INROADS

Real Analysis Exchange
Vol. 44(2), 2019, pp. 403–426
DOI: 10.14321/realanalexch.41.1.0403

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VERIFYING DIFFERENTIABILITY WITHOUT CALCULATING THE DERIVATIVE

Abstract

We study various real-variable techniques for determining whether a function is differentiable without actually calculating a derivative. These include:

- approximation theory
- Fourier analysis
- Sobolev spaces
- Poisson integral
- finite differences
- Campanato-Morrey theory
- Landau's inequalities

In most cases complete proofs are given.

1 Introduction

The derivative is one of the oldest ideas in modern analysis. It is natural, if one wants to determine whether a function is differentiable, to endeavor to calculate the desired derivative. But there are many contexts in which

Mathematical Reviews subject classification: Primary: 26A24
Key words: Fourier transform, Fourier series, Poisson integral, Campanato-Morrey theory, Laplace transform, Taylor expansion, derivative, finite differences, approximation theory
Received by the editors January 21, 2019
Communicated by: Brian S. Thomson

this is not feasible, and it is desirable to find other means to determine the differentiability of a function. In this paper we explore these other means.

As instances of these ideas, the best way to characterize the duals of the Hardy spaces is by way of the Campanato-Morrey spaces, which we treat below. One of the most convenient ways to study the regularity for the Laplacian is by way of finite differences, which we treat below. A useful way to study joint smoothness of functions of several variables is by way of approximation theory, which we treat below.

The good news is that most of the alternative techniques presented here are rather accessible. They may be understood by a relative neophyte in analysis. The even better news is that some of these techniques are profound and far-reaching.

In this paper we shall, for simplicity, usually restrict attention to $\mathbb{R}^1$, Euclidean 1-dimensional space. At the expense of some notation, it is not difficult to extend the results to all dimensions. We shall also concentrate on the existence of the first derivative. Higher derivatives are handled by similar but trickier arguments.

2 The Calculus of Finite Differences

Let f be a function on $\mathbb{R}$. The classical first difference operator $\triangle f$ is defined by

$$\triangle_h f(x) \equiv f(x+h) - f(x-h)\,.$$

We sometimes also denote this operation by $\triangle_h^1 f(x)$. The second difference operator is

$$\triangle_h^2 f(x) \equiv \triangle_h \circ \triangle_h f(x) = f(x+2h) + f(x-2h) - 2f(x)\,.$$

Most mathematicians will encounter these difference operators in the context of Lipschitz spaces. Let $0 < \alpha \leq 1$. All the functions f that we treat here are maps from the reals to the reals. Then the classical Lipschitz space of order α is given by

$$\mathrm{Lip}_\alpha = \left\{ f : \sup_{h\neq 0, x} | \triangle_h^1 f(x)|/|h|^\alpha + \|f\|_{\sup} \equiv \|f\|_{\mathrm{Lip}_\alpha} < \infty \right\}$$

Note that, if $f \in \mathrm{Lip}_\alpha$, then $| \triangle_h^2 f(x)| \leq C|h|^\alpha$ also.

For $1 < \alpha < 2$ we let

$$\mathrm{Lip}_\alpha = \left\{ f \in C^1 : \sup_{h \neq 0, x} | \triangle_h^1 f'(x)|/|h|^{\alpha-1} + \|f'\|_{\sup} + \|f\|_{\sup} \equiv \|f\|_{\mathrm{Lip}_\alpha} < \infty \right\}$$

In both these definitions, C is a constant independent of x and h. Also $'$ denotes the derivative.

For reasons that will be made clear as the paper develops, many of them arising from harmonic analysis, it is useful to replace the space Lip_1 by the Zygmund space Λ_1 defined by

$$\Lambda_1 = \left\{ f : \sup_{h \neq 0, x} | \triangle_h^2 f(x)|/|h| + \|f\|_{\sup} \equiv \|f\|_{\mathrm{Lip}_1} < \infty \right\}$$

In practice we use the notation Λ_α to denote the traditional Lipschitz spaces when $0 < \alpha < 2$, $\alpha \neq 1$ and to denote the Zygmund space when $\alpha = 1$.

3 Approximation Theory

Let $\varphi \in C_c^\infty(\mathbb{R})$ satisfy these properties:

- $\operatorname{supp} \varphi \subseteq [-1, 1]$;
- $0 \leq \varphi \leq 1$;
- φ is even;
- $\int \varphi \, dx = 1$.

Set $\varphi_j(x) = 2^j \cdot \varphi(2^j x)$ and $\psi_j(x) = \varphi_{j+1}(x) - \varphi_j(x)$ for $j = 1, 2, \ldots$. Let $f : \mathbb{R} \to \mathbb{R}$ be a locally integrable function. Now let us define

$$f_j(x) = f * \psi_j(x) \quad \text{for } j = 1, 2, \ldots$$

and

$$f_0(x) = f * \varphi_1(x) \, .$$

If f is continuous, then it follows from standard results (see [15], [17], [7]) that

$$f(x) = \sum_{j=0}^{\infty} f_j(x) \, ,$$

with convergence uniform on compact sets.

In fact it is quite standard to approximate real functions in this fashion (see [15], [17]). Such approximations are useful in harmonic analysis and the theory of partial differential equations. They also come up naturally in the modern theory of wavelets (see [7] or [9] for a quick introduction).

Our first significant result of the paper is the following.

Theorem 1. *Let $f \in \Lambda_\alpha$, $0 < \alpha < 2$. Then*

$$|f_j(x)| \leq C 2^{-j\alpha} \quad \text{for all } j\,.$$

Conversely, if

$$|f_j(x)| \leq C 2^{-j\alpha} \quad \text{for all } j\,, \tag{3.1.1}$$

then $f \in \Lambda_\alpha$.

In particular, we see for $\alpha > 1$ that the condition (3.1.1) is sufficient for f to be continuously differentiable. And that condition is defined in terms of an integral—*not* a derivative.

We begin now with a lemma.

Lemma 2. *Let $1 < \alpha < 2$. If $f \in \Lambda_\alpha$, then*

$$|f(x+h) + f(x-h) - 2f(x)| \leq C|h|^\alpha$$

for all $x, h \in \mathbb{R}$.

PROOF. We write

$$\begin{aligned} |f(x+h) + f(x-h) - 2f(x)| &= |[f(x+h) - f(x)] - [f(x) - f(x-h)]| \\ &= |f'(x+\xi) \cdot h - f'(x-\eta) \cdot h|, \end{aligned}$$

where ξ, η exist by the mean value theorem. Note that $|\xi| \leq |h|$ and $|\eta| \leq |h|$. Now this last is

$$\begin{aligned} &= |h| \cdot |f'(x+\xi) - f'(x-\eta)| \\ &\leq C \cdot |h| \cdot |h|^{\alpha-1} \\ &= C|h|^\alpha. \end{aligned}$$

That is the desired result. □

PROOF OF THEOREM 1. The case $j = 0$ is trivial. For $j \geq 1$, we write

$$\begin{aligned} |f_j(x)| &= \left| \int f(x-t)\psi_j(t)\,dt \right| \\ &= \frac{1}{2} \left| \int [f(x-t) + f(x+t)]\psi_j(t)\,dt \right| \end{aligned}$$

by the evenness of ψ_j. But, by the fact that $\int \psi_j \, dt = 0$, this last equals

$$\begin{aligned}
&= \frac{1}{2} \left| \int [f(x-t) + f(x+t) - 2f(x)] \psi_j(t) \, dt \right| \\
&\leq \frac{1}{2} \left| \int |t|^\alpha |\psi_j(t)| \, dt \right| \\
&\leq C \cdot 2^{-j\alpha} \cdot 2^{j+1} \cdot \int |\varphi(2^j t) + \varphi(2^{j+1}(t)| \, dt \\
&\leq C \cdot 2^{-j\alpha} . \qquad (3.1.2)
\end{aligned}$$

For the converse direction, we assume that $|f_j(x)| \leq C \cdot 2^{-j\alpha}$ for all j. We now prove that this implies that

$$\left| \frac{d}{dx} f_j \right| \leq C \cdot 2^{-j(\alpha-1)} .$$

We write

$$\begin{aligned}
f_j &= f * \varphi_{j+1} - f * \varphi_j \\
&= \varphi_j * (f * \varphi_{j+1} - f * \varphi_j) \\
&\qquad + \varphi_{j+1} * (f - f * \varphi_j) \\
&\qquad + \varphi_j * (f - f * \varphi_j) .
\end{aligned}$$

Thus

$$\begin{aligned}
\left| \frac{d}{dx} f_j \right| &\leq \left\| \frac{d}{dx} \varphi_j \right\|_{L^1} \cdot \| f * \psi_j \|_{L^\infty} \\
&\qquad + \left\| \frac{d}{dx} \varphi_{j+1} \right\|_{L^1} \cdot \| f - f * \varphi_j \|_{L^\infty} \\
&\qquad + \left\| \frac{d}{dx} \varphi_j \right\|_{L^1} \cdot \| f - f * \varphi_j \|_{L^\infty} \\
&\equiv I + II + III .
\end{aligned}$$

Now $I \le C \cdot 2^j \cdot \|f * \psi_j\|_{L^\infty} \le C \cdot 2^j \cdot 2^{-j\alpha} = C \cdot 2^{-j(\alpha-1)}$. Next,

$$\begin{aligned} II &\le C \cdot 2^j \cdot \|f - f * \varphi_j\|_{L^\infty} \\ &\le C \cdot 2^j \cdot \sum_{\ell=j}^{\infty} \|f * \psi_\ell\|_{L^\infty} \\ &= C \cdot 2^j \cdot \sum_{\ell=j}^{\infty} \|f_\ell\|_{L^\infty} \\ &\le C \cdot 2^j \cdot \sum_{\ell=j}^{\infty} 2^{-\alpha\ell} \\ &= C \cdot 2^j \cdot 2^{-\alpha j} \cdot \sum_{\ell=0}^{\infty} 2^{-\alpha\ell} \\ &= C \cdot 2^{-j(\alpha-1)} . \end{aligned}$$

The estimate of III is similar.

We note that a similar argument shows that

$$\left| \frac{d^2}{dx^2} f_j \right| \le C \cdot 2^{-j(\alpha-2)} .$$

And now we need to verify that $f \in \Lambda_\alpha$. For $0 < \alpha < 1$ we see that

$$\begin{aligned} |f(x+h) - f(x)| &= \left| \sum_j f_j(x+h) - f_j(x) \right| \\ &\le \sum_{j=1}^{|\log_2 |h||} |f_j(x+h) - f_j(x)| \\ &\quad + \sum_{j=|\log_2 |h||+1}^{\infty} |f_j(x+h) - f_j(x)| \\ &\le \sum_{j=1}^{|\log_2 |h||} |f_j'(x+\xi) \cdot h| + \sum_{j=|\log_2 |h||+1}^{\infty} 2\|f_j\|_{L^\infty} \\ &\le \sum_{j=1}^{|\log_2 |h||} 2^{-j(\alpha-1)} \cdot |h| + \sum_{j=|\log_2 |h||+1}^{\infty} 2 \cdot 2^{-j\alpha} \\ &\le C \cdot |h|^{\alpha-1} \cdot |h| + C \cdot |h|^\alpha \\ &= C \cdot |h|^\alpha . \end{aligned}$$

For $1 < \alpha < 2$ we apply a similar argument to $f_j'(x+h) - f_j'(x)$.
For $\alpha = 1$, we give a similar argument but we use the estimate on f_j''. □

4 Fourier Theory and Sobolev Theory

If $f \in L^1(\mathbb{R})$ then we define the *Fourier transform* of f to be

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{ix\xi}\, dx\,.$$

The Fourier transform dates back to Jean-Baptiste Joseph Fourier (1768–1830), who developed basic Fourier theory in his book *The Analytical Theory of Heat*. Indeed, Fourier was the first to present the standard formulas for calculating the Fourier series coefficients of an "arbitrary" function. Although his arguments were ludicrously incorrect, the work still had considerable impact—for it answered a question that had been in the air for fifty or more years. It took Fourier a good many years to get his book published—in fact he finally published it himself when he was the secretary of the French National Academy. Fourier treated the Fourier transform on the line in a later work.

Lemma 3. *If $f \in L^1$ and $f' \in L^1$ then*

$$\left|\widehat{f}(\xi)\right| \le \frac{C}{1+|\xi|}\,.$$

Proof. First assume that $f \in C_c^\infty$. For $\xi \neq 0$ we write

$$\left|\widehat{f}(\xi)\right| = \left|\int_{\mathbb{R}} f(t)e^{it\xi}\,dt\right| \le \left| f(t)\frac{e^{it\xi}}{i\xi}\right|_{-\infty}^{\infty}\right| + \left|\int_{\mathbb{R}} f'(t)\frac{e^{it\xi}}{i\xi}\,d\xi\right| = \frac{1}{|\xi|}\left|\widehat{f'}(\xi)\right|\,.$$

In the first inequality we used integration by parts. The result now follows with the extra assumption. We prove the general case by an approximation argument. □

Now we have a sort of converse.

Proposition 4. *Let $f \in L^1(\mathbb{R})$. Suppose that $\widehat{f}(\xi)| \le 1/(1+|\xi|)^{2+\epsilon}$ for some small $\epsilon > 0$. Then f can be corrected on a set of measure 0 to be continuously differentiable.*

Proof. Using Fourier inversion, we may write

$$f(t) = c\int_{\mathbb{R}} \widehat{f}(\xi)e^{-it\xi}\,d\xi\,.$$

Here c is some universal constant (see [7] for the exact value). Since $\xi\widehat{f}(\xi) \in L^1$, we may differentiate under the integral sign to obtain

$$f'(t) = -c\int_{\mathbb{R}} \widehat{f}(\xi)e^{-it\xi} i\xi \, d\xi \, .$$

The continuity of the derivative follows from the Riemann-Lebesgue lemma. □

A similar result may be proved, by similar techniques, for Fourier series.

A convenient definition of the L^2 Sobolev spaces is the following:

Definition 5. Let $f \in L^2(\mathbb{R})$. We say that $f \in W^s$ if

$$\int |\widehat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi < \infty \, .$$

A basic form of the Sobolev embedding theorem is this. If $f \in W^{3/2+\epsilon}$ for some $\epsilon > 0$, then f is continuously differentiable. For a proof, examine the inequalities

$$\begin{aligned} |f'(x)| &= \left| \int \xi e^{ix\xi} \widehat{f}(\xi) \, d\xi \right| \\ &\le \left| \int \frac{(|\xi|^2+1)^{3/4+\epsilon/2}}{(|\xi|^2+1)^{1/4+\epsilon/2}} \cdot |\widehat{f}(\xi)| \, d\xi \right| \\ &\le \int (|\xi|^2+1)^{3/2+\epsilon} |\widehat{f}(\xi)|^2 \, d\xi^{1/2} \cdot \int \frac{1}{(|\xi|^2+1)^{1/2+\epsilon}} \, d\xi^{1/2} \\ &\le C \cdot \|f\|_{W^{3/2+\epsilon}} \, . \end{aligned}$$

Similar statements may be formulated and proved about Bessel spaces, Nikol'skii spaces, and other Triebel-Lizorkin spaces (see [GRA], for instance).

5 Convergence of the Poisson Integral

In this section we briefly describe a method of recognizing smooth functions using an idea that dates back to G. H. Hardy. In fact Hardy's result was about holomorphic functions on the disc. But the ideas presented here are quite similar. Departing from the usual paradigm in this paper, we now work in $\mathbb{R}^2$.

In fact Hardy's classical theorem has inspired a number of modern works—including [16], [8], [14]. It is an elegant and powerful idea that connects real and complex analysis in an effective manner.

Theorem 6. *Let Ω be a smoothly bounded domain in $\mathbb{R}^2$. Let f be a continuous function on $\partial\Omega$ and let u be the solution of the classical Dirichlet problem on Ω with boundary data f. Let $0 < \alpha < 2$ and assume that k is an integer greater than α. We suppose that, for $x \in \Omega$,*

$$|\nabla^k u(x)| \leq C \cdot \delta(x)^{\alpha - k} ,$$

where $\delta(x)$ is the distance of x to the boundary. Then $u \in \Lambda_\alpha(\overline{\Omega})$.

PROOF. We first treat the case $0 < \alpha < 1$. Afterward we shall commente on the case $\alpha \geq 1$.

Let $x, x + h \in \Omega$. We estimate $u(x + h) - u(x)$ by looking at a box in Ω. We only need consider $x, x+h$ near the boundary. If $p \in \Omega$ is a point near the boundary then let ν_p be the unit outward normal vector at p. Now we have

$$\begin{aligned}
|u(x+h) - u(x)| &\leq |u(x+h) - u((x+h) - |h|\nu_{x+h})| \\
&\quad + |u((x+h) - |h|\nu_{x+h}) - u(x - |h|\nu_x)| \\
&\quad + |u(x - |h|\nu_x) - u(x)| \\
&\leq \int_0^{|h|} |\nabla u(x + h - t\nu_{x+h})| \, dt \\
&\quad + \int_0^{|h|} |\nabla u(x - |h|\nu_{(x+h} + th)| \, dt \\
&\quad + \int_{|h|}^0 |\nabla u(x - t\nu_x)| \, dt \\
&\leq \int_0^{|h|} t^{\alpha-1} \, dt + \int_0^{|h|} |h|^{\alpha-1} \, dt + \int_0^{|h|} t^{\alpha-1} \, dt \\
&\leq C \cdot |h|^\alpha .
\end{aligned}$$

That is the desired result.

We handle the case $1 \leq \alpha < 2$ by applying the preceding argument to ∇f and to $\alpha - 1$.

The case $\alpha = 1$ is best handled by the technique of interpolation of operators. ☐

6 Campanato-Morrey Theory

The Campanato-Morrey spaces arose originally in the study of partial differential equations. They first appeared in the papers [1], [2], but were also explored in [12], [13]. See also [11]. These spaces have also proved useful in the context of harmonic analysis—in particular in the characterization of the dual spaces of the H^p spaces. In any event, they are a nice and intuitive way to think about smoothness of functions. The savvy reader will want to compare the definitions of the Campanato-Morrey spaces to the definition of BMO. Here the space BMO is defined as follows:

$$BMO(\mathbb{R})=\left\{ f \text{ locally integrable} : \sup_I \frac{1}{|I|} \int_I |f(x) - f_I| \, dx \equiv \|f\|_{BMO} < \infty \right\}.$$

Here I ranges over all open intervals in $\mathbb{R}$ and f_I is the average of f over the interval.

The space BMO was invented by John and Nirenberg [5] for applications in partial differential equations. But it was really put on the map by C. Fefferman [3] who proved that BMO is the dual of the Hardy space H^1.

We work as usual in $\mathbb{R} = \mathbb{R}^1$. We begin by defining the Campanato-Morrey spaces. If k is a nonnegative integer, then let $\mathcal{P}_k$ denote the polynomials of degree not exceeding k. If $x \in \mathbb{R}$, then $(x - r, x + r)$ is the usual open interval centered at x and having radius r.

Definition 7. Let $1 \le q < \infty$. Let $0 < \lambda \in \mathbb{R}$ and also let $0 \le k \in \mathbb{Z}$. We define the Campanato-Morrey space

$$\mathcal{L}_k^{(q,\lambda)}(\mathbb{R}) = \left\{ f \in L^q(\mathbb{R}) : \sup_{\substack{x\in\mathbb{R}\\ r>0}} \left[r^{-\lambda} \inf_{P\in\mathcal{P}_k} \int_{(x-r,x+r)} |f(t) - P(t)|^q \, dt \right]^{1/q} + \|f\|_{L^q} \equiv \|f\|_{\mathcal{L}_k^{(q,\lambda)}} < \infty \right\}.$$

And now our main result about these spaces is the following:

Theorem 8. *Let $0 < \alpha \in \mathbb{R}$ and $\alpha < k \in \mathbb{Z}$. Suppose that $1 \le q < \infty$ and $\lambda > 1$ and set $\alpha = (\lambda - 1)/q$. Then any element of $\mathcal{L}_k^{(q,\lambda)}$ can be corrected on a set of measure zero so that it lies in Λ_α. Furthermore, the injection*

$$\mathcal{L}_k^{(q,\lambda)} \to \Lambda_\alpha$$

is continuous.

Finally, for any fixed $\varphi \in C_c^\infty$ and any $k > \alpha$, the map $f \mapsto \varphi \cdot f$ is continuous from Λ_α to $\mathcal{L}_k^{(q,\lambda)}$.

This result can be thought of as one of the main results of the present paper. Certainly its proof is the longest. The theorem is proved by way of a sequence of lemmas. Most of these lemmas are quite transparent.

Lemma 9. *Fix an* $\eta \in C_c^\infty$. *Let* $0 < \alpha \in \mathbb{R}$ *and* $\alpha < k \in \mathbb{Z}$. *Suppose that* $1 \le q < \infty$ *and* $\lambda > 1$ *and set* $\alpha = (\lambda - 1)/q$. *Then the map* $f \mapsto \eta \cdot f$ *is bounded from* Λ_α *to* $\mathcal{L}_k^{(q,\lambda)}$.

Proof. Let $f \in \Lambda_\alpha$. Fix a point $x_0 \in \mathbb{R}$. Let $r > 0$. Let φ, ψ be as in the section on approximation theory. Define $f_j = f * \psi_j$ as we did there. Then we can be sure that

$$\left| \frac{d^\ell}{dx^\ell} f_j \right| \le C \cdot (2^{-j})^{\alpha - \ell}$$

as long as $\ell > \alpha$. Let p be the kth order Taylor polynomial of f_j expanded about x_0. Then

$$\begin{aligned} |f_j(x) - p(x)| &\le C \cdot \|f_j\|_{C^{k+1}} \cdot |x - x_0|^{k+1} \\ &\le C \cdot (2^{-j})^{\alpha - k - 1} \cdot |x - x_0|^{k+1} . \end{aligned} \tag{6.9.1}$$

As a result,

$$\begin{aligned} \left[r^{-\lambda} \int_{(x_0 - r, x_0 + r)} |f(x) - p(x)|^q \right]^{1/q} &\le \left[r^{-\lambda} \int_{(x_0 - r, x_0 + r)} |f_j(x) - p(x)|^q \right]^{1/q} \\ &\quad + \left[r^{-\lambda} \int_{(x_0 - r, x_0 + r)} |f(x) - f_j(x)|^q \right]^{1/q} \\ &\equiv I + II . \end{aligned}$$

By (6.9.1) we see that

$$\begin{aligned} I &\le C \cdot \left[r^{-\lambda} \int_{(x_0 - r, x_0 + r)} \left((2^{-j})^{\alpha - k - 1} \cdot r^{k+1} \right)^q \, dx \right]^{1/q} \\ &\le C \cdot r^{-\lambda/q} \cdot r^{1/q} \cdot r^{k+1} \cdot (2^{-j})^{\alpha - k - 1} \\ &\le C \end{aligned}$$

by the choice of α, λ, q, j.

On the other hand,

$$\begin{aligned} II &\le C \cdot \left[r^{-\lambda} \int_{(x_0 - r, x_0 + r)} (2^{-j})^{q\alpha} \, dx \right]^{1/q} \\ &\le C \cdot r^{-\lambda/q} \cdot r^{1/q} \cdot (2^{-j})^{\alpha} \\ &= C . \end{aligned}$$

Finally, we note that $f \mapsto \eta \cdot f$ is a continuous map from L^∞ to L^q. That completes the proof. □

Lemma 10 (Gagliardo). *Let $0 \le \ell \in \mathbb{Z}$. Then there is a constant $C = C(k,\ell)$ so that*

$$|\nabla^1 p(x)| \le C(k,\ell) \cdot \left[r^{-1-\ell q} \int_{(x-r,x+r)} |p(t)|^q \, dt \right]^{1/q} \tag{6.10.1}$$

$$| \triangle_h^\ell \, p(x)| \le C(k,\ell) \cdot |h|^\ell \cdot \left[r^{-1-\ell q} \int_{(x-r,x+r)} |p(t)|^q \, dt \right]^{1/q} \tag{6.10.2}$$

for all $x \in \mathbb{R}$, $r > 0$, $|h| \le r$, and $p \in \mathcal{P}_k$.

PROOF. By change of scale and translation, we may suppose that $r = 1$ and $x = 0$. Observe that the righthand side of (6.10.1) is a norm on the finite-dimensional vector space $\mathcal{P}_k$ while the lefthand side is a seminorm on that space. Since there is only one norm (up to equivalence) on a finite-dimensional vector space, the result is immediate.

The proof of (6.10.2) is similar. □

Lemma 11. *Let $f \in \mathcal{L}_k^{(q,\lambda)}$ and $x \in \mathbb{R}$ and $r > 0$ be fixed. Then there is a unique polynomial $p(x, r, \,\cdot\,)$ in $\mathcal{P}_k$ which minimizes*

$$\left[r^{-\lambda} \int_{(x-r,x+r)} |f(t) - p(t)|^q \, dt \right]^{1/q} \tag{6.11.1}$$

over all $p \in \mathcal{P}_k$.

PROOF. Choose a sequence $p_j \in \mathcal{P}_k$ so that

$$\left[r^{-\lambda} \int_{(x-r,x+r)} |f(t) - p_j(t)|^q \, dt \right]^{1/q}$$

tends to the infimum. Then, by the convexity of the unit ball in L^q,

$$\left[r^{-\lambda} \int_{(x-r,x+r)} |p_j(t) - p_\ell(t)|^q \, dt \right]^{1/q}$$

tends to 0 as j, ℓ tend to ∞. If $y \in (x - r/2, x + r/2)$ and m is any positive integer, then

$$\left[r^{-mq-1} \int_{(y-r/2,y+r/2)} |p_j(t) - p_\ell(t)|^q \, dt\right]^{1/q}$$

tends to 0 as j, ℓ tend to ∞. By Gagliardo's lemma,

$$\left|\frac{d^m}{dt^m} p_j(t) - \frac{d^m}{dt^m} p_\ell(t)\right| \to 0$$

uniformly on $(x - r/2, x + r/2)$ for any index m. It follows that $\{p_j\}$ has a limit function which is a polynomial and which minimizes (6.11.1).

The uniqueness part follows because, if $\widetilde{p}_j$ is another sequence that minimizes (6.11.1), then $p_j - \widetilde{p}_j$ tends to 0 as in the first half of the proof. □

Lemma 12. *Let $x \in \mathbb{R}$, $r > 0$, and $t \in (x - r/2, x + r/2)$. Then*

$$|p(t, r, \ \cdot\ ) - p(x, r, \ \cdot\ )| \leq C\|f\|_{\mathcal{L}_k^{(q,\lambda)}} \cdot r^{(\lambda-1)/q} . \tag{6.12.1}$$

PROOF. By Gagliardo's lemma, the left side of (6.12.1) is

$$\begin{aligned}
&\leq C \cdot \left[r^{-1} \int_{t-r/2,t+r/2)} |p(t, r, s) - p(x, r, s)|^q \, ds\right]^{1/q} \\
&\leq C \cdot \left[(r/2)^{-1} \int_{(t-r/2,t+r/2)} |p(t, r, s) - f(s)|^q \, ds\right]^{1/q} \\
&\quad + \left[r^{-1} \int_{(t-r/2,t+r/2)} |p(x, r, s) - f(s)|^q \, ds\right]^{1/q} \\
&\leq C \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \cdot r^{(\lambda-1)/q} \\
&\quad + \left[r^{-1} \int_{(x-r,x+r)} |p(x, r, s) - f(s)|^q \, ds\right]^{1/q}
\end{aligned}$$

since $(t - r/2, t + r/2) \subseteq (x - r, x + r)$. But the last line is

$$\leq C \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \cdot r^{(\lambda-1)/q} .$$

□

Lemma 13. *If $x \in \mathbb{R}$, $r > 0$, and $j \in \mathbb{Z}^+$, then*

$$\left[(2^{-j-1}r)^{-1} \int_{(x-2^{-j-1}r, x+2^{-j-1}r)} |p(x, 2^{-j}r, s) - p(x, 2^{-j-1}r, s)|^q \, ds\right]^{1/q} \leq C \cdot \|f\|_{\mathcal{L}_k^{q,\lambda}} \cdot (2^{-j-1}r)^{(\lambda-1)/q} . \tag{6.13.1}$$

Proof. The lefthand side of (6.13.1) does not exceed

$$\begin{aligned}
& C \cdot \left[(2^{-j-1}r)^{-1} \cdot \int_{(x-2^{-j-1}r, x+2^{-j-1}r)} |p(x, 2^{-j}r, s) - f(s)|^q \, ds\right]^{1/q} \\
& \quad + C \cdot \left[(2^{-j-1}r)^{-1} \cdot \int_{(x-2^{-j-1}r, x+2^{-j-1}r)} |f(s) - p(x, 2^{-j-1}r, s)|^q \, ds\right]^{1/q} \\
& \leq C \cdot \left[(2^{-j}r)^{-1} \cdot \int_{(x-2^{-j}r, x+2^{-j}r)} |p(x, 2^{-j}r, s) - f(s)|^q \, ds\right]^{1/q} \\
& \quad + C \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \cdot (2^{-j-1}r)^{(\lambda-1)/q} \\
& \leq C \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \cdot (2^{-j-1}r)^{(\lambda-1)/q} . \qquad (1)
\end{aligned}$$

□

Lemma 14. *Let $x \in \mathbb{R}$, $r > 0$, $\ell \in \mathbb{Z}^+$. Then*

$$|p(x, r, x) - p(x, 2^{-\ell}r, x)| \leq C \cdot \|f\|_{\mathcal{L}_k^{q,\lambda}} \cdot \sum_{n=0}^{\ell-1} r^{(\lambda-1)/q} \cdot 2^{-n(\lambda-1)/q} . \tag{6.14.1}$$

Proof. The lefthand side of (6.14.1) does not exceed

$$\begin{aligned}
& \sum_{n=0}^{\ell-1} |p(x, 2^{-n}r, x) - p(x, 2^{-n-1}r, x)| \\
& \leq \quad C \cdot \sum_{n=0}^{\ell-1} \left[(2^{-n-1}r)^{-1} \int_{(x-2^{-n-1}r, x+2^{-n-1}r)} |p(x, 2^{-n}r, s) - p(x, 2^{-n-1}r, s))|^q \, ds\right]^{1/q}
\end{aligned}$$

(by Gagliardo's lemma) which is

$$\leq C \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \cdot \sum_{n=0}^{\ell-1} (2^{-n-1}r)^{(\lambda-1)/q}$$

by the preceding lemma. This gives the desired result. □

Lemma 15. *For each $x \in \mathbb{R}$ there is a $v(x) \in \mathbb{R}$ such that, for any $r > 0$, we have*

$$|p(x,r,x) - v(x)| \le C \cdot \|f\|_{\mathcal{L}_k^{q,\lambda}} \cdot r^{(\lambda-1)/q} .$$

PROOF. The proof consists of two steps:

I. We find $v(x)$ as the limit of a sequence $p(x, r_j, x)$.

II. We show that the limit in **I** is independent of the sequence $\{r_j\}$.

Step I: Let $\delta > 0$ and $x \in \mathbb{R}$ be fixed. Define $r_j \equiv 2^{-j}\delta$, $j = 0, 1, 2, \ldots$. Let us estimate, for $m > j$, the expression

$$|p(x, r_j, x) - p(x, r_m, x)| \tag{6.15.1}$$

by applying the preceding lemma with $\ell = m - j > 0$ and $r = 2^{-j}\delta$. Then (6.15.1) does not exceed

$$C \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \cdot \sum_{n=0}^{m-j-1} (2^{-j}\delta)^{(\lambda-1)/q} \cdot (2^{-n})^{(\lambda-1)/q}$$
$$\le C \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \cdot \sum_{n=j}^{m-1} \delta^{(\lambda-1)/q} \cdot (2^{-n})^{(\lambda-1)/q} .$$

Since $(\lambda - 1)/q = \alpha > 0$, we see that $\{p(x, r_j, x)\}$ is a Cauchy sequence.

Step II: Let $0 < \delta_1 \le \delta_2$. We wish to see that $\{p(x, 2^{-j}\delta_1, x)\}_{j=0}^{\infty}$ and $\{p(x, 2^{-j}\delta_2, x)\}_{j=0}^{\infty}$, both Cauchy sequences, have the same limit. But, for any $\ell \in \mathbb{Z}^+$, we have

$$|p(x, 2^{-\ell}\delta_1, x) - p(x, 2^{-\ell}\delta_2, x)|$$
$$\le C \cdot \left[(2^{-\ell}\delta_1)^{-1} \int_{(x-2^{-\ell}\delta_1, x+2^{-\ell}\delta_1)} |p(x, 2^{-\ell}\delta_1, s) - p(x, 2^{-\ell}\delta_2, s)|^q \, ds \right]^{1/q}$$

by Gagliardo's lemma. This in turn is

$$\le C \cdot \left[(2^{-\ell}\delta_1)^{-1} \int_{(x-2^{-\ell}\delta_1, x+2^{-\ell}\delta_1)} |p(x, 2^{-\ell}\delta_1, s) - f(x)|^q \, ds \right]^{1/q}$$
$$+ C \cdot \left[(2^{-\ell}\delta_1)^{-1} \int_{(x-2^{-\ell}\delta_2, x+2^{-\ell}\delta_1)} |p(x, 2^{-\ell}\delta_2, s) - f(s)|^q \, ds \right]^{1/q}$$
$$\equiv I + II .$$

It remains to estimate I and II.

Now

$$I \le C \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \cdot (2^{-\ell}\delta_1)^{(\lambda-1)/q}\,.$$

Also

$$\begin{aligned} II &\le C\cdot(\delta_2/\delta_1)^{1/q}\cdot\left[(2^{-\ell}\delta_2)^{-1}\int_{(x-2^{-\ell}\delta_2,x+2^{-\ell}\delta_2)}|p(x,2^{-\ell}\delta_2,s)-f(s)|^q\,ds\right]^{1/q} \\ &\le C\cdot(\delta_2/\delta_1)^{1/q}\cdot\|f\|_{\mathcal{L}_k^{(q,\lambda)}}\cdot(2^{-\ell}\delta_2)^{(\lambda-1)/q}\,. \end{aligned}$$

As a result,

$$|p(x,2^{-\ell}\delta_1,x)-p(x,2^{-\ell}\delta_2,x)| \le C(\delta_1,\delta_2)\cdot\|f\|_{\mathcal{L}_k^{(q,\lambda)}}\cdot(2^{-\ell})^{(\lambda-1)/q}\,.$$

That completes Step II. Thus v exists and is well defined. □

Our next, and most important, goal is to show that v equals f almost everywhere and that v has the desired boundedness and smoothness properties.

Lemma 16. *Let $x \in \mathbb{R}$ and $r > 0$. Then*

$$\left|\triangle_r^m p(x,r,\;\cdot\;)\big|_x - \triangle_r^m v(x)\right| \le C\cdot\|f\|_{\mathcal{L}_k^{(q,\lambda)}}\,.$$

Proof. By Lemma 12, for any $t \in (x-r/2, x+r/2)$,

$$|p(t,r,t)-p(x,r,t)| \le C\cdot\|f\|_{\mathcal{L}_k^{(q,\lambda)}}\cdot r^{(\lambda-1)/q}\,.$$

But now, letting $m = [\alpha]+1$ and $\widetilde{r} = r/(2m)$, we have that

$$\left|\triangle_{\widetilde{r}}^m p(\;\cdot\;,r,\;\cdot\;)\big|_x - \triangle_{\widetilde{r}}^m p(x,r,\;\cdot\;)\big|_x\right| \le C\cdot\|f\|_{\mathcal{L}_k^{(q,\lambda)}}\,. \tag{6.16.1}$$

The preceding lemma now tells us that

$$|p(t,r,t)-v(t)| \le C\cdot\|f\|_{\mathcal{L}_k^{(q,\lambda)}}\cdot r^{(\lambda-1)/q}\,.$$

As a consequence,

$$\left|\triangle_r^m p(\;\cdot\;,r,\;\cdot\;)\big|_x - \triangle_r^m v(\;\cdot\;)\big|_x\right| \le C\cdot\|f\|_{\mathcal{L}_k^{(q,\lambda)}}\cdot r^{(\lambda-1)/q}\,. \tag{6.16.2}$$

Finally, (6.16.1) and (6.16.2) and the triangle inequality give the result. □

Lemma 17. *Let $x \in \mathbb{R}$ and $r > 0$. Then*

$$\left| \triangle_r^m p(x, r, \ \cdot \ )\big|_x \right| \le C \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \cdot r^{(\lambda-1)/q} .$$

PROOF. By Gagliardo's lemma, for any $\rho > 0$, $n \in \mathbb{Z}^+$, and $0 < r \le 2^{-n}\rho$,

$$\begin{aligned}
&\left| \triangle_r^m \left\{ p(x, 2^{-n}\rho, \ \cdot \ ) - p(x, 2^{-n-1}\rho, \ \cdot \ ) \right\}\big|_x \right| \\
&\quad \le C \cdot r^m (2^{-n}\rho)^{-m-1/q} \\
&\qquad \int_{(x-2^{-n-1}\rho, x+2^{-n-1}\rho)} |p(x, 2^{-n}\rho, s) - p(x, 2^{-n-1}\rho, s)|^q \, dx^{1/q} \\
&\quad \le C \cdot r^m \cdot (2^{-n}\rho)^{-m+(\lambda-1)/q} \|f| \mathcal{L}_k^{(q,\lambda} .
\end{aligned}$$

In the last line we have used the idea of Lemma 13.

Now choose M so that $2^{-M-1} \le r \le 2^{-M}$. Write $r = 2^{-M}\widetilde{r}$ with $1/2 < \widetilde{r} \le 1$. Then

$$\begin{aligned}
&\left| \triangle_r^m p(x, \widetilde{r}, \ \cdot \ )\big|_x - \triangle_r^m p(x, 2^{-M}\widetilde{r}, \ \cdot \ )\big|_x \right| \\
&\quad \le \sum_{n=0}^{M-1} \left| \triangle_r^m p(x, 2^{-n}\widetilde{r}, \ \cdot \ )\big|_x - \triangle_r^m p(x, 2^{-n-1}\widetilde{r}, \ \cdot \ )\big|_x \right| \\
&\quad \le \sum_{n=0}^{M-1} C \cdot r^m (2^{-n}\widetilde{r})^{-m+(\lambda-1)/q} \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \\
&\quad \le C \cdot r^m \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \cdot 2^{-M(-m+(\lambda-2)/q)} \\
&\quad \le C \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \cdot r^{(\lambda-1)/q} . \qquad (6.17.1)
\end{aligned}$$

At last, by using Gagliardo's lemma and the fact that $1/2 < \widetilde{r} \le 1$, we see that

$$\begin{aligned}
\left| \triangle_r^m p(x, \widetilde{r}, \ \cdot \ )\big|_x \right| &\le C \cdot r^m \cdot (\widetilde{r})^{-M-1/q} \int_{(x-\widetilde{r}, x+\widetilde{r})} |p(x, \widetilde{r}, s)|^q \, ds^{1/q} \\
&\le C \cdot r^m \widetilde{r}^{-\lambda/q} \int_{(x-\widetilde{r}, x+\widetilde{r})} |p(x, \widetilde{r}, s) - f(x)|^q \, ds^{1/q} \\
&\quad + C \cdot r^m \int_{(x-\widetilde{r}, x+\widetilde{r})} |f(s)|^q \, ds^{1/q} \\
&\le C \cdot r^m \cdot \left(\|f\|_{\mathcal{L}_k^{(q,\lambda)}} + \|f\|_{L^q} \right) . \qquad (6.17.2)
\end{aligned}$$

Now (6.17.1), (6.17.2), and the triangle inequality tell us that

$$\left|\triangle_r^m p(x, r, \ \cdot\ )\big|_x\right| \leq C \cdot (r^m + r^{(\lambda-1)/q}) \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \leq C \cdot r^{(\lambda-1)/q} \|f\|_{\mathcal{L}_k^{(q,\lambda)}}$$

for $m > (\lambda - 1)/q$ and $0 < r < 1$. □

Lemma 18. *The function v is bounded.*

PROOF. Let $0 \leq n \in \mathbb{Z}$. Let $x \in \mathbb{R}$. By Step I of the proof of Lemma 15,

$$\begin{aligned} |p(x, 2^{-n}, x) - p(x, 1, x)| &\leq \sum_{j=0}^{n-1} 1^{(\lambda-1)/q} \cdot (2^{-j})^{(\lambda-1)/q} \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \\ &\leq C \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} \end{aligned} \tag{6.18.1}$$

since $(\lambda - 1)/q > 0$.

Now, by Gagliardo's lemma,

$$\begin{aligned} |p(x, 1, x)| &\leq C \cdot 1^{-1/q} \cdot \int_{(x-1,x+1)} |p(x, 1, s)|^q \, ds^{1/q} \\ &\leq C \cdot \int_{(x-1,x+1)} |p(x, 1, s) - f(s)|^q \, ds^{1/q} + C \cdot \int_{(x-1,x+1)} |f(s)|^q \, ds^{1/q} \\ &\leq C \cdot \|f\|_{\mathcal{L}_k^{(q,\lambda)}} . \end{aligned} \tag{6.18.2}$$

Combining now (6.18.1) and (6.18.2), we see that

$$|p(x, 2^{-n}, x)| \leq C \cdot \|f\|_{\mathcal{L}_x^{(q,\lambda)}} .$$

By definition of v, the result follows. □

Lemma 19. *It holds that $v = f$ almost everywhere. More precisely, $v(x) = f(x)$ at every point x where*

$$\lim_{r \to 0} r^{-1/q} \int_{(x-r,x+r)} |f(x) - f(s)|^q \, ds^{1/q} = 0 \, , \tag{6.19.1}$$

that is, at each Lebesgue point of f.

PROOF. Let x be a Lebesgue point for f. Then f is defined by equation (6.19.1). Then, for any $0 < r < 1$,

$$|f(x) - v(x)| \leq |f(x) - p(x,r,x)| + |p(x,r,x) - v(x)| \,.$$

The second term here of course tends to 0 with r by the definition of v.

For x fixed, the function $f(x) - p(x,r,\ \cdot\ )$ is a polynomial. So we may apply Gagliardo's lemma to see that

$$\begin{aligned}
|f(x) - p(x,r,x)| &\leq C \cdot r^{-1/q} \int_{(x-r,x+r)} |f(x) - p(x,r,s)|^q \, ds^{1/q} \\
&\leq C \cdot r^{-1/q} \int_{(x-r,x+r)} |f(x) - f(s)|^q \, ds^{1/q} \\
&\quad + C \cdot r^{-1/q} \int_{(x-r,x+r)} |f(s) - p(x,r,s)|^q \, ds^{1/q} \\
&\leq o(1) + O(r^{(\lambda-1)/q}) \,.
\end{aligned}$$

□

COMPLETION OF THE PROOF OF THEOREM 8: First suppose that $0 < \alpha = (\lambda - 1)/q < 1$. Let $m = 1$. Then Lemmas 16 and 17 yield that

$$| \triangle_r^1 v(x)| \leq C \cdot r^{(\lambda-1)/q} \,.$$

Since v is bounded, we conclude that $v \in \Lambda_{(\lambda-1)/q}$. By Lemma 19 and the Lebesgue differentiation theorem, we conclude that $v = f$ almost everywhere. So the proof in this case is complete.

In case $\alpha = (\lambda - 1)/q \geq 1$, the first case applies *a fortiori*. So v is continuous. But then Lemmas 16 and 17 tell us that

$$| \triangle_r^m v(x)| \leq C \cdot r^{(\lambda-1)/q} \,.$$

Since v is bounded, we see that $v \in \Lambda_{(\lambda-1)/q}$. That completes the proof. □

7 Landau's Inequalities

We begin this treatment with some notation.

Definition 20. Let f be a function on $\mathbb{R}$ and let $\delta > 0$. We define

$$\omega_f^0(\delta) = \omega_f(\delta) = \sup_{\substack{|h|<\delta \\ x\in\mathbb{R}}} |f(x+h) - f(x)|$$

and

$$\omega_f^k(\delta) = \omega_{d^k/dx^k f}(\delta) \,.$$

Although the results presented in this section are not strictly results about the existence of derivatives (without actually taking any derivatives), they are still of some interest in the context of theorems about interpolation of operators and other contexts of harmonic analysis. They can be thought of as *a priori* estimates for derivatives.

Lemma 21. *Let $0 < k \in \mathbb{Z}$. Let $0 \le \ell \le k$ be another integer. Then there are constants $a_0(k,\ell)$, $a_1(k,\ell)$, ..., $a_k(k,\ell)$ and integers $s_0(k,\ell)$, $s_1(k,\ell)$, ..., $s_k(k,\ell)$ such that, for all $f \in C^k(\mathbb{R})$ and all $h, k \in \mathbb{R}$ with $|h| \le 1$ we have*

$$h^\ell f^{(\ell)}(x) = \sum_{p=0}^{k} a_p \cdot f(x + s_p h) + O(\omega_f^k(k|h|))\,.$$

Here $f^{(\ell)}$ denotes the ℓth derivative of f.

PROOF. This is just a linear algebra problem. For $p = 0, 1, \ldots, k$ we write, using Taylor's expansion,

$$f(x + ph) = \sum_{q=0}^{k} \frac{f^{(q)}(x)}{q!}(ph)^q + O(\omega_f^k(p|h|))\,.$$

Hence

$$\sum_{p=0}^{k} a_p \cdot f(x + ph) = \sum_{q=0}^{k}\sum_{p=0}^{k} \left(a_p \cdot \frac{p^q}{q!}\right) h^q f^{(q)}(x) + O(\omega_f^k(k|h|))\,.$$

Thus we can take our integers $s_0, s_1, \ldots, s_k$ to be $0, 1, 2, \ldots, k$ provided we can find $a_0, a_1, \ldots, a_k$ satisfying

$$\sum_{p=0}^{k} a_p \cdot \frac{p^q}{q!} = 0 \quad \text{for } 0 \le q \le k\ ,\ q \ne \ell$$

and

$$\sum_{p=0}^{k} a_p \cdot \frac{p^\ell}{\ell!} = 1\,.$$

Since the coefficients of this system form a Vandermonde matrix, the system can certainly be solved. □

Landau's result, which we treat below, is classical (see [10]). There is an entire industry devoted to calculating best constants in various Landau inequalities.

Theorem 22 (Landau)**.** *Let $n_0 < n_1$ be nonnegative integers. Then there is a constant $K = K(n_0, n_1)$ such that, if n is an integer strictly between n_0 and n_1 and $g \in C^n$, then*

$$\|g^{(n)}\|_{\sup} \le K \cdot \|g^{(n_0)}\|_{\sup}^{(n_1-n)/(n_1-n_0)} \cdot \|g^{(n_1)}\|_{\sup}^{(n-n_0)/(n_1-n_0)} .$$

PROOF. First we assume that

$$\|g^{(n_0)}\|_{\sup} = \|g^{(n_1)}\|_{\sup} = 1 .$$

Let us apply the preceding lemma with $k = n_1 - n_0 = 1$, $\ell = n - n_0$, $f = g^{(n_0)}$. Then there are constants $a_0, a_1, \ldots, a_k$ and $s_0, s_1, \ldots, s_k$ such that

$$h^{n-n_0} g^{(n)}(x) = \sum_{j=0}^{k} a_j g^{(n_0)}(x + s_j h) + O(\omega_g^{n_1-1}(k|h|) , \quad \text{any } h \in \mathbb{R} .$$

Of course the a_j, s_j are independent of g, x, h. Set $h = 1$ to obtain

$$\begin{aligned} \left| \frac{d^n}{dx^n} g(x) \right| &\le \sum_{j=0}^{k} |a_j| + O(\omega_g^{n_1-1}(k|h|)) \\ &\le \sum_{j=0}^{k} |a_j| + c\|g^{(n_1)}\|_{\sup} \\ &\le K , \end{aligned}$$

since $\|g^{(n_0)}\|_{\sup} = \|g^{(n_1)}\|_{\sup} = 1$.

For the general case, let $m = \|g^{(n_0)}\|_{\sup}$ and $M = \|g^{(n_1)}\|_{\sup}$. Define

$$\widetilde{g}(x) = \frac{1}{m} \cdot \left(\frac{M}{m} \right)^{n_0/(n_1-n_0)} \cdot g\left(x \cdot \left(\frac{m}{M} \right)^{1/(n_1-n_0)} \right) .$$

Then

$$\|\widetilde{g}^{(n_0)}\|_{\sup} = 1$$

and

$$\|\widetilde{g}^{(n_1)}\|_{\sup} = 1 .$$

By the first part of the proof, we may then conclude that

$$\|\widetilde{g}^{(n)}\|_{\sup} \le K .$$

But this just says that

$$\left\| \frac{1}{m} \left(\frac{M}{m}\right)^{n_0/(n_1-n_0)} \left(\frac{m}{M}\right)^{n/(n_1-n_0)} g^{(n)} \right\|_{\sup} \le K\,.$$

In other words,

$$\|g^{(n)}\|_{\sup} \le K \cdot m^{(n-n_0)/(n_1-n_0)} \cdot M^{(n_1-n)/(n_1-n_0)}\,,$$

as was to be proved. □

We conclude this discussion by presenting a version of Landau's theorem that is adapted to the Lipschitz spaces Λ_α. This result is due to the present author.

Theorem 23. *Let $0 < n_0 < n_1$ be integers. There is a constant $K = K(n_0, n_1)$ such that, if $g \in C^{n_1}$ and $n_0 < \alpha < n_1$ then, for any $n_1 \le \ell \in \mathbb{Z}$,*

$$\sup_{x,h\in\mathbb{R}} \left| \frac{\triangle_h^k g(x)}{|h|^\alpha} \right| \le K \cdot \|g^{(n_0)}\|_{\sup}^{(n_1-\alpha)/(n_1-n_0)} \cdot \|g^{(n_1)}\|_{\sup}^{(\alpha-n_0)/(n_1-n_0)}\,.$$

PROOF. Fix $x, h \in \mathbb{R}$ and consider $\triangle_h^{n_1} g(x)$. On the one hand,

$$\begin{aligned} |\triangle_h^{n_1} g(x)| &= |\triangle_h^1 (\triangle_h^{n_1-1} g(x))| \\ &= |h| \cdot |\triangle_h^{n_1-1} g'(\xi_1)| \end{aligned}$$

with $x - h < \xi_1 < x + h$ by the mean value theorem. And this last, after n_0 iterations, equals

$$|h|^{n_0} \cdot |g^{(n_0)}(\xi_{n_0})|\,.$$

On the other hand,

$$|\triangle_h^{n_1} g(x)| = |h|^{n_1} \cdot |g^{(n_1)}(\xi_{n_1})|\,.$$

As a result,

$$\begin{aligned} |\triangle_h^{n_1} g(x)| &= |\triangle_h^{n_1} g(x)|^{(n_1-n)/(n_1-n_0)} \cdot |\triangle_h^{n_1} g(x)|^{(n-n_0)/(n_1-n_0)} \\ &\le |h|^\alpha \cdot \|g^{(n_0)}\|_{\sup}^{n_1-n)/(n_1-n_0)} \cdot \|g^{(n_1)}\|_{\sup}^{n-n_0)/(n_1-n_0)}\,. \end{aligned}$$

In conclusion,

$$\sup_{h,x} \left| \frac{\triangle_h^\ell g(x)}{|h|^\alpha} \right| \le C \cdot \|g^{(n_0)}\|_{\sup}^{(n_1-n)/(n_1-n_0)} \cdot \|g^{(n_1)}\|_{\sup}^{(n-n_0)/(n_1-n_0)}\,,$$

any $\ell \ge n_1$. That is the desired conclusion. □

8 Concluding Remarks

In many different contexts in analysis it is desirable to study the existence of, and other properties of, derivatives of functions without actually calculating the derivatives in the traditional sense. We have endeavored in this paper to present several different methods, taken from many different contexts, for doing so.

This is an open-ended discussion, and we look forward to further developments in these directions in the future.

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