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A BRIDGE BETWEEN THE UNIT SQUARE AND SINGLE INTEGRALS FOR REAL FUNCTIONS OF THE FORM $f(x \cdot y)$

Abstract

Sondow and co-workers have employed a key change of variables in order to evaluate double integrals over the unit square $[0,1] \times [0,1]$ in exact closed-form. Motivated by their results, I introduce here a change of variables which creates a 'bridge' between integrals of the form $\int_0^1\int_0^1 f(x \cdot y)\ dx\, dy$ and single integrals of the form $\int_0^1 f(p)\, \ln p\ dp$. This allows for prompt closed-form evaluations of several interesting integrals, including some of those investigated recently by Sampedro [Ramanujan J. **40**, 541 (2016)]. I also show that the bridge holds when the intervals of integration are changed from $[0,1]$ to $[1,\infty)$. Finally, a generalization for higher dimensions is proved, which reveals an interesting link of those integrals to Mellin's transform.

1 Introduction

In [6, 15], Sondow and co-workers investigated exact closed-form evaluations of certain unit square integrals, most of which were solved by applying the change of variables (see [6, Theor. 3.1])

$$x = \frac{u}{v} \quad \text{and} \quad y = v\,, \tag{1}$$

as, e.g.,

$$\int_0^1\int_0^1 \frac{1}{1-xy}\ dx\, dy = \zeta(2) \tag{2}$$

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and

$$\int_0^1\!\!\int_0^1 \frac{1}{1+x^2y^2}\, dx\, dy = G\,, \tag{3}$$

where $G := \sum_{n\ge 0} (-1)^n/(2n+1)^2$ is Catalan's constant and $\zeta(s) := \sum_{n\ge 1} 1/n^s$, $\Re(s) > 1$, is the Riemann zeta function. Other unit square integrals were solved by those authors, including the more general results

$$\int_0^1\!\!\int_0^1 \frac{(xy)^{u-1}}{\ln xy}\, dx\, dy = -\,\frac{1}{u}\,,$$
$$\int_0^1\!\!\int_0^1 \frac{(xy)^{u-1}}{(1+xy)\,\ln xy}\, dx\, dy = -\,\frac{1}{2}\left[\psi\left(\frac{u+1}{2}\right) - \psi\left(\frac{u}{2}\right)\right], \tag{4}$$

where $\psi(u) := \Gamma'(u)/\Gamma(u)$ is the digamma function, $\Gamma(u) := \int_0^\infty t^{u-1}\, e^{-t}\, dt$ being the classical gamma function,

$$\int_0^1\!\!\int_0^1 \frac{(-\ln xy)^n}{1-xyz}\, dx\, dy = (n+1)!\, \frac{\mathrm{Li}_{n+2}(z)}{z}\,, \tag{5}$$

valid for $z \in \mathbb{C}\backslash[1,\infty), n \ge -1$ (or $z = 1, n \ge 0$), where $\mathrm{Li}_n(z) := \sum_{k\ge 1} z^k/k^n$ is the polylogarithm function, and

$$\int_0^1\!\!\int_0^1 \frac{(-\ln xy)^s}{1-xy}\, dx\, dy = \Gamma(s+2)\,\zeta(s+2)\,, \quad \Re(s) > -1\,, \tag{6}$$
$$\int_0^1\!\!\int_0^1 \frac{(-\ln xy)^s}{1+x^2y^2}\, dx\, dy = \Gamma(s+2)\,\beta(s+2)\,, \quad \Re(s) > -2\,, \tag{7}$$

where $\beta(s) := \sum_{n\ge 0} (-1)^n/(2n+1)^s$, $\Re(s) > 0$, is the Dirichlet beta function. align On investigating other similar change of variables, I have found one which allows for a direct conversion of unit square integrals into single integrals, valid when the integrand $f(x,y)$ has the form $f(x\cdot y)$. Many examples are given, some of them involving certain identities for $\mathrm{Li}_2(r)$, $r \in \mathbb{Q}$, discovered by Ramanujan.

2 The 'bridge'

The bridge mentioned above comes from the following change of variables:

$$x = u\,v\,, \quad y = \frac{u}{v}\,. \tag{8}$$

As shown in the theorem below, this change allows for a direct conversion of unit square integrals of real functions of the form $f(x,y) = f(x\cdot y)$ into single integrals involving just $f(p)\ln p$, where p stands for the *product* $x\,y$. For Lebesgue integrals, only the requirement that $f(p)\cdot\ln p$ is integrable on $[0,1]$ is needed to prove the next theorem, but for simplicity, we restrict ourselves to the Riemann integrable case.

Theorem 1 (Bridge between unit square and single integrals)**.** *Let* $f:[0,1]\to\mathbb{R}$ *be a function such that* $f(p)\ \ln p$ *is Riemann-integrable on* $[0,1]$ *and* $f(x\cdot y)$ *is Riemann-integrable on the unit square* $[0,1]\times[0,1]$*. Then*

$$\int_0^1\int_0^1 f(x\cdot y)\,dx\,dy = -\int_0^1 f(p)\ \ln p\ dp\,.$$

Proof. Since $f(x\cdot y)$ is an integrable real function, we can define

$$I := \int_0^1\int_0^1 f(x\cdot y)\ dx\,dy.$$

On changing the variables according to Eq. (8), one finds

$$I = \iint_S f(x(u,v)\cdot y(u,v))\ |\det J(u,v)|\ du\,dv\,, \tag{9}$$

where S is the domain of integration in the uv-plane, as seen in Fig. 1, and

$$J(u,v) = \begin{pmatrix} v & u \\ 1/v & -u/v^2 \end{pmatrix}$$

is the Jacobian matrix. From Fig. 1, it is clear that

$$\begin{aligned} I &= \iint_S f\left(u^2\right)\ \left|\frac{-2\,u}{v}\right|\ du\,dv \\ &= 2\iint_S f\left(u^2\right)\ \frac{u}{v}\ du\,dv \\ &= 2\int_0^1 u\,f\big(u^2\big)\left[\int_u^{1/u}\frac{1}{v}\ dv\right]du \\ &= 2\int_0^1 u\,f\big(u^2\big)\left[\ln\left(\frac{1}{u}\right)-\ln u\right]du \\ &= 2\int_0^1 u\,f\big(u^2\big)\ (-2\ \ln u)\ du \\ &= -4\int_0^1 u\,f\big(u^2\big)\ \ln u\ du\,. \end{aligned} \tag{10}$$

This change of variables in (9) is a C^1 diffeomorphism since the functions $x(u,v) = uv$ and $y(u,v) = u/v$, as well as their inverses $u(x,y) = \sqrt{xy}$ and $v(x,y) = \sqrt{x/y}$, are differentiable at all points of the corresponding open domains. The substitution $p = u^2$ now completes the proof. □

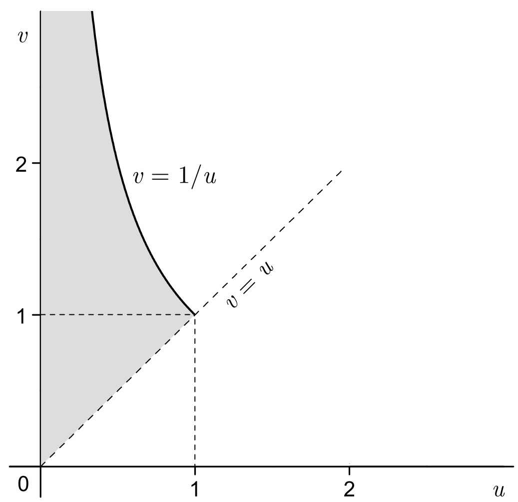

Figure 1: The hachured region (bounded by the v-axis, the dashed-line $v = u$, and above by the curve $v = 1/u$) is the domain of integration S, in the uv-plane, corresponding to the change of variables in Eq. (8) taken over the unit square $(x,y) \in [0,1] \times [0,1]$.

It should be mentioned that the change of variables introduced by Guillera and Sondow, as given in our Eq. (1), for which the Jacobian determinant equals $1/v$, also leads to this bridge, but those authors missed the interesting bridge we are emphasizing here. In fact, their change of variables promptly yields $\int_0^1 \int_0^1 f(x \cdot y)\, dx\, dy = \int_0^1 f(u) \int_u^1 (1/v)\; dv = -\int_0^1 f(u)\; \ln u\; du$.

2.1 Some examples

Let us apply the above theorem to some functions $f(x \cdot y)$ in order to illustrate how our bridge works. Most of the integrals below are hard (if not impossible) to be solved by other means.

For instance, the function $f(x,y) = 1/\ln(xy)$ yields

$$\int_0^1\!\!\int_0^1 \frac{1}{\ln(xy)}\,dx\,dy = -\int_0^1 \frac{1}{\ln p}\,\ln p\,dp = -\int_0^1 dp = -1\,. \tag{11}$$

Similarly,

$$\int_0^1\!\!\int_0^1 \frac{1}{(1+xy)\,\ln(xy)}\,dx\,dy = -\int_0^1 \frac{1}{1+p}\,dp = -\ln 2\,. \tag{12}$$

A less trivial example is as follows. Given any $a \geq 1$,

$$\begin{aligned}
\int_0^1\!\!\int_0^1 \frac{1}{a-xy}\,dx\,dy &= -\int_0^1 \frac{\ln p}{a-p}\,dp \\
&= -\frac{1}{a}\int_0^1 \frac{\ln p}{1-p/a}\,dp \\
&= -\frac{1}{a}\int_0^{1/a} \frac{\ln(a\,u)}{1-u}\,a\,du \\
&= -\int_0^{1/a} \frac{\ln u + \ln a}{1-u}\,du \\
&= -\int_0^{1/a} \frac{\ln u}{1-u}\,du - \ln a \int_0^{1/a} \frac{1}{1-u}\,du\,,
\end{aligned} \tag{13}$$

where we substituted $u = p/a$. On taking into account the integral definition for the dilogarithm function, namely $\mathrm{Li}_2(z) := -\int_0^z \frac{\ln(1-s)}{s}\,ds$ [8], the above integrals are reduced to

$$\begin{aligned}
\int_0^1\!\!\int_0^1 \frac{1}{a-xy}\,dx\,dy &= \mathrm{Li}_2(1) - \mathrm{Li}_2\left(1-\frac{1}{a}\right) - \ln a\,\left[-\ln\left(1-\frac{1}{a}\right)\right] \\
&= \mathrm{Li}_2(1) - \mathrm{Li}_2\left(1-\frac{1}{a}\right) + \ln a\;\ln\left(1-\frac{1}{a}\right) \\
&= \mathrm{Li}_2\left(\frac{1}{a}\right),
\end{aligned} \tag{14}$$

the last step being an application of Euler's reflection formula (see [9, Eq.(1.5)]). In particular, for $a = 1$ one has

$$\int_0^1\!\!\int_0^1 \frac{1}{1-xy}\,dx\,dy = \mathrm{Li}_2(1) = \zeta(2)\,, \tag{15}$$

which agrees with Eq. (2), a nice unit square integral taken into account by Apostol to show that $\zeta(2) = \pi^2/6$ [2].

When the domain of the integral in Eq. (13) is generalized to $[0,\alpha] \times [0,\alpha]$, $0 < \alpha \le 1$, one finds

$$\begin{aligned} \int_0^\alpha \int_0^\alpha \frac{1}{1-xy}\, dx\, dy &= \int_0^1 \int_0^1 \frac{1}{1-\alpha^2 XY}\, \alpha^2\, dX\, dY \\ &= \int_0^1 \int_0^1 \frac{1}{1/\alpha^2 - XY}\, dX\, dY \\ &= \mathrm{Li}_2\big(\alpha^2\big)\,, \end{aligned} \tag{16}$$

where we substituted $X = x/\alpha$ and $Y = x/\alpha$, and the dilogarithm result comes from Eq. (14). This corresponds to the 2D case of a general result obtained by McCartney in a recent paper, see [12, Eq. (27)].

Another interesting result related to the integrals investigated by McCartney is obtained by choosing $g_1(x) = \alpha\, x^2$ and $g_2(y) = \alpha\, y^2$, $0 < \alpha \le 1$, there in the main result of [12]. This leads to

$$\begin{aligned} G_2(g) :&= \int_0^1\!\!\int_0^1 \frac{1}{1 - g_1(x)\, g_2(y)}\, dx\, dy \\ &= \int_0^1\!\!\int_0^1 \frac{1}{1-\alpha^2 x^2 y^2}\, dx\, dy \\ &= \frac{1}{\alpha^2} \int_0^1\!\!\int_0^1 \frac{1}{1/\alpha^2 - x^2 y^2}\, dx\, dy \\ &= -\,\frac{1}{\alpha^2} \int_0^1 \frac{\ln p}{1/\alpha^2 - p^2}\, dp \\ &= -\,\frac{1}{\alpha^2} \left(\frac{\alpha}{2} \int_0^1 \frac{\ln p}{1/\alpha + p}\, dp + \frac{\alpha}{2} \int_0^1 \frac{\ln p}{1/\alpha - p}\, dp \right) \\ &= -\,\frac{1}{2\,\alpha} \left[\mathrm{Li}_2(-\alpha) - \mathrm{Li}_2(\alpha) \right] \\ &= \frac{\mathrm{Li}_2(\alpha)}{\alpha} - \frac{\mathrm{Li}_2\big(\alpha^2\big)}{4\,\alpha}\,, \end{aligned} \tag{17}$$

where the last step follows from the identity $\frac{1}{2}\,\mathrm{Li}_2(z^2) = \mathrm{Li}_2(z) + \mathrm{Li}_2(-z)$ (see Eq. (1.15) of Ref. [8]). On the other hand, McCartney has shown in [12, Eqs. 19 and 20], that

$$\begin{aligned} G_2(g) &= \sum_{n=0}^{\infty} \int_0^1 g_1^n(x)\ dx \int_0^1 g_2^n(y)\ dy \\ &= \sum_{n=0}^{\infty} \left[\int_0^1 \left(\alpha\, x^2\right)^n\ dx \right]^2 \\ &= \sum_{n=0}^{\infty} \alpha^{2n} \left[\int_0^1 x^{2n}\ dx \right]^2 \\ &= \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n+1)^2} \\ &= \frac{\chi_2(\alpha)}{\alpha}\,, \end{aligned} \tag{18}$$

where $\chi_2(z) := \sum_{n=0}^{\infty} z^{2n+1}/(2n+1)^2$ is the Legendre chi function. On equating the above results for $G_2(g)$, one finds

$$\chi_2(\alpha) = \mathrm{Li}_2(\alpha) - \frac{1}{4}\,\mathrm{Li}_2\!\left(\alpha^2\right), \tag{19}$$

a nice functional identity stated in [8, Eq. 23].

A similar integral that can be solved with our approach is

$$\int_0^{\alpha}\!\!\int_0^{\beta} \frac{1}{1-x^2y^2}\ dx\, dy = \chi_2(\alpha\,\beta)\,, \quad 0 < \alpha, \beta \le 1\,. \tag{20}$$

For this, it is enough to substitute $x = \beta\, X,\ y = \alpha Y$, which leads to

$$\alpha\,\beta \int_0^1\!\!\int_0^1 \frac{1}{1-\alpha^2\beta^2X^2Y^2}\ dX\, dY\,. \tag{21}$$

Now, note that our Eqs. (17) and (18) imply that

$$\alpha \int_0^1\!\!\int_0^1 \frac{1}{1-\alpha^2x^2y^2}\ dx\, dy = \chi_2(\alpha)\,. \tag{22}$$

The change $\alpha \to \alpha\,\beta$ completes the proof of Eq. (20). A much more complex proof, in fact the only known one, is found in [7].

Another interesting example comes from Theorem 5 of a recent work by Sampedro in Ref. [14], namely

$$-\int_0^1\!\!\int_0^1 \frac{x^{b-1}\,y^{g-1}}{(1-rx^cy^h)\,\ln(x^ay^d)}\ dx\, dy = \sum_{n=0}^{\infty} r^n\, \frac{\ln(d/a) - \ln\left[(hn+g)/(cn+b)\right]}{d\,(cn+b) - a\,(hn+g)}, \tag{23}$$

where a, b, c, d, h, g, and r are any real numbers such that the series converges. In particular, when $d \to a$, $g \to b$, and $h \to c$, Eq. (23) reduces to

$$-\frac{1}{a}\int_0^1\!\!\int_0^1 \frac{(xy)^{b-1}}{[1-r(xy)^c]\,\ln(xy)}\,dx\,dy = \lim_{\epsilon\to 0}\sum_{n=0}^{\infty} r^n\,\frac{\ln\!\left(1+\frac{\epsilon}{a}\right)-\ln\!\left(1+\frac{n+1}{cn+b}\,\epsilon\right)}{(cn+b)\,\epsilon - a\,(n+1)\,\epsilon}$$
$$= \sum_{n=0}^{\infty} r^n \lim_{\epsilon\to 0}\frac{\ln\!\left(1+\frac{\epsilon}{a}\right)-\ln\!\left(1+\frac{n+1}{cn+b}\,\epsilon\right)}{(cn+b)\,\epsilon - a\,(n+1)\,\epsilon}. \tag{24}$$

On applying our bridge to the left-hand side and the L'Hôpital rule to the right-hand side, one finds

$$\int_0^1 \frac{p^{b-1}}{1-r\,p^c}\,dp = \frac{1}{c}\sum_{n=0}^{\infty}\frac{r^n}{n+b/c}\,. \tag{25}$$

Let us show that the left-hand side reduces to a hypergeometric value of the kind ${}_2F_1\!\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\,;\,r\right) := \sum_{n=0}^{\infty}\frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n}\,\frac{r^n}{n!}$, where $(\alpha)_n := \Gamma(\alpha+n)/\Gamma(\alpha)$ is the Pochhammer symbol. For this, put $\alpha = 1$, $\beta = b/c$, and $\gamma = 1 + b/c$ in Euler's integral representation

$${}_2F_1\!\left(\begin{matrix}\alpha,\,\beta\\ \gamma\end{matrix}\,;\,r\right) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\int_0^1 \frac{t^{\beta-1}\,(1-t)^{\gamma-\beta-1}}{(1-rt)^{\alpha}}\,dt\,,$$

and then substitute $t = p^c$. Meanwhile, the right-hand side of Eq. (25) can be written in terms of the Lerch transcendent $\Phi(z,s,u) := \sum_{n=0}^{\infty} z^n/(n+u)^s$, which converges for any real number $u > 0$ if z and s are complex numbers with $|z| < 1$. This reduces Eq. (25) to the nice identity

$${}_2F_1\!\left(\begin{matrix}1,\,\tilde b\\ 1+\tilde b\end{matrix}\,;\,r\right) = \tilde b\;\Phi\!\left(r,1,\tilde b\right), \quad \tilde b > 0\,, \tag{26}$$

where $\tilde b := b/c$.

In [14, Ex. 13], Sampedro uses special values of the Lerch transcendent to show that

$$-\int_0^1\!\!\int_0^1 \frac{1}{\sqrt[4]{(xy)^3}\,(1+xy)\,\ln(xy)}\,dx\,dy = \frac{\pi + 2\,\coth^{-1}\!\left(\sqrt{2}\,\right)}{\sqrt{2}}\,. \tag{27}$$

For this unit square integral, our bridge yields

$$
\begin{aligned}
&\int_0^1 \frac{1}{p^{3/4}\,(1+p)}\,dp \\
&= \sqrt{2}\left[\tan^{-1}\left(\sqrt{2}\sqrt[4]{p}+1\right) - \tan^{-1}\left(1-\sqrt{2}\sqrt[4]{p}\right) + \coth^{-1}\left(\frac{\sqrt{p}+1}{\sqrt{2}\sqrt[4]{p}}\right)\right]_0^1 \\
&= \sqrt{2}\left[\tan^{-1}\left(\sqrt{2}+1\right) - \tan^{-1}\left(1-\sqrt{2}\right) + \coth^{-1}\left(\frac{2}{\sqrt{2}}\right)\right] \\
&= \sqrt{2}\left[\frac{\pi}{2} + \coth^{-1}\left(\sqrt{2}\right)\right], \qquad (28)
\end{aligned}
$$

which can be viewed as an alternative, *elementary proof* of Sampedro's result. Here, by elementary we mean that our proof does not involve Fourier series, complex analysis techniques (e.g., Cauchy's Residue Theorem), or Plancherel's Identity.

In [6, Ex. 3.6], by using special values of the Lerch transcendent Guillera and Sondow showed that

$$\int_0^1\int_0^1 \frac{9+xy}{9-x^2y^2}\,dx\,dy = \frac{\pi^2}{6} - \frac{\ln^2 3}{2}\,. \qquad (29)$$

For this unit square integral, our bridge yields

$$
\begin{aligned}
-&\int_0^1 \frac{9+p}{9-p^2}\,\ln p\,dp \\
&\quad= \left\{2\,\mathrm{Li}_2\left(\frac{p}{3}\right) - \mathrm{Li}_2\left(-\frac{p}{3}\right) + \ln p\left[2\ln(3-p) - \ln(9+3p)\right]\right\}_0^1 \\
&\quad= 2\,\mathrm{Li}_2\left(\frac{1}{3}\right) - \mathrm{Li}_2\left(-\frac{1}{3}\right) \\
&\quad= 3\,\mathrm{Li}_2\left(\frac{1}{3}\right) - \frac{1}{2}\mathrm{Li}_2\left(\frac{1}{9}\right), \qquad (30)
\end{aligned}
$$

in which we have substituted $u = p/3$ in the integral and the identity $\mathrm{Li}_2(z^2) = 2\left[\mathrm{Li}_2(z) + \mathrm{Li}_2(-z)\right]$ was applied in the last step. On taking into account one of the Ramanujan's dilogarithm identity, namely [3, p. 324]

$$\mathrm{Li}_2\left(\frac{1}{3}\right) - \frac{1}{6}\,\mathrm{Li}_2\left(\frac{1}{9}\right) = \frac{\pi^2}{18} - \frac{\ln^2 3}{6}\,, \qquad (31)$$

the result in Eq. (29) promptly follows. Again, this can be viewed as a simpler proof of Eq. (29).

2.2 Crossing the 'bridge' in the opposite direction

The bridge also works in the opposite direction; i.e., we can make use of a known unit square integral to deduce an exact closed-form expression for the corresponding single integral.

The simplest example is found by taking into account the constant function $f(x \cdot y) = c$, $c \neq 0$. For this function, one has

$$-\int_0^1 c \ln p \; dp = \int_0^1\!\!\int_0^1 c \; dx\, dy = c \int_0^1\!\!\int_0^1 dx\, dy = c\,, \tag{32}$$

which promptly yields $\int_0^1 \ln p \; dp = -1$. This improper integral can also be solved by parts, as done, within the rigour of mathematical analysis, by Tavares [11], but the solution is considerably lengthier than ours.

A less obvious example involves Catalan's constant and it comes from entry 2 of Adamchik's list of representations for G, where one finds $-\int_0^1 \ln p/(1+p^2) \; dp = G$ [1]. Our bridge transforms this integral into

$$\int_0^1\!\!\int_0^1 \frac{1}{1+x^2y^2} \; dx\, dy = G\,. \tag{33}$$

This is a simple alternative proof for Eq. (3). Interestingly, we can use this unit square integral to explore other integral representations for G, e.g. by applying the non-trivial change of variables (see, e.g., [10])

$$x = \frac{\sin u}{\cosh v}\,, \quad y = \frac{\sinh v}{\cos u}\,, \tag{34}$$

whose Jacobi determinant is $\det J = 1 + x^2\, y^2 = 1 + \tan^2 u \, \tanh^2 v$. This yields $\iint_S du\, dv = G$, where S is the region in the uv-plane bounded by the coordinate axes, and above by the curve $v(u) = \text{arcsinh}(\cos u)$. Since this double integral equals the area of S, then

$$\int_0^{\pi/2} \text{arcsinh}(\cos u)\, du = G. \tag{35}$$

Although this result is not found in the literature, it resembles an integral found in [1, entry 17] namely, $\int_0^{\pi/2} \sinh^{-1}(\sin u)\; du = G$. Analogously, by calculating the area of S via $\int_0^b u(v)\; dv$, with $u(v) = \arccos(\sinh v)$ and $b = \text{arcsinh}(1)$, one finds

$$\int_0^{\ln\left(1+\sqrt{2}\right)} \arccos(\sinh v)\; dv = G\,, \tag{36}$$

which is also new.

A more complex example related to Catalan's constant will reveal the power of our bridge. From entry 33 of Adamchik's webpage, one knows that

$$G=\int_0^1\left(\frac{2}{p^2-4p+8}-\frac{3}{p^2+2p+2}\right)\ln p\;dp\,. \tag{37}$$

For this integral, our bridge yields

$$G=-\int_0^1\!\!\int_0^1\left(\frac{2}{x^2y^2-4xy+8}-\frac{3}{x^2y^2+2xy+2}\right)\,dx\,dy\,, \tag{38}$$

for which *Mathematica* (release 11) returns a closed-form expression with some dilogarithm values. After some algebra, that result simplifies to

$$\begin{aligned} 6\,G = & \quad -i\,\mathrm{Li}_2\!\left(e^{i(\pi-b)}\right)-3\,i\,\mathrm{Li}_2\!\left(e^{ib}\right)-2\,i\,\mathrm{Li}_2\!\left(e^{ia}\right)\\ + & \quad i\left[\frac{\pi^2}{4}-\frac{\pi}{4}\arctan\!\left(\frac{336}{527}\right)-\frac{a^2}{2}-\frac{3}{2}\,ab+b\,\arctan(3)\right], \end{aligned} \tag{39}$$

where $a=\arctan(3/4)$ and $b=\arctan(4/3)=\pi/2-a$. Fortunately, all the dilogarithms above are for points on the unit-circle in the complex plane, for which the expansion $\mathrm{Li}_2\left(e^{i\theta}\right)=g(\theta)+i\,\mathrm{Cl}_2(\theta)$ holds, where $g(\theta):=\pi^2/6+\theta^2/4-\pi\,|\theta|/2$ and $\mathrm{Cl}_2(\theta):=\sum_{n=1}^{\infty}\sin(n\theta)/n^2=-\int_0^{\theta}\ln(2\,\sin(\theta/2))\,d\theta$ is the Clausen integral. On equating the real part of each side of Eq. (39), one finds the identity

$$\mathrm{Cl}_2(\pi-b)+3\,\mathrm{Cl}_2(b)+2\,\mathrm{Cl}_2\!\left(\frac{\pi}{2}-b\right)=6\,G\,. \tag{40}$$

Our bridge then yields a simple alternative proof for [8, Eq. 20].

Another interesting example comes from [9, Eq. 16.21], namely

$$-\,q^2\int_0^1\frac{t^{\,p-1}}{1-t^q}\,\ln t\;dt=\psi\,'\!\left(\frac{p}{q}\right), \tag{41}$$

where p, q are positive integers and $\psi\,'(u):=d\psi/du$ is the trigamma function. This also appears as $\zeta(2,l/m)$ in [13, p. 289], where $\zeta(s,x):=\sum_{n\ge 0}1/(n+x)^s$ is the Hurwitz-zeta function. There the result is justified by the well-known identity $\psi\,'(x)=\zeta(2,x)$, which is valid for all $x>0$, which in turn follows from the series expansion of $\psi\,'(x)$. But via our bridge, we have that

$$q^2\int_0^1\!\!\int_0^1\frac{(xy)^{p-1}}{1-(xy)^q}\,dx\,dy=\psi\,'\!\left(\frac{p}{q}\right)\,, \tag{42}$$

a result that seems hard to be obtained by other methods. This unit square integral generalizes that in [6, Eg. 41], namely

$$\int_0^1\!\!\int_0^1 \frac{(xy)^{u-1}}{1-xy}\,dx\,dy = \psi\,'(u)\,, \tag{43}$$

for which our bridge yields

$$-\int_0^1 \frac{p^{u-1}}{1-p}\,\ln p\,dp = \psi\,'(u)\,, \tag{44}$$

which can be confirmed by substituting $p = e^{-t}$ in $\psi\,'(u) = \int_0^\infty t\,e^{-ut}/(1-e^{-t})\,dt$, an integral representation obtained from $\psi(u) = \int_0^\infty [\,e^{-t}/t - e^{-ut}/\,(1-e^{-t})]\,dt$ [5].

3 The 'bridge' for integrals over $[1,\infty)$

Let us show that our bridge remains valid (apart from a minus sign) when the intervals of integration are changed from $[0,1]$ to $[1,\infty)$.

Theorem 2 (Bridge for integrals over $[1,\infty)$)**.** *Let $f\colon [1,\infty) \to \mathbb{R}$ be a function such that $f(p)\,\ln p$ is Riemann-integrable on $[1,\infty)$ and $f(x\cdot y)$ is Riemann-integrable on $[1,\infty)\times[1,\infty)$. Then*

$$\int_1^\infty\!\!\int_1^\infty f(x\cdot y)\,dx\,dy = \int_1^\infty f(p)\,\ln p\,dp\,.$$

Proof. Let $K := \int_1^\infty\!\int_1^\infty f(x\cdot y)\,dx\,dy$. On changing the variables according to Eq. (8), one finds

$$K = \iint_{\widetilde{S}} f(x(u,v)\cdot y(u,v))\;|\det J(u,v)|\;du\,dv = 2\iint_{\widetilde{S}} f\big(u^2\big)\;\frac{u}{v}\,du\,dv\,, \tag{45}$$

where $\widetilde{S}$ is the domain of integration in the uv-plane bounded below by the curve $v = 1/u$ and above by the line $v = u$. Therefore,

$$\begin{aligned} K &= 2\int_1^\infty u\,f\big(u^2\big)\left[\int_{1/u}^u \frac{1}{v}\,dv\right]du \qquad (46)\\ &= 2\int_1^\infty u\,f\big(u^2\big)\left[\ln u - \ln\left(\frac{1}{u}\right)\right]du \\ &= 4\int_1^\infty u\,f\big(u^2\big)\,\ln u\;du\,. \qquad (47)\end{aligned}$$

The substitution $p = u^2$ completes the proof. □

Alternatively, we could have applied the change of variables $x = 1/X$, $y = 1/Y$ to $\int_0^1\int_0^1 f(x \cdot y)\,dx\,dy$ in order to get a simple proof that the above theorem follows from our Theorem 1, but we have preferred to exhibit a proof independent of that theorem.

3.1 Examples

Theorem 2 allows for the direct evaluation of many double integrals.

For instance, we begin with the 'harmless' double integral

$$\int_1^\infty\int_1^\infty \frac{1}{(1+x^2y^2)\ln(xy)}\,dx\,dy \tag{48}$$

which *Mathematica* (release 11) cannot solve analytically. Our bridge promptly reduces it to

$$\begin{aligned}\int_1^\infty \frac{1}{1+p^2}\,dp &= \lim_{b\to\infty}\int_1^b \frac{1}{1+p^2}\,dp \\ &= \lim_{b\to\infty} \arctan(b) - \arctan(1) \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\,.\end{aligned} \tag{49}$$

Other double integrals related to classical mathematical constants are

$$\int_1^\infty\int_1^\infty \frac{1}{1+x^2y^2}\,dx\,dy = \int_1^\infty \frac{\ln p}{1+p^2}\,dp = G \tag{50}$$

and

$$\int_1^\infty\int_1^\infty \frac{\ln(xy)}{1-x^2y^2}\,dx\,dy = \int_1^\infty \frac{\ln^2 p}{1-p^2}\,dp = -\frac{7}{4}\,\zeta(3)\,, \tag{51}$$

in which the Apéry's constant $\zeta(3)$ makes an appearance.

For the improper double integral

$$\int_1^\infty\int_1^\infty \frac{e^{-xy}}{xy\,\ln(xy)}\,dx\,dy\,, \tag{52}$$

which *Mathematica* (release 11) cannot solve, our bridge promptly yields

$$\int_1^\infty \frac{e^{-p}}{p}\,dp = -\operatorname{Ei}(-1) = 0.21938393\ldots\,, \tag{53}$$

where $\operatorname{Ei}(z) := -\int_{-z}^\infty e^{-t}/t\,dt$ is the Euler integral. It's interesting that the same result is found when our bridge is applied to $\int_1^\infty\int_1^\infty e^{-xy}\,dx\,dy$, since, using integration by parts, $\int_1^\infty e^{-p}\,\ln p\,dp$ also reduces to $-\operatorname{Ei}(-1)$.

4 Generalization to higher dimensions and a connection to Mellin's transform

Our Theorem 1 and the several examples presented above suggest that it could exist a similar bridge linking multiple integrals over $[0,1]^N$ to a single integral over $[0,1]$, where $N > 2$ is an integer. This generalization is established below.

Theorem 3 (Bridge for multiple integrals)**.** *Let* $f\colon [0,1] \to \mathbb{R}$ *be a continuous function such that* $f(p)\,\ln p$ *is Riemann-integrable on* $[0,1]$*. Then, for any integer* $N > 1$*,*

$$\int_0^1 \cdots \int_0^1 f(x_1\, x_2 \cdots x_N)\; dx_1\, dx_2 \cdots dx_N = \frac{(-1)^{N-1}}{(N-1)!} \int_0^1 f(p)\,(\ln p)^{N-1}\, dp\,.$$

Proof. First, let us show that the theorem holds for $f(x) = x^m$, $m \ge 0$ being an integer. For this simple function, one finds

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 f(x_1\, x_2 \cdots x_N)\; dx_1\, dx_2 \cdots dx_N \\ &\quad = \prod_{n=1}^{N} \int_0^1 x_n^m\; dx_m = \frac{1}{(m+1)^N}\,, \end{aligned} \tag{54}$$

whereas

$$\begin{aligned} \frac{(-1)^{N-1}}{(N-1)!} \int_0^1 f(p)\,(\ln p)^{N-1}\, dp &= \frac{1}{(N-1)!} \int_0^1 p^m\,(-\ln p)^{N-1}\, dp \\ &= \frac{1}{(N-1)!} \int_0^\infty e^{-(m+1)\,t}\; t^{N-1}\, dt \\ &= \frac{1}{(N-1)!\;(m+1)^N} \int_0^\infty e^{-u}\; u^{N-1}\, du \\ &= \frac{1}{(m+1)^N\;(N-1)!}\;\Gamma(N) \\ &= \frac{1}{(m+1)^N}. \end{aligned} \tag{55}$$

The well-known identity $\Gamma(k) = (k-1)!$, valid for all integers $k \ge 1$, was taken into account in the last step. Since the integral is a linear operator, clearly the theorem also holds for any polynomial $f(p) = \sum_{m=0}^{M} a_m\, p^m$, where $M \ge 0$ is an integer and the coefficients a_m are arbitrary real numbers.

Now, suppose that $f\colon [0,1] \to \mathbb{R}$ is a continuous function. By the Weierstrass approximation theorem, we know that for every $\epsilon > 0$ there is a polynomial $P_M(x)$ such that $|f(x) - P_M(x)| < \epsilon$ for all $x \in [0,1]$. As the theorem holds for any polynomial, one has

$$\int_0^1 \cdots \int_0^1 P_M(x_1\, x_2 \cdots x_N)\; dx_1\, dx_2 \cdots dx_N = \frac{(-1)^{N-1}}{(N-1)!} \int_0^1 P_M(p)\, (\ln p)^{N-1}\, dp. \tag{56}$$

The Weierstrass theorem then yields

$$\begin{aligned} &\left| \int_0^1 \cdots \int_0^1 [f(x_1\, x_2 \cdots x_N) - P_M(x_1\, x_2 \cdots x_N)]\; dx_1\, dx_2 \cdots dx_N \right| \\ &\leq \int_0^1 \cdots \int_0^1 |f(x_1\, x_2 \cdots x_N) - P_M(x_1\, x_2 \cdots x_N)|\; dx_1\, dx_2 \cdots dx_N \\ &< \epsilon\,. \end{aligned} \tag{57}$$

Similarly,

$$\begin{aligned} &\left| \frac{(-1)^{N-1}}{(N-1)!} \int_0^1 f(y)\, (\ln y)^{N-1}\, dy - \frac{(-1)^{N-1}}{(N-1)!} \int_0^1 P_M(y)\, (\ln y)^{N-1}\, dy \right| \\ &\qquad \leq \frac{1}{(N-1)!} \int_0^1 |f(y) - P_M(y)|\; (-\ln y)^{N-1}\, dy \\ &\qquad < \epsilon\, \frac{1}{(N-1)!} \int_0^1 (-\ln y)^{N-1}\, dy \\ &\qquad = \epsilon\,. \end{aligned} \tag{58}$$

Finally, from the triangle inequality it follows that the two real numbers

$$\int_0^1 \cdots \int_0^1 f(x_1\, x_2 \cdots x_N)\; dx_1\, dx_2 \cdots dx_N$$

and

$$\frac{1}{(N-1)!} \int_0^1 f(p)\, (-\,\ln p)^{N-1}\, dp$$

differ by at most $2\,\epsilon$. □

The above theorem has an interesting consequence for the Mellin transform of continuous functions.

Theorem 4 (Mellin transform of a continuous function)**.** *Let* $f\colon [0,1] \to \mathbb{R}$ *be a continuous function such that* $f(p)\, \ln p$ *is Riemann-integrable on* $[0,1]$*. Then,*

$$\sum_{N=1}^{\infty} C_N\, z^{N-1} = \int_0^1 f(p)\, p^{-z}\, dp\,,$$

holds for all complex z with $|z| < 1$, where
$C_N := \int_0^1 \cdots \int_0^1 f(x_1\, x_2 \cdots x_N)\; dx_1\, dx_2 \cdots dx_N$.

Proof. Since the function $f(x)$ is continuous in $[0,1]$, then it is bounded in this interval. Hence the real numbers C_N are also bounded in $[0,1]$. This means that if $B > 0$ is such that $|f(x)| \le B$ for all $x \in [0,1]$, then

$$\begin{aligned} |C_N| &= \left| \int_0^1 \cdots \int_0^1 f(x_1\, x_2 \cdots x_N)\; dx_1\, dx_2 \cdots dx_N \right| \\ &\le \int_0^1 \cdots \int_0^1 |f(x_1\, x_2 \cdots x_N)|\; dx_1\, dx_2 \cdots dx_N \\ &\le \int_0^1 \cdots \int_0^1 B\; dx_1\, dx_2 \cdots dx_N \\ &= B\,. \end{aligned} \tag{59}$$

Now, let z be a complex variable. Because the numbers C_N are bounded, the power series

$$F(z) := \sum_{N=1}^{\infty} C_N\; z^{N-1} \tag{60}$$

converges for all z with $|z| < 1$, a domain in which it defines an analytic function. Of course, the analytic function $F(z)$ is also given by

$$F(z) = \sum_{N=1}^{\infty} \frac{\int_0^1 f(p)\,(-\ln p)^{N-1}\, dp}{(N-1)!}\; z^{N-1}\,. \tag{61}$$

Again, using the fact that f is continuous and $|f(p)|$ is bounded by B, according to the dominated convergence theorem the series and the integral can be interchanged. This yields

$$\begin{aligned} F(z) &= \int_0^1 f(p) \left[\sum_{N=1}^{\infty} \frac{(-z\, \ln p)^{N-1}}{(N-1)!} \right] dp \\ &= \int_0^1 f(p)\; e^{-z\, \ln p}\; dp \\ &= \int_0^1 f(p)\; p^{-z}\; dp\,, \end{aligned} \tag{62}$$

which completes the proof. □

From the above proof, we have

$$F(z) = \sum_{N=1}^{\infty} C_N \, z^{N-1} = \int_0^1 f(y)\, y^{-z} \, dy \, ,$$

which holds for all complex z with $|z| < 1$. The series in the above theorem is obviously a power series, and the integral on the right is an example of a Mellin transform,an integral transform closely related to the theory of Dirichlet series, Laplace transform, and Fourier transform, as well as the theory of the gamma function and allied special functions (see, e.g., [4]). More precisely, the function $F(z)$ defined by the power series is the Mellin transform of the continuous function $f(y)$, $y \in [0, 1]$.

Clearly, many other interesting integrals can be explored with the 'bridges' put forward here in this paper. At last, I leave for the readers a conjecture that I have found to be true for many special cases, but I could not find the suitable conditions, nor a rigorous proof.

Conjecture (Bridge for integrals over $[0, \infty)$)**.** *Let* $f\colon [0,\infty) \to \mathbb{R}$ *be a real function. Under certain conditions (to be determined),*

$$\int_0^{\infty} f(p) \, \ln p \; dp \stackrel{?}{=} - \int_0^1 \! \int_0^1 f(x \cdot y) \, dx \, dy + \int_1^{\infty} \! \int_1^{\infty} f(x \cdot y) \, dx \, dy \, .$$

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