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Decay of correlations for some partially hyperbolic diffeomorphisms

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Abstract. In this paper we study a $C^{1+\alpha}$-partially hyperbolic diffeomorphism f of which restriction on one dimensional center unstable direction behaves as Manneville-Pomeau map. We show that f admits a unique ergodic SRB measure with polynomial upper bounds on correlations for Hölder continuous functions.

Key words: almost Anosov diffeomophism, SRB measure, polynomial upper bounds on correlations, first return map.

1. Introduction

Let M be a d-dimensional closed manifold ($d \geq 2$) and f be a diffeomorphism of M. It is well known that any C^2-transitive Anosov diffeomorphism f admits a unique invariant measure μ which has absolutely continuous conditional measures on unstable manifolds ([25]). This result holds for any Axiom A diffeomorphism f and (f, μ) has exponential decay of correlations for Hölder continuous functions ([6], [22]).

An invariant probability measure μ is said to be a *Sinai–Ruelle–Bowen* measure (abbrev. *SRB* measure) if (i) μ has positive Lyapunov exponents and (ii) μ has *absolutely continuous conditional measures on unstable manifolds* (see Section 2 for the precise definition). The existence of SRB measures with exponential decay of correlations is discussed in [5] for Hénon maps, and in [7] for some partially hyperbolic diffeormophisms.

$f : M \circlearrowleft$ is called an *almost Anosov diffeomorphism with uniformly contracting direction* if there exist a norm $\|\cdot\|$ on M, $0 < \lambda < 1$ and a $D_x f$-invariant decomposition of the tangent space $\mathrm{T}_x M = E^s(x) \oplus E^u(x)$ such that the set $S := \{x \in M \mid \|D_x f^{-n}|_{E^u(x)}\| = 1 \ \ (n \geq 0)\}$ is finite and consists of fixed points for f and such that

$$\|D_x f|_{E^s(x)}\| \leq \lambda \quad (x \in M), \qquad \|D_x f^{-1}|_{E^u(x)}\| < 1 \quad (x \in M \setminus S).$$

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This paper shows that there exist $C^{1+\alpha}$-almost Anosov diffeomorphisms f of M with uniformly contracting direction such that f admits a unique SRB measure with polynomial upper bounds on correlations (Theorem), which is related to [20] and [28]. More precisely, we impose on f the following Conditions 1–4.

Condition 1 f is a $C^{1+\alpha}$-almost Anosov diffeomorphism ($0 < \alpha < 1$) of M with co-dimension–one uniformly contracting direction.

Given $0 < \varepsilon \leq 1$, let D^u_ε and D^s_ε be the closed balls of radius ε centered at the origin in $\mathbb{R}$ and $\mathbb{R}^{d-1}$ respectively. Let $\mathrm{Emb}^r(D^\sigma_\varepsilon, M)$ ($r \geq 1$) denote the set of C^r-embeddings of D^σ_ε into M with the C^r-topology for $\sigma = s, u$. By Condition 1, it follows from Theorem 5.5 in [11] (see also [23] Theorem IV.1) that there exist two continuous maps $\phi^s : M \to \mathrm{Emb}^1(I_1, M)$ and $\phi^u : M \to \mathrm{Emb}^1(I_1, M)$ with $\phi^\sigma(\{x\} \times 0) = x$ ($x \in M$, $\sigma = s, u$) such that for any $\varepsilon \in (0, 1]$ the *local stable* and *local center unstable manifolds* $V^s_\varepsilon(x) := \phi^s(\{x\} \times I_\varepsilon)$ and $V^u_\varepsilon(x) := \phi^u(\{x\} \times I_\varepsilon)$ satisfy $T_x V^\sigma_\varepsilon(x) = E^\sigma(x)$ for $\sigma = s, u$ (for more details, see Sections 2 and 4).

Condition 2 ϕ^u is a continuous map from M to $\mathrm{Emb}^2(D^u_1, M)$.

Since $p \in S$ is a fixed point for f, we have that $f^{-1}(V^u_\varepsilon(p)) \subset V^u_\varepsilon(p)$. Then f restricted to $V^u_\varepsilon(p)$, $f|_{V^u_\varepsilon(p)}$, is a map from $f^{-1}(V^u_\varepsilon(p))$ to $V^u_\varepsilon(p)$. By using ϕ^u we can identify $D^u_\varepsilon = [-\varepsilon, \varepsilon]$ with $V^u_\varepsilon(x)$ for any $x \in M$. Then p corresponds to the origin 0 in D^u_ε, and thus 0 is a fixed point for $f|_{V^u_\varepsilon(p)}$.

Condition 3 For any $p \in S$ the graph of $f|_{V^u_\varepsilon(p)}$ can be represented as

$$f|_{V^u_\varepsilon(p)}(x) = \begin{cases} x + x^{1+\alpha} + O(x^2) & (x \geq 0), \\ x - |x|^{1+\alpha} - O(x^2) & (x < 0). \end{cases}$$

f is called *topologically mixing* if for any open sets $U, V \subset M$ there exists $N > 0$ such that $f^{-n}(U) \cap V \neq \emptyset$ ($n \geq N$).

Condition 4 f is topologically mixing.

Let $\mathcal{H}_\eta$ be the set of Hölder continuous functions of M with Hölder exponent η. We say that (f, μ) has *polynomial upper bounds on correlations* for functions in $\mathcal{H}_\eta$ *with exponent* $\tau > 0$ if for any φ, $\psi \in \mathcal{H}_\eta$ there exists $C' = C'(\varphi, \psi) > 0$ such that

$$\mathrm{Cor}_n(\varphi, \psi; \mu) = \left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \le C' n^{-\tau} \quad (n \ge 1).$$

Theorem *Let $f : M \circlearrowleft$ be a diffeomorphism satisfying Conditions* 1–4. *Then f admits a unique ergodic SRB measure ν and (f, ν) has polynomial upper bounds on correlations for functions in $\mathcal{H}_\eta$ with exponent* $\min\{(\alpha')^{-1} - 1, \alpha^{-1}\eta\}$ *for any $\alpha' \in (\alpha, 1)$.*

In order to establish Theorem, we prove Key Lemma below. In fact, assume that $f : M \circlearrowleft$ satisfies Condition 1. Then f is expansive and satisfies shadowing property (Lemma 2.3). Here f is said to be *expansive* if there exists $\delta > 0$ such that if $x, y \in M$ and $d(f^i(x), f^i(y)) < \delta$ $(i \in \mathbb{Z})$ then $x = y$. A sequence $\{x_i\}_{i \in \mathbb{Z}} \subset M$ is called a *δ-pseudo orbit* for f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. A point $x \in M$ is said to be an *ς-shadowing point* for a δ-pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ if $d(f^i(x), x_i) < \varsigma$ $(i \in \mathbb{Z})$. We say that f satisfies *shadowing property* if for any $\varsigma > 0$ there exists $\delta > 0$ such that for any δ-pseudo orbit there exists at least one ς-shadowing point.

Thus it follows from Theorem 4.2.8 in [3] that f has a *Markov partition* $\mathcal{Q} = \{\mathcal{Q}_i\}_{i=1}^r$ (see [6] for the definition) with arbitrarily small diameter. By Condition 1, f is uniformly hyperbolic on

$$\Lambda := M \setminus \mathcal{P} \quad \text{where} \quad \mathcal{P} := \mathrm{int}\Bigg(\bigcup_{i : \mathcal{Q}_i \cap S \neq \emptyset} \mathcal{Q}_i \Bigg).$$

Here $\mathrm{int}(A)$ is the interior of a set A. Let $R(x)$ be the smallest positive integer $n \ge 1$ such that $f^n(x) \in \Lambda$ for $x \in \Lambda$. For any $\mathcal{Q}_i \in \mathcal{Q}$ and $x \in \mathrm{int}(\mathcal{Q}_i)$, let $\gamma^\sigma(x) := V_\varepsilon^\sigma(x) \cap \mathcal{Q}_i$ $(\sigma = s, u)$. Since $f(\gamma^s(x)) \subset \gamma^s(f(x))$ for any $x \in M$, R is constant on each $\gamma^s(x)$. We define *the first return map* f^R : $\Lambda \circlearrowleft$ by $(f^R)(x) = f^{R(x)}(x)$ for $x \in \Lambda$. Let m denote the Lebesgue measure on M. Since the points x such that $R(x) = \infty$ lie only on $f^{-1}(\gamma^s(p)) \setminus \gamma^s(p)$ $(p \in S)$ by Condition 1, $R(x) < \infty$ for m-a.e. $x \in \Lambda$. Thus f^R is well defined for m-a.e. x.

Define

$$\Lambda_i := \{y \in \Lambda \mid R(y) = i\} \quad (i \ge 1),$$

then $f^R(x) = f^i(x)$ for any $x \in \Lambda_i$. Let $\mathcal{Q}'_i := \mathcal{Q}_i \setminus \cup_{j=0}^{i-1} \mathcal{Q}_j$ $(1 \le i \le r)$. Then $\mathcal{Q}' := \{\mathcal{Q}'_i\}_{i=1}^r$ is a partition of M. $\mathcal{D}_0 := \{\Lambda_i^j \,|\, i \ge 1,\, 1 \le j \le r\}$

(where $\Lambda_i^j := \Lambda_i \cap \mathcal{Q}'_j$) is a partition of Λ. For $x, y \in \Lambda$, the *separation time* $s(x, y)$ is defined as the smallest $n \geq 0$ such that $(f^R)^n(x)$ and $(f^R)^n(y)$ belong to distinct Λ_i^j's. For any submanifold $\gamma \subset M$ let m_γ denote the Lebesgue measure on γ. Let f^u denote the restriction of f to the local unstable manifolds.

Key Lemma *Let $f : M \circlearrowleft$ be a diffeomorphism satisfying Conditions* 1, 2 *and* 4. *Assume further that f satisfies the following properties*:
(K-1) *There exist $C_1 > 0$ and $0 < \beta_1 < 1$ such that*

$$\log \frac{|det(D_x(f^u)^i)|}{|det(D_y(f^u)^i)|} \leq C_1 \beta_1^{s(f^i(x), f^i(y))}$$

for any $i \geq 1$, $1 \leq j \leq r$, $x \in \Lambda_i^j$ and $y \in \Lambda_i^j \cap \gamma^u(x)$.
(K-2) *There exists $\tau > 1$ such that*

$$m_{\gamma^u(x)}(\{y \in M \mid R(y) > n\}) = O(n^{-\tau})$$

for any $x \in \Lambda$.
Then f admits a unique ergodic SRB measure ν and (f, ν) has polynomial upper bounds on correlations for functions in $\mathcal{H}_\eta$ with exponent $\min\{\tau' - 1, \tau\eta\}$ *for any $\tau' \in (1, \tau)$.*

It will be shown in Appendix B that Conditions 2 and 3 imply (K-1) and (K-2) for $\tau = \alpha^{-1}$ (Lemmas 5.2 and 5.3).

This paper is organized as follows: In Section 2, we collect definitions and preliminary results (Lemmas 2.1–2.6) to show Key Lemma. Proofs are postponed to Appendix A. In Section 3, we prove Key Lemma. The strategy is the following: Using the argument as in [27] (cf. [10]) we construct a tower map F conjugating to f and a quotient tower map $\overline{F}$ by collasping the local stable manifolds. We estimate that the correlation function for f is approximated by that for $\overline{F}$ with polynomially error (Lemmas 3.6 and 3.7). Then we apply the result of [17] to $\overline{F}$ (Lemma 3.8), and obtain polynomial upper bounds on correlations for f.

2. Preliminaries

In this section we give definitions and preliminary results to show Key Lemma. An invariant probability measure μ is said to be an *SRB measure* if (i) μ has non-zero Lyapunov exponents and (ii) μ has *absolutely continuous conditional measures on unstable manifolds* (abbrev. *accm*) whose notion is defined as follows: Assume that μ has non-zero Lyapunov exponents. Describe the *unstable manifold* at x as

$$W^u(x) := \left\{ y \in M \;\middle|\; \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0 \right\}$$

([19]). Here d is the Riemannian metric on M. Let $\mathcal{B}$ be the Borel σ algebra of M. Let ξ be a measurable partition of M and $\mathcal{B}_\xi$ be the set of all Borel subsets which consist of the unions of the elements of ξ. Then there exists a family of conditional probability measures $\{\mu_x^\xi\}$ (μ-a.e.x) such that for μ-a.e.x and $B \in \mathcal{B}$, $\mu_x^\xi(B)$ is a $\mathcal{B}_\xi$-measurable function of x and

$$\mu(E \cap B) = \int_E \mu_x^\xi(B) d\mu(x) \quad (E \in \mathcal{B}_\xi)$$

(see [21]). Then μ has *accm* if for any measurable partition ξ^u such that $\xi^u(x) \subset W^u(x)$ and contains an open neighborhood of x in $W^u(x)$ for μ–a.e. x, the canonical system $\{\mu_x^u\}$ (μ-a.e.x) of conditional measures of μ w.r.t. ξ^u is absolutely continuous w.r.t. $m_{W^u(x)}$ ($\mu_x^u \ll m_{W^u(x)}$) ([15], [9]).

We use the following lemmas. Assume that f satisfies Condition 1. For any $x \in M$ and $\varepsilon \in (0,1]$, let $W_\varepsilon^\sigma(x) := V_\varepsilon^\sigma(x) \cap B_\varepsilon(x)$ ($\sigma = s, u$) where $B_\varepsilon(x)$ is the ball centered at x with radius ε. Then there exist $L > 0$ and $\lambda < \lambda_s < 1$ such that

$$d(f^n(x), f^n(y)) \le L \cdot \lambda_s^n d(x,y) \quad (n \ge 0) \tag{2.1}$$

for any $y \in W_\varepsilon^s(x)$ ($x \in M$) (see [23] p. 79).

Lemma 2.1 *For any $x \in M$ and $y \in W_\varepsilon^u(x)$, $T_y W_\varepsilon^u(x) = E^u(y)$.*

The following lemma ensures the local uniqueness of the local unstable manifolds.

Lemma 2.2 *For any $x \in M$, there exists a unique $W_\varepsilon^u(x)$ such that*

$T_x W^u_\varepsilon(x) = E^u(x)$.

Lemma 2.3 (1) *f is expansive and satisfies shadowing property,* (2) *f has a Markov partition with arbitrarily small diameter.*

Lemma 2.4 *There exist $C_2 > 0$ and $0 < \beta_2 < 1$ such that for any $x \in \Lambda$, $y \in \gamma^u(x)$ with $s(x,y) < \infty$ and $0 \le k \le n \le s(x,y) - 1$,*

$$\sum_{i=k}^{n} \log \frac{|det(D_{f^i(x)} f^u)|}{|det(D_{f^i(y)} f^u)|} \le C_2 \beta_2^{s(x,y)-n}$$

Lemma 2.5 *There exist $C_3 > 0$ and $0 < \beta_3 < 1$ such that for any $x \in M$, $y \in \gamma^s(x)$ and $n \ge 1$*

$$\sum_{i=n}^{\infty} \log \frac{|det(D_{f^i(x)} f^u)|}{|det(D_{f^i(y)} f^u)|} \le C_3 \beta_3^n.$$

Let (X_1, m_1) and (X_2, m_2) be finite measure spaces. We say that a measurable bijection $T : (X_1, m_1) \to (X_2, m_2)$ is *absolutely continuous* (or *nonsingular*) if it maps m_1-measure 0 to sets of m_2-measure 0. If T is absolutely continuous, then there exists the Jacobian $J(T) = J_{m_1,m_2}(T)$ of T w.r.t. m_1 and m_2 which is the Radon-Nykodym derivative $\frac{d(T_*^{-1} m_2)}{dm_1}$.

Let

$$\Gamma^\sigma := \{\gamma^\sigma(x) \,|\, x \in \mathcal{Q}_i, \mathcal{Q}_i \cap \mathcal{P} = \emptyset\} \quad (\sigma = s, u).$$

For any $\gamma, \gamma' \in \Gamma^u$ the *holonomy map* $\Theta_{\gamma,\gamma'} : \gamma \cap \Lambda \to \gamma' \cap \Lambda$ is defined by $\Theta_{\gamma,\gamma'}(x) = \gamma^s(x) \cap \gamma'$. Then $\Theta_{\gamma,\gamma'}$ is bijective.

Lemma 2.6 *If f satisfies (K-1), then for any $\gamma, \gamma' \in \Gamma^u$ the holonomy map $\Theta_{\gamma,\gamma'}$ is absolutely continuous and*

$$J(\Theta_{\gamma,\gamma'})(x) = \prod_{i=0}^{\infty} \frac{|det(D_{f^i(x)} f^u)|}{|det(D_{f^i(\Theta_{\gamma,\gamma'}(x))} f^u)|}.$$

3. Proof of Key Lemma

Throughout this section let f be a $C^{1+\alpha}$-diffeomorphism of M. Assume that f satsfies Conditions 1, 2, 4, (K-1) and (K-2). To show the existence of SRB measures, we need the arguments used in the proof of Theorem 1 in [27].

Lemma 3.1 *f^R admits an invariant probability measure μ such that $\mu_x^u \ll m_{\gamma^u(x)}$ with $\mu_x^u(\omega) \leq c_0 \cdot m_{\gamma^u(x)}(\omega)$ for μ-a.e.$x \in \Lambda$ and any Borel set $\omega \subset \gamma^u(x)$. Here $c_0 > 0$ is a global constant.*

Proof. For any $\gamma_0 \in \Gamma^u$, let m_{γ_0} denote the Lebesgue measure on γ_0. Define a probability measure on Λ by

$$\mu_n := \frac{1}{n}\sum_{j=0}^{n-1}\frac{(f^R)^j_* m_{\gamma_0}}{m_{\gamma_0}(\gamma_0)} \quad (n \geq 1).$$

Then there exist a subsequence $\{\mu_{n_j}\}_{j\geq 1} \subset \{\mu_n\}_{n\geq 1}$ such that $\{\mu_{n_j}\}$ converges to a f^R invariant probability measure μ.

For any $j \geq 1$ let ρ_j be the densities of $(f^R)^j_*(m_{\gamma_0})$ on the components of $(f^R)^j(\gamma_0) \cap \mathcal{Q}'_k$ for $1 \leq k \leq r$. Using the argument in [26] and [27] (see also [1], [16], [24]), (K-1) ensures that there exists $K > 0$ such that

$$\frac{1}{K} \leq \frac{\rho_j(x)}{\rho_j(y)} \leq K$$

for any $j \geq 1$ and x, y which belong to the same component of $(f^R)^j(\gamma_0) \cap \mathcal{Q}'_k$. Therefore we have the lemma for μ by [27] (cf. [8], [10]). □

Let μ_0 be as in Lemma 3.1 and define an f-invariant measure by

$$\mu' := \sum_{i=1}^{\infty} f_*^{i-1}(\mu_0|_{\{R\geq i\}}).$$

Since $\int_\gamma R dm_\gamma < \infty$ ($\gamma \in \Gamma^u$) by (K-2), μ' is a finite measure by Lemmas 2.6 and 3.1, and so normalize μ' (we denote it by ν). Clearly ν satisfies accm by Lemma 3.1, and furthermore ν is an SRB measure by the following Lemma 3.2.

For any f–invariant ergodic probability measure μ let $B(\mu)$ be the set of points x such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x))=\int\varphi d\mu$$

for any continuous function φ. Then by Birkhoff's ergodic theorem we have $\mu(B(\mu))=1$.

Lemma 3.2 *ν is a unique ergodic SRB measure.*

Proof. Put $\mathrm{Sat}(\mathcal{Q}_i):=\cup\{\gamma^s\in\Gamma^s\,|\,\gamma^s\cap f^n(\mathcal{Q}_i)\neq\emptyset,\,n\in\mathbb{Z}\}$ for any $1\leq i\leq r$. Then $f(\mathrm{Sat}(\mathcal{Q}_i))\subset\mathrm{Sat}(\mathcal{Q}_i)$. Discard those $\mathcal{Q}_i$ with $\nu(\mathcal{Q}_i)=0$. Let ν_i be the normalization of $\nu|_{\mathrm{Sat}(\mathcal{Q}_i)}$. To establish the lemma, let us show that (i) ν_i is ergodic and (ii) ν_i has non-zero Lyapunov exponents.

We show (i). Since ν is as in Lemma 3.1, Condition 1 and Lemma 2.1 allow us to apply arguments from [4] (pages 118,119) to $(f,\,\nu_i)$ to establish that the forward Birkhoff average of any ν_i–integrable function ψ is a constant on $\mathrm{Sat}(\mathcal{Q}_i)$ (mod ν_i). This implies that $\nu_i(A)=0$ or 1 for any f–invariant set A.

To prove (ii) it suffices to establish that ν_i has only positive Lyapunov exponents along E^u. The ergodicity of ν_i implies that for ν_i–a.e. x there exists $n_0\geq 1$ such that $\sharp\{0\leq i\leq n-1\,|\,f^i(x)\in M\setminus\mathcal{P}\}\geq n\frac{\nu_i(M\setminus\mathcal{P})}{2}$ for any $n\geq n_0$. This combined with Condition 1 implies (ii).

Let ν_i be as above. Assume that f admits another ergodic SRB measure μ. By Condition 4, it follows from arguments in [12] that there exist $\mathcal{Q}_i\in\mathcal{Q}$ and $x,y\in\mathcal{Q}_i$ such that $m_{\gamma^u(x)}(\gamma^u(x)\cap B(\nu_i))=m_{\gamma^u(x)}(\gamma^u(x))$ and $m_{\gamma^u(y)}(\gamma^u(y)\cap B(\mu))>0$. By Lemma 2.6, we can find $z\in\gamma^u(x)\cap B(\nu_i)$ such that $\gamma^s(z)\cap\gamma^u(y)\subset B(\mu)$. Therefore we have $\nu_i=\mu$. □

3.1. The tower map F.

To obtain polynomial upper bounds on correlation for $(f,\,\nu)$, we check it for the tower system conjugating to f which is described by L.-S. Young ([26], [27]).

A tower Δ is a union of the ℓ-th floors Δ_ℓ for $\ell\in\mathbb{Z}^+$ where $\mathbb{Z}^+:=\{0\}\cup\mathbb{N}$. The base Δ_0 is a finite measure space (Λ,m). Δ is defined by a countable partition $\mathcal{D}_0:=\{\Lambda_i^j\}_{i\geq 1,\,1\leq j\leq r}$ of Λ (mod m) and a function R with $R_i:=R|_{\Lambda_i^j}$ for $\Lambda_i^j\in\mathcal{D}_0$. Here Λ, $\mathcal{D}_0$ and R be as in Introduction. Let Δ_ℓ be a copy of a part of Λ by

$$\Delta_\ell := \{(x,\ell)\,|\,x \in \Lambda,\ \ell < R(x)\}.$$

Let $\Delta^j_{\ell,i}$ be a copy of Λ^j_i by

$$\Delta^j_{\ell,i} := \{(x,\ell)\,|\,x \in \Lambda^j_i,\ \ell < R(x)\}.$$

Then a system F on the tower $\Delta = \cup_{\ell \geq 0}\Delta_\ell$ is defined by

$$F(x,\ell) := \begin{cases} (x,\ell+1) & \text{if } \ell+1 < R(x) \\ (f^R(x),0) & \text{if } \ell+1 = R(x). \end{cases}$$

Here $f^R : \Lambda \to \Lambda$ is the first return map. Identifying Λ with Δ_0, we can define the map $F^R : \Delta_0 \to \Delta_0$ by $F^R(x) := (f^R(x),0)$ for any $x \in \Delta_0$. Define the partition of Δ by

$$\mathcal{D} := \{\Delta^j_{\ell,i}\}_{\ell \geq 0, i \geq 1, 1 \leq j \leq r}. \tag{3.1}$$

For any $\Delta^j_{0,i} \in \mathcal{D}$ and $\gamma^u \in \Gamma^u$, the F^R-image of each component of $\gamma^u \cap \Delta^j_{0,i}$ is a union of some elements in Γ^u. Thus $\mathcal{D}$ is a Markov partition for F in the usual sense.

3.2. Quotienting map $\overline{F}$

Define a relation $x \sim y$ if $y \in \gamma^s(x)$. By this relation we define the quotient space $\overline{\Delta} := \Delta/\sim$ by identifying points on each $\gamma^s \in \Gamma^s$. $\overline{\Delta}_\ell$ and $\overline{\Delta}^j_{\ell,i}$ are defined similary. Since F sends $\gamma \in \Gamma^s$ to $\gamma' \in \Gamma^s$, a quotinent map $\overline{F} : \overline{\Delta} \to \overline{\Delta}$ is well defined.

We define a reference measure $\overline{m}$ on $\overline{\Delta}_0 := \Delta_0/\sim$ by the following way which is introduced in [27]. If it is done, we can define the measure $\overline{m}|_{\overline{\Delta}_\ell}$ by using the natural identification of $\overline{\Delta}_\ell$ with a subset of $\overline{\Delta}_0$ such that $J(\overline{F}) \equiv 1$ except on $\overline{F}^{-1}(\overline{\Delta}_0)$, where $J(\overline{F}) = J(\overline{f^R} \circ \overline{F}^{-(R-1)})$. Here $\overline{f^R}$ is defined by the similar way as above.

Take an arbitrary $\hat{\gamma} \in \Gamma^u$. Let $\hat{x} := \gamma^s(x) \cap \hat{\gamma}$ for any $x \in \Lambda$ and define

$$\Phi_n(x) := \sum_{i=0}^{n-1}(\psi^u(f^i(x)) - \psi^u(f^i(\hat{x})))$$

where $\psi^u(z) := \log|\det(D_x f^u)|$ for any $z \in M$. By Lemma 2.5 there exists a function Φ such that Φ_n converges uniformly to Φ as $n \to \infty$. On each $\gamma \in \Gamma^u$ define $\overline{m}_\gamma = e^\Phi m^u_\gamma$. If $f^R(\Lambda^j_i \cap \gamma) \subset \gamma'$ holds, then for $x \in \Lambda^j_i \cap \gamma$ we write $J(f^R)(x) = J_{\overline{m}_\gamma, \overline{m}_{\gamma'}}(f^{R_i}|_{\Lambda^j_i \cap \gamma})(x)$.

By (1) of the following Lemma 3.3 the measure $\overline{m}$ on $\overline{\Delta}_0$ whose representative on each $\gamma \in \Gamma^u$ is $\overline{m}_\gamma$ is well defined. By (2) of the lemma $J(f^R)$ is also well defined w.r.t. $\overline{m}$.

Lemma 3.3 (1) *For any $\gamma, \gamma' \in \Gamma^u$ let $\Theta = \Theta_{\gamma,\gamma'} : \gamma \to \gamma'$ be the sliding map along the local stable manifolds. Then $\Theta_* \overline{m}_\gamma = \overline{m}_{\gamma'}$,*

(2) *$J(f^R)(x) = J(f^R)(y)$ for any $y \in \gamma^s(x)$,*

(3) *There exist $C_4 > 1$ and $0 < \beta_4 < 1$ such that*

$$\left| \frac{J(f^R)(x)}{J(f^R)(y)} - 1 \right| \leq C_4 \beta_4^{s(f^R(x), f^R(y))}$$

for any $i \geq 1$, $1 \leq j \leq r$, $\gamma \in \Gamma^u$ and $x, y \in \gamma \cap \Lambda^j_i$.

Proof. By the same argument from [27] (see also [10] Lemma 3.4), we have (1) and (2). To show (3), we estimate $|\Phi(x) - \Phi(y)|$ for any $x, y \in \Lambda^j_i \cap \gamma^u$ as follows: Choose $\frac{1}{3}s(x,y) \leq k \leq \frac{1}{2}s(x,y)$. We have that $|\Phi(x) - \Phi(y)| \leq$ (I) + (II) where

$$\text{(I)} = \left| \sum_{j=0}^{k-1} \big(\psi^u(f^j(x)) - \psi^u(f^j(y))\big) - \sum_{j=0}^{k-1} \big(\psi^u(f^j(\hat{x})) - \psi^u(f^j(\hat{y}))\big) \right|,$$

$$\text{(II)} = \left| \sum_{j=k}^{\infty} \big(\psi^u(f^j(x)) - \psi^u(f^j(\hat{x}))\big) - \sum_{j=k}^{\infty} \big(\psi^u(f^j(y)) - \psi^u(f^j(\hat{y}))\big) \right|.$$

Using Lemma 2.4, (I) $\leq C_2 \beta_2^{\frac{s(x,y)}{2}}$. By Lemma 2.5, (II) $\leq C_3 \beta_3^{\frac{s(x,y)}{3}}$. Thus

$$\text{(I)} + \text{(II)} \leq 2(C_3 + C_2)\beta_4^{s(f^R(x), f^R(y))}. \tag{3.2}$$

Here $0 < \beta_4 = \max\{\beta_1, \beta_2^{1/2}, \beta_3^{1/3}\} < 1$. By the similar argument as above we have

$$\left|\Phi(f^R(x)) - \Phi(f^R(y))\right| \leq 2(C_3 + C_2)\beta_4^{s(f^R(x), f^R(y))}. \tag{3.3}$$

Therefore, by (3.2), (3.3) and (K-1), there exists $C_1' > 0$ such that

$$\log \frac{J(f^R)(x)}{J(f^R)(y)} = \log \frac{|\det(D_x f^i|_{E^u(x)}|}{|\det(D_y f^i|_{E^u(y)}|} + \Phi(f^R(x)) \\ - \Phi(f^R(y)) - (\Phi(x) - \Phi(y)) \\ \leq (C_1' + 4(C_3 + C_2))\beta_4^{s(f^R(x), f^R(y))}$$

which concludes the lemma. □

(K-2) and Lemmas 2.5 and 2.6 imply that $\overline{m}(\{R \circ \overline{\pi}^{-1} > n\}) = O(n^{-\tau})$ for some $\tau > 1$. Here $\overline{\pi} : \Delta \to \overline{\Delta}$ is the projection such that $\overline{\pi} \circ F = \overline{F} \circ \overline{\pi}$. Then we have that $\int_{\overline{\Delta}_0} R \circ \overline{\pi}^{-1} d\overline{m} < \infty$. We summarize the properties of $\overline{F} : \overline{\Delta} \to \overline{\Delta}$ as follows:

(a) $\overline{F^R} : \overline{\Delta}^j_{0,i} \to \overline{F^R}(\overline{\Delta}^j_{0,i})$ is bijective and $\overline{F^R}(\overline{\Delta}^j_{0,i})$ is a union of some $\overline{\Delta}^q_{0,p}$'s (mod $\overline{m}$), and furthermore there exists $\eta > 0$ such that $\inf_{i \geq 1, 1 \leq j \leq r}\{\overline{m}(\overline{F^R}(\overline{\Delta}^j_{0,i}))\} \geq \eta$,

(b) $\overline{\mathcal{D}} := \{\overline{\Delta}^j_{\ell,i}\}_{\ell \geq 0, i \geq 1, 1 \leq j \leq r}$ is a partition such that $\vee_{j=0}^{\infty} \overline{F}^{-j}(\overline{\mathcal{D}})$ is the partition into points,

(c) $\overline{m}(A) = \overline{m}(\overline{F}(A))$ for any $A \subset \overline{\Delta}^j_{\ell,i}$ with $\overline{F}(A) \subset \overline{\Delta}^j_{\ell+1,i}$, and

(d) for any $i \geq 1$ and $1 \leq j \leq r$, $\overline{F^R}|_{\overline{\Delta}^j_{0,i}}$ and its inverse are nonsingular w.r.t. $\overline{m}$.

We redefine the separation time $\bar{s}(\cdot, \cdot)$ on $\bar{\Delta}$ as follows: Firstly for any $\bar{x}, \bar{y} \in \bar{\Delta}_0$, $\bar{s}(\bar{x}, \bar{y})$ is defined by $s(x, y)$ where $(x, 0) \in \bar{\pi}^{-1}(\bar{x})$ and $(y, 0) \in \bar{\pi}^{-1}(\bar{y})$. Secondly for any $\bar{x}, \bar{y} \in \bar{\Delta}_\ell$, $\bar{s}(\bar{x}, \bar{y})$ is defined by $\bar{s}(\bar{x}_0, \bar{y}_0)$ where $\bar{x}_0, \bar{y}_0 \in \bar{\Delta}_0$ are the unique preimages of $\bar{x}, \bar{y}$ by $\bar{F}^\ell$, i.e. $\bar{F}^\ell(\bar{x}_0) = \bar{x}$ and $\bar{F}^\ell(\bar{y}_0) = \bar{y}$. Otherwise $\bar{s}(\bar{x}, \bar{y}) = 0$.

(e) $J(\overline{F^R})$ satisfies that $\left| \frac{J(\overline{F^R})(\bar{x})}{J(\overline{F^R})(\bar{y})} - 1 \right| \leq C_4 \beta_4^{\bar{s}(\overline{F^R}(\bar{x}), \overline{F^R}(\bar{y}))}$ for any $i \geq 1$ and $\bar{x}, \bar{y} \in \overline{\Delta}^j_{0,i}$, and

(f) for any ℓ, $\ell' \geq 0$, i, $i' \geq 1$ there exists $N > 0$ such that $\overline{F}^{-n}(\overline{\Delta}_{\ell,i}) \cap \overline{\Delta}_{\ell',i'} \neq \emptyset$ for any $n \geq N$ (by Condition 4).

Let $C_{\beta_4}(\overline{\Delta}) := \{\overline{\varphi} : \overline{\Delta} \to \mathbb{R} \,|\, \exists C_{\overline{\varphi}} > 0 \text{ s.t. } |\overline{\varphi}(\bar{x}) - \overline{\varphi}(\bar{y})| \leq C_\varphi \beta_4^{\bar{s}(\bar{x}, \bar{y})}$ $(\forall \bar{x}, \bar{y} \in \overline{\Delta}^j_{\ell,i})\}$.

Lemma 3.4 *$\overline{F}$ admits a mixing invariant probability measure $\overline{\nu}$ such that $d\overline{\nu} = \overline{\varrho} d\overline{m}$. Here $\overline{\varrho}$ satisfies $\overline{c_0}^{-1} \leq \overline{\varrho} \leq \overline{c_0}$ for $\overline{c_0} > 0$ with*

$$|\overline{\varrho}(\bar{x}) - \overline{\varrho}(\bar{y})| \leq \overline{c_0} \beta_4^{\overline{s}(\bar{x},\bar{y})} \quad (\bar{x}, \bar{y} \in \overline{\Delta}_{\ell,i}^{j}).$$

Proof. Applying the arguments as in [1], [13], [16], [17], [26], [27], [29], [10] gives the lemma. □

We define the *transfer operator* associated with $\overline{F}$ and the measure $\overline{m}$ by

$$\bar{\mathcal{L}}(\bar{\varphi})(\bar{x}) := \sum_{\bar{x}' : \bar{F}(\bar{x}') = \bar{x}} \frac{\bar{\varphi}(\bar{x}')}{J(\bar{F})(\bar{x}')} \tag{3.4}$$

for $\bar{\varphi} \in L^2(\bar{m})$ and $\bar{x} \in \bar{\Delta}$. Then $\bar{\mathcal{L}}(\overline{\varrho}) = \overline{\varrho}$ where $\overline{\varrho}$ is as in Lemma 3.4. Let $L^\infty(\overline{m})$ be the set of functions which are essentially bounded w.r.t. $\overline{m}$. We denote the essential sup norm w.r.t. $\overline{m}$ by $\| \cdot \|_\infty$. For any $\overline{\psi} \in C_{\beta_4}(\overline{\Delta})$ we define $\|\overline{\psi}\| := \max\{\|\overline{\psi}\|_\infty, C_{\overline{\psi}}\}$.

Proposition 3.5 ([17] Proposition 3.13, Corollary 3.15) *Let $w := \{w(\ell)\}_{\ell \in \mathbb{Z}^+}$ be a positive increasing sequence such that (i) $\sum_{\ell=1}^{\infty} w(\ell)\, \bar{m}(\bar{\Delta}_\ell) < \infty$ and (ii) the sequence $\{\frac{w(\ell)}{w(\ell+1)}\}_{\ell=1}^{\infty}$ is also increasing. Then there exist $k_1 = k_1(w) \in \mathbb{N}$ and $C_5 = C_5(w, k_1) > 0$ such that for any $\bar{\phi} \in C_\beta(\bar{\Delta})$ with $\int \bar{\phi} d\bar{m} = 1$, any $n \in \mathbb{N}$ with $n = k_1 j + r$ for some $j \in \mathbb{N}$ and $r \in \{0, \dots, k_1 - 1\}$, and any $\ell \in \mathbb{Z}^+$,*

$$\sup_{\bar{x} \in \bar{\Delta}_\ell} \left| \bar{\mathcal{L}}^n(\bar{\phi})(\bar{x}) - \bar{\varrho}(\bar{x}) \right| \leq C_5 \, \|\bar{\phi}\| \, \frac{w(\ell)}{w(k_1 j)}$$

where $\bar{\varrho}$ is as in Lemma 3.4.

Throughout this section, we fix a positive increasing function $v : \mathbb{R}^+ \to \mathbb{R}$ such that (i) for any $\gamma \in \Gamma^u$, $\sum_{\ell=1}^{\infty} v(\ell)\, m_\gamma(\{R > \ell\}) < \infty$, and (ii) the sequence $\{\frac{v(\ell)}{v(\ell+1)}\}_{\ell=1}^{\infty}$ is also increasing. By Lemma 3.3(1) we have that $\sum_{\ell=0}^{\infty} v(\ell)\, \bar{m}(\bar{\Delta}_\ell) < \infty$. Then we let $k_1 = k_1(v) \in \mathbb{N}$ and $C_5 = C_5(v, k_1) > 0$ as in Proposition 3.5. Then by Proposition 3.5 we have that for any $n \geq 1$ and $\overline{\psi} \in C_{\beta_4}(\overline{\Delta})$ with $\int \overline{\psi} d\overline{m} = 1$,

$$\sup_{\bar{x}\in\overline{\Delta}_\ell} \left|\bar{\mathcal{L}}^n(\overline{\psi})(\bar{x}) - \overline{\varrho}(\bar{x})\right| \le C_5'(\overline{\psi})\frac{v(\ell)}{v(\frac{n}{2})} \quad (\ell \ge 0) \tag{3.5}$$

where

$$C_5'(\overline{\psi}) := \max_{0\le j\le k_1-1}\left\{\frac{v\left(\frac{k_1}{2}\right)}{v(0)}\left(\|\bar{\mathcal{L}}^j(\overline{\psi})\|_\infty + \|\overline{\varrho}\|_\infty\right), C_5\|\overline{\psi}\|\right\}. \tag{3.6}$$

3.3. Polynomial upper bounds on correlations for F

Let $\pi_1 : \Delta \to M$ be the natural projection by $\pi_1(x,\ell) = f^\ell(x)$ for $(x,\ell) \in \Delta$. Then we have $f\circ\pi_1 = \pi_1\circ F$. For any function $\varphi : M \to \mathbb{R}$, let $\tilde{\varphi}$ be the lift of φ to Δ defined by $\tilde{\varphi} = \varphi\circ\pi_1$. Define an F-invariant probability measure $\tilde{\nu}$ by $\tilde{\nu} = \nu\circ\pi_1$, and the correlation function for $(F,\tilde{\nu})$ by $\mathrm{Cor}_n(\tilde{\varphi},\tilde{\psi};\tilde{\nu}) = \left|\int(\tilde{\varphi}\circ F^n)\tilde{\psi}d\tilde{\nu} - \int\tilde{\varphi}d\tilde{\nu}\int\tilde{\psi}d\tilde{\nu}\right|$. Then we have that $\mathrm{Cor}_n(\varphi,\psi;\nu) = \mathrm{Cor}_n(\tilde{\varphi},\tilde{\psi};\tilde{\nu})$. We define $\mathcal{D}_j := \vee_{i=0}^j F^{-i}(\mathcal{D})$ for any $j\ge 0$ where $\mathcal{D}$ is as in (3.1). For any $x\in\Delta$ let $\mathcal{D}_j(x)$ denote the element of $\mathcal{D}_j$ which containts x.

Lemma 3.6 *There exists $C_6 > 0$ such that for any $k\ge 1$ and $x\in\Delta$, $diam(\pi_1\circ F^k(\mathcal{D}_{2k}(x))) \le C_6 k^{-\tau}$.*

Proof. Put $\hat{y} = \gamma^s(y_1)\cap\gamma^u(y_2)$ for $y_1, y_2 \in \mathcal{D}_{2k}(x)$. Assume that $\mathcal{D}_{2k}(x) \subset \Delta_\ell$ for some $l\ge 0$. Then there exist y_1^0, y_2^0 and $\hat{y}^0 \in \Delta_0$, such that $F^\ell(y_1^0) = y_1$, $F^\ell(y_2^0) = y_2$ and $F^\ell(\hat{y}^0) = \hat{y}$. Since $f\circ\pi_1 = \pi_1\circ F$, we have

$$d\left(\pi_1\circ F^k(\hat{y}), \pi_1\circ F^k(y_2)\right) = d\left(f^{k+\ell}(\hat{y}^0), f^{k+\ell}(y_2^0)\right).$$

Using Condition 2, (K-2) and Lemma 5.1 we have that (see [10] Lemma 4.12) there exists $K_1 > 0$ such that

$$d(f^{k+\ell}(\hat{y}^0), f^{k+\ell}(y_2^0)) \le K_1 k^{-\tau}. \tag{3.7}$$

Note that $\pi_1\circ F^{-\ell}(\hat{y}) \in \gamma^s(\pi_1\circ F^{-\ell}(y_1))$. Then we have that $d(\pi_1\circ F^{-\ell}(\hat{y}), \pi_1\circ F^{-\ell}(y_1)) \le 1$, and thus by (2.1)

$$\begin{aligned} d\left(\pi_1\circ F^k(y_1), \pi_1\circ F^k(\hat{y})\right) &\le d\left(f^{k+\ell}\circ\pi_1\circ F^{-\ell}(y_1), f^{k+\ell}\circ\pi_1\circ F^{-\ell}(\hat{y})\right) \\ &\le L\lambda_s^k. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we estimate that $d(\pi_1 \circ F^k(y_1), \pi_1 \circ F^k(y_2)) \leq K_1 k^{-\tau} + L\lambda_s^k$. This concludes the proof. □

For any continuous function φ define a function $\overline{\varphi}^k : \Delta \to \mathbb{R}$ by $\overline{\varphi}^k | A = \inf\{\tilde{\varphi}(x) \,|\, x \in F^k(A)\}$ for $A \in \mathcal{D}_{2k}$. Put $\tilde{\varphi}_k = d(F_*^k(\overline{\varphi}_k \tilde{\nu}))/d\tilde{\nu}$.

Lemma 3.7 *For any $\varphi, \psi \in \mathcal{H}_\eta$ there exists $C_7 = C_7(\varphi, \psi) > 0$ such that for any $1 \leq k \leq n$,*

$$\left| Cor_n(\tilde{\varphi}, \tilde{\psi}; \tilde{\nu}) - Cor_{n-k}(\overline{\varphi}^k, \tilde{\psi}_k; \tilde{\nu}) \right| \leq C_7 k^{-\tau\eta}.$$

Proof. Using the same argument as in [27] it follows from Lemma 3.6 (see also [10] Lemma 3.9) that there exists $K_2 = K_2(\varphi, \psi) > 0$ such that

$$\left|\mathrm{Cor}_{n-k}(\tilde{\varphi} \circ F^k, \tilde{\psi}; \tilde{\nu}) - \mathrm{Cor}_{n-k}(\overline{\varphi}^k, \tilde{\psi}; \tilde{\nu})\right| \leq K_2 k^{-\tau\eta},$$
$$\left|\mathrm{Cor}_{n-k}(\overline{\varphi}^k, \tilde{\psi}; \tilde{\nu}) - \mathrm{Cor}_{n-k}(\overline{\varphi}^k, \tilde{\psi}_k; \tilde{\nu})\right| \leq K_2 k^{-\tau\eta} \quad (1 \leq k \leq n).$$

This concludes the proof. □

Let $k \in \mathbb{N}$ be such that $k \in [\frac{n}{6}, \frac{n}{4}]$. Since $\overline{\psi}^k$ and $\overline{\varphi}^k$ are constant on $\gamma^s \in \Gamma^s$ and $\overline{\pi} \circ F = \overline{F} \circ \overline{\pi}$, we have by Lemma 3.4 that

$$\int \left(\overline{\varphi}^k \circ F^{n-k}\right) \cdot \tilde{\psi}_k d\tilde{\nu} = \int \left(\overline{\varphi}^k \circ \overline{\pi}^{-1} \circ \overline{F}^n\right) \cdot \overline{\psi}^k \circ \overline{\pi}^{-1} d\overline{\nu}$$
$$= \int \overline{\varphi}^k \circ \overline{\pi}^{-1} \cdot \bar{\mathcal{L}}^n \big(\overline{\psi}^k \circ \overline{\pi}^{-1} \cdot \bar{\varrho}\big) d\overline{m},$$

where $\bar{\varrho}$ is as in Lemma 3.4. By the similar argument as above, we have that $\int \overline{\varphi}^k d\tilde{\nu} = \int \overline{\varphi}^k \circ \overline{\pi}^{-1} \cdot \bar{\varrho} d\overline{m}$ and $\int \overline{\psi}^k d\tilde{\nu} = \int \overline{\psi}^k \circ \overline{\pi}^{-1} d\overline{\nu}$. Thus we have that

$$\mathrm{Cor}_{n-k}\big(\overline{\varphi}^k, \tilde{\psi}_k; \tilde{\nu}\big)$$
$$= \left| \int \overline{\varphi}^k \circ \overline{\pi}^{-1} \cdot \bar{\mathcal{L}}^n \big(\overline{\psi}^k \circ \overline{\pi}^{-1} \cdot \bar{\varrho}\big) d\overline{m} - \int \overline{\varphi}^k \circ \overline{\pi}^{-1} \cdot \bar{\varrho} d\overline{m} \int \overline{\psi}^k \circ \overline{\pi}^{-1} d\overline{\nu} \right|.$$

Let $a_\psi := 2 \max |\psi| + 1$. Then it follows from the argument in [26] (page 175) (cf. [10] Lemma 3.10) that

$$\begin{aligned}
&\mathrm{Cor}_{n-k}\big(\overline{\varphi}^k, \tilde{\psi}_k; \tilde{\nu}\big) \\
&\quad \le 2a_\psi \max|\varphi| \sum_{\ell=0}^{\infty} \bar{m}(\bar{\Delta}_\ell) \sup_{\bar{x}\in\bar{\Delta}_\ell} \left| \bar{\mathcal{L}}^{n-2k} \circ \bar{\mathcal{L}}^{2k} \left(\frac{(\bar{\psi}_k + a_\psi)\,\bar{\varrho}}{\int (\bar{\psi}_k + a_\psi) d\bar{\nu}} \right)(\bar{x}) - \bar{\varrho}(\bar{x}) \right|
\end{aligned} \tag{3.9}$$

where $\bar{\mathcal{L}}$ is as in (3.4). Since $\mathrm{Cor}_n(\varphi,\psi;\nu) = \mathrm{Cor}_n(\tilde{\varphi},\tilde{\psi};\tilde{\nu})$, Lemma 3.7 implies that

$$\mathrm{Cor}_n(\varphi,\psi;\nu) \le \mathrm{Cor}_{n-k}\big(\overline{\varphi}^k, \tilde{\psi}_k; \tilde{\nu}\big) + C_7 k^{-\tau\eta}. \tag{3.10}$$

To apply Proposition 3.5 to $\bar{\mathcal{L}}^{2k}\Big(\frac{(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)\overline{\varrho}}{\int(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)d\overline{\nu}}\Big)$ in (3.9), we need to show that $\bar{\mathcal{L}}^{2k}\Big(\frac{(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)\overline{\varrho}}{\int(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)d\overline{\nu}}\Big) \in C_{\beta_4}(\overline{\Delta})$ and the constant $C_5'\Big(\bar{\mathcal{L}}^{2k}\Big(\frac{(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)\overline{\varrho}}{\int(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)d\overline{\nu}}\Big)\Big)$ as in (3.6) is bounded above by some constant independent of k.

Lemma 3.8 *There exists $C_8 = C_8(\psi,\overline{\varrho}) > 0$ such that*

(1) $$\left\| \bar{\mathcal{L}}^{j+2k}\left(\frac{(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)\overline{\varrho}}{\int(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)d\overline{\nu}} \right) \right\|_\infty \le C_8 \qquad (0 \le j \le k_1),$$

(2) $$\left| \bar{\mathcal{L}}^{2k}\left(\frac{(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)\overline{\varrho}}{\int(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)d\overline{\nu}} \right)(x) - \bar{\mathcal{L}}^{2k}\left(\frac{(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)\overline{\varrho}}{\int(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)d\overline{\nu}} \right)(y) \right| \le C_8 \beta_4^{\overline{s}(x,y)} \qquad (x,y \in \overline{\Delta}_{\ell,i}).$$

Here k_1 is the number as in Proposition 3.5.

Proof. We note that $\bar{\mathcal{L}}^\ell(\overline{\psi})(\bar{x}) = \sum_{\bar{x}':\overline{F}^\ell(\bar{x}')=\bar{x}} \frac{\overline{\psi}}{J(\overline{F}^\ell)}(\bar{x}')$ for $\overline{\psi} : \overline{\Delta} \to \mathbb{R}$ and $\ell \ge 1$. By Lemma 3.3 it follows from the same argument in Theorem 1 in [26] (see also [10] Lemma 3.5(2)) that there exists $K_3 > 0$ (independent of ℓ) such that $\sum_{\bar{x}':\overline{F}^\ell(\bar{x}')=\bar{x}} \frac{1}{J(\overline{F}^\ell)(\bar{x}')} \le K_3$. Since ψ is Hölder continuous, there exists $C_\psi > 0$ such that $|\overline{\psi}^k\circ\overline{\pi}^{-1}(\bar{x})| \le C_\psi$ for any $x \in \overline{\Delta}$. Since $\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi \ge 1$, by Lemma 3.3 we have that $\left|\bar{\mathcal{L}}^{j+2k}\Big(\frac{(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)\overline{\varrho}}{\int(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)d\overline{\nu}}\Big)(\bar{x})\right| \le K_3(C_\psi + a_\psi)\bar{c}_0$ for any $x \in \overline{\Delta}$. (1) is shown.

We prove (2). For any $\bar{x},\bar{y} \in \bar{\Delta}_{\ell,i}$, let $\{\bar{x}_j'\}_{j\in\mathbb{N}}, \{\bar{y}_j'\}_{j\in\mathbb{N}}$ be the paired preimages of $\bar{x}, \bar{y}$ by $\bar{F}^{2k}$, i.e. for any $j \in \mathbb{N}$, $\bar{F}^{2k}(\bar{x}'_j) = \bar{x}$ and $\bar{F}^{2k}(\bar{y}'_j) = \bar{y}$, and for each $j \in \mathbb{N}$ and every $h \in \{0, \dots, 2k\}$, $\bar{F}^h(\bar{x}'_j)$ and $\bar{F}^h(\bar{y}'_j)$ belong to the same element of $\bar{\mathcal{D}}$. Using that $\bar{\psi}_k + a_\psi \geq 1$, we have that

$$\left| \bar{\mathcal{L}}^{2k} \left(\frac{(\bar{\psi}_k + a_\psi)\, \bar{\varrho}}{\int (\bar{\psi}_k + a_\psi) d\bar{\nu}} \right)(\bar{x}) - \bar{\mathcal{L}}^{2k} \left(\frac{(\bar{\psi}_k + a_\psi)\, \bar{\varrho}}{\int (\bar{\psi}_k + a_\psi) d\bar{\nu}} \right)(\bar{y}) \right| \leq \text{(III)} + \text{(IV)}$$

where

$$\text{(III)} = \sum_{j=1}^{\infty} \frac{1}{J(\bar{F}^{2k})(\bar{x}'_j)} \left| (\bar{\psi}_k(\bar{x}'_j) + a_\psi)\, \bar{\varrho}(\bar{x}'_j) - (\bar{\psi}_k(\bar{y}'_j) + a_\psi)\, \bar{\varrho}(\bar{y}'_j) \right|,$$

$$\text{(IV)} = \sum_{j=1}^{\infty} \left| (\bar{\psi}_k(\bar{y}'_j) + a_\psi)\, \bar{\varrho}(\bar{y}'_j) \right| \frac{1}{J(\bar{F}^{2k})(\bar{y}'_j)} \left| \frac{J(\bar{F}^{2k})(\bar{y}'_j)}{J(\bar{F}^{2k})(\bar{x}'_j)} - 1 \right|.$$

We estimate (III). By the definition of $\bar{\psi}_k$, $\bar{\psi}_k$ is constant on each element of $\bar{\mathcal{D}}_{2k}$ where $\bar{\mathcal{D}}_j := \vee_{i=0}^{j} \bar{F}^{-i}(\bar{\mathcal{D}})$. Since $\bar{x}'_j$ and $\bar{y}'_j$ are belong to the same element of $\bar{\mathcal{D}}_{2k}$, we have that for any $j \in \mathbb{N}$, $(\bar{\psi}_k)(\bar{x}'_j) = (\bar{\psi}_k)(\bar{y}'_j)$. So we estimate that

$$\begin{aligned}
&\left| (\bar{\psi}_k(\bar{x}'_j) + a_\psi)\, \bar{\varrho}(\bar{x}'_j) - (\bar{\psi}_k(\bar{y}'_j) + a_\psi)\, \bar{\varrho}(\bar{y}'_j) \right| \\
&\quad = \left| (\bar{\psi}_k(\bar{x}'_j) + a_\psi)\, (\bar{\varrho}(\bar{x}'_j) - \bar{\varrho}(\bar{y}'_j)) \right| \\
&\quad \leq 2\, a_\psi \left| \bar{\varrho}(\bar{x}'_j) - \bar{\varrho}(\bar{y}'_j) \right| \\
&\quad \leq 2\, a_\psi\, \overline{c_0}\, \beta^{\bar{s}(\bar{x}'_j, \bar{y}'_j)} \quad (\because \text{Lemma 3.4}) \\
&\quad \leq 2\, a_\psi\, \overline{c_0}\, \beta^{\bar{s}(\bar{x}', \bar{y}')} \quad (\because \bar{s}(\bar{x}'_j, \bar{y}'_j) \geq \bar{s}(\bar{x}, \bar{y})).
\end{aligned}$$

Substituting this into (III) and using the inequality $\sum_{\bar{x}' : \overline{F}^{2k}(\bar{x}') = \bar{x}} \cdot \frac{1}{J(\overline{F}^{2k})(\bar{x}')} \leq K_3$ ([10] Lemma 3.5), we have that

$$\text{(III)} \leq 2 a_\psi \overline{c_0} K_3 \beta^{\bar{s}(\bar{x}, \bar{y})}.$$

By Lemma 3.3(3) we have that for any $\bar{x}, \bar{y} \in \bar{D} \in \bar{\mathcal{D}}_{2k}$, $\left| \frac{J(\bar{F}^{2k})(\bar{x})}{J(\bar{F}^{2k})(\bar{y})} - 1 \right| \leq C_1\, \beta^{\bar{s}(\bar{F}^{2k}(\bar{x}), \bar{F}^{2k}(\bar{y}))}$ (see [10] Lemma 3.5(1)). Using this inequality and Lemma 3.4, we esimate that (IV) $\leq (C_\psi + a_\psi) \overline{c_0} K_3 C_4 \beta_4^{\bar{s}(\bar{x}, \bar{y})}$. Therefore

$$\text{(III)} + \text{(IV)} \le 2K_3\overline{c_0}(C_\psi + a)C_4\beta_4^{\overline{s}(\bar{x},\bar{y})}.$$

(2) is proved. □

Let $\tau > 1$ be as in (K-2). For any $1 < \tau' < \tau$, let $v(t) = 1$ $(0 \le t < 1)$ and $v(t) = t^{\tau'-1}$ $(t \ge 1)$. Then by Lemmas 2.5 and 2.6, we have that $\sum_{\ell=0}^{\infty} v(\ell)\overline{m}(\overline{\Delta}_\ell) < \infty$. By Lemma 3.8(2), $\bar{\mathcal{L}}^{2k}\Big(\frac{(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)\overline{\varrho}}{\int(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)d\overline{\nu}}\Big) \in C_{\beta_4}(\overline{\Delta})$. Then by (3.5) we have that

$$\begin{aligned}\sup_{\bar{x}\in\overline{\Delta}_\ell}&\left|\bar{\mathcal{L}}^{n-2k}\circ\bar{\mathcal{L}}^{2k}\left(\frac{(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)\overline{\varrho}}{\int(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)d\overline{\nu}}\right)(\bar{x}) - \overline{\varrho}(\bar{x})\right|\\ &\le C_5'\left(\bar{\mathcal{L}}^{2k}\left(\frac{(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)\overline{\varrho}}{\int(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)d\overline{\nu}}\right)\right)\cdot\frac{v(\ell)}{v\left(\frac{n-2k}{2}\right)}\quad(n\ge 1).\end{aligned}\tag{3.11}$$

By (3.6) and Lemmas 3.4 and 3.8, we have that $C_5'\Big(\bar{\mathcal{L}}^{2k}\Big(\frac{(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)\overline{\varrho}}{\int(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)d\overline{\nu}}\Big)\Big)$ $\le C_9$ where $C_9 := \max\Big\{\frac{v(\frac{k_1}{2})}{v(0)}(C_8+\overline{c}_0), C_5C_8\Big\}$ (see [10] Lemma 3.12). Thus by (3.11) we have that

$$\sup_{\bar{x}\in\overline{\Delta}_\ell}\left|\bar{\mathcal{L}}^{n-2k}\circ\bar{\mathcal{L}}^{2k}\left(\frac{(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)\overline{\varrho}}{\int(\overline{\psi}^k\circ\overline{\pi}^{-1}+a_\psi)d\overline{\nu}}\right)(x)-\overline{\varrho}(x)\right| \le C_9\frac{v(\ell)}{v\left(\frac{n-2k}{2}\right)}\quad(n\ge 1).\tag{3.12}$$

By (3.9), (3.10) and (3.12) we have that

$$\mathrm{Cor}_n(\varphi,\psi;\nu) \le C_{10}\frac{1}{v\left(\frac{n-2k}{2}\right)} + C_7k^{-\tau\eta}\quad(n\ge 1)$$

where $C_{10} = 2a_\psi\max|\varphi|C_9\sum_{\ell=0}^{\infty}v(\ell)\overline{m}(\overline{\Delta}_\ell)$. Since $k \in [\frac{n}{6},\frac{n}{4}]$ and $v(t)$ increases with t, we have that $v(\frac{n}{4}) \le v(\frac{n-2k}{2})$, and obtain that

$$\mathrm{Cor}_n(\varphi,\psi;\nu) \le C_{10}\frac{1}{v\left(\frac{n}{4}\right)} + C_7k^{-\tau\eta} \le C_{10}\left(\frac{4}{n}\right)^{\tau'-1} + 3^{\tau\eta}C_7n^{-\tau\eta}.$$

The proof of Key Lemma is complete.

4. Appendix A: Proofs of Lemmas 2.1–2.6

In this section we prove Lemmas 2.1–2.6. Throughout this section we assume that $f : M \circlearrowleft$ is an almost Anosov diffeomorphism with uniformly contracting direction (not necessarily co-dimension one uniformly contracting direction). We begin by noting the basic properties of the local stable and center unstable manifolds. We have the following ([23]):

(i) $T_x W^\sigma_\varepsilon(x) = E^\sigma(x)$ for $\sigma = s, u$,
(ii) $W^s_\varepsilon(x) \subset \{y \in M \mid d(f^n(x), f^n(y)) \leq \varepsilon$ for any $n \in \mathbb{Z}^+\}$,
(iii) $f(W^s_\varepsilon(x)) \subset W^s_\varepsilon(f(x))$,
(iv) $f(W^u_\varepsilon(x)) \supset W^u_{\varepsilon'}(f(x))$ for some $\varepsilon' \in (0, \varepsilon]$,
(v) there exists $L_1 > 0$ independent of x and ε such that for any $y \in W^\sigma_\varepsilon(x)$ $(\sigma = s, u)$,

$$d^\sigma(y, x) < L_1 d(y, x) \tag{4.1}$$

where d^σ denotes the Riemannian distance measured along $W^\sigma_\varepsilon(x)$.

Since the correspondence $x \mapsto W^u_\varepsilon(x)$ is continuous w.r.t. the C^1 metric, there exists $\delta_1 > 0$ such that if x, y satisfies $d(x, y) < \delta_1$, then $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$ have a single transeverse intersection point, so write

$$[x, y] = W^s_\varepsilon(x) \cap W^u_\varepsilon(y) \quad (x, y \in M \text{ with } d(x, y) < \delta_1). \tag{4.2}$$

Proof of Lemma 2.1. It suffices to show that for any $x \in M$ and $y \in W^u_{\delta_1}(x)$, $T_y W^u_\varepsilon(x) = E^u(y)$. Arguing by contradiction, assume that there exist $x \in M$ and $y \in W^u_{\delta_1}(x)$ such that $T_y W^u_\varepsilon(x) \neq E^u(y)$. Then there exist $z \in B_{\delta_1}(x) \setminus W^u_{\delta_1}(x)$ and a piecewise C^1-curve $\mathcal{C} : [0, 1] \to M$ with $\mathcal{C}(0) = x$ and $\mathcal{C}(1) = z$ such that (a) the length of $\mathcal{C}([0, 1])$ is less than δ_1 and (b) $\frac{d\mathcal{C}}{dt}(t) \in E^u(\mathcal{C}(t))$ for any $0 \leq t \leq 1$. Indeed, since the set $A := \{z \in W^u_{\delta_1}(x) \mid T_z W^u_\varepsilon(x) = E^u(z)\}$ is closed in $W^u_\varepsilon(x)$, the set A^c is open. We let $y' \in \mathrm{Cl}(A^c)$ be such that $d^u(x, y') = d^u(x, \mathrm{Cl}(A^c))$. We take a C^1–curve $\gamma_1 : [0, \frac{1}{2}] \to W^u_{\delta_1}(x)$ with $\gamma_1(0) = x$ and $\gamma_1(\frac{1}{2}) = y'$ such that $\frac{d\gamma_1}{dt}(t) \in E^u(\gamma_1(t))$ for any $0 \leq t \leq \frac{1}{2}$, and the length of γ_1 is less than $\delta_1/2$. Then we can find a C^1–curve $\gamma_2 : [\frac{1}{2}, 1] \to B_{\delta_1}(y')$ with $\gamma_2(\frac{1}{2}) = y'$ and $z := \gamma_2(1) \notin W^u_\varepsilon(x)$ such that $\frac{d\gamma_2}{dt}(t) \in E^u(\gamma_2(t))$ for any $\frac{1}{2} \leq t \leq 1$, and the length of γ_2 is less than $\delta_1/2$. It is clear that a piecewise C^1-curve $\mathcal{C} : [0, 1] \to W^u_\varepsilon(x)$ defined by $\mathcal{C}_{[0, \frac{1}{2}]} := \gamma_1$ and $\mathcal{C}_{[\frac{1}{2}, 1]} := \gamma_2$ is desired. Since

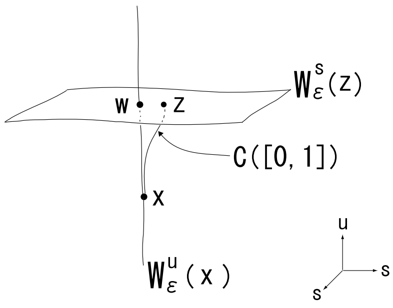

Figure 1. a figure of $\mathcal{C}([0,1])$

$d(z,x)$ is less than the length of $\mathcal{C}([0,1])$, by (a) we have that $d(z,x) < \delta_1$. Thus $W^s_\varepsilon(z)$ and $W^u_\varepsilon(x)$ have a single transeverse intersection, $w = [z,x]$ (see Figure 1).

Since $\|D_q f^{-n}|_{T_q\mathcal{C}([0,1])}\| \le \|D_q f^{-n}|_{E^u(q)}\| \le 1$ for any $q \in \mathcal{C}([0,1])$ and $n \in \mathbb{Z}^+$ by (b), we have that for any $n \in \mathbb{Z}^+$,

$$d_{f^{-n}\mathcal{C}}(f^{-n}(z), f^{-n}(x)) \le d_{\mathcal{C}}(z,x)(< \delta_1) \tag{4.3}$$

where $d_{f^{-n}\mathcal{C}}$ is the Riemannian metric on $f^{-n}(\mathcal{C}([0,1]))$. Then we have that for any $n \in \mathbb{Z}^+$, $d(f^{-n}(z), f^{-n}(x)) < \delta_1$, and so we can define for any $n \in \mathbb{Z}^+$,

$$[f^{-n}(z), f^{-n}(x)] = W^s_\varepsilon(f^{-n}(z)) \cap W^u_\varepsilon(f^{-n}(x)).$$

Since w is the single transeverse intersection of $W^s_\varepsilon(z)$ and $W^u_\varepsilon(x)$, we have $f^{-1}(w) = [f^{-1}(z), f^{-1}(x)]$. Repeating this manner, we have that for any $n \in \mathbb{Z}^+$,

$$f^{-n}(w) = [f^{-n}(z), f^{-n}(x)], \quad f^{-n}(w) \ne f^{-n}(z). \tag{4.4}$$

Thus, for any $n \in \mathbb{Z}^+$, $f^{-n}(w) \in W^s_\varepsilon(f^{-n}(z))$, and then $d(f^{-n}(z), f^{-n}(w)) < \varepsilon$. By (2.1) we have that for any $n \in \mathbb{Z}^+$,

$$d(z,w) \le L_1 \lambda_s^n d(f^{-n}(z), f^{-n}(w)) \le L\lambda_s^n \varepsilon.$$

Therefore, taking the limit as $n \to \infty$, we obtain that $d(z,w) = 0$. This is a contradiction since $z \ne w$. □

Proof of Lemma 2.2. Since the correspondence $x \mapsto W^\sigma_\varepsilon(x)$ is continuous w.r.t. the C^1 topology ($\sigma = s, u$), there exists $\delta' > 0$ such that if $d(y, y') < \delta'$, then $d(y, [y, y']) < \delta_1/2$ and $d(y, [y', y]) < \delta_1/2$. Assume, by a contradiction, that there exist $y, y' \in B_{\delta'}(x)$ with $[x, y] \neq [x, y']$ such that $W^u_\varepsilon(y) \cap W^u_\varepsilon(y') \cap B_{\delta'}(x) \neq \emptyset$. Then there exist $z \in W^u_\varepsilon(y) \cap W^u_\varepsilon(y') \cap B_{\delta'}(x)$ and $z' \in (W^u_\varepsilon(y') \setminus W^u_\varepsilon(y)) \cap B_{\delta'}(x)$. By definition of δ', $d(z, z') < \delta_1$. By (4.2), $W^u_\varepsilon(z)$ and $W^s_\varepsilon(z')$ have a single transeverse intersection, $w := [z', z]$. We have that

$$f^{-n}(w) = [f^{-n}(z'), f^{-n}(z)], \quad f^{-n}(w) \neq f^{-n}(z') \quad (n \geq 0).$$

This implies that $f^{-n}(w) \in W^s_\varepsilon(f^{-n}(z'))$ for any $n \geq 0$. Since $d(f^{-n}(w), f^{-n}(z')) < \varepsilon$, we have by (2.1) that $d(w, z') \leq L\lambda^n_s \varepsilon$ ($n \geq 0$) and thus $d(w, z') = 0$. This is a contradiction with the fact that $w \neq z'$. □

For any $\eta > 0$ and $x \in M$ let $\overline{W}^u_\eta(x) := \{y \in M \mid d(f^{-n}(y), f^{-n}(x)) \leq \eta \, (n \geq 0)\}$ and $\overline{W}^s_\eta(x) := \{y \in M \mid d(f^n(y), f^n(x)) \leq \eta \, (n \geq 0)\}$. For the proof of Lemma 2.3 we need the following Lemma 4.1.

Lemma 4.1 *For any $x \in M$, $\overline{W}^\sigma_{\delta_1/2}(x) \subset W^\sigma_\varepsilon(x) \subset \overline{W}^\sigma_{L_1\varepsilon}(x)$ ($\sigma = s, u$).*

Proof. We show that $\overline{W}^u_{\delta_1/2}(x) \subset W^u_\varepsilon(x)$. Take $z \in \overline{W}^u_{\delta_1/2}(x)$ and assume $z \notin W^u_\varepsilon(x)$. Then $W^s_\varepsilon(z)$ and $W^u_\varepsilon(x)$ have a single transeverse intersection, $w = [z, x]$. By the same argument as in (4.4) we have $f^{-k}(w) = [f^{-k}(z), f^{-k}(x)]$, $f^{-k}(w) \neq f^{-k}(z)$ ($k \geq 0$). Using (2.1) we have $d(w, z) = 0$. This contradicts that $z \neq w$.

To prove that $W^u_\varepsilon(x) \subset \overline{W}^u_{L_1\varepsilon}(x)$, it is enough to show that for any $y \in W^u_\varepsilon(x)$ and $n \geq 0$, $d^u(f^{-n}(x), f^{-n}(y)) < L_1\varepsilon$. Since $d^u(x, y) < L_1\varepsilon$ for $y \in W^u_\varepsilon(x)$, it is obvious the case when $n = 0$. Assume that $d^u(f^{-n}(x), f^{-n}(y)) < L_1\varepsilon$. Then we have that

$$\begin{aligned} &d^u(f^{-n-1}(x), f^{-n-1}(y)) \\ &\quad \leq \sup\left\{\|D_w(f^{-1})^u\| \mid w \in f^{-n}(W^u_\varepsilon(x))\right\} d^u(f^{-n}(x), f^{-n}(y)). \end{aligned}$$

Since $\|D_w(f^{-1})^u\| = \|D_w f^{-1}|_{E^u(f^{-n}(x))}\| \leq 1$ for any $w \in W^u_\varepsilon\,(f^{-n}(x))$ by Lemma 2.1, we have that $d^u(f^{-n-1}(x), f^{-n-1}(y)) < L_1\varepsilon$. Therefore our disire is proved for $n+1$. The case $\sigma = u$ is proved. The similar arguments as above works for the case $\sigma = s$. □

f has *canonical coordinates* if for any $\eta > 0$ there exists $\delta(\eta) > 0$ such that $d(x, y) \le \delta(\eta)$ implies $\overline{W}^s_\eta(x) \cap \overline{W}^u_\eta(y) \ne \emptyset$.

Proof of Lemma 2.3. We prove that f is expansive. Indeed take $x, y \in M$ with $d(f^i(x), f^i(y)) < \delta_1/2$ ($i \in \mathbb{Z}$). This implies that $y \in \overline{W}^s_{\delta_1/2}(x) \cap \overline{W}^u_{\delta_1/2}(x)$. By Lemma 4.1 we have that $y \in W^s_\varepsilon(x) \cap W^u_\varepsilon(x)$. Thus by (4.2), $y = x$.

Let us show that f has canonical coordinates. Then the shadowing property of f follows from Theorem in [18]. If $\rho \in (0, 1]$, then there exists $\delta'' \in (0, 1)$ such that (4.2) holds with replacing ε by ρ/L_1 and δ by δ''. Let x, y be such that $d(x, y) \le \delta''$. By definition of δ'', $W^s_{\rho/L_1}(x)$ and $W^u_{\rho/L_1}(y)$ have a single transeverse intersection point. On the one hand by Lemma 4.1 we have that $W^s_{\rho/L_1}(x) \subset \overline{W}^s_\rho(x)$ and $W^u_{\rho/L_1}(y) \subset \overline{W}^u_\rho(y)$. Combining the arguments above we have that $\overline{W}^s_\rho(x) \cap \overline{W}^u_\rho(y) \ne \emptyset$. If $\rho > 1$, then there exists $\delta''' \in (0, 1)$ such that (4.2) holds with replacing ε by $1/L_1$ and δ by δ'''. Then using the similar arguments as above allows us to have the disired result. (1) is proved. By (1), (2) follows from Theorem 4.2.8 in [3]. □

Let dist be the distance in the Grassmannian bundle generated by the Riemannian metric. An 1-dimensional distribution E is (δ, L, ξ)-*Hölder continuous* if for any $x, y \in M$ with $d(x, y) < \delta$, $\mathrm{dist}(E(x), E(y)) \le L d(x, y)^\xi$. It follows from [4] (Theorem 2.3.2) that there exist $\delta_2 \in (0, \delta_1)$, $L_2 > 0$ and $\xi_1 \in (0, \alpha]$ such that the distributions E^s and E^u are (δ_2, L_2, ξ_1)-Hölder continuous.

Lemma 4.2 *There exist $C_{11} > 0$ and $\delta_3 \in (0, \delta_2)$ such that the following hold for any $x, y, z \in M$:*

(1) *For any $y, z \in W^\sigma_{\delta_3}(x)$ ($\sigma = s, u$), $\log \frac{|det(D_y f^u)|}{|det(D_z f^u)|} \le C_{11} d(y, z)^{\xi_1}$,*
(2) *Especially if f satisfies Conditions 1 and 2, then for any $y, z \in W^u_{\delta_3}(x)$, $\log \frac{|det(D_y f^u)|}{|det(D_z f^u)|} \le C_{11} d(y, z)^\alpha$.*

Proof. Using the arguments in [4] (page 104) together with [14] (Lemma 3.2 (page 49)), there exist $K_4 > 0$ and $\varsigma_1 \in (0, \delta_2)$ such that for any $x \in M$ and $y, z \in B_{\varsigma_1}(x)$,

$$\left|\det(D_y f^u) - \det(D_z f^u)\right| \le K_4(\mathrm{dist}(E^u(y), E^u(z)) + d(y, z)^\alpha).$$

Then (1) follows from the fact that E^u is (δ_2, C_{10}, ξ_1)-Hölder continuous. Condition 2 combined with Lemma 2.1 tells us that the correspondence $w \mapsto E^u(w)$ is C^1 on each $W^u_{\delta_1}(x)$. So there exist $K_5 > 0$ and $\varsigma_2 \in (0, \varsigma_1)$ such that for any $x \in M$ and $y, z \in W^u_{\varsigma_2}(x)$, $\mathrm{dist}(E^u(y), E^u(z)) \le K_5 d(y, z)$. This combined with the ineqality above gives the proof of (2). □

By Lemma 2.3 (2) we may assume that the diameter of the Markov partition $\{\mathcal{Q}_i\}$ is less than δ_3. To show Lemma 2.4, we need the following Lemma 4.3.

Lemma 4.3 *There exist $C_{12} > 0$ and $0 < \beta_5 < 1$ such that for any $x \in \Lambda$, $y \in \gamma^u(x)$ with $s(x, y) < \infty$, and $0 \le k \le s(x, y) - 1$, $d(f^k(y), f^k(x)) \le C_{12}\beta_5^{s(x,y)-k}$.*

Proof. We put $\lambda_u := \max\{\|D_x f^{-1}|_{E^u(x)}\| \,|\, x \in \Lambda\}(< 1)$. Let $x \in \Lambda$, $y \in \gamma^u(x)$ with $s(x, y) < \infty$, and $0 \le k \le s(x, y) - 1$. There exist $\{n_i\}_{i\ge1}$ and $\{m_i\}_{i\ge1}$ with $0 = m_0 = n_0 \le n_1 < m_1 < n_2 < m_2 < \cdots n_\ell < m_\ell < \cdots$ such that $f^{n_i+j}(x), f^{n_i+j}(y) \notin \mathcal{P}$ $\quad(0 \le j \le m_i - n_i - 1,\ i \ge 1)$, and $f^{m_i+j}(x)$, $f^{m_i+j}(y) \in \mathcal{P}$ $\quad(0 \le j \le n_{i+1} - m_i - 1,\ i \ge 0)$. Then one of the following two cases holds:

(i) $f^k(x)$, $f^k(y) \in \mathcal{P}$, i.e. $m_{i-1} \le k \le n_i - 1$ $(i \ge 1)$,
(ii) $f^k(x)$, $f^k(y) \notin \mathcal{P}$, i.e. $n_i \le k \le m_i - 1$.

Let γ_ℓ denote the curve of the minimum length in $W^u_\varepsilon((f^R)^\ell(x))$ which connects between $(f^R)^\ell(x)$ and $(f^R)^\ell(y)$ for any $0 \le \ell \le s(x, y) - 1$. We denote $\ell(\gamma_\ell)$ the length of γ_ℓ. We deal with case (i). Case (ii) is estimated similarly as case (i). Since $s(f^{n_i}(x), f^{n_i}(y)) = s(x, y) - \{i + \sum_{j=0}^{i-1}(m_j - n_j)\}$ holds, by Condition 1 and Lemma 2.1 we have

$$
\begin{aligned}
&d(f^k(x), f^k(y)) \\
&\quad\le \sup\left\{\|D_z f^{-s(f^{n_i}(x), f^{n_i}(y))+1}|_{E^u(z)}\| \,\middle|\, z \in \gamma_{s(f^{n_i}(x), f^{n_i}(y))-1}\right\} \\
&\qquad \cdot \ell(\eta_{s(f^{n_i}(x), f^{n_i}(y))-1}) \\
&\quad\le \lambda_u^{s(f^{n_i}(x), f^{n_i}(y))-1} = \frac{1}{\lambda_u}\lambda_u^{s(x,y)-\{i+\sum_{j=0}^{i-1}(m_j-n_j)\}}.
\end{aligned}
$$

Since $k \ge i + \sum_{j=0}^{i-1}(m_j - n_j)$, the last term above is bounded above by $\le \frac{1}{\lambda_u}\lambda_u^{s(x,y)-k}$. □

Proof of Lemma 2.4. Let $x, y \in \gamma \in \Gamma^u$ be such that $s(x,y) < \infty$. Since $d(f^i(x), f^i(y)) < \delta_3$ for any $0 \le i \le s(x,y) - 1$, Lemmas 4.2(1) for case $\sigma = u$ and 4.3 conclude the proof. □

Proof of Lemma 2.5. Since the diameter of the Markov parition is less than δ', for any $x, y \in \gamma^s \in \Gamma^s$, $d(x,y) < \delta_3$. Then Lemma 4.2(1) for case $\sigma = s$ and (2.1) conclude the proof. □

Proof of Lemma 2.6. For any γ, $\gamma' \in \Gamma^u$ let $\Theta = \Theta_{\gamma,\gamma'} : \gamma \cap \Lambda \to \gamma' \cap \Lambda$ be the holonomy map. To show the lemma, it suffices to prove that there exists $K_6 > 0$ such that for any $x \in \gamma$ and any $r > 0$,

$$\left| \frac{m_{\gamma'}(\Theta(B(x,r)))}{m_\gamma(B(x,r))} - 1 \right| \le K_6 d(\gamma, \gamma')^\zeta \tag{4.5}$$

for m_γ-a.e.$x \in \gamma$. Here $d(\gamma, \gamma') = \sup\{d(x, \Theta(x)) \,|\, x \in \gamma\}$. If (4.5) is proved, then the same arguments as in [4] (p. 110) allows us to have the desired result. Since $x \mapsto W^u_\varepsilon(x)$ is C^1–continuous by Condition 1, we can find partitions $\{\gamma_i\}_{i\ge1}$ of $\gamma \cap B(y,r)$ (mod m_γ) and $\{\gamma'_i\}_{i\ge1}$ of $\gamma' \cap \Theta(B(y,r))$ (mod $m_{\gamma'}$) with the following properties:

(a) γ_i and γ'_i are intervals such that $\gamma'_i = \Theta(\gamma_i)$,

(b) for any $i \ge 1$ there exists $n_i \ge 1$ such that $(f^R)^{n_i}(\gamma_i)$ and $(f^R)^{n_i}(\gamma'_i)$ are intervals such that $(f^R)^{n_i}(\gamma'_i)$ is the Θ_i image of $(f^R)^{n_i}(\gamma_i)$. Here $\overline{\gamma_i}, \overline{\gamma_i}' \in \Gamma^u$ satisfy $(f^R)^{n_i}(\gamma_i) \subset \overline{\gamma_i} \in \Gamma^u$, $(f^R)^{n_i}(\gamma'_i) \subset \overline{\gamma_i}'$, and $\Theta_i : \overline{\gamma_i} \to \overline{\gamma_i}'$ is a holonomy map sliding along stable disks,

(c) for any $x, y \in \gamma_i$, $\beta^{s((f^R)^{n_i}(x),(f^R)^{n_i}(y))} < d(\gamma, \gamma')$, and the same holds for $x, y \in \gamma'_i$, and

(d) there exists $K_7 > 0$ such that $\left| \frac{m_{\overline{\gamma_i}}((f^R)^{n_i}(\gamma_i))}{m_{\overline{\gamma_i}'}((f^R)^{n_i}(\gamma'_i))} - 1 \right| \le K_7 d(\gamma, \gamma')$.

Then by (c) and (K-1) for any $x, y \in \gamma_i$,

$$\sum_{i=0}^{n_i-1} \log \left| \frac{\det(D_{(f^R)^i(x)}(f^R)^u}{\det(D_{(f^R)^i(y)}(f^R)^u} \right| \le \frac{C_1}{1-\beta_1} d(\gamma, \gamma'). \tag{4.6}$$

Then by the same estimation as above, (4.6) holds with γ'_i instead of γ_i. By Lemma 4.2(1) for case $\sigma = s$ and (2.1) we have that for any $z \in \gamma$,

$$\sum_{i=0}^{n_i-1} \log \left| \frac{\det(D_{(f^R)^i(z)}(f^R)^u}{\det(D_{(f^R)^i(\Theta(z)}(f^R)^u} \right| \le \frac{C_{11}L^\zeta}{1-\lambda_s^\zeta} d(\gamma,\gamma')^\zeta. \tag{4.7}$$

Combining (d), (4.6) and (4.7) we have (4.5). □

5. Appendix B: Verifying (K-1) and (K-2) under Conditions 1–3

In this section we show that Conditions 1–4 imply (K-1) and (K-2) of Key Lemma. Throughout this section we assume that f satisfies Conditions 1–4. We say that I is an *interval* belonged to $W^u_\varepsilon(x)$ if there exists an interval $J \subset D^u_\varepsilon$ such that $\phi^u(x)(J) = I$. For any interval $I \subset W^u_\varepsilon(x)$, let $\ell(I)$ denote the length of I, and for any $x, q \in M$ with $d(x,q) < \delta$, we put $[I,q] = \{[y,q] \,|y \in I\}$.

Lemma 5.1 *There exists $C_{13} > 0$ such that*

$$C_{13}^{-1}\ell(J) \le \ell([J,q]) \le C_{13}\ell(J).$$

for any interval $J \subset W^u_\varepsilon(y)$ and any $y, q \in M$ with $d(y,q) < \delta$.

Proof. By Conditions 1, 4 and Lemma 4.2 the same argument from [12] (Proposition 2.5) allows us to have the desired result. □

By Conditions 2, 3 and Lemma 5.1 we easily have the following Lemma 5.2, which implies (K-2).

Lemma 5.2 *There exists $C_{14} > 0$ such that for any $\gamma \in \Gamma^u$,*

$$m_\gamma(\{R > n\}) \le C_{14} n^{-\frac{1}{\alpha}} \quad (n \ge 1).$$

The next Lemma 5.3 implies (K-1).

Lemma 5.3 *There exist $C_{15} > 0$ such that*

$$\left| \frac{|det(D_x(f^i)^u|}{|det(D_y(f^i)^u|} - 1 \right| \le C_{15} d^u(f^i(x), f^i(y))^\alpha$$

for any $i \ge 1$, $1 \le j \le r$, $x \in \Lambda_i^j$ and $y \in \Lambda_i^j \cap \gamma^u(x)$.

Proof. To show the lemma, it suffices to prove that there exists $K_8 = K_8(\alpha) > 0$ such that

$$\frac{1}{K_8} \leq \frac{|\det(D_x(f^\ell)^u|}{|\det(D_{x'}(f^\ell)^u|} \leq K_8 \quad (1 \leq \ell \leq i) \tag{5.1}$$

for any $i \geq 1$, $1 \leq q \leq r$, $x \in \Lambda_i^q$ and $x' \in \Lambda_i^q \cap \gamma^u(x)$.

Let $x \in \Lambda_i^q$ and $x' \in \Lambda_i^q \cap \gamma^u(x)$. Let $I_{x,i}$ be the connected component of $\gamma^u(x) \cap \Lambda_i^q$ which contains $x \in \Lambda_i^q$. Fix $p \in S$. Let Θ_p be a holonomy map to $f(\gamma^u(p))$ by sliding along stable disks. We denote $I_i = \Theta_p \circ f^i(I_x)$, and $I = f(\gamma^u(p)) \setminus \gamma^u(p)$. Then we have that $I_i \subset I$ for any $i \geq 1$. By Lemma 5.1 we have that $\ell(f^j(I_{x,i})) \leq C_{13}\ell(f^{j-i}(I_i))$ for $0 \leq j \leq i$. Then we have that

$$d^u(f^j(x), f^j(x')) \leq C_{13}\ell(f^{j-i}(I_i)) \quad (0 \leq j \leq i). \tag{5.2}$$

Using Conditions 2 and 3 we have that $\ell(f^{-k}(I_i)) \leq K_9 k^{-\frac{1}{\alpha}-1}$ for some $K_9 > 0$ ([26], see also [10] Lemma 4.6), from which $\sum_{k\geq 1} \ell(f^{-k}(I_i))^\alpha < \infty$. Noting that $d(f^j(x), f^j(y)) < \delta_3$ for any $0 \leq j \leq i-1$, and combining the arguments as above with Lemma 4.2(2), we estimate that

$$\begin{aligned} \sum_{j=0}^{\ell} \log \frac{|\det(D_{f^j(x)} f^u)|}{|\det(D_{f^j(x')} f^u)|} &\leq C_{11} \sum_{j=0}^{i-1} d(f^j(x), f^j(x'))^\alpha \\ &\leq C_{11} C_{13}^\alpha K_9^\alpha \sum_{k\geq 1} k^{-1-\alpha}, \end{aligned} \tag{5.3}$$

which proves (5.1) for $K_{10} = \exp\{C_{11}C_{13}^\alpha K_9^\alpha \sum_{k\geq 1} k^{-1-\alpha}\}$.

By (5.1), (5.2) and Lemma 5.1 we have that

$$\frac{d^u(f^j(x), f^j(y))}{\ell(f^{j-i}(I_i^q))} \leq K_8 C_{13}^2 \frac{d^u(f^i(x), f^i(y))}{\ell(I_i^q)} \quad (0 \leq j \leq i-1).$$

Substituting this into (5.3), we conclude the proof. □

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