(t)\|_{L^2}),$$

the problem is reduced to getting the large time asymptotics of $\alpha(t,\,\xi) := \mathcal{F}(\mathcal{U}^{-1}u(t))(\xi)$ if we have a good control of L^2 norm of $\partial_x^j J^k u$ ($j + k \le 3$, $k \le 1$). On the other hand, setting $\mathcal{W} = \mathcal{FMF}^{-1}$, we see that

$$\begin{aligned}
i\partial_t\alpha &= \mathcal{FU}^{-1}\Big(i\partial_t + \frac{1}{2}\partial_x^2\Big)u \\
&= \mathcal{FU}^{-1}N(u,\,u_x) \\
&= \mathcal{W}^{-1}\mathcal{D}^{-1}\mathcal{M}^{-1}N(\mathcal{MDW}\alpha,\,\mathcal{MDW}(i\xi\alpha)) \\
&= \frac{1}{t}\mathcal{W}^{-1}N(\mathcal{W}\alpha,\,\mathcal{W}(i\xi\alpha)) \\
&= \frac{1}{t}N(1,\,i\xi)|\alpha|^2\alpha + \rho,
\end{aligned}$$

where

$$\rho = \frac{1}{t}\Big\{\mathcal{W}^{-1}N(\mathcal{W}\alpha,\,\mathcal{W}(i\xi\alpha)) - N(\alpha,\,i\xi\alpha)\Big\}.$$

Using the estimate $\|(\mathcal{W}^{\pm 1} - 1)\psi\|_{L^\infty} \le Ct^{-1/4}\|\psi\|_{H^1}$, we can show that ρ is a harmless remainder when $t \gg 1$. This suggests that $\alpha(t,\,\xi)$ behaves like $A(\log t,\,\xi)$ in the large time, where $A(s,\,\xi)$ solves (2.6). It is justified in [17]. $\square$

We finish the discussion on NLS by mentioning the case of quadratic nonlinearities in two space dimensions. In view of the decay rate of the free solution, it might be natural to expect analogous conclusions. However, much less is known in this case and the author has no result at present. Difficulties in 2D quadratic NLS are explained in [11] (see also the papers by Hayashi and Naumkin cited there).

4. Nonlinear Klein-Gordon equations

Quite analogous assertions have been establshed in the Klein-Gordon case. We will mention them in what follows. Let us consider

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + u = F(u,\, \partial_t u,\, \partial_x u), & t > 0,\ x \in \mathbb{R}, \\ u(0,\, x) = \varepsilon\varphi(x), \quad \partial_t u(0,\, x) = \varepsilon\psi(x), & x \in \mathbb{R}, \end{cases} \tag{4.1}$$

where φ, ψ are real-valued C^∞ functions with compact support, and F denotes a cubic homogeneous polynomial in $(u,\, u_t,\, u_x)$ with real coefficients. We introduce $H \in (0,\, +\infty]$ by

$$\frac{1}{H} = \sup_{|y|<1} \big(2|\Phi(y)|^2 \operatorname{Im} N(y)\big), \tag{4.2}$$

where

$$N(y) = \frac{1}{2\pi\omega_0(y)} \int_0^{2\pi} e^{-i\theta} F(\cos\theta,\, -\omega_0(y)\sin\theta,\, -\omega_1(y)\sin\theta) d\theta,$$
$$\omega_0(y) = \frac{1}{\sqrt{1-y^2}}, \quad \omega_1(y) = \frac{-y}{\sqrt{1-y^2}}$$

for $|y| < 1$, and $\Phi(y)$ is given by

$$\Phi(y) = \begin{cases} e^{i\pi/4}\langle\xi\rangle^{3/2}\Big[\hat{\varphi}(\xi) - \dfrac{i}{\langle\xi\rangle}\hat{\psi}(\xi)\Big]\,\Big|_{\xi=-y/\sqrt{1-y^2}} & |y| < 1, \\ 0 & |y| \geq 1. \end{cases}$$

Note that the corresponding free solution $u_0(t,\, x)$ can be approximately written as

$$u_0(t,\, x) = \frac{\varepsilon}{\sqrt{t}} \operatorname{Re}\Big[\Phi\Big(\frac{x}{t}\Big) e^{i\sqrt{t^2-x^2}}\Big] + \cdots$$

as $t \to \infty$ (see Section 7.2 of [21]). According to [9], [10], [45] (see also [47]), we have the following

Theorem 4.1 (1) *Let T_ε be the supremum of all $T > 0$ such that a unique classical solution of* (4.1) *exists for $0 \le t \le T$. Then we have*

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \ge H,$$

where H is given by (4.2).
(2) *Suppose that*

$$\operatorname{Im} N(y) \le 0 \quad \text{for all } |y| < 1.$$

Then $T_\varepsilon = +\infty$ for sufficiently small ε. Moreover, the following asymptotic expression is valid as $t \to +\infty$ in the sense of L^∞_x:

$$u(t, x) = \frac{1}{\sqrt{t}} \operatorname{Re}\Big[A\Big(\log t, \frac{x}{t}\Big) e^{i\sqrt{x^2 - t^2}}\Big] + o((t \log t)^{-1/2}),$$

where $A(s, y)$ solves

$$i\partial_s A = N(y)|A|^2 A.$$

As in the case of NLS, let us focus on the following three conditions:
(i) $\sup_{|y|<1} \operatorname{Im} N(y) < 0$,
(ii) $\operatorname{Im} N(y) \equiv 0$,
(iii) $N(y) \equiv 0$.
(Remember again that $H = +\infty$ implies either $\Phi(y)$ vanishes or $\operatorname{Im} N(y) \le 0$.)

- The nonlinear damping term $-u_t^3$ is a typical example of N which satisfies (i). (Indeed, $\operatorname{Im} N(y) = -3\omega_0(y)^2/8$.) For Klein-Gordon equation with the nonlinear damping term, the energy decay was previously considered by Nakao [39] and Mochizuki-Motai [37]. Theorem 4.1 covers their results although the restriction on the data is stronger. Actually, Theorem 4.1 implies that the solution of (4.1) gains an additional logarithmic time decay (in the sense of L^p_x for $p \le 2 \le \infty$) if (i) is satisfied (see [45] for the detail).
- The condition (ii) contains the power type nonlinearity λu^3 with $\lambda \in \mathbb{R}$. (Actually, we see that $\operatorname{Im} N(y) \equiv 0$ while $\operatorname{Re} N(y) = 3\lambda/8\omega_0(y)$.) As in the NLS case, there must be some logarithmic modifications in the phase. This was first proved by J.-M. Delort [10] (see also [16], [34], [35], etc.).
- Under the condition (iii), we can verify that the nonlinear term must

be written as a linear combination of

$$(-u^2 + 3u_t^2 - 3u_x^2)u,\ (-3u^2 + u_t^2 - u_x^2)u_t,\ (-3u^2 + u_t^2 - u_x^2)u_x.$$

It should be remarkable that the normal form arguments (cf. [43]) are effective only for these three nonlinearities. Based on this fact, Moriyama [38] and Katayama [25] proved small data global existence for (4.1) with these nonlinearities (in fact, quasilinear equations were considered in [25], [38]).

Finally we must mention an example due to Yordanov, who proved that the solution for (4.1) with the nonlinearity $u_t^2 u_x$ blows up in finite time with the estimate $T_\varepsilon \leq \exp(C/\varepsilon^2)$ (see Proposition 7.8.8 of [21]). Theorem 4.1 does not cover this example, but they are compatible because $\operatorname{Im} N(y) = 3\omega_0(y)\omega_1(y)/8$ if $F = u_t^2 u_x$ (whence $\operatorname{Im} N(y) > 0$ for $y < 0$). It would be an interesting open problem whether or not $H < \infty$ implies small data blow-up in general.

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