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Paley's inequality of integral transform type

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Abstract. Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers with Hadamard gap. For an analytic function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit disc satisfying $\sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|\, d\theta < \infty$, the inequality $(\sum_{k=1}^{\infty} |a_{n_k}|^2)^{1/2} < \infty$ holds, which is familiar as Paley's inequality. In this paper, an integral transform version of this inequality is established.

Key words: Paley's inequality, Hankel transform.

1. Introduction and Results

A well-known inequality of Paley says in terms of the real Hardy space $H^1(\mathbf{T})$ on the torus $\mathbf{T}$ that there exists a constant C such that

$$\left\{ \sum_{k=1}^{\infty} (|c_{n_k}|^2 + |c_{-n_k}|^2) \right\}^{1/2} \leq C \|f\|_{H^1(\mathbf{T})},$$

for $f(\theta) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$ in $H^1(\mathbf{T})$, where $\{n_k\}_{k=1}^{\infty}$ is a Hadamard sequence, that is, a sequence of positive integers such that $n_{k+1}/n_k \geq \rho$ with a constant $\rho > 1$.

Kanjin and Sato [3] obtained the Paley-type inequality with respect to the Jacobi expansions, and Sato [4] proved the inequality of the same type in the Fourier-Bessel expansions.

The main purpose of this paper is to establish Paley's inequality with respect to the Hankel transform for the real Hardy space on the half line $(0, \infty)$.

The Hankel transform $H_\nu f$ of order $\nu > -1$ of a function f on $(0, \infty)$ is defined by

$$H_\nu f(y) = \int_0^{\infty} f(t) \sqrt{yt} J_\nu(yt)\, dt, \quad y > 0,$$

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where J_ν is the Bessel function of the first kind of order ν. We remark that the Hankel transforms $H_{-1/2}f(y)$ and $H_{1/2}f(y)$ are the cosine and the sine transforms:

$$H_{-1/2}f(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos yt \, dt, \quad H_{1/2}f(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin yt \, dt.$$

From now on, we let the order ν of the Hankel transform be greater than or equal to $-1/2$ unless otherwise stated explicitly.

It is known that the Hankel transform $H_\nu f$ of f is continuous for $f \in L^1(0,\infty)$, and $|H_\nu f(y)| \le C_\nu \|f\|_{L^1(0,\infty)}$, $y > 0$, where C_ν is a constant depending only on ν. Further, the following facts are known: The Hankel transform H_ν, initially defined on $L^1(0,\infty) \cap L^2(0,\infty)$, extends uniquely to an isometry of $L^2(0,\infty)$ (Parseval's identity for the Hankel transform), $H_\nu H_\nu = I$ (The inversion formula for the Hankel transform) where I is the identity operator of $L^2(0,\infty)$, and

$$\int_0^\infty f(t)g(t) \, dt = \int_0^\infty H_\nu f(y) H_\nu g(y) \, dy$$

for $f, g \in L^2(0,\infty)$ (Plancherel's theorem for the Hankel transform). For these facts, see [6, Chapter VIII], [5].

Let $H^1(\mathbf{R})$ be the real Hardy space on the real line $\mathbf{R}$. We shall work on the space $H^1(0,\infty)$ defined by

$$H^1(0,\infty) = \{h|_{(0,\infty)} \; ; \; h \in H^1(\mathbf{R}), \; \operatorname{supp} h \subset [0,\infty)\},$$

where $[0,\infty)$ is the closed half line, and we endow the space with the norm $\|f\|_{H^1(0,\infty)} = \|h\|_{H^1(\mathbf{R})}$, where $h \in H^1(\mathbf{R})$, $\operatorname{supp} h \subset [0,\infty)$ and $f = h|_{(0,\infty)}$. We remark that $H^1(0,\infty) = \{h|_{(0,\infty)} \; ; \; h \in H^1(\mathbf{R}), \text{ even}\}$ and $c_1\|h\|_{H^1(\mathbf{R})} \le \|f\|_{H^1(0,\infty)} \le c_2\|h\|_{H^1(\mathbf{R})}$ with positive constants c_1 and c_2, where $f = h|_{(0,\infty)}$ and $h \in H^1(\mathbf{R})$ is even. For this fact, see [1, Chapter III, Lemma 7.40].

Our theorem is as follows:

Theorem *Let $\nu \ge -1/2$. Let $L > 0$. Then, the Hankel transform $H_\nu f$ of a function $f \in H^1(0,\infty)$ satisfies*

$$\left(\sum_{k=1}^{\infty}\int_{n_k\le y\le n_k+L}|H_\nu f(y)|^2\,dy\right)^{1/2}\le C\|f\|_{H^1(0,\infty)}, \tag{1}$$

where C is independent of f.

As a corollary, we state here that the same type of result holds with respect to the Fourier transform.

Corollary *Under the same assumptions of the theorem, there exists a constant C such that*

$$\left(\sum_{k=1}^{\infty}\int_{n_k\le |\xi|\le n_k+L}|\mathcal{F}h(\xi)|^2\,d\xi\right)^{1/2}\le C\|h\|_{H^1(\mathbf{R})}$$

for $h\in H^1(\mathbf{R})$, where $\mathcal{F}h$ is the Fourier transform of h:

$$\mathcal{F}h(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h(x)e^{-i\xi x}\,dx,\quad \xi\in\mathbf{R}.$$

The corollary follows from the following simple relations between the Fourier transform and the Hankel transforms:

$$\mathcal{F}h(\xi)=\begin{cases}H_{-1/2}(R[h_e])(\xi)+i\,H_{-1/2}(R[(\mathcal{H}h)_e])(\xi), & \text{a.e. } \xi>0,\\ H_{-1/2}(R[h_e])(-\xi)-i\,H_{-1/2}(R[(\mathcal{H}h)_e])(-\xi), & \text{a.e. } \xi<0.\end{cases}$$

Here, $\mathcal{H}h$ is the Hilbert transform of h, and $R[h]$ is the restriction of h to the half interval $(0,\infty)$, and h_e is the even part of h. If $h\in H^1(\mathbf{R})$, then $\mathcal{H}h\in H^1(\mathbf{R})$ and $R[(\mathcal{H}h)_e]\in H^1(0,\infty)$. Therefore, the inequality (1) with $\nu=-1/2$ implies the corollary.

Applying an interpolation method to the theorem, we have the $L^p, 1<p\le 2$ case which is an integral transform version of Zygmund's Fourier series case [8, (7.6)]. Further, we obtain that in the theorem we can not replace the space $H^1(0,\infty)$ with $L^1(0,\infty)$. We precisely state these as a proposition.

Proposition *Under the same assumptions of the theorem, the following* (i) *and* (ii) *hold.*

(i) *Let $1<p\le 2$. Then, the Hankel transform $H_\nu f$ of a function $f\in L^p(0,\infty)$ satisfies*

$$\left(\sum_{k=1}^{\infty}\int_{n_k\le y\le n_k+L}|H_\nu f(y)|^2\,dy\right)^{1/2}\le C\|f\|_{L^p(0,\infty)}, \tag{2}$$

where C is independent of f.

(ii) *There exists a function $f\in L^1(0,\infty)$ such that*

$$\sum_{k=1}^{\infty}\int_{n_k\le y\le n_k+L}|H_\nu f(y)|^2\,dy=\infty.$$

A proof of the theorem will be given in the next section. The (H^1, BMO)-duality will play an essential role in our proof. In the last section, we shall give a proof of the proposition.

2. Proof of the theorem

We shall prove the theorem. The letter C will be used to denote positive constants not necessarily the same at each occurrence.

Let $\{n_k\}_{k=1}^{\infty}$ be a Hadamard sequence, that is, $n_{k+1}/n_k\ge\rho>1$. Let $L>0$ and let $\chi_k(y)$ be the characteristic function of the interval $[n_k, n_k+L]$ for every positive integer k. If we show the following inequality

$$\left|\int_0^\infty\sum_{k=1}^{N}\chi_k(y)H_\nu f(y)g(y)\,dy\right|\le C\|f\|_{H^1(0,\infty)}\|g\|_{L^2(0,\infty)} \tag{3}$$

for $N=1,2,\ldots$, $f\in H^1(0,\infty)\cap L^2(0,\infty)$ and $g\in L^2(0,\infty)$, where C is independent of N, f and g, then we have

$$\int_0^\infty\left|\sum_{k=1}^{N}\chi_k(y)H_\nu f(y)\right|^2dy\le C\|f\|^2_{H^1(0,\infty)}. \tag{4}$$

Since $\{n_k\}_{k=1}^{\infty}$ is a Hadamard sequence, we may suppose that the intervals $[n_k, n_k+L]$ are non-overlapping. Letting $N\to\infty$ in (4), we have

$$\sum_{k=1}^{\infty}\int_{n_k\le y\le n_k+L}|H_\nu f(y)|^2\,dy\le C\|f\|^2_{H^1(0,\infty)}$$

for $f \in H^1(0,\infty)\cap L^2(0,\infty)$. Since $H^1(0,\infty)\cap L^2(0,\infty)$ is dense in $H^1(0,\infty)$, the standard density argument allows us to obtain the theorem. Therefore, it is enough to prove the inequality (3).

Let $f \in H^1(0,\infty) \cap L^2(0,\infty)$ and $g \in L^2(0,\infty)$. We set

$$\begin{aligned} G_N(y) &= H_\nu\Big(\sum_{k=1}^{N} \chi_k g\Big)(y) = \sum_{k=1}^{N} H_\nu(\chi_k g)(y) \\ &= \sum_{k=1}^{N} \int_{n_k \le t \le n_k+L} g(t)\sqrt{yt}J_\nu(yt)\,dt \end{aligned}$$

for $N = 1, 2, \dots$. Then, by Plancherel's theorem we have

$$\begin{aligned} \int_0^\infty \sum_{k=1}^{N} \chi_k(y) H_\nu f(y) g(y)\,dy &= \int_0^\infty f(y) G_N(y)\,dy \\ &= \frac{1}{2}\int_{-\infty}^{\infty} E[f](x)E[G_N](x)\,dx, \end{aligned}$$

where we denote by $E[g]$ the even extension of a function g on $(0,\infty)$ to the whole line $(-\infty,\infty)$. By the (H^1, BMO)-duality, we have

$$\left|\int_{-\infty}^{\infty} E[f](x)E[G_N](x)\,dx\right| \le C\|E[f]\|_{H^1(\mathbf{R})}\|E[G_N]\|_*,$$

where $\|\cdot\|_*$ is the BMO-norm. By the inequality $\|E[f]\|_{H^1(\mathbf{R})} \le C\|f\|_{H^1(0,\infty)}$ and the definition of BMO-norm, we see that to show (3) it is enough to prove that for every interval I of $(-\infty,\infty)$ there exists a constant c such that

$$\frac{1}{|I|}\int_I |E[G_N](x) - c|\,dx \le C\|g\|_{L^2(0,\infty)}, \tag{5}$$

where C is independent of N, g and I. We may assume that $I \subset [0,\infty)$, and it suffices to show that there exists a constant c such that

$$\frac{1}{|I|}\int_I |G_N(y) - c|\,dy \le C\|g\|_{L^2(0,\infty)}. \tag{6}$$

For, if $I \subset (-\infty, 0]$, then (5) follows from (6) since we treat the even extension. If $I = [-a_1, a_2]$, $a_1, a_2 > 0$, then

$$\begin{aligned}\frac{1}{|I|}\int_I |E[G_N](x) - c|\, dx &= \frac{1}{|I|}\left\{\int_0^{a_1} |G_N(y) - c|\, dy + \int_0^{a_2} |G_N(y) - c|\, dy\right\} \\ &\le \frac{2}{a}\int_0^a |G_N(y) - c|\, dy\end{aligned}$$

for any constant c, where $a = \max\{a_1, a_2\}$. Thus, if we can prove (6), then (5) is obtained.

Now we turn to a proof of (6). Let $I = [y_0, y_1]$, $y_1 > y_0 \ge 0$. If $|I| > 1/n_1$, then we have by Scwarz's inequality and Parseval's identity for the Hankel transform that

$$\begin{aligned}\frac{1}{|I|}\int_I |G_N(y)|\, dy &\le \left(\frac{1}{|I|}\int_I |G_N(y)|^2\, dy\right)^{1/2} \\ &\le n_1^{1/2}\left(\int_0^\infty |G_N(y)|^2\, dy\right)^{1/2} \\ &= n_1^{1/2}\left\|\sum_{k=1}^N \chi_k g\right\|_{L^2(0,\infty)} \le n_1^{1/2}\|g\|_{L^2(0,\infty)},\end{aligned}$$

that is, we have (6) with $c = 0$.

Suppose that $1/n_{M+1} < |I| \le 1/n_M$ with a positive integer M. We first deal with the case $N \le M$. In this case, we shall show (6) with $c = G_N(y_0)$. It follows that

$$\begin{aligned}|G_N(y) - G_N(y_0)|^2 &= \left|\int_0^\infty g(t)\left\{\sum_{k=1}^N \chi_k(t)(\phi_\nu(yt) - \phi_\nu(y_0 t))\right\} dt\right|^2 \\ &\le \|g\|^2_{L^2(0,\infty)} \sum_{k=1}^N \int_{n_k \le t \le n_k + L} |\phi_\nu(yt) - \phi_\nu(y_0 t)|^2\, dt,\end{aligned} \tag{7}$$

where $\phi_\nu(u) = \sqrt{u} J_\nu(u)$.

We need to estimate the quantity $|\phi_\nu(yt) - \phi_\nu(y_0 t)|$. We shall show that there exists a constant C depending only on ν such that

$$|\phi_\nu(u_2) - \phi_\nu(u_1)| \le C|u_2 - u_1|^\delta \tag{8}$$

for $u_2, u_1 > 0$, where $\delta = \nu + 1/2$ for $-1/2 < \nu < 1/2$, and $\delta = 1$ for $\nu = -1/2$ or $1/2 \le \nu$. The case $\nu = -1/2$ is obvious since $\phi_{-1/2}(u) = (2/\pi)^{1/2} \cos u$ is a smooth function. We assume $\nu > -1/2$. By the facts $J_\nu(z) \sim z^\nu$ $(z \to +0)$ and $J_\nu(z) = O(z^{-1/2})$ $(z \to +\infty)$, we have $\sup_{u \ge 0} |\phi_\nu(u)| \le C$. Thus, it suffices to show (8) for $0 \le u_1 < u_2$ and $u_2 - u_1 \le 1$. The formula $J'_\nu(z) = (\nu/z)J_\nu(z) - J_{\nu+1}(z)$ leads to $(d/du)\phi_\nu(u) = (\nu + (1/2))u^{-1/2}J_\nu(u) - u^{1/2}J_{\nu+1}(u)$, and $\sup_{1 \le u} |(d/du)\phi_\nu(u)| \le C$. It follows from this that (8) holds when $1 \le u_1 < u_2$ and $u_2 - u_1 \le 1$. Since we can divide the matter into two parts at the point 1, it is enough to deal with the case $0 \le u_1 < u_2 \le 1$. It follows from the series definition of the Bessel function that $\phi_\nu(u) = u^{\nu+1/2}h_\nu(u)$, where

$$h_\nu(u) = 2^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n (u/2)^{2n}}{n!\Gamma(\nu + n + 1)},$$

which is an entire function. We have

$$\begin{aligned} |\phi_\nu(u_2) - \phi_\nu(u_1)| &\le |u_2^{\nu+1/2}||h_\nu(u_2) - h_\nu(u_1)| + |u_2^{\nu+1/2} - u_1^{\nu+1/2}||h_\nu(u_1)| \\ &\le |u_2 - u_1| \sup_{0 \le u \le 1} |h'_\nu(u)| + C|u_2 - u_1|^\delta \sup_{0 \le u \le 1} |h_\nu(u)| \\ &\le C|u_2 - u_1|^\delta, \end{aligned}$$

and obtain the inequality (8).

Let us go back to estimating (7). It follows from (8) that $|\phi_\nu(yt) - \phi_\nu(y_0 t)| \le C_\nu |y - y_0|^\delta t^\delta$, with which (7) leads to

$$\begin{aligned} |G_N(y) - G_N(y_0)|^2 &\le C\|g\|^2_{L^2(0,\infty)} |y - y_0|^{2\delta} \sum_{k=1}^{N} \int_{n_k \le t \le n_k + L} t^{2\delta}\, dt \\ &\le K_L \|g\|^2_{L^2(0,\infty)} |y - y_0|^{2\delta} \sum_{k=1}^{N} n_k^{2\delta}, \end{aligned}$$

where K_L depends only on ν and L. Since the sequence $\{n_k\}_{k=1}^{\infty}$ has a Hadamard gap, $n_{k+1}/n_k \geq \rho > 1$, it follows that $\sum_{k=1}^{N} n_k^{2\delta} \leq C n_N^{2\delta}$ with a constant C depending only on ν and ρ. For $y \in I = [y_0, y_1]$, we have $|y - y_0| n_N \leq |I| n_N \leq 1$ for $N \leq M$ by the choice of M. Thus, we have $|G_N(y) - G_N(y_0)|^2 \leq C\|g\|_{L^2(0,\infty)}^2$ for $y \in I$ and $N \leq M$ with C depending only on ν, ρ and L. Applying Schwarz's inequality to the left-hand side of the inequality (6) and using this inequality, we see that (6) with $c = G_N(y_0)$ holds in the case $N \leq M$.

Remark The constant K_L satisfies $CL \leq K_L$, where C depends only on ν. It is crucial for our proof to take the lengths of the intervals $[n_k, n_k + L]$ so as to be constant L. In other words, our proof do not allow to treat the intervals $[n_k, n_k + L_k]$ with $L_k \to \infty$ as $k \to \infty$.

Let us deal with the case $M < N$. We write

$$\begin{aligned} G_N(y) &= G_M(y) + \sum_{k=M+1}^{N} \int_{n_k \leq t \leq n_k + L} g(t)\sqrt{yt} J_\nu(yt)\, dt \\ &= G_M(y) + R_{M,N}(y), \qquad \text{say.} \end{aligned}$$

In this case, for $I = [y_0, y_1]$ we shall show that (6) with $c = G_M(y_0)$ holds. We have that

$$\begin{aligned} &\frac{1}{|I|} \int_I |G_N(y) - G_M(y_0)|\, dy \\ &\quad \leq \frac{1}{|I|} \int_I |G_M(y) - G_M(y_0)|\, dy + \frac{1}{|I|} \int_I |R_{M,N}(y)|\, dy. \end{aligned}$$

By the case $N \leq M$ we just proved, we see that the first term on the right-hand side of the above inequality is bounded by $C\|g\|_{L^2(0,\infty)}$. Thus, it is enough to show that the second term on the right-hand side is bounded by $C\|g\|_{L^2(0,\infty)}$, that is,

$$\frac{1}{|I|} \int_I |R_{M,N}(y)|\, dy \leq C\|g\|_{L^2(0,\infty)}. \tag{9}$$

Let us estimate $((1/|I|) \int_I |R_{M,N}(y)|\, dy)^2$. It follows that

$$\left(\frac{1}{|I|}\int_I |R_{M,N}(y)|\,dy\right)^2 \le \frac{1}{|I|}\int_I |R_{M,N}(y)|^2\,dy$$
$$\le \int_0^\infty \int_0^\infty |g(t)||g(s)| \sum_{k,j=M+1}^{N} \chi_k(t)\chi_j(s)K_I(t,s)\,dtds,$$

where

$$K_I(t,s) = \frac{1}{|I|}\left|\int_I \sqrt{yt}J_\nu(yt)\sqrt{ys}J_\nu(ys)\,dy\right| = \frac{1}{|I|}\left|\int_I \phi_\nu(yt)\phi_\nu(ys)\,dy\right|.$$

We state an estimate for $K_I(t,s)$ as a lemma, which will be proved after finishing the proof of the theorem.

Lemma *Let I be a subinterval of $[0,\infty)$, and let M be a positive integer such that $1/n_{M+1} < |I| \le 1/n_M$. Then the inequalities*

$$K_I(t,s) \le C\gamma^{|k-j|}, \quad t \in [n_k, n_k+L], \quad s \in [n_j, n_j+L] \tag{10}$$

hold for $k,j = M+1, M+2, \dots,$ where C is a positive constant depending only on ν, ρ and L, and γ is a constant with $0<\gamma<1$ depending only on ν and ρ.

By the lemma, we have

$$\left(\frac{1}{|I|}\int_I |R_{M,N}(y)|\,dy\right)^2 \le C \sum_{k,j=M+1}^{\infty} B_kB_j\gamma^{|k-j|}, \quad B_k = \int_0^\infty |g(t)|\chi_k(t)\,dt.$$

By using Schwarz's inequality, we see that

$$\sum_{k,j=M+1}^{\infty} B_kB_j\gamma^{|k-j|}$$
$$= \sum_{k=M+1}^{\infty} B_k^2 + 2\gamma \sum_{k=M+1}^{\infty} B_{k+1}B_k + \cdots + 2\gamma^m \sum_{k=M+1}^{\infty} B_{k+m}B_k + \cdots$$
$$\le (1+2\gamma+\cdots+2\gamma^m+\cdots)\sum_{k=M+1}^{\infty} B_k^2.$$

This leads to

$$\left(\frac{1}{|I|}\int_I |R_{M,N}(y)|\,dy\right)^2 \le C\sum_{k=M+1}^{\infty} B_k^2$$
$$\le CL\sum_{k=M+1}^{\infty}\int_{n_k\le t\le n_k+L} |g(t)|^2\,dt \le C\|g\|^2_{L^2(0,\infty)}$$

with a constant C not depending on M, N, I and g, which implies (9). Therefore, we complete the proof of the theorem.

We turn to the proof of the lemma. Let $t \in [n_k, n_k + L]$ and $s \in [n_j, n_j + L]$ be fixed, and let $I = [y_0, y_1]$. We may assume that $j \ge k$. Denote by K the greatest non-negative integer such that $2\pi K/s \le y_1 - y_0$, and put $a_p = y_0 + 2\pi p/s$ for $p = 0, 1, 2, \ldots, K$ and $a_{K+1} = y_1$. We note that $a_{p+1} - a_p \le 2\pi/n_j$ for $p = 0, 1, \ldots, K$. We write

$$\int_I \phi_\nu(yt)\phi_\nu(ys)\,dy = \sum_{p=0}^{K}\ \{A_p^{(1)} + A_p^{(2)}\},$$

where

$$A_p^{(1)} = \int_{a_p}^{a_{p+1}} (\phi_\nu(yt) - \phi_\nu(a_p t))\phi_\nu(ys)\,dy, \quad A_p^{(2)} = \phi_\nu(a_p t)\int_{a_p}^{a_{p+1}} \phi_\nu(ys)\,dy.$$

Combining (8) and the fact $|\phi_\nu(ys)| \le C$ for $\nu \ge -1/2$, we have that

$$|A_p^{(1)}| \le Ct^\delta\left(\frac{2\pi}{n_j}\right)^\delta (a_{p+1} - a_p) \le C\left(\frac{n_k}{n_j}\right)^\delta (a_{p+1} - a_p),$$

which leads to

$$\sum_{p=0}^{K}|A_p^{(1)}| \le C\left(\frac{n_k}{n_j}\right)^\delta |I| \le C\left(\frac{1}{\rho^\delta}\right)^{j-k}|I|, \tag{11}$$

since $1 < \rho \le n_{i+1}/n_i$, $i = 1, 2, \ldots$. Let us estimate $A_p^{(2)}$. For $A_0^{(2)}$ and $A_K^{(2)}$, we see by $n_k|I| \ge 1$, $k = M + 1, \ldots$ that

$$\big|A_p^{(2)}\big| \le \frac{C}{n_j} = C\frac{n_k}{n_j n_k |I|}|I| \le C\left(\frac{1}{\rho}\right)^{j-k}|I|, \quad p = 0, K. \tag{12}$$

We deal with $A_p^{(2)}$, $p = 1, 2, \dots, K-1$. We may assume $K \ge 2$. We use the following well-known asymptotic formula:

$$J_\nu(z) = \sqrt{2/(\pi z)}\cos(z - (2\nu+1)\pi/4) + O(z^{-3/2}), \quad z \to +\infty. \tag{13}$$

For $y \in [a_p, a_{p+1}]$, $p = 1, 2, \dots, K-1$, it follows from $ys \ge 2\pi$ that

$$\phi_\nu(ys) = \sqrt{2/\pi}\cos(ys - (2\nu+1)\pi/4) + R(ys), \quad |R(ys)| \le C(ys)^{-1},$$

where C depends only on ν. This leads to

$$\big|A_p^{(2)}\big| \le C\left|\int_{a_p}^{a_{p+1}} \big\{\sqrt{2/\pi}\cos(ys - (2\nu+1)\pi/4) + R(ys)\big\}\,dy\right|$$

for $p = 1, 2, \dots, K-1$. Since $\int_{a_p}^{a_{p+1}} \cos(ys - (2\nu+1)\pi/4)\,dy = 0$, it follows that

$$\big|A_p^{(2)}\big| \le \frac{C}{s}\int_{a_p}^{a_{p+1}} \frac{1}{y}\,dy = \frac{C}{s}(\log a_{p+1} - \log a_p),$$

and $\sum_{p=1}^{K-1} |A_p^{(2)}| \le (C/s)\log K$. By the choice of K, we have $\log K \le \log(s|I|)$. Let a constant η be fixed such that $0 < \eta < 1$. Then there exists a positive constant C depending only on η satisfying $(1/x)\log x \le Cx^{-\eta}$ for $x \ge 2$. Thus we have

$$\sum_{p=1}^{K-1} \big|A_p^{(2)}\big| \le C|I|\left(\frac{1}{s|I|}\right)^{\eta}.$$

We note that $n_k|I| > 1$ since $k \ge M+1$. It follows that

$$\frac{1}{s|I|} \le \frac{1}{n_k|I|}\frac{n_k}{n_j} \le \frac{n_k}{n_j} \le \left(\frac{1}{\rho}\right)^{j-k}.$$

Thus we have

$$\sum_{p=1}^{K-1} \left|A_p^{(2)}\right| \le C|I| \left(\frac{1}{\rho^\eta}\right)^{j-k}. \tag{14}$$

Combining (11), (12) and (14), we have the desired inequality (10), which completes the proof of the lemma.

3. Proof of the proposition

Let us prove (i) of the proposition. We first note that the Hankel transform $H_\nu f$ is well-defined for a function $f \in L^p(0,\infty)$ with $1 \le p \le 2$. For, if $1 \le p \le 2$, then the Hausdorff-Young inequality $\|H_\nu f\|_{L^q(0,\infty)} \le C\|f\|_{L^p(0,\infty)}$ of the Hankel transform holds, where $1/p + 1/q = 1$.

The case $p = 2$ is trivial. Let $1 < p < 2$ and let $f \in L^p(0,\infty)$. We denote by $E[f]$ the even extension of f to $(-\infty,\infty)$. We use the result [7, XIV, Proposition 5.1], which says that given $\lambda > 0$, there exist functions $E[f]^\lambda \in H^1(\mathbf{R})$ and $E[f]_\lambda \in L^2(\mathbf{R})$ such that $E[f] = E[f]^\lambda + E[f]_\lambda$ and

$$\left\|E[f]^\lambda\right\|_{H^1(\mathbf{R})} \le C\lambda^{1-p}\|E[f]\|_{L^p(\mathbf{R})}^p,$$

$$\left\|E[f]_\lambda\right\|_{L^2(\mathbf{R})}^2 \le C\lambda^{2-p}\|E[f]\|_{L^p(\mathbf{R})}^p$$

with C independent of $E[f]$ and λ. Let χ_+ be the characteristic function of $(0,\infty)$. Then we have that $f = E[f]_e\chi_+$, where $E[f]_e$ is the even part of $E[f]$. This leads to $f = (E[f]^\lambda)_e\chi_+ + (E[f]_\lambda)_e\chi_+$. By applying the result [1, III, Lemma 7.39], we see that $(E[f]^\lambda)_e\chi_+ \in H^1(\mathbf{R})$ and $\|(E[f]^\lambda)_e\chi_+\|_{H^1(\mathbf{R})} \le C\|E[f]^\lambda\|_{H^1(\mathbf{R})}$, which implies that $(E[f]^\lambda)_e\chi_+ \in H^1(0,\infty)$ and

$$\left\|(E[f]^\lambda)_e\chi_+\right\|_{H^1(0,\infty)} \le C\left\|E[f]^\lambda\right\|_{H^1(\mathbf{R})} \le C\lambda^{1-p}\|f\|_{L^p(0,\infty)}^p. \tag{15}$$

Also, we have

$$\left\|(E[f]_\lambda)_e\chi_+\right\|_{L^2(0,\infty)} \le C\left\|E[f]_\lambda\right\|_{L^2(\mathbf{R})} \le C\lambda^{(2-p)/2}\|f\|_{L^p(0,\infty)}^{p/2}. \tag{16}$$

The left-hand side of (2) is equal to $\|\sum_{k=1}^\infty \chi_k H_\nu f\|_{L^2(0,\infty)}$, where χ_k is the characteristic function of $[n_k, n_k + L]$, since we may assume that the intervals $[n_k, n_k + L]$ are non-overlapping. By the theorem and Parseval's identity, we have

$$\begin{aligned}
&\Big\|\sum_{k=1}^{\infty}\chi_k H_\nu f\Big\|_{L^2(0,\infty)}\\
&\quad\le \Big\|\sum_{k=1}^{\infty}\chi_k H_\nu\big((E[f]^\lambda)_e\chi_+\big)\Big\|_{L^2(0,\infty)}+\Big\|\sum_{k=1}^{\infty}\chi_k H_\nu\big((E[f]_\lambda)_e\chi_+\big)\Big\|_{L^2(0,\infty)}\\
&\quad\le C\big\|(E[f]^\lambda)_e\chi_+\big\|_{H^1(0,\infty)}+\big\|(E[f]_\lambda)_e\chi_+\big\|_{L^2(0,\infty)}\\
&\quad\le C\big(\lambda^{1-p}\|f\|^p_{L^p(0,\infty)}+\lambda^{(2-p)/2}\|f\|^{p/2}_{L^p(0,\infty)}\big).
\end{aligned}$$

The last inequality follows from (15) and (16). Choosing λ so as $\lambda = \|f\|_{L^p(0,\infty)}$, we obtain the desired inequality (2), which completes the proof of (i).

We now turn to proving (ii) of the proposition. Suppose that the series on the left-hand side of (2) converges for every $f \in L^1(0,\infty)$. Then, by the closed graph theorem we have

$$\left(\sum_{k=1}^{\infty}\int_{n_k\le y\le n_k+L}|H_\nu f(y)|^2\,dy\right)^{1/2}\le C\|f\|_{L^1(0,\infty)}$$

for $f \in L^1(0,\infty)$ with C independent of f. Let t_0 be a fixed positive number. For every $j = 1, 2, \dots$, we define the function f_j by $f_j(t) = j$ $(t_0 \le t \le t_0 + 1/j)$ and $f_j(t) = 0$ (otherwise). Then, $\|f_j\|_{L^1(0,\infty)} = 1$ for every j and $\lim_{j\to\infty} H_\nu f_j(y) = \sqrt{yt_0}\,J_\nu(yt_0)$. The above inequality and Fatou's lemma lead to

$$\begin{aligned}
&\sum_{k=1}^{\infty}\int_{n_k\le y\le n_k+L}\big|\sqrt{yt_0}J_\nu(yt_0)\big|^2\,dy\\
&\qquad\le \liminf_{j\to\infty}\sum_{k=1}^{\infty}\int_{n_k\le y\le n_k+L}|H_\nu f_j(y)|^2\,dy\le C.
\end{aligned}$$

By the asymptotic formula (13), we have

$$\big|\sqrt{t_0y}J_\nu(yt_0)\big|^2\ge C_1|\cos(yt_0-(2\nu+1)\pi/4)|^2-C_2y^{-1}$$

for $y \geq 1$, where positive constants C_1 and C_2 are independent of k, but may depend on t_0 and ν. It follows that $\sum_{k=1}^{\infty} \int_{n_k}^{n_n+L} y^{-1}\,dy \leq L \sum_{k=1}^{\infty} n_k^{-1} \leq Ln_1^{-1}\rho(\rho-1)^{-1}$. Thus we have the inequality

$$\sum_{k=1}^{\infty} \int_{n_k \leq y \leq n_k+L} \left|\cos(yt_0 - (2\nu+1)\pi/4)\right|^2 dy \leq C \tag{17}$$

with a positive constant C.

On the other hand, there exists a point t_0 such that the set of points $\{\langle n_k t_0/\pi\rangle\}_{k=1}^{\infty}$ is dense in $(0,1)$ (cf. [2, Theorem 1.40]), where $\langle t\rangle$ denotes the fractional part of t. For such a t_0, the integral in the sum of (17) is larger than $\int_0^{L/2} (\cos t_0 y)^2\,dy$ for infinitely many k's and hence (17) is impossible. We complete the proof of (ii), and the proof of the proposition.

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