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A DESCRIPTIVE DEFINITION OF THE BACKWARDS ITÔ-HENSTOCK INTEGRAL

Abstract

In this paper, we introduced the backwards derivative of a Hilbert space-valued function and formulate a version of Fundamental Theorem for the backwards Itô-Henstock integral of an operator-valued stochastic process with respect to a Hilbert space-valued Wiener process.

1 Introduction.

In 1950s, a Riemann-type integral was discovered independently by R. Henstock and J. Kurzwiel. This integral includes Riemann and that of Lebesgue. This integral is now known as Henstock-Kurzwiel or HK integral. In this paper, however, we will call this integral simply as Henstock integral. In contrast to Riemann integral, the Henstock integral used non-uniform meshes. This technique is now known as the Henstock approach.

In dealing with random functions such as functions of a Brownian motion (BM), it is impossible to define stochastic integrals using Riemann approach since Brownian motion moves so rapidly and irregularly that almost all of

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its sample paths are nowhere differentiable. The Riemann integral cannot handle highly oscillating functions like that of a Brownian motion, see [18]. For the same reason, it is not even possible to define the stochastic integral as a Riemann-Stieltjes integral, see [10]. In the classical approach to stochastic integration, the stochastic integral of a real-valued adapted process is obtained from the mean square limit of stochastic integrals of simple processes, see [12]. This is approach to stochastic integration which is almost similar in defining the Lebesgue integral of a measurable function. Hence, Henstock approach to stochastic integration have been studied in several papers (see [5], [9], [11], [16], [17], and [18]), since it gives more explicit definition, reduces the technicalities in the classical way of defining the stochastic integral, and is less measure theoretic. In [15], the class of all Henstock-Kurzweil-Itô integrable processes has been characterized by its primitive processes.

In [6], [7], and [13], the concept of stochastic integral has been extended to infinite-dimensional spaces, namely Hilbert and Banach spaces. In a Hilbert space, the stochastic integral is presented in a manner similar to the real-valued case. The integrator is Q-Wiener process, a Hilbert space-valued Wiener process which is dependent on a symmetric nonnegative trace-class operator Q and the integrand is an operator-valued stochastic process. In a general Banach space, however, there seems to be no unifying treatment of stochastic integration.

In [19], a stochastic integral called the *backwards Itô-Henstock integral* is introduced. This integral was defined using the Henstock approach.

In this paper, we formulate a descriptive definition of the backwards Itô-Henstock integral of an operator-valued stochastic process with respect to a Hilbert space-valued Wiener process.

2 Preliminaries.

Throughout this note, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_0^+$ denotes the set of nonnegative real numbers, $\mathbb{N}$ the set of positive integers and $\{\Omega, \mathcal{G}, \mathbb{P}\}$ denotes a probability space.

Definition 1. Let $\{\mathcal{G}_t : 0 \leq t \leq T\}$ be a family of sub σ-field of $\mathcal{G}$. Then $\{\mathcal{G}_t : 0 \leq t \leq T\}$ is called a *backwards filtration* if $\mathcal{G}_t \subseteq \mathcal{G}_s$ for all $0 \leq s \leq t \leq T$. If in addition, $\{\mathcal{G}_t : 0 \leq t \leq T\}$ satisfies the following condition:

(i) $\mathcal{G}_T$ contains all sets of $\mathbb{P}$-measure zero in $\mathcal{G}$; and

(ii) for each $t \in [0, T]$, $\mathcal{G}_t = \mathcal{G}_{t-} := \bigcap_{s<t} \mathcal{G}_s$.

Then $\{\mathcal{G}_t : 0 \leq t \leq T\}$ is called a *standard backwards filtration.*

We often write $\{\mathcal{G}_t\}$ instead of $\{\mathcal{G}_t : 0 \le t \le T\}$. Some terminologies above can be found in [1].

Definition 2. Let H be a separable Banach space. A *stochastic process* f or simply *process* is a function $f : [0,T] \times \Omega \to H$, where $[0,T]$ is an interval in $\mathbb{R}_0^+$ and $f(\cdot,t)$ is $\mathcal{G}_t$-measurable for each $t \in [0,T]$. A process $f = \{f_t : t \in [0,T]\}$ is said to be *backwards adapted* to a standard backwards filtration $\{\mathcal{G}_t\}$ if f_t is $\mathcal{G}_t$-measurable for each $t \in [0,T]$.

Let U and V be separable Hilbert spaces. Denote $L(U,V)$ the space of all bounded linear operators from U to V, $L(U) := L(U,U)$, $Qu := Q(u)$ if $Q \in L(U,V)$, and $L^2(\Omega, V)$ the space of all square-integrable random variables from Ω to V.

Definition 3. An operator $Q \in L(U)$ is said to be *self-adjoint* or *symmetric* if for all $u, u' \in U$, $\langle Qu, u' \rangle_U = \langle u, Qu' \rangle_U$ and is said to be *nonnegative definite* if for every $u \in U$, $\langle Qu, u \rangle_U \ge 0$.

Let $\{e_j\}_{j=1}^{\infty}$, or simply $\{e_j\}$, be an orthonormal basis (abbrev. as ONB) in U. If $Q \in L(U)$ is nonnegative definite, then the trace of Q is defined by tr $Q = \sum_{j=1}^{\infty} \langle Qe_j, e_j \rangle_U$. It is shown in [14] that tr Q is well-defined and may be defined in terms of an arbitrary ONB. Moreover, there exists a unique operator $Q^{\frac{1}{2}} \in L(U)$ such that $Q^{\frac{1}{2}}$ is nonnegative definite and $(Q^{\frac{1}{2}})^2 = Q$.

Definition 4. An operator $Q : U \to U$ is said to be *trace-class* if tr $[Q] :=$ tr $(QQ^*)^{\frac{1}{2}} < \infty$, where Q^* is the adjoint or dual of Q.

If $Q \in L(U)$ is a symmetric nonnegative definite trace-class operator, then there exists an ONB $\{e_j\} \subset U$ and a sequence of nonnegative real numbers $\{\lambda_j\}$ such that $Qe_j = \lambda_j e_j$ for all $j \in \mathbb{N}$, and $\lambda_j \to 0$ as $j \to \infty$ [14, p.203]. We shall call the sequence of pairs $\{\lambda_j, e_j\}$ an *eigensequence defined by* Q. The subspace $U_Q := Q^{\frac{1}{2}}U$ of U equipped with the inner product $\langle u, v \rangle_{U_Q} = \left\langle Q^{-1/2}u, Q^{-1/2}v \right\rangle_U$, where $Q^{1/2}$ is being restricted to $[\mathrm{Ker} Q^{1/2}]^{\perp}$ is a separable Hilbert space with $\left\{ \sqrt{\lambda_j} e_j \right\}$ as its ONB, see [6, p.90], [7, p.23].

Definition 5. Let $\{f_j\}$ be an ONB in U_Q. An operator $S \in L(U_Q, V)$ is said to be *Hilbert-Schmidt* if $\sum_{j=1}^{\infty} \|Sf_j\|_V^2 = \sum_{j=1}^{\infty} \langle Sf_j, Sf_j \rangle_V < \infty$.

Denote by $L_2(U_Q, V)$ the space of all Hilbert-Schmidt operators from U_Q to V, which is known [13, p.112] to be a separable Hilbert space with norm $\|S\|_{L_2(U_Q,V)} = \sqrt{\sum_{j=1}^{\infty} \|Sf_j\|_V^2}$. The Hilbert-Schmidt operator $S \in L_2(U_Q, V)$ and the norm $\|S\|_{L_2(U_Q,V)}$ may be defined in terms of an arbitrary ONB, see [6, p.418], [13, p.111]. It is shown in [7, p.25] that $L(U, V)$ is properly contained in $L_2(U_Q, V)$.

We fix an element $Q \in L(U)$, symmetric nonnegative definite trace-class operator.

Definition 6. A U-valued stochastic process W_t, $t \in [0, T]$, on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is called a *Q-Wiener process in U* if:

1. $W(0, \omega) = 0_U$ for each $\omega \in \Omega$,
2. W has $\mathbb{P}$-almost surely (abbrev. as $\mathbb{P}$-a.s.) continuous trajectories; i.e.,
$$W(\cdot, \omega) : [0, T] \to U \quad \text{is } \mathbb{P}\text{-a.s. continuous}$$
3. the increments of W are independent; i.e. the random variables
$$W_{t_1}, W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \ldots, W_{t_n} - W_{t_{n-1}}$$
are independent for all $0 \leq t_1 < \cdots < t_n \leq T$, $n \in N$, and
4. the increments have the following Gaussian laws:
$$\mathbb{P} \circ (W_t - W_s)^{-1} = \mathcal{N}(0, (t - s)Q) \quad \text{for all } 0 \leq s \leq t \leq T.$$

By Proposition 4.2 (see [6, p.88]), such a Q-Wiener process exists.

Let $W = \{W_t : t \in [0, T]\}$ be a U-valued Q-Wiener process. Define

$$\mathcal{N} := \{A \in \mathcal{G} \mid \mathbb{P}(A) = 0\}, \quad \tilde{\mathcal{G}}_t := \sigma(W_T - W_s \mid t \leq s \leq T), \quad \tilde{\mathcal{G}}_t^0 := \sigma(\tilde{\mathcal{G}}_t \cup \mathcal{N})$$

and $\mathcal{G}_t := \bigcap_{s<t} \tilde{\mathcal{G}}_s^0, \quad t \in [0, T]$. Since $\mathcal{N} \subseteq \tilde{\mathcal{G}}_s^0$ for all $s \in [0, T]$ and $\{\mathcal{G}_t\}_{0 \leq t \leq T}$ is decreasing, then $\mathcal{G}_t$ is a standard backwards filtration. It is shown in [19] that $W_t - W_s$ is independent of $\mathcal{G}_t$ for all $0 \leq s \leq t \leq T$.

From now onwards, the backwards filtered probability $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, \mathbb{P})$ shall mean a filtered probability space such that W_t is adapted to $\mathcal{G}_t$ and $W_t - W_s$ is independent of $\mathcal{G}_t$ for all $0 \leq s \leq t \leq T$.

3 Backwards Itô-Henstock Integral, Backwards Derivative.

In this section, we shall present the backwards Itô-Henstock integral and some related results.

Definition 7. [2] Let δ be a positive function on $(0, T]$.

1. A *finite* collection $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ of interval-point pairs is said to be a *backwards partial division* of $[0, T]$ if $\{(u_i, \xi_i]\}_{i=1}^n$ is a finite collection of disjoint subintervals of $(0, T]$.

2. An interval-point pair $((u, \xi], \xi)$ is said to be *backwards δ-fine* if $(u, \xi] \subseteq (\xi - \delta(\xi), \xi]$, whenever $(u, \xi] \subseteq (0, T]$ and $\xi \in (0, T]$.

3. We call $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ a *backwards δ-fine partial division* of $[0, T]$ if D is a backwards partial division of $[0, T]$ and for each i, the interval-point pair $((u_i, \xi_i], \xi_i)$ is backwards δ-fine.

The term *partial* is used in Definition 7 since the finite collection of disjoint subintervals of $(0, T]$ may not cover the entire $(0, T]$. Using the Vitali covering theorem, the following concept can be defined.

Definition 8. [2] Given $\eta > 0$, a given backwards δ-fine partial division $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ is said to be *backwards (δ, η)-fine partial division* of $[0, T]$ if it fails to cover $(0, T]$ by at most length η, that is,

$$\left| T - (D)\sum_{i=1}^{n} (\xi_i - u_i) \right| \leq \eta.$$

Throughout the following discussions, assume that U and V are separable Hilbert spaces, $Q : U \to U$ is a symmetric nonnegative definite trace-class operator, $\{\lambda_j, e_j\}$ is an eigensequence defined by Q, and W is a U-valued Q-Weiner process. The backwards Itô-Henstock integral is defined as follows.

Definition 9. Let $f : [0, T] \times \Omega \to L(U, V)$ be a backwards adapted process. Then f is said to be *backwards Itô-Henstock integrable*, or $\mathcal{IH}_B$-integrable, on $[0, T]$ with respect to W if there exists $A \in L^2(\Omega, V)$ such that for every $\varepsilon > 0$, there is a positive function δ on $(0, T]$ and a positive number η such that for any backwards (δ, η)-fine partial division $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ of $[0, T]$, we have

$$\mathbb{E}\left[\| S(f, D, \delta, \eta) - A \|_V^2 \right] < \varepsilon$$

where

$$S(f, D, \delta, \eta) := (D)\sum f_\xi (W_\xi - W_u) := \sum_{i=1}^{n} f_{\xi_i} (W_{\xi_i} - W_{u_i}).$$

In this case, f is $\mathcal{IH}_B$-integrable to A on $[0, T]$ and A is called the $\mathcal{IH}_B$-integral of f which will be denoted by $(\mathcal{IH}_B) \int_0^T f_t \, dW_t$ or $(\mathcal{IH}_B) \int_0^T f \, dW$.

It is worth noting that the backwards Itô-Henstock integral possesses the following standard properties of an integral. The proofs of the following results are standard in Henstock-Kurzweil integration, hence omitted.

(1) The backwards Itô-Henstock integral is uniquely determined, in the sense that if A_1 and A_2 are two backwards Itô-Henstock integrals of f in Definition 9, then $\|A_1 - A_2\|_{L^2(\Omega,V)} = 0$.

(2) Let $\alpha \in \mathbb{R}$. If f and g are $\mathcal{IH}_B$-integrable on $[0,T]$, then

(i) $f+g$ is $\mathcal{IH}_B$-integrable on $[0,T]$, and

$$(\mathcal{IH}_B)\int_0^T (f+g)\ dW = (\mathcal{IH}_B)\int_0^T f\ dW + (\mathcal{IH}_B)\int_0^T g\ dW;$$

(ii) αf is $\mathcal{IH}_B$-integrable on $[0,T]$, and

$$(\mathcal{IH}_B)\int_0^T (\alpha f)\ dW = \alpha \cdot (\mathcal{IH}_B)\int_0^T f\ dW.$$

(3) If $f : [0,T]\times\Omega \to L(U,V)$ is $\mathcal{IH}_B$-integrable on $[0,c]$ and $[c,T]$ where $c \in (0,T)$, then f is $\mathcal{IH}_B$-integrable on $[0,T]$ and

$$(\mathcal{IH}_B)\int_0^T f\ dW = (\mathcal{IH}_B)\int_0^c f\ dW + (\mathcal{IH}_B)\int_c^T f\ dW.$$

(4) If $f : [0,T]\times\Omega \to L(U,V)$ is $\mathcal{IH}_B$-integrable on $[0,T]$, then f is also $\mathcal{IH}_B$-integrable on every subinteval $[c,d]$ of $[0,T]$.

(5) (Sequential Definition). A process $f : [0,T]\times\Omega \to L(U,V)$ is $\mathcal{IH}_B$--integrable on $[0,T]$ if and only if there exist $A \in L^2(\Omega,V)$, a decreasing sequence $\{\delta_n\}$ of positive functions defined on $(0,T]$, and a decreasing sequence of positive numbers $\{\eta_n\}$ such that for any backwards (δ_n,η_n)-fine partial division D_n of $[0,T]$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\|S(f,D_n,\delta_n,\eta_n) - A\|_V^2\right] = 0.$$

In this case,

$$A = (\mathcal{IH}_B)\int_0^T f_t\ dW_t.$$

(6) (Cauchy criterion). A process $f : [0,T]\times\Omega \to L(U,V)$ is $\mathcal{IH}_B$-integrable on $[0,T]$ if and only if for every $\varepsilon > 0$, there exist a positive function δ on $(0,T]$ and a positive number η such that for any two backwards (δ,η)-fine partial divisions D and D' of $[0,T]$, we have

$$\mathbb{E}\left[\|S(f,D,\delta,\eta) - S(f,D',\delta,\eta)\|_V^2\right] < \varepsilon.$$

In the paper of Cao [4], he generalized the definition of the Henstock integral for real-valued functions taking values in Banach spaces. He also proved that the Saks-Henstock lemma (strong version) no longer holds for Banach-valued functions, that is, the summation symbol cannot be put outside the norm. He then considered the weak version of the lemma. The next result is a weak version of Saks-Henstock lemma for the $\mathcal{IH}_B$-integral.

Lemma 10 (Weak Version of Saks-Henstock Lemma)**.** *Let f be $\mathcal{IH}_B$-integrable on $[0,T]$ and $F(u,v] := (\mathcal{IH}_B)\int_u^v f_t\, dW_t$ for any $(u,v] \subseteq [0,T]$. Then for every $\varepsilon > 0$, there exist a positive function δ on $(0,T]$ and a positive number η such that for any backwards (δ,η)-fine partial division D of $[0,T]$, we have*

$$\mathbb{E}\left[\left\|(D)\sum\{f_\xi(W_\xi - W_v) - F(v,\xi]\}\right\|_V^2\right] < \varepsilon.$$

Before we proceed with the Itô isometry, we need to consider the backwards Henstock integral defined in [3], which is equivalent to the Lebesgue integral.

Definition 11. [3] A real-valued function f defined on $[0,T]$ is said to be *backwards Henstock integrable* to $A \in \mathbb{R}$ if given $\varepsilon > 0$, there exists a positive function δ on $(0,T]$ and a real constant $\eta > 0$ such that

$$\left|(D)\sum f(\xi)(\xi - v) - A\right| < \varepsilon$$

whenever D is a backwards δ-fine partial division of $[0,T]$ with $(D)\sum(\xi - v) > T - \eta$.

The following result can be proved using the sequential definition of $\mathcal{IH}_B$ and Definition 11.

Lemma 12 (Itô Isometry)**.** *Let f be $\mathcal{IH}_B$-integrable on $[0,T]$. Then $\mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]$ is Lebesgue integrable on $[0,T]$ and*

$$\mathbb{E}\left[\left\|(\mathcal{IH}_B)\int_0^T f_t\, dW_t\right\|_V^2\right] = (\mathcal{L})\int_0^T \mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right] dt.$$

Example 1. *Let U be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_U$, $W : [0,T] \times \Omega \to U$ be a Q-Weiner process, and $\langle W, \cdot \rangle_U : [0,T] \times \Omega \to L(U,\mathbb{R})$ be a process such that $\langle W, \cdot \rangle_U (t,\omega) := \langle W(t,\omega,\cdot) \rangle := \langle W_t, \cdot \rangle_U$ is a bounded linear operator from U to $\mathbb{R}$. Then $\langle W_t, \cdot \rangle_U$ is $\mathcal{IH}_B$-integrable on $[0,T]$ and*

$$(\mathcal{IH}_B) \int_0^T \langle W_t, \cdot \rangle_U \, dW_t = \frac{1}{2}(||W_T||_U^2 + T(trQ)).$$

Next, we define the backwards derivative of a Hilbert space-valued function. Throughout the following, denote by $\mathcal{J}$, the collection of all half-closed intervals $(u,v] \subset [0,T]$. In the following definition, when no confusion arises, we may refer to $F((u,v],\cdot)$ or $F((u,v],\omega)$ as simply $F(u,v]$.

Definition 13. A function $F : \mathcal{J} \times \Omega \to V$ is said to be *backwards differentiable* at $\xi \in (0,T]$ if there exists a random variable $f_\xi : \Omega \to L(U,V)$ such that for every $\varepsilon > 0$, there exists a positive function δ on $(0,T]$ such that for any backwards δ-fine subinterval $(v,\xi]$ of $[0,T]$, we have

$$\mathbb{E}\left[\|f_\xi(W_\xi - W_v) - F(v,\xi]\|_V^2\right] < \varepsilon(\xi - v).$$

The random variable f_ξ is called the *backwards derivative* of F at the point $\xi \in (0,T]$and is denoted by DF_ξ.

The next result shows that the backwards derivative follows the rule of linearity and can be easily verified.

Theorem 14. *Let $\alpha \in \mathbb{R}$. If $F : \mathcal{J} \times \Omega \to V$ and $G : \mathcal{J} \times \Omega \to V$ are backwards differentiable at $\xi \in (0,T]$ with backwards derivatives f_ξ and g_ξ, respectively, then*

(i) $F + G$ is backwards differentiable at $\xi \in (0,T]$ with backwards derivative $f_\xi + g_\xi$, and

(ii) αF is backwards differentiable at $\xi \in (0,T]$ with backwards derivative αf_ξ.

Definition 15. A function $F : \mathcal{J} \times \Omega \to V$

(i) is said to be $AC^2[0,T]$ if for every $\varepsilon > 0$, there exists $\eta > 0$ such that for any finite collection $D = \{(v,\xi]\}$ of disjoint subintervals $(v,\xi] \in \mathcal{J}$ with $(D)\sum(\xi - v) < \eta$, we have $\mathbb{E}\left[\|(D)\sum F(v,\xi]\|_V^2\right] < \varepsilon$.;

(ii) has the *orthogonal increment property* if for all disjoint intervals $(a,b]$, $(u,v] \subset [0,T]$, $\mathbb{E}\left[\langle F(a,b], F(u,v]\rangle_V\right] = 0$.

Remark 1. *Let* $\{(v_i, \xi_i]\}$ *be a collection of disjoint subintervals of* $[0, T]$*. If* $F : \mathcal{J} \times \Omega \to V$ *has orthogonal increment property, then*

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n} F(v_i, \xi_i]\right\|_V^2\right] = \sum_{i=1}^{n} \mathbb{E}\left[||F(v_i, \xi_i]||_V^2\right].$$

Before proving Theorem 18, we shall consider first the following results.

Proposition 16. [13] *Let* $(E_1, \mathcal{E}_1)$ *and* $(E_2, \mathcal{E}_2)$ *be two measurable spaces and* $\Psi : E_1 \times E_2 \to \mathbb{R}$ *be a bounded measurable function. Let* X_1 *and* X_2 *be two random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* $(E_1, \mathcal{E}_1)$ *and* $(E_2, \mathcal{E}_2)$*, respectively, and let* $\mathcal{G} \subseteq \mathcal{F}$ *be a fixed* σ*-field. Assume that* X_1 *is* $\mathcal{G}$*-measurable and* X_2 *is independent of* $\mathcal{G}$*. Then*

$$\mathbb{E}\left[\Psi(X_1, X_2)\,|\mathcal{G}\right] = \widehat{\Psi}(X_1),$$

where $\widehat{\Psi}(x_1) = \mathbb{E}[\Psi(x_1, X_2)]$*,* $x_1 \in E_1$*.*

In the following lemma, one may refer to [8, Lemma 3.5 and Lemma 3.6] for analogous proof. For completeness, we shall present its proof.

Lemma 17. *Let* $f : [0, T] \times \Omega \to L(U, V)$ *be a backwards adapted process and* $\{(v_i, \xi_i]\}_{i=1}^{n}$ *be a finite collection of disjoint subintervals of* $[0, T]$*. Then*

$$(i)\ \ \mathbb{E}\left[\sum_{i<j} \left\langle f_{\xi_i}(W_{\xi_i} - W_{v_i},), f_{\xi_j}(W_{\xi_j} - W_{v_j})\right\rangle_V\right] = 0;$$

$$(ii)\ \ \mathbb{E}\left[\left\|\sum_{i=1}^{n} f_{\xi_i}(W_{\xi_i} - W_{v_i})\right\|_V^2\right] = \sum_{i=1}^{n} (\xi_i - v_i)\mathbb{E}\left[\|f_{\xi_i}\|_{L_2(U_Q, V)}^2\right].$$

Proof. (i) It is enough to show that

$$\mathbb{E}\left[\left\langle f_{\xi_i}(W_{\xi_i} - W_{v_i}), f_{\xi_j}(W_{\xi_j} - W_{v_j})\right\rangle_V\right] = 0 \quad \text{for } i < j.$$

Since $W_{\xi_i} - W_{v_i}$ is independent of $\mathcal{G}_{\xi_i}$ and $f_{\xi_i}^* f_{\xi_j}(W_{\xi_j} - W_{v_j})$ is $\mathcal{G}_{\xi_i}$-measurable, then by Proposition 16,

$$\begin{aligned}&\mathbb{E}\left[\left\langle W_{\xi_i} - W_{v_i}, f_{\xi_i}^* f_{\xi_j}(W_{\xi_j} - W_{v_j}\right\rangle_U \middle| \mathcal{G}_{\xi_i}\right](\omega) \\ &\quad = \mathbb{E}\left[\left\langle W_{\xi_i} - W_{v_i}, f_{\xi_i}^*(\omega) f_{\xi_j}(\omega)(W_{\xi_j}(\omega) - W_{v_j}(\omega)\right\rangle_U\right] = 0\end{aligned}$$

since $\mathbb{E}\left[\langle W_t - W_s, u\rangle_U\right] = 0$ for all $u \in U$. Thus,

$$\mathbb{E}\left[\left\langle f_{\xi_i}(W_{\xi_i} - W_{v_i}), f_{\xi_j}(W_{\xi_j} - W_{v_j})\right\rangle_V\right] = 0.$$

(ii) By (i),

$$\begin{aligned}\mathbb{E}\left[\left\|\sum_{i=1}^{n} f_{\xi_i}(W_{\xi_i} - W_{v_i})\right\|_V^2\right] &= \mathbb{E}\left[\sum_{i=1}^{n} \langle f_{\xi_i}(W_{\xi_i} - W_{v_i}), f_{\xi_i}(W_{\xi_i} - W_{v_i})\rangle_V\right. \\ &\qquad \left. +2\sum_{i<j} \left\langle f_{\xi_i}(W_{\xi_i} - W_{v_i}), f_{\xi_j}(W_{\xi_j} - W_{v_j})\right\rangle_V\right] \\ &= \sum_{i=1}^{n} \mathbb{E}\left[\|f_{\xi_i}(W_{\xi_i} - W_{v_i})\|_V^2\right].\end{aligned}$$

Let $S_m = \sum_{l=1}^{m} \langle f_{\xi_i}(W_{\xi_i} - W_{v_i}), b_l\rangle_V^2$, where $\{b_l\}$ is an ONB in V. Note that

$$S_m \longrightarrow \sum_{l=1}^{\infty} \langle f_{\xi_i}(W_{\xi_i} - W_{v_i}), b_l\rangle_V^2 := S \quad \text{as } m \to \infty$$

and $S_m(\omega) \leq S_{m+1}(\omega)$, for all $m \in \mathbb{N}$. By the monotone convergence theorem for Lebesgue integral, we have

$$\int_\Omega S(\omega)\ d\mathbb{P} = \lim_{m\to\infty} \int_\Omega S_m(\omega)\ d\mathbb{P}$$

so that

$$\begin{aligned}\mathbb{E}\left[\sum_{l=1}^{\infty} \langle f_{\xi_i}(W_{\xi_i} - W_{v_i}), b_l\rangle_V^2\right] &= \lim_{m\to\infty} \mathbb{E}\left[\sum_{l=1}^{m} \langle f_{\xi_i}(W_{\xi_i} - W_{v_i}), b_l\rangle_V^2\right] \\ &= \sum_{l=1}^{\infty} \mathbb{E}\left[\mathbb{E}\left[\left\langle W_{\xi_i} - W_{v_i}, f_{\xi_i}^* b_l\right\rangle_U^2 \middle| \mathcal{G}_{\xi_i}\right]\right].\end{aligned}$$

Using Proposition 16,

$$\mathbb{E}\left[\left\langle W_{\xi_i} - W_{v_i}, f_{\xi_i}^* b_l\right\rangle_U^2 \middle| \mathcal{G}_{\xi_i}\right](\omega) = \mathbb{E}\left[\langle W_{\xi_i} - W_{v_i}, f_{\xi_i}(\omega)^* b_l\rangle_U^2\right].$$

Since $\mathbb{E}\left[\langle W_t - W_s, u\rangle_U^2\right] = (t-s)\langle Qu, u\rangle$ for all $u \in U$, we obtain

$$\mathbb{E}\left[\langle W_{\xi_i} - W_{v_i}, f_{\xi_i}(\omega)^* b_l\rangle_U^2\right] = (\xi_i - v_i)\langle Qf_{\xi_i}(\omega)^* b_l, f_{\xi_i}(\omega)^* b_l\rangle_U.$$

It follows that

$$\mathbb{E}\left[\sum_{l=1}^{\infty}\langle f_{\xi_i}(W_{\xi_i}-W_{v_i}),b_l\rangle_U^2\right]=\sum_{l=1}^{\infty}(\xi_i-v_i)\mathbb{E}\left[\left\langle Qf_{\xi_i}^*b_l,f_{\xi_i}^*b_l\right\rangle_U\right]. \tag{1}$$

Let $\{\lambda_j,e_j\}$ be an eigensequence defined by Q. Then

$$\mathbb{E}\left[\left\langle Qf_{\xi_i}^*b_l,f_{\xi_i}^*b_l\right\rangle_U\right]=\mathbb{E}\left[\sum_{j=1}^{\infty}\lambda_j\left\langle f_{\xi_i}^*b_l,e_j\right\rangle_U^2\right]=\mathbb{E}\left[\sum_{j=1}^{\infty}\left\langle f_{\xi_i}\left(\sqrt{\lambda_j}e_j\right),b_l\right\rangle_U^2\right]. \tag{2}$$

Thus, using (1) and (2), we have

$$\begin{aligned}&\mathbb{E}\left[\sum_{i=1}^{n}\langle f_{\xi_i}(W_{\xi_i}-W_{v_i}),f_{\xi_i}(W_{\xi_i}-W_{v_i})\rangle_V\right]\\&=\sum_{i=1}^{n}\sum_{l=1}^{\infty}(\xi_i-v_i)\mathbb{E}\left[\sum_{j=1}^{\infty}\left\langle f_{\xi_i}\left(\sqrt{\lambda_j}e_j\right),b_l\right\rangle_V^2\right]\\&=\sum_{i=1}^{n}(\xi_i-v_i)\mathbb{E}\left[\sum_{j=1}^{\infty}\left\|f_{\xi_i}\left(\sqrt{\lambda_j}e_j\right)\right\|_V^2\right]\\&=\sum_{i=1}^{n}(\xi_i-v_i)\mathbb{E}\left[\|f_{\xi_i}\|_{L_2(U_Q,V)}^2\right],\end{aligned}$$

which completes the proof. □

Theorem 18. *Let f be $\mathcal{IH}_B$-integrable on $[0,T]$ and define*

$$F(v,\xi]:=(\mathcal{IH}_B)\int_v^{\xi}f_t\,dW_t$$

for all $(v,\xi]\in\mathcal{J}$. Then F is $AC^2[0,T]$ and has the orthogonal increment property.

Proof. F is $AC^2[0,T]$ follows from [19, Theorem 4]. Next, we show that F has the orthogonal increment property. Let $(a,b]$ and $(u,v]$ be disjoint intervals in $[0,T]$. From the sequential definition of $\mathcal{IH}_B$ integral, there exist a decreasing sequence $\{\delta_n\}$ of positive functions defined on $(0,T]$ and a decreasing sequence $\{\eta_n\}$ of positive numbers such that for any backwards

(δ_n,η_n)-fine partial divisions $D_n[a,b]=\{((u_i^{(n)},\xi_i^{(n)}],\xi_i^{(n)})\}_{i=1}^m$ and $D_n[u,v]=\{((u_j^{(n)},\xi_j^{(n)}],\xi_j^{(n)})\}_{j=1}^p$ of $[a,b]$ and $[u,v]$ respectively, we have

$$\mathbb{E}\left[\|S(f,D_n[a,b],\delta_n,\eta_n)-F(a,b]\|\right]\to 0 \text{ as } n\to\infty$$

and

$$\mathbb{E}\left[\|S(f,D_n[u,v],\delta_n,\eta_n)-F(u,v]\|\right]\to 0 \text{ as } n\to\infty.$$

In view of Lemma 17 (i), for all $n\in\mathbb{N}$,

$$\mathbb{E}\left[\sum_{i=1}^m\sum_{j=1}^p\left\langle f_{\xi_i^{(n)}}(W_{\xi_i^{(n)}}-W_{v_i^{(n)}},),f_{\xi_j^{(n)}}(W_{\xi_j^{(n)}}-W_{v_j^{(n)}})\right\rangle_V\right]=0.$$

Since $\mathbb{E}\left[\langle S(f,D_n[a,b],\delta_n,\eta_n),S(f,D_n[u,v],\delta_n,\eta_n)\rangle_V\right]\to\mathbb{E}\left[\langle F(a,b],F(u,v]\rangle_V\right]$ as $n\to\infty$, it follows that $\mathbb{E}\left[\langle F(a,b],F(u,v]\rangle_V\right]=0$. □

Theorem 19. [19, Theorem 5] *Let $f:[0,T]\times\Omega\to L(U,V)$ be a backwards process. Then f is $\mathcal{IH}_B$-integrable on $[0,T]$ if and only if there exists an $AC^2[0,T]$ function F such that for every $\varepsilon>0$, there exist a positive function δ on $(0,T]$ such that whenever $D=\{((v,\xi],\xi)\}$ is a backwards δ-fine partial division of $[0,T]$, we have*

$$\mathbb{E}\left[\left\|(D)\sum\{f_\xi(W_\xi-W_v)-F(v,\xi]\}\right\|_V^2\right]<\varepsilon.$$

Lemma 20. *Let $f:[0,T]\times\Omega\to L(U,V)$ be a backwards adapted process and let $F:\mathcal{J}\times\Omega\to V$. Then for any disjoint subintervals $(a,b]$ and $(c,d]$ of $[0,T]$, we have*

$$\mathbb{E}\left[\langle F(a,b],f_d(W_d-W_c)\rangle_V\right]=0.$$

Proof. Since $f_d^*(F(a,b])$ is $\mathcal{G}_d$-measurable and W_d-W_c is independent of $\mathcal{G}_d$, then by Proposition 16, for each $\omega\in\Omega$,

$$\mathbb{E}\left[\langle f_d^*(F(a,b]),W_d-W_c\rangle_U|\,\mathcal{G}_d\right](\omega)=\mathbb{E}\left[\langle f_d^*(\omega)(F(a,b],(\omega)),W_d-W_c\rangle_U\right].$$

Since $\mathbb{E}[\langle W_t-W_s,u\rangle_U]=0$ for all $0\le s<t\le T$ and for all $u\in U$,

$$\mathbb{E}\left[\langle f_d^*(\omega)(F(a,b],(\omega)),W_d-W_c\rangle_U\right]=0.$$

Hence, for each $\omega\in\Omega$, $\mathbb{E}\left[\left\langle f_d^*(F(a,b]),W_d-W_c\right\rangle_U\middle|\,\mathcal{G}_d\right](\omega)=0$. This implies that $\mathbb{E}\left[\mathbb{E}\left[\left\langle f_d^*(F(a,b]),W_d-W_c\right\rangle_U\middle|\,\mathcal{G}_d\right]\right]=0$. Thus,

$$\mathbb{E}\left[\langle f_d^*(F(a,b]),W_d-W_c\rangle_U\right]=0.$$

It follows that $\mathbb{E}\left[\langle F(a,b],f_d(W_d-W_c)\rangle_V\right]=0$. □

In view of Lemma 17 and Lemma 20, we have the following result.

Lemma 21. *Let $f : [0,T] \times \Omega \to L(U,V)$ be a backwards adapted process, $F : \mathcal{J} \times \Omega \to V$ with orthogonal property, and $\{(v_i, \xi_i]\}_{i=1}^n$ be a finite collection of disjoint subintervals of $[0,T]$. Then*

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n}\{f_{\xi_i}(W_{\xi_i} - W_{v_i}) - F(v_i,\xi_i]\}\right\|_V^2\right] = \sum_{i=1}^{n}\mathbb{E}\left[\|f_{\xi_i}(W_{\xi_i} - W_{v_i}) - F(v_i,\xi_i]\|_V^2\right].$$

By Lemma 10, Theorem 18, and Lemma 21, we have the strong version of Saks-Henstock Lemma as follows.

Lemma 22 (Strong Version of Saks-Henstock Lemma)**.** *Let f be $\mathcal{IH}_B$-integrable on $[0,T]$ and $F(u,v] := (\mathcal{IH}_B)\int_u^v f_t\, dW_t$ for any $(u,v] \subset [0,T]$. Then for every $\varepsilon > 0$, there exists a positive function δ on $(0,T]$ such that for any backwards δ-fine partial division $D = \{(v,\xi],\xi\}$ of $[0,T]$, we have*

$$(D)\sum \mathbb{E}\left[\|f_\xi(W_\xi - W_v) - F(u,v]\|_V^2\right] < \varepsilon.$$

4 Descriptive Definition of Backwards Itô-Henstock Integral.

We shall now prove the main result of this paper. Here we will show that an antiderivative of a process f is the backwards Itô-Henstock integral of f under some specific conditions. Recall that F is an *antiderivative* of f if $DF = f$, *a.e.* In the proofs, denote by μ^* and μ, the Lebesgue outer measure and Lebesgue measure, respectively.

Theorem 23. *Let $f : [0,T] \times \Omega \to L(U,V)$ be $\mathcal{IH}_B$-integrable on $[0,T]$ with $F(u,v] = (\mathcal{IH}_B)\int_u^v f_t\, dW_t$ for all $(u,v] \subset [0,T]$. Then*

(i) F is $AC^2[0,T]$ and has orthogonal increment property; and

(ii) $DF_\xi = f_\xi$ a.e. on $(0,T]$.

Proof. We note that (i) follows directly from Theorem 18. We are left to show that $DF_\xi = f_\xi$ *a.e.* on $(0,T]$. Let

$$S = \{s \in (0,T] : DF_s \text{ does not exist or } DF_s \neq f_s\}.$$

We will show that the Lebesgue measure of S, $\mu(S)$, is zero. Let $\xi \in S$. Then there exists $\gamma(\xi) > 0$ such that for every positive function δ on $(0,T]$, there exists a backwards δ-fine subinterval $(v,\xi] \subset [0,T]$ with

$$\mathbb{E}\left[\|f_\xi(W_\xi - W_v) - F(u,\xi]\|_V^2\right] \geq \gamma(\xi)(\xi - v). \tag{3}$$

For each $k \in \mathbb{N}$, let $S_k = \{s \in S : \gamma(s) \geq \frac{1}{k}\}$. Then $S = \cup_{k\in\mathbb{N}} S_k$. Let Γ be the collection of point-interval pairs $((v,\xi],\xi)$ such that $\xi \in S_k$ and $(v,\xi]$ is a backwards δ-fine subinterval of $[0,T]$ that satisfies (3). Then for every $\xi \in S_k$ and any $\varepsilon_1 > 0$ (constant function δ), there exists a point-interval pair $((v,\xi],\xi)$ in Γ such that $\xi - v < \varepsilon_1$. This means that Γ covers S_k in the sense of Vitali. By the strong version of Saks-Henstock Lemma and (3), for each $\varepsilon > 0$, there exists a positive function δ_1 on $(0,T]$ such that for any backwards δ_1-fine partial division $D_1 = \{((v_i,\xi_i],\xi_i)\}_{i=1}^n \subset \Gamma$ of $[0,T]$, we have

$$\frac{1}{k}\sum_{i=1}^n (\xi_i - v_i) \leq (D_1)\sum_{i=1}^n \mathbb{E}\left[\|f_{\xi_i}(W_{\xi_i} - W_{v_i}) - F(v_i,\xi_i]\|_V^2\right] < \frac{\varepsilon}{2k}.$$

Therefore

$$\sum_{i=1}^n (\xi_i - v_i) < \frac{\varepsilon}{2}.$$

Using the Vitali covering lemma, we can find a partial division

$$D = \{((v,\xi],\xi)\} \subset \Gamma \quad \text{such that} \quad \mu^*(S_k) < (D)\sum(\xi - v) + \frac{\varepsilon}{2} < \varepsilon.$$

Since ε is arbitrary, $\mu^*(S_k) = 0$. Thus, $\mu^*(S) = 0$, since S is the countable union of S_k. Hence, $\mu(S) = 0$. □

The following result is the converse of the above theorem.

Theorem 24. *Let $f : [0,T] \times \Omega \to L(U,V)$ be a backwards adapted process on $[0,T]$. Suppose that*

(i) $F : \mathcal{J} \times \Omega \to V$ be $AC^2[0,T]$ with orthogonal increment property, and

(ii) $DF_\xi = f_\xi$ a.e. on $(0,T]$.

Then f is $\mathcal{IH}_B$-integrable on $[0,T]$ with $F(v,\xi] = (\mathcal{IH}_B)\int_v^\xi f_t\,dW_t$.

Proof. Let $S = \{s \in (0,T] : DF_s \text{ does not exists or } DF_s \neq f_s\}$. Then $\mu(S) = 0$. Let $\xi \in S^c = [0,T]\backslash S$. Then for every $\varepsilon > 0$, there exists a positive function δ_1 on $(0,T]$ such that for any backwards δ_1-fine subinterval $(v,\xi] \subset [0,T]$, we have

$$\mathbb{E}\left[\|f_\xi(W_\xi - W_v) - F(v,\xi]\|_V^2\right] < \frac{\varepsilon(\xi - v)}{4T}.$$

Let $D_1 = \{((v_i,\xi_i],\xi_i)\}_{i=1}^n$ be a backwards δ_1-fine partial division on $[0,T]$ with $\xi_i \in S^c$. Then by Lemma 21

$$\begin{aligned}\mathbb{E}&\left[\left\|\sum_{i=1}^n \{f_{\xi_i}(W_{\xi_i} - W_{v_i}) - F(v_i,\xi_i]\}\right\|_V^2\right] \\ &= \sum_{i=1}^n \mathbb{E}\left[\|f_{\xi_i}(W_{\xi_i} - W_{v_i}) - F(v_i,\xi_i]\|_V^2\right] \\ &< \frac{\varepsilon}{4T}\sum_{i=1}^n (\xi_i - v_i) \le \frac{\varepsilon}{4}.\end{aligned}$$

If $S = \varnothing$, then we are done. Suppose that $S \neq \varnothing$. Let $\xi \in S$. Note that for $v < \xi$,

$$\mathbb{E}\left[||f_\xi(W_\xi - W_v)||_V^2\right] = (\xi - v)\mathbb{E}\left[\|f_\xi\|_{L_2(U_Q,V)}^2\right].$$

Let $G_m = \sum_{j=1}^m \langle f_\xi(W_\xi - W_v), g_j\rangle_V^2$, where $\{g_j\}$ is an ONB in V. Since

$$G_m \to G := \sum_{j=1}^\infty \langle f_\xi(W_\xi - W_v), g_j\rangle_V^2 \quad \text{as} \quad m \to \infty$$

and $G_m \le G_{m+1}$, by the monotone convergence theorem for Lebesgue integral, we have

$$\begin{aligned}\lim_{m\to\infty} \mathbb{E}\left[\sum_{j=1}^m \langle f_\xi(W_\xi - W_v), g_j\rangle_V^2\right] &= \mathbb{E}\left[\sum_{j=1}^\infty \langle f_\xi(W_\xi - W_v), g_j\rangle_V^2\right] \\ &= \mathbb{E}\left[||f_\xi(W_\xi - W_v)||_V^2\right] < \infty.\end{aligned}$$

This implies that there exists $N \in \mathbb{N}$ such that $N-1 \le \mathbb{E}\left[\|f_\xi\|_{L_2(U_Q,V)}^2\right] < N$. Since F is $AC^2[0,T]$, there exists $\eta > 0$ with $\eta \le \frac{\varepsilon}{N\cdot 2^4}$ such that for any finite collection of disjoint subintervals $\{(v,\xi]\}$ of $[0,T]$ with $\sum(\xi - v) < \eta$, we have

$$\mathbb{E}\left[\left\|\sum F(v,\xi]\right\|_V^2\right] < \frac{\varepsilon}{2^4}.$$

Since S is a set of Lebesgue measure zero, there exists an open set $O \supset S$ such that $\mu(O) < \eta$. Now, we define a function δ_2 on S as follows: let $\xi \in S$ and define $\delta_2(\xi) > 0$ such that whenever $((v,\xi],\xi)$ is a backwards δ_2-fine with $\xi \in S$, we have $(v,\xi] \subset O$. Then by Lemma 17 (ii) and Remark 1, for every backwards δ_2-fine partial division $D_2 = \{((v,\xi],\xi)\}$ of $[0,T]$ with $\xi \in S$, we have

$$\begin{aligned}
\mathbb{E}&\left[\left\|(D_2)\sum\{f_\xi(W_\xi - W_v) - F(v,\xi]\}\right\|_V^2\right] \\
&\le 2\mathbb{E}\left[\left\|(D_2)\sum f_\xi(W_\xi - W_v)\right\|_V^2\right] + 2\mathbb{E}\left[\left\|(D_2)\sum F(v,\xi]\right\|_V^2\right] \\
&= 2(D_2)\sum(\xi - v)\mathbb{E}\left[\|f_\xi\|_{L_2(U_Q,V)}^2\right] + 2(D_2)\sum \mathbb{E}\left[\|F(v,\xi]\|_V^2\right] \\
&< 2N \cdot \frac{\varepsilon}{N \cdot 2^4} + 2 \cdot \frac{\varepsilon}{2^4} = \frac{\varepsilon}{4}.
\end{aligned}$$

If $\xi \in S$, choose a positive function δ defined on S such that $\delta(\xi) = \delta_2(\xi)$ and if $\xi \notin S$, choose $\delta(\xi) = \delta_1(\xi)$. Let $D = \{((v,\xi],\xi)\}$ be a backwards δ-fine partial division of $[0,T]$. Then

$$\begin{aligned}
\mathbb{E}&\left[\left\|(D)\sum\{f_\xi(W_\xi - W_v) - F(v,\xi]\}\right\|_V^2\right] \\
&\le 2\mathbb{E}\left[\left\|\sum_{\xi \in S^c}\{f_\xi(W_\xi - W_v) - F(v,\xi]\}\right\|_V^2\right] \\
&\qquad + 2\mathbb{E}\left[\left\|\sum_{\xi \in S}\{f_\xi(W_\xi - W_v) - F(v,\xi]\}\right\|_V^2\right] \\
&< 2\left(\frac{\varepsilon}{4}\right) + 2\left(\frac{\varepsilon}{4}\right) = \varepsilon.
\end{aligned}$$

By Theorem 19, f is $\mathcal{IH}_B$-integrable on $[0,T]$. □

Combining Theorem 18, Theorem 23, and Theorem 24, we get the following result, which is referred to as the descriptive definition of the backwards Itô-Henstock integral.

Theorem 25. *Let $f : [0,T] \times \Omega \to L(U,V)$ be a backwards adapted process on $[0,T]$. Then f is $\mathcal{IH}_B$-integrable on $[0,T]$ if and only if there exists an $AC^2[0,T]$ function $F : \mathcal{J} \times \Omega \to V$ with orthogonal increment property and $DF_\xi = f_\xi$ a.e. on $(0,T]$.*

5 Conclusion and Recommendations.

In this paper, we formulate a version of Fundamental Theorem for the backwards Itô-Henstock integral of an operator-valued stochastic process with respect to a Hilbert space-valued Q-Wiener process. We use the notion of backwards derivative and $AC^2[0, T]$-property, a version of absolute continuity, to attain this objective. A worthwhile direction for further investigation is to formulate an equivalent definition of this type of integral using double Lusin condition and $AC^2[0, T]$-property.

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References

[1] D. Applebaum, *Lévy processes and stochastic calculus*, Cambridge University Press, New York, 2004.

[2] J. Arcede and E. Cabral, *An equivalent definition for the backwards Itô integral*, Thai Journal of Mathematics, **9** (2011), 619–630.

[3] J. Arcede and E. Cabral, *Backwards Henstock integral*, Matimyas Matematika, Journal of the Mathematical Society of the Philippines, **32** (2009), 1–12.

[4] S. Cao, *The Henstock integral for Banach-valued functions*, Southeast Asian Bull. Math., **16** (1992), 35–40.

[5] T. S. Chew T. L. Toh and J. Y. Tay, *The non-uniform Riemann approach to Itô's integral*, Real Anal. Exchange, **27** (2002-2003), 495–514.

[6] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[7] L. Gawarecki and V. Mandrekar, *Stochastic Differential Equations in Infinite Dimensions with Applications to Stochastic Partial Differential Equations*, Springer, Berlin, 2011.

[8] M. Labendia, E. De Lara-Tuprio and T. R. Teng, *Itô-Henstock integral and Itô's formula for the operator-valued stochastic process*, Mathematica Bohemica, **143** (2018), 135–160.

[9] E. J. McShane, *Stochastic Integrals and Stochastic Functional Equations*, SIAM J. Appl. Math., **17** (1969), 287–306.

[10] T. Mikosch, *Elementary Stochastic Calculus with Finance in view*, World Scientific, Singapore, 1998.

[11] Z. R. Pop-Stojanovic, *On McShane's belated stochastic integral*, SIAM J. Appl. Math., **22** (1972), 87–92.

[12] B. Øksendal, *Stochastic Differential Equations (5th edition)*, Springer-Verlag, New York, 1998.

[13] C. Prévôt and M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Springer, Berlin, 2007.

[14] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, London, 1980.

[15] T. L. Toh and T. S. Chew, *On belated differentiation and a characterization of Henstock-Kurzweil-Itô integrable processes*, Mathematica Bohemica, **130** (2005), 63–72.

[16] T. L. Toh and T. S. Chew, *On Itô-Kurzweil-Henstock integral and integration-by-part formula*, Czechoslovak Mathematical Journal, **55** (2005), 653–663.

[17] T. L. Toh and T. S. Chew, *On the Henstock-Fubini theorem for multiple stochastic integrals*, Real Anal. Exchange, **30** (2004-2005), 295–310.

[18] T. L. Toh and T. S. Chew, *The Riemann approach to stochastic integration using non-uniform meshes*, J. Math. Anal. Appl., **280** (2003), 133–147.

[19] R. F. Rulete and M. A. Labendia, *Backwards Itô-Henstock integral for the Hilbert-Schmidt-valued stochastic process*, European Journal of Pure and Applied Mathematics, **12** (2019), 58–78.