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Instability of standing waves for the Klein-Gordon-Schrödinger system

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Abstract. We study the orbital instability of standing wave solutions for the Klein-Gordon-Schrödinger system in three space dimensions. It is proved that the standing wave is unstable if the frequency is sufficiently small.

Key words: Klein-Gordon-Schrödinger, standing waves, instability.

1. Introduction

We consider the Klein-Gordon-Schrödinger system with Yukawa coupling in three space dimensions:

$$\begin{cases} i\partial_t u + \Delta u = -2uv, & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\ \partial_t^2 v - \Delta v + v = |u|^2, & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \end{cases} \tag{1}$$

where $u\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ and $v\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$. We study the orbital instability of standing wave solutions $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ of (1), where $\omega > 0$ and $(\varphi_\omega, \psi_\omega)$ is a ground state of

$$\begin{cases} -\Delta\varphi + \omega\varphi = 2\varphi\psi, & x \in \mathbb{R}^3, \\ -\Delta\psi + \psi = |\varphi|^2, & x \in \mathbb{R}^3. \end{cases} \tag{2}$$

In our previous paper [15], we proved that the standing wave solution $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ of (1) is orbitally stable for sufficiently large $\omega > 0$. In the present paper, we will prove that $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ is orbitally unstable for sufficiently small $\omega > 0$.

It is known that the Cauchy problem for (1) is globally well-posed in the energy space $X = H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{R}) \times L^2(\mathbb{R}^3, \mathbb{R})$ (see [2] and also [1, 3, 6, 13, 14, 23]). That is, for any $(u_0, v_0, v_1) \in X$, there exists a unique global solution $(u, v, \partial_t v) \in C(\mathbb{R}, X)$ of (1) with $(u(0), v(0), \partial_t v(0)) = (u_0, v_0, v_1)$. Moreover, the solution satisfies the conservation laws:

$$E(u(t), v(t), \partial_t v(t)) = E(u_0, v_0, v_1), \quad \|u(t)\|_2 = \|u_0\|_2, \quad t \in \mathbb{R},$$

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where $E(u, v, w) = J(u, v) + \|w\|_2^2$, and

$$J(u, v) = \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|v\|_2^2 - 2\int_{\mathbb{R}^3} |u|^2 v dx.$$

Next, we consider the stationary problem (2) in the space $H^1_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{C}) \times H^1_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{R})$ of radially symmetric functions. For $\omega > 0$, we put

$$S_\omega(u, v) = J(u, v) + \omega\|u\|_2^2$$

for $(u, v) \in H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{R})$. Then, $(\varphi, \psi) \in H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{R})$ is a solution of (2) if and only if $S'_\omega(\varphi, \psi) = 0$. We denote the set of nontrivial radial solutions of (2) by $\mathcal{A}_\omega$, and the set of ground states of (2) by $\mathcal{G}_\omega$:

$$\begin{aligned}\mathcal{A}_\omega &= \{(u, v) \in H^1_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{C}) \times H^1_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{R}) \colon \\ &\qquad\qquad S'_\omega(u, v) = 0,\ (u, v) \neq (0, 0)\}, \\ \mathcal{G}_\omega &= \{(\varphi, \psi) \in \mathcal{A}_\omega \colon S_\omega(\varphi, \psi) \leq S_\omega(u, v) \text{ for all } (u, v) \in \mathcal{A}_\omega\}.\end{aligned}$$

We will prove existence of a ground state of (2) in Section 2. We do not know uniqueness (modulo symmetries) of the ground state, but the uniqueness is not needed in the following argument.

Definition Let $(\varphi_\omega, \psi_\omega) \in \mathcal{G}_\omega$. We say that the standing wave solution $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ of (1) is *orbitally stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(u_0, v_0, v_1) \in X$ and $\|(u_0, v_0, v_1) - (\varphi_\omega, \psi_\omega, 0)\|_X < \delta$, then the solution $(u(t), v(t))$ of (1) with $(u(0), v(0), \partial_t v(0)) = (u_0, v_0, v_1)$ satisfies

$$\inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^3} \|(u(t), v(t), \partial_t v(t)) - (e^{i\theta}\varphi_\omega(\,\cdot\, + y), \psi_\omega(\,\cdot\, + y), 0)\|_X < \varepsilon$$

for all $t \geq 0$. Otherwise, $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ is said to be *orbitally unstable.*

Now, we state our main result in this paper.

Theorem 1 *Let $\omega > 0$ and $(\varphi_\omega, \psi_\omega) \in \mathcal{G}_\omega$. Then, there exists a constant $\omega^* > 0$ such that the standing wave solution $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ of (1) is orbitally unstable for any $\omega \in (0, \omega^*)$.*

Remark Recall that for every $(u_0, v_0, v_1) \in X$, the solution $(u(t), v(t), \partial_t v(t))$ of the Cauchy problem for (1) with $(u(0), v(0), \partial_t v(0)) = (u_0, v_0, v_1)$ is global and $\sup\{\|(u(t), v(t), \partial_t v(t))\|_X \colon t \in \mathbb{R}\} < \infty$ (see, e.g., Lemma 1 of [3]). Thus, we can not expect strong instability of the standing wave solutions for (1). Concerning strong instability of standing waves, see Berestycki and Cazenave [4] for nonlinear Schrödinger equations, and see Shatah [24] and Ohta and Todorova [22] for nonlinear Klein-Gordon equations.

The proof of Theorem 1 is based on the scaling $(\lambda^{3/2}u(\lambda x),\, \lambda^3 v(\lambda x))$. We remark that $\|\lambda^{3/2}u(\lambda\cdot)\|_2 = \|u\|_2$ for $\lambda > 0$, and

$$\partial_\lambda J(\lambda^{3/2}u(\lambda\cdot),\, \lambda^3 v(\lambda\cdot))|_{\lambda=1} = P(u,\, v), \tag{3}$$

where

$$P(u,\, v) = 2\|\nabla u\|_2^2 + 5\|\nabla v\|_2^2 + 3\|v\|_2^2 - 6\int_{\mathbb{R}^3} |u|^2 v dx.$$

Moreover, solutions $(u(t),\, v(t))$ of (1) formally satisfy the identity:

$$\frac{d}{dt}\Big\{\mathrm{Im}\int_{\mathbb{R}^3} \overline{u} x\cdot\nabla u dx - 2\int_{\mathbb{R}^3} \partial_t v(x\cdot\nabla v + 3v)dx\Big\} = P(u,\, v) - 3\|\partial_t v\|_2^2. \tag{4}$$

We use a local version of the identity to prove the following sufficient condition for instability (see Lemma 6 below).

Theorem 2 *Let $(\varphi_\omega,\, \psi_\omega) \in \mathcal{G}_\omega$. Assume that*

$$\partial_\lambda^2 J(\lambda^{3/2}\varphi_\omega(\lambda\cdot),\, \lambda^3\psi_\omega(\lambda\cdot))|_{\lambda=1} < 0. \tag{5}$$

Then, the standing wave solution $(e^{i\omega t}\varphi_\omega,\, \psi_\omega)$ of (1) *is orbitally unstable.*

The proof of Theorem 2 is based on that of Theorem 3 in Ohta [21], which is a modification of the original idea of Shatah and Strauss [25] (see Section 4 in [25]). See also Gonçalves Rebeiro [10] for another modification of [25], and see [8, 9, 17] for applications of [21]. Note that it seems difficult to apply the abstract theory by Grillakis, Shatah and Strauss [11, 12] to the problems studied in [8, 9, 10, 17, 21] directly (see also [5, 7, 16] for related problems). We believe that once we could find an appropriate identity like (4), our approach is much simpler than others. Another advantage of our approach is that the assumption (5) in Theorem 2 can be easily checked in the following way. Since $P(\varphi_\omega,\, \psi_\omega) = 0$ and $\|\nabla\psi_\omega\|_2^2 + \|\psi_\omega\|_2^2 = \int_{\mathbb{R}^3} |\varphi_\omega|^2\psi_\omega dx$, we have

$$\partial_\lambda^2 J(\lambda^{3/2}\varphi_\omega(\lambda\cdot), \lambda^3\psi_\omega(\lambda\cdot))|_{\lambda=1}$$

$$= 2\|\nabla\varphi_\omega\|_2^2 + 20\|\nabla\psi_\omega\|_2^2 + 6\|\psi_\omega\|_2^2 - 12\int_{\mathbb{R}^3} |\varphi_\omega|^2\psi_\omega dx$$
$$= 9\|\nabla\psi_\omega\|_2^2 - 3\|\psi_\omega\|_2^2.$$

Moreover, by the scaling

$$\varphi_\omega(x) = \omega^{1/2}\tilde{\varphi}_\omega(\omega^{1/2}x), \quad \psi_\omega(x) = \omega\tilde{\psi}_\omega(\omega^{1/2}x), \tag{6}$$

we have

$$\partial_\lambda^2 J(\lambda^{3/2}\varphi_\omega(\lambda\cdot), \lambda^3\psi_\omega(\lambda\cdot))|_{\lambda=1} = 3\omega^{1/2}\big(3\omega\|\nabla\tilde{\psi}_\omega\|_2^2 - \|\tilde{\psi}_\omega\|_2^2\big). \tag{7}$$

Note that $(\tilde{\varphi}_\omega, \tilde{\psi}_\omega)$ satisfies

$$\begin{cases} -\Delta\varphi + \varphi = 2\varphi\psi, & x \in \mathbb{R}^3, \\ -\omega\Delta\psi + \psi = |\varphi|^2, & x \in \mathbb{R}^3, \end{cases} \tag{8}$$

and passing to the limit as $\omega \to 0$, (8) reduces formally to the equation

$$-\Delta\varphi + \varphi = 2|\varphi|^2\varphi, \quad x \in \mathbb{R}^3. \tag{9}$$

To show that (5) is satisfied for sufficiently small $\omega > 0$, it suffices to prove the following Proposition.

Proposition 3 *Let $\omega > 0$, $(\varphi_\omega, \psi_\omega) \in \mathcal{G}_\omega$, and define $(\tilde{\varphi}_\omega, \tilde{\psi}_\omega)$ by (6). Then we have*

$$\lim_{\omega\to 0} \omega\|\nabla\tilde{\psi}_\omega\|_2^2 = 0, \quad \inf_{\omega>0} \|\tilde{\psi}_\omega\|_2^2 > 0.$$

In conclusion, Theorem 1 follows from Theorem 2, (7) and Proposition 3. The rest of the paper is organized as follows. In Section 2, we show the existence and the variational characterization of ground states of (2). In Section 3, we give the proof of Theorem 2. In Section 4, using the variational characterizations of ground states, we prove Proposition 3.

2. Existence of ground states

In this section, we assume $\omega > 0$. We put $\mathbf{H}^1_{\mathrm{rad}} = H^1_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{C}) \times H^1_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{R})$, and $\|(u, v)\|^2_{\mathbf{H}^1} = \|\nabla u\|_2^2 + \omega\|u\|_2^2 + \|\nabla v\|_2^2 + \|v\|_2^2$ for $(u, v) \in \mathbf{H}^1_{\mathrm{rad}}$. Furthermore, we put

$$K_\omega(u, v) = \|(u, v)\|^2_{\mathbf{H}^1} - 3\int_{\mathbb{R}^3} |u|^2 v dx,$$

$$
\begin{aligned}
d(\omega) &= \inf\{S_\omega(u,\, v)\colon \\
&\qquad\qquad (u,\, v) \in \mathbf{H}^1_{\mathrm{rad}},\ K_\omega(u,\, v) = 0,\ (u,\, v) \neq (0,0)\}, \quad (10)\\
\mathcal{M}_\omega &= \{(\varphi,\, \psi) \in \mathbf{H}^1_{\mathrm{rad}}\colon \\
&\qquad\qquad S_\omega(\varphi,\, \psi) = d(\omega),\ K_\omega(\varphi, \psi) = 0,\ (\varphi,\, \psi) \neq (0,\, 0)\}.
\end{aligned}
$$

Note that $\partial_\lambda S_\omega(\lambda u,\, \lambda v)|_{\lambda=1} = 2K_\omega(u,\, v)$, and that if $K_\omega(u,\, v) = 0$ then

$$
S_\omega(u,\, v) = \int_{\mathbb{R}^3} |u|^2 v dx = \frac{1}{3}\|(u,\, v)\|^2_{\mathbf{H}^1}.
$$

Lemma 1 *If* $(u,\, v) \in \mathbf{H}^1_{\mathrm{rad}}$ *and* $K_\omega(u,\, v) < 0$*, then* $d(\omega) < \int_{\mathbb{R}^3} |u|^2 v dx$*.*

Proof. If $K_\omega(u,\, v) < 0$, then $(u,\, v) \neq (0,\, 0)$, and there exists $\lambda_0 \in (0,\, 1)$ such that $K_\omega(\lambda_0 u,\, \lambda_0 v) = 0$. Thus, we have

$$
d(\omega) \leq S_\omega(\lambda_0 u,\, \lambda_0 v) = \lambda_0^3 \int_{\mathbb{R}^3} |u|^2 v dx < \int_{\mathbb{R}^3} |u|^2 v dx.
$$

This completes the proof. □

Lemma 2 *For any* $\omega > 0$*,* $d(\omega) > 0$*.*

Proof. Let $(u,\, v) \in \mathbf{H}^1_{\mathrm{rad}}$, $K_\omega(u,\, v) = 0$ and $(u,\, v) \neq (0,\, 0)$. Then, by the Young and the Sobolev inequalities, we have

$$
\|(u,\, v)\|^2_{\mathbf{H}^1} = 3\int_{\mathbb{R}^3} |u|^2 v dx \leq 2\|u\|_3^3 + \|v\|_3^3 \leq C\|(u,\, v)\|^3_{\mathbf{H}^1}.
$$

Since $(u, v) \neq (0, 0)$, we have $\|(u, v)\|_{\mathbf{H}^1} \geq 1/C$, which implies $d(\omega) > 0$. □

Lemma 3 *For any* $\omega > 0$*, the set* $\mathcal{M}_\omega$ *is not empty.*

Proof. Let $\{(u_n,\, v_n)\}$ be a minimizing sequence for (10). Then, $\{(u_n,\, v_n)\}$ is bounded in $\mathbf{H}^1_{\mathrm{rad}}$, and there exist a subsequence of $\{(u_n,\, v_n)\}$ (we denote it by the same letter) and $(\varphi,\, \psi) \in \mathbf{H}^1_{\mathrm{rad}}$ such that $(u_n,\, v_n) \rightharpoonup (\varphi,\, \psi)$ weakly in $\mathbf{H}^1_{\mathrm{rad}}$. Since the embedding $H^1_{\mathrm{rad}}(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is compact for $2 < q < 6$ ([26]), we see that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |\varphi|^2 \psi dx = \lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^2 v_n dx = \lim_{n\to\infty} S_\omega(u_n,\, v_n) = d(\omega),\\
&K_\omega(\varphi,\, \psi) \leq \liminf_{n\to\infty} K_\omega(u_n,\, v_n) = 0.
\end{aligned}
$$

By Lemma 1, we have $K_\omega(\varphi,\, \psi) = 0$ and $S_\omega(\varphi,\, \psi) = d(\omega)$. Moreover, by Lemma 2, we see that $(\varphi,\, \psi) \neq (0,\, 0)$. Thus, $(\varphi,\, \psi) \in \mathcal{M}_\omega$. □

Lemma 4 *For any $\omega > 0$, $\mathcal{G}_\omega = \mathcal{M}_\omega$.*

Proof. First, let $(\varphi, \psi) \in \mathcal{M}_\omega$. Then, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $S'_\omega(\varphi, \psi) = \lambda K'_\omega(\varphi, \psi)$, and we have

$$0 = 2K_\omega(\varphi, \psi) = \langle S'_\omega(\varphi, \psi), (\varphi, \psi)\rangle = \lambda\langle K'_\omega(\varphi, \psi), (\varphi, \psi)\rangle.$$

Since $K_\omega(\varphi, \psi) = 0$, we have

$$\langle K'_\omega(\varphi, \psi), (\varphi, \psi)\rangle = 2\|(\varphi, \psi)\|^2_{\mathbf{H}^1} - 9\int_{\mathbb{R}^3} |\varphi|^2\psi dx = -\|(\varphi, \psi)\|^2_{\mathbf{H}^1}.$$

Thus, we have $\lambda = 0$, which implies that $(\varphi, \psi) \in \mathcal{A}_\omega$. Moreover, for any $(u, v) \in \mathcal{A}_\omega$, we have $K_\omega(u, v) = 0$. Thus, by (10), we have $S_\omega(\varphi, \psi) = d(\omega) \le S_\omega(u, v)$, which shows that $(\varphi, \psi) \in \mathcal{G}_\omega$. Therefore, $\mathcal{M}_\omega \subset \mathcal{G}_\omega$. On the other hand, let $(\varphi, \psi) \in \mathcal{G}_\omega$. By Lemma 3, we can take $(u, v) \in \mathcal{M}_\omega$. Then, we have $d(\omega) = S_\omega(u, v)$ and $(u, v) \in \mathcal{G}_\omega$. Moreover, since $(\varphi, \psi) \in \mathcal{G}_\omega$, we see that $d(\omega) = S_\omega(u, v) = S_\omega(\varphi, \psi)$. Furthermore, since $K_\omega(\varphi, \psi) = 0$ and $(\varphi, \psi) \neq (0, 0)$, we have $(\varphi, \psi) \in \mathcal{M}_\omega$. Therefore, $\mathcal{G}_\omega \subset \mathcal{M}_\omega$. This completes the proof. □

3. Proof of Theorem 2

Let $\eta \in C^2([0, \infty))$ be a non-negative function such that

$$\eta(r) = \begin{cases} 3 \text{ for } 0 \le r \le 1, \\ 0 \text{ for } r \ge 2, \end{cases} \qquad \eta'(r) \le 0 \text{ for } 1 \le r \le 2,$$

and let

$$\zeta(r) = \frac{1}{r^3}\int_0^r s^2\eta(s)ds.$$

For $n \in \mathbb{N}$ and $(u, v, w) \in X$, we put

$$\eta_n(x) = \eta(|x|/n), \quad \zeta_n(x) = \zeta(|x|/n), \quad x \in \mathbb{R}^3,$$
$$I_n(u, v, w) = \operatorname{Im}\int_{\mathbb{R}^3} \overline{u}x \cdot \nabla u\zeta_n dx - 2\int_{\mathbb{R}^3} w(x \cdot \nabla v\zeta_n + v\eta_n)dx.$$

Lemma 5 *For $r \ge 0$, we have*

$$r\zeta'(r) + 3\zeta(r) = \eta(r), \quad \zeta'(r) \le 0, \quad 0 \le \zeta(r) \le \min\Big\{1, \frac{8}{r^3}\Big\}.$$

Moreover, for $n \in \mathbb{N}$ and $x \in \mathbb{R}^3$, we have

$$\begin{aligned}
&x \cdot \nabla\zeta_n(x) + 3\zeta_n(x) = \eta_n(x),\\
&\eta_n(x) = 3, \quad \zeta_n(x) = 1 \textit{ for } 0 \le |x| \le n,\\
&0 \le \eta_n(x) \le 3, \quad 0 \le \zeta_n(x) \le 1,\\
&|\Delta\eta_n(x)| \le \frac{C}{n^2}, \quad x \cdot \nabla\zeta_n(x) \le 0,\\
&|x\zeta_n(x)| \le \min\{|x|, 8n^3/|x|^2\} \le 2n.
\end{aligned}$$

Proof. By integration by parts, we have

$$3\int_0^r s^2\eta(s)ds = r^3\eta(r) - \int_0^r s^3\eta'(s)ds \ge r^3\eta(r),$$

which implies that $r\zeta'(r) = \eta(r) - 3\zeta(r) \le 0$ for $r \ge 0$. Thus, we have $\zeta'(r) \le 0$ for $r \ge 0$. The other properties can be easily proved. □

Lemma 6 *Let $(u(t), v(t))$ be a radially symmetric solution of (1) with initial data in X_{rad}. Then, there exists a positive constant C_0 independent of n such that*

$$\begin{aligned}
&\frac{d}{dt}I_n(u(t), v(t), \partial_t v(t))\\
&\le P(u(t), v(t)) + 6\int_{|x|\ge n}|u|^2|v|(t, x)dx + \frac{C_0}{n^2}(\|u(t)\|_2^2 + \|v(t)\|_2^2)
\end{aligned}$$

for $t \ge 0$.

Proof. By simple computations and Lemma 5, we have the following identity (see [18, 19, 20, 22, 24]):

$$\begin{aligned}
&\frac{d}{dt}I_n(u(t), v(t), \partial_t v(t))\\
&= 2\int_{\mathbb{R}^3}|\nabla u|^2\zeta_n dx + \int_{\mathbb{R}^3}|\nabla v|^2(2\zeta_n + \eta_n)dx\\
&\quad + \int_{\mathbb{R}^3} v^2\eta_n dx - 2\int_{\mathbb{R}^3}|u|^2 v\eta_n dx - \int_{\mathbb{R}^3}(\partial_t v)^2\eta_n dx\\
&\quad + 2\,\mathrm{Re}\int_{\mathbb{R}^3}(\nabla\zeta_n \cdot \nabla u)(x \cdot \nabla\overline{u})dx - \frac{1}{2}\int_{\mathbb{R}^3}|u|^2\Delta\eta_n dx\\
&\quad + 2\int_{\mathbb{R}^3}(\nabla\zeta_n \cdot \nabla v)(x \cdot \nabla v)dx - \int_{\mathbb{R}^3} v^2\Delta\eta_n dx.
\end{aligned}$$

Since $u(t)$, $v(t)$ and ζ_n are radially symmetric, we have

$$\mathrm{Re}\int_{\mathbb{R}^3}(\nabla\zeta_n\cdot\nabla u)(x\cdot\nabla\overline{u})dx=\int_{\mathbb{R}^3}x\cdot\nabla\zeta_n|\nabla u|^2dx,$$
$$\int_{\mathbb{R}^3}(\nabla\zeta_n\cdot\nabla v)(x\cdot\nabla v)dx=\int_{\mathbb{R}^3}x\cdot\nabla\zeta_n|\nabla v|^2dx.$$

Thus, by Lemma 5, we obtain the desired inequality. □

In the following, as in Section 2, we put

$$\mathbf{H}^1_{\mathrm{rad}}=H^1_{\mathrm{rad}}(\mathbb{R}^3,\,\mathbb{C})\times H^1_{\mathrm{rad}}(\mathbb{R}^3,\,\mathbb{R}).$$

Lemma 7 *Let $\omega>0$ and $(\varphi_\omega,\psi_\omega)\in\mathcal{G}_\omega$. Then*

$$S_\omega(\varphi_\omega,\psi_\omega)=\inf\Big\{S_\omega(u,\,v)\colon (u,\,v)\in\mathbf{H}^1_{\mathrm{rad}},\ \int_{\mathbb{R}^3}|u|^2v\,dx=\int_{\mathbb{R}^3}|\varphi_\omega|^2\psi_\omega dx\Big\}.$$

Proof. By Lemmas 1 and 4, if $(u,\,v)\in\mathbf{H}^1_{\mathrm{rad}}$ and $K_\omega(u,\,v)<0$, then

$$\int_{\mathbb{R}^3}|\varphi_\omega|^2\psi_\omega dx=d(\omega)<\int_{\mathbb{R}^3}|u|^2v\,dx.$$

Thus, if $(u,\,v)\in\mathbf{H}^1_{\mathrm{rad}}$ satisfies $\int_{\mathbb{R}^3}|u|^2vdx=\int_{\mathbb{R}^3}|\varphi_\omega|^2\psi_\omega dx$, then we have $K_\omega(u,\,v)\ge 0$, and

$$S_\omega(\varphi_\omega,\,\psi_\omega)=\int_{\mathbb{R}^3}|\varphi_\omega|^2\psi_\omega dx=\int_{\mathbb{R}^3}|u|^2v\,dx\le S_\omega(u,\,v).$$

This completes the proof. □

Lemma 8 *Under the assumption in* Theorem 2, *there exist positive constants ε_0 and δ_1 such that*

$$J(\varphi_\omega,\,\psi_\omega)\le E(u,\,v,\,w)+(\Lambda(u,\,v)-1)P(u,\,v),$$
$$|\Lambda(u,\,v)-1|<\delta_1$$

for any $(u,\,v,\,w)\in\mathcal{N}_{\varepsilon_0}$ satisfying $\|u\|_2=\|\varphi_\omega\|_2$, where

$$\Lambda(u,\,v)=\Big(\int_{\mathbb{R}^3}|\varphi_\omega|^2\psi_\omega dx\Big/\int_{\mathbb{R}^3}|u|^2v\,dx\Big)^{1/3},\tag{11}$$
$$\mathcal{N}_\varepsilon=\{(u,\,v,\,w)\in X_{\mathrm{rad}}\colon \inf_{\theta\in\mathbb{R}}\|(u,\,v,\,w)-(e^{i\theta}\varphi_\omega,\,\psi_\omega,\,0)\|_X<\varepsilon\}.$$

Proof. Since

$$J(\lambda^{3/2}u(\lambda\cdot),\, \lambda^3 v(\lambda\cdot)) \\ = \lambda^2\|\nabla u\|_2^2 + \lambda^5\|\nabla v\|_2^2 + \lambda^3\|v\|_2^2 - 2\lambda^3\int_{\mathbb{R}^3}|u|^2 v\,dx,$$

we see that the function $(\lambda, u, v) \mapsto \partial_\lambda^2 J(\lambda^{3/2}u(\lambda\cdot), \lambda^3 v(\lambda\cdot))$ is continuous on $(0,\infty)\times H^1_{\mathrm{rad}}(\mathbb{R}^3,\mathbb{C})\times H^1_{\mathrm{rad}}(\mathbb{R}^3,\mathbb{R})$. Thus, by the assumption (5), there exist positive constants ε_1 and δ_1 such that $\partial_\lambda^2 J(\lambda^{3/2}u(\lambda\cdot), \lambda^3 v(\lambda\cdot)) < 0$ if $|\lambda-1|<\delta_1$ and $(u, v, 0)\in\mathcal{N}_{\varepsilon_1}$. Thus, the Taylor expansion at $\lambda = 1$ and (3) imply that

$$J(\lambda^{3/2}u(\lambda\cdot),\, \lambda^3 v(\lambda\cdot)) \le J(u, v) + (\lambda-1)P(u, v) \tag{12}$$

if $|\lambda-1|<\delta_1$ and $(u, v, 0)\in\mathcal{N}_{\varepsilon_1}$. Moreover, by (11), we can take $\varepsilon_0\in(0,\varepsilon_1)$ such that $|\Lambda(u, v)-1|<\delta_1$ if $(u, v, 0)\in\mathcal{N}_{\varepsilon_0}$. Let $(u, v, w)\in\mathcal{N}_{\varepsilon_0}$ and $\|u\|_2=\|\varphi_\omega\|_2$. Since $\|\lambda^{3/2}u(\lambda\cdot)\|_2=\|u\|_2=\|\varphi_\omega\|_2$ and

$$\int_{\mathbb{R}^3}|\lambda^{3/2}u(\lambda\cdot)|^2\lambda^3 v(\lambda\cdot)\,dx = \int_{\mathbb{R}^3}|\varphi_\omega|^2\psi_\omega\,dx$$

when $\lambda=\Lambda(u, v)$, it follows from Lemma 7 that

$$J(\varphi_\omega, \psi_\omega) \le J(\lambda^{3/2}u(\lambda\cdot), \lambda^3 v(\lambda\cdot))|_{\lambda=\Lambda(u,v)}. \tag{13}$$

Therefore, by (12) and (13), we have

$$J(\varphi_\omega, \psi_\omega) \le E(u, v, w) + \{\Lambda(u, v)-1\}P(u, v)$$

for $(u, v, w)\in\mathcal{N}_{\varepsilon_0}$ satisfying $\|u\|_2=\|\varphi_\omega\|_2$. □

In Lemmas 9 and 10 below, we assume the assumption in Theorem 2, and let ε_0 be the positive constant given in Lemma 8. For $(u_0, v_0, v_1)\in\mathcal{N}_{\varepsilon_0}$, we define the exit time $T_0(u_0, v_0, v_1)$ from $\mathcal{N}_{\varepsilon_0}$ by

$$T_0(u_0, v_0, v_1) = \sup\{T>0\colon (u(t), v(t), \partial_t v(t))\in\mathcal{N}_{\varepsilon_0} \\ \text{for any } 0\le t\le T\},$$

where $(u(t), v(t), \partial_t v(t))$ is the solution of (1) with $(u(0), v(0), \partial_t v(0)) = (u_0, v_0, v_1)$. Moreover, we put

$$\mathcal{R}_0 = \{(u, v, w)\in\mathcal{N}_{\varepsilon_0}\colon E(u, v, w) < J(\varphi_\omega, \psi_\omega), \\ \|u\|_2=\|\varphi_\omega\|_2,\ P(u, v)<0\}.$$

Lemma 9 *For any* $(u_0, v_0, v_1) \in \mathcal{R}_0$*, there exists a constant* $\delta_0 > 0$ *such that*

$$P(u(t), v(t)) \le -2\delta_0, \quad t \in [0, T_0(u_0, v_0, v_1)),$$

where $(u(t), v(t))$ *is the solution of* (1) *with* $(u(0), v(0), \partial_t v(0)) = (u_0, v_0, v_1)$.

Proof. Let $(u_0, v_0, v_1) \in \mathcal{R}_0$, and put

$$T_0 := T_0(u_0, v_0, v_1), \quad \delta_2 := J(\varphi_\omega, \psi_\omega) - E(u_0, v_0, v_1).$$

Then, by Lemma 8 and by the conservations of E and $\|u\|_2$, we have

$$0 < \delta_2 \le \{\Lambda(u(t), v(t)) - 1\} P(u(t), v(t)), \quad t \in [0, T_0).$$

Since the function $t \mapsto P(u(t), v(t))$ is continuous and $P(u_0, v_0) < 0$, we have $P(u(t), v(t)) < 0$ and $1 - \delta_1 < \Lambda(u(t), v(t)) < 1$ for $t \in [0, T_0)$. Thus, we have

$$-P(u(t), v(t)) \ge \frac{\delta_2}{1 - \Lambda(u(t), v(t))} \ge \frac{\delta_2}{\delta_1}, \quad t \in [0, T_0),$$

which completes the proof. □

Lemma 10 *If* $(u_0, v_0, v_1) \in \mathcal{R}_0$*, then* $T_0(u_0, v_0, v_1) < \infty$.

Proof. Suppose that there exists $(u_0, v_0, v_1) \in \mathcal{R}_0$ such that $T_0(u_0, v_0, v_1) = \infty$. Let $(u(t), v(t))$ be the solution of (1) with $(u(0), v(0), \partial_t v(0)) = (u_0, v_0, v_1)$. By Lemma 6, we have

$$\begin{aligned}&\frac{d}{dt} I_n(u(t), v(t), \partial_t v(t)) \\ &\le P(u(t), v(t)) + 6 \int_{|x| \ge n} |u|^2 |v|(t, x) dx + \frac{C_0}{n^2} (\|u(t)\|_2^2 + \|v(t)\|_2^2)\end{aligned}$$

for $t \ge 0$. Since $(u(t), v(t), \partial_t v(t)) \in \mathcal{N}_{\varepsilon_0}$ for $t \ge 0$,

$$M := \sup_{t \ge 0} \|(u(t), v(t), \partial_t v(t))\|_X < \infty.$$

Furthermore, since $u(t)$ and $v(t)$ are radially symmetric, by the radial lemma of Strauss [26], there exists a constant $C_1 > 0$ such that

$$6 \int_{|x| \ge n} |u|^2 |v|(t, x) dx \le 6 \|v(t)\|_{L^\infty(|x| \ge n)} \|u(t)\|_2^2$$

$$\leq \frac{C_1}{n}\|u(t)\|_2^2\|v(t)\|_{H^1}, \quad t \geq 0.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&6\int_{|x|\geq n_0} |u|^2|v|(t,\,x)dx + \frac{C_0}{n_0^2}(\|u(t)\|_2^2 + \|v(t)\|_2^2) \\
&\leq \frac{C_1 M^3}{n_0} + \frac{C_0 M^2}{n_0^2} \leq \delta_0, \quad t \geq 0,
\end{aligned}$$

and by Lemmas 8 and 9, we have

$$\frac{d}{dt}I_{n_0}(u(t),\,v(t),\,\partial_t v(t)) \leq -\delta_0, \quad t \geq 0,$$

which implies $I_{n_0}(u(t),\,v(t),\,\partial_t v(t)) \to -\infty$ as $t \to \infty$. On the other hand, by Lemma 5, there exists a constant $C_2 > 0$ such that

$$|I_{n_0}(u(t),\,v(t),\,\partial_t v(t))| \leq C_2 n_0 \|(u(t),\,v(t),\,\partial_t v(t))\|_X^2 \leq C_2 n_0 M^2, \qquad t \geq 0.$$

This contradiction proves the Lemma. □

Proof of Theorem 2. Since

$$\begin{aligned}
&\partial_\lambda J(\lambda^{3/2}\varphi_\omega(\lambda\cdot),\,\lambda^3\psi_\omega(\lambda\cdot))|_{\lambda=1} = P(\varphi_\omega,\,\psi_\omega) = 0, \\
&\partial_\lambda^2 J(\lambda^{3/2}\varphi_\omega(\lambda\cdot),\,\lambda^3\psi_\omega(\lambda\cdot))|_{\lambda=1} < 0, \\
&\lambda\partial_\lambda J(\lambda^{3/2}\varphi_\omega(\lambda\cdot),\,\lambda^3\psi_\omega(\lambda\cdot)) = P(\lambda^{3/2}\varphi_\omega(\lambda\cdot),\,\lambda^3\psi_\omega(\lambda\cdot)), \\
&\|\lambda^{3/2}\varphi_\omega(\lambda\cdot)\|_2 = \|\varphi_\omega\|_2,
\end{aligned}$$

we see that $(\lambda^{3/2}\varphi_\omega(\lambda\cdot),\,\lambda^3\psi_\omega(\lambda\cdot),\,0) \in \mathcal{R}_0$ for $\lambda > 1$ sufficiently close to 1. Moreover, since $(\lambda^{3/2}\varphi_\omega(\lambda\cdot),\,\lambda^3\psi_\omega(\lambda\cdot),\,0) \to (\varphi_\omega,\,\psi_\omega,\,0)$ in X as $\lambda \to 1$, the orbital instability of $(e^{i\omega t}\varphi_\omega,\,\psi_\omega)$ follows from Lemma 10. □

4. Proof of Proposition 3

Let $\phi \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ be a unique positive radial solution of (9), and we put

$$\tilde{K}_\omega(u,\,v) = \|\nabla u\|_2^2 + \|u\|_2^2 + \omega\|\nabla v\|_2^2 + \|v\|_2^2 - 3\int_{\mathbb{R}^3} |u|^2 v dx.$$

Then, since $\|\nabla\phi\|_2^2+\|\phi\|_2^2=2\|\phi\|_4^4$ and $\phi\in L^\infty(\mathbb{R}^3)$, we see that

$$\begin{aligned}\tilde{K}_\omega(\sqrt{2}\phi,\,2\phi^2)&=2\|\nabla\phi\|_2^2+2\|\phi\|_2^2+8\omega\|\phi\nabla\phi\|_2^2-8\|\phi\|_4^4\\&=-4\|\phi\|_4^4+8\omega\|\phi\nabla\phi\|_2^2<0\end{aligned}$$

if $0<\omega<\omega_0$, where $\omega_0=\|\phi\|_4^4/(2\|\phi\nabla\phi\|_2^2)$. Thus, by Lemmas 1 and 4, we have

$$\int_{\mathbb{R}^3}|\tilde{\varphi}_\omega|^2\tilde{\psi}_\omega dx\le 4\|\phi\|_4^4,\quad \omega\in(0,\,\omega_0).$$

Since $\tilde{K}_\omega(\tilde{\varphi}_\omega,\,\tilde{\psi}_\omega)=0$, we have

$$\|\tilde{\varphi}_\omega\|_{H^1}^2+\omega\|\nabla\tilde{\psi}_\omega\|_2^2+\|\tilde{\psi}_\omega\|_2^2\le 12\|\phi\|_4^4,\quad \omega\in(0,\,\omega_0). \tag{14}$$

By the first equation of (8) and by the Hölder and the Gagliardo-Nirenberg-Sobolev inequalities, we have

$$\begin{aligned}&\|\tilde{\varphi}_\omega\|_{H^2}=\|(1-\Delta)\tilde{\varphi}_\omega\|_2=2\|\tilde{\varphi}_\omega\tilde{\psi}_\omega\|_2\le 2\|\tilde{\varphi}_\omega\|_6\|\tilde{\psi}_\omega\|_3\\&\le C\|\nabla\tilde{\varphi}_\omega\|_2\|\tilde{\psi}_\omega\|_2^{1/2}\|\nabla\tilde{\psi}_\omega\|_2^{1/2}.\end{aligned} \tag{15}$$

Moreover, by the second equation of (8), we have

$$\begin{aligned}&\|\nabla\tilde{\psi}_\omega\|_2=\|(1-\omega\Delta)^{-1}\nabla|\tilde{\varphi}_\omega|^2\|_2\le\|\nabla|\tilde{\varphi}_\omega|^2\|_2\\&\le 2\|\tilde{\varphi}_\omega\|_\infty\|\nabla\tilde{\varphi}_\omega\|_2\le C\|\tilde{\varphi}_\omega\|_{H^2}\|\nabla\tilde{\varphi}_\omega\|_2.\end{aligned} \tag{16}$$

Therefore, by (14), (15) and (16), we have

$$\omega\|\nabla\tilde{\psi}_\omega\|_2^2\le C\omega\|\nabla\tilde{\varphi}_\omega\|_2^4\|\tilde{\psi}_\omega\|_2\|\nabla\tilde{\psi}_\omega\|_2\le C\omega^{1/2},\quad \omega\in(0,\,\omega_0),$$

which implies $\lim_{\omega\to 0}\omega\|\nabla\tilde{\psi}_\omega\|_2^2=0$.

Finally, by the first equation of (8), we have

$$\|\tilde{\varphi}_\omega\|_{H^1}^2=2\int_{\mathbb{R}^3}|\tilde{\varphi}_\omega|^2\tilde{\psi}_\omega dx\le 2\|\tilde{\varphi}_\omega\|_4^2\|\tilde{\psi}_\omega\|_2\le C\|\tilde{\varphi}_\omega\|_{H^1}^2\|\tilde{\psi}_\omega\|_2.$$

Since $\tilde{\varphi}_\omega\ne 0$, we have $\|\tilde{\psi}_\omega\|_2\ge 1/C$ for $\omega>0$. This completes the proof of Proposition 3. □

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