INROADS

Real Analysis Exchange
Vol. 44(2), 2019, pp. 391–402
DOI: 10.14321/realanalexch.41.1.0391

Savita Bhatnagar, Centre for Advanced Study in Mathematics, Panjab University, Chandigarh, 160014, INDIA. email: bhsavita@pu.ac.in

THE RADON NIKODYM PROPERTY AND MULTIPLIERS OF $\mathcal{HK}$-INTEGRABLE FUNCTIONS

Abstract

We study the space of vector valued multipliers of strongly Henstock-Kurzweil ($\mathcal{SHK}$) integrable functions. We prove that if X is a commutative Banach algebra, with identity e of norm one, satisfying Radon-Nikodym property and $g : [a, b] \to X$ is of strong bounded variation, then the multiplication operator defined by $M_g(f) = fg$ maps $\mathcal{SHK}$ to $\mathcal{SHK}$. We also investigate the problems when the domain is $\mathcal{HK}$ or when X satisfies weak Radon-Nikodym property.

1 Introduction

A function φ is called a multiplier if the product φf is integrable for every integrable function f. For the Lebesgue integral, every essentially bounded measurable function is a multiplier. Surprisingly, for the real Henstock-Kurzweil integral, real continuous functions need not be multipliers, even on intervals of finite length. In fact, a function φ is a multiplier for the class of Henstock-Kurzweil integrable functions on $[a, b]$ if and only if it is equal almost everywhere to a function of bounded variation on $[a, b]$. See [8, Theorem 6.1.5 and Theorem 6.1.9] or [7, Theorem 12.9].

The aim of this paper is to strengthen the study of vector valued multipliers of the family of vector valued Henstock-Kurzweil integrable functions

Mathematical Reviews subject classification: Primary: 26A39; Secondary: 28B05
Key words: $\mathcal{HK}$-integrable, Banach Algebra, Multiplier, Henstock-Kurzweil-Pettis integrable, strong bounded variation
Received by the editors December 13, 2018
Communicated by: Luisa Di Piazza

carried out in [16]. The case of scalar valued multipliers for strongly Henstock-Kurzweil integrable functions is already known [5]. Let X be a commutative Banach algebra with identity e of norm one. In [16], we proved that if $g : [a,b] \to X$ is of strong bounded variation, then the multiplication operator M_g defined by $M_g(f) = fg$ maps $\mathcal{SHK}$ to $\mathcal{HK}$ and the associated operator T_g defined by $T_g(f) = \int_a^b fg$ maps $\mathcal{SHK}$ to X which is the vector analogue of Riesz Representation theorem. For the converse, we put an additional condition that X satisfy Radon-Nikodym property ($\mathcal{RNP}$). We proved that if $M : \mathcal{SHK} \to \mathcal{HK}$ is a bounded linear multiplication operator and X has $\mathcal{RNP}$, then there exists a function g of weak bounded variation such that $\tau(M(f)) = \tau(f)\tau(g)$ for all $f \in \mathcal{SHK}$ and every multiplicative linear functional τ of the Banach algebra X. Analogously, if $T : \mathcal{SHK} \to X$ is a bounded linear operator then there exists a function g of weak bounded variation such that $\tau(T(f)) = \int_a^b \tau(fg)$ for all $f \in \mathcal{SHK}$ and every multiplicative linear functional τ of the Banach algebra X.

In this note, we show that if X satisfies $\mathcal{RNP}$ and $g : [a,b] \to X$ is of strong bounded variation, then the multiplication operator $M_g(f) = fg$ maps $\mathcal{SHK}$ to $\mathcal{SHK}$. We also show that under the hypothesis of X satisfying $\mathcal{RNP}$, we have for $f \in \mathcal{HK}$ and $g : [a,b] \to X$ of strong bounded variation there exists $h \in \mathcal{SHK}$ such that $\tau(fg) = \tau(h)$. Moreover, if X satisfies $\mathcal{WRNP}$, then the function h is in $\mathcal{HKP}$.

Section 2 contains preliminaries and the main results are in Section 3.

2 Preliminaries

This section contains the preliminary material from which we shall draw throughout the rest of the paper.

Let $[a,b]$ be a compact real interval, $\mathcal{I}$ be the family of compact subintervals of $[a,b]$, $\mathcal{L}$ be the σ-algebra of all Lebesgue measurable subsets of $[a,b]$, m stand for Lebesgue measure on $[a,b]$ and X be a commutative Banach algebra with identity e such that $\|e\| = 1$.

A set function $F : \mathcal{I} \to X$ is said to be *additive* if $F(J \cup K) = F(J) + F(K)$, for all non-overlapping intervals $J, K \in \mathcal{I}$ such that $J \cup K \in \mathcal{I}$.

A collection $\{(t_i, J_i); i = 1, \ldots, k\}$ of point-interval pairs is called a *tagged-partition* of the interval $[a,b]$ if each $t_i \in J_i$ and $\{J_i : i = 1, \ldots, k\}$ are pairwise non-overlapping compact subintervals of $[a,b]$ with $[a,b] = \bigcup_{i=1}^k J_i$.

Any positive function $\delta : [a,b] \to (0,\infty)$ is called a *gauge* on $[a,b]$ and the above tagged partition is said to be *δ-fine* if $J_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, for every $i = 1, \ldots, k$.

A function $f:[a,b]\to X$ is said to be *strongly Henstock-Kurzweil integrable* on $[a,b]$ if there is an additive function $F:\mathcal{I}\to X$ such that for every $\epsilon>0$ there exists a gauge δ on $[a,b]$ such that the inequality

$$\sum_{i=1}^{k}\|f(t_i)\ \ m(J_i)-F(J_i)\|_X<\epsilon, \tag{1}$$

is satisfied, for every δ-fine tagged partition $\{(t_i,J_i):i=1,\dots,k\}$ of $[a,b]$.

The $\mathcal{HK}$-integral is defined on the same lines, except the summation sign in (1) comes inside the norm sign. In that case $F([a,b])$ is known as the Henstock-Kurzweil integral of f over an interval $[a,b]$ and is denoted by $(\mathcal{HK})\int_a^b f\ \ dm$. For more details on these integrals, see ([3], [6], [10], [11], [12]).

The classes of the strongly Henstock-Kurzweil and the Henstock-Kurzweil integrable functions from $[a,b]$ to X are denoted, respectively, by $\mathcal{SHK}$ and $\mathcal{HK}$. If X is a finite dimensional space then $\mathcal{SHK}=\mathcal{HK}$. In general, $\mathcal{SHK}\subseteq\mathcal{HK}$.

A function $f:[a,b]\to X$ is said to be *Pettis integrable* on $[a,b]$ if for all $x^*\in X^*$ the function $x^*(f)$ is Lebesgue integrable and for all $E\in\mathcal{L}$ there exists $w_E\in X$ such that

$$<x^*,w_E>=\int_E x^*(f)dm.$$

We call w_E the Pettis integral of f over E and we write $w_E=(P)\int_E fdm$. The class of Pettis integrable functions from $[a,b]$ to X is denoted by $\mathcal{P}$.

A function $f:[a,b]\to X$ is said to be *scalarly Henstock-Kurzweil integrable* on $[a,b]$ if for all $x^*\in X^*$ the function $x^*(f)$ is Henstock-Kurzweil integrable. A scalarly Henstock-Kurzweil integrable function is *Henstock-Kurzweil-Pettis* integrable if for all $I\in\mathcal{I}$ there exists $w_I\in X$ such that

$$<x^*,w_I>=(\mathcal{HK})\int_I x^*(f)dm.$$

We call w_I the Henstock-Kurzweil-Pettis integral of f over I and we write $w_I=(\mathcal{HKP})\int_I fdm$. The class of the Henstock-Kurzweil-Pettis integrable functions from $[a,b]$ to X is denoted by $\mathcal{HKP}$.

Given an additive interval function $\phi:\mathcal{I}\to X$, a gauge δ and a set $E\subset[a,b]$, we define

$$Var(\phi,\delta,E)=$$
$$sup\left\{\sum_{i=1}^{p}\parallel\phi(I_i)\parallel:\{(I_i,t_i):i=1,..,p\}\delta-\text{fine partition anchored on }E\right\}$$

Then we set $V_\phi(E) = inf\{Var(\phi, \delta, E) : \delta\text{-a gauge on } E\}$. We call V_ϕ the *variational measure generated by* ϕ. It is known [15] that V_ϕ is a metric outer measure on $[a, b]$. In particular, V_ϕ is a measure over all Borel sets of $[a, b]$.

For more details see [15].

Definitions:

1. Let $f : [a, b] \to X$ be a given function. f is said to be of *strong bounded variation ($\mathcal{BV}$)* on $[a, b]$ if

$$\sup \sum_i \|f(d_i) - f(c_i)\| < \infty,$$

   where the supremum is taken over all finite collections of non-overlapping intervals $\{[c_i, d_i]\}$ in $[a, b]$.

2. Let (Ω, Σ) be a measurable space and $F : \Sigma \to X$ be a vector measure. The *variation* of F is the extended nonnegative function $|F|$ whose value on a set $E \in \Sigma$ is given by $|F|(E) = sup \sum \|F(E_i)\|$ where the sup is taken over all finite partitions of E into a finite number of pairwise disjoint members of Σ. If $|F|(\Omega) < \infty$, then F is said to be a measure of strong bounded variation [4, page 2].

3. A Banach space X is said to have the *Radon-Nikodym property ($\mathcal{RNP}$)* [4, page 61] with respect to the finite measure space (Ω, Σ, μ) if for each μ-continuous vector measure $F : \Sigma \to X$ of strong bounded variation, there exists $g \in L^1(\mu, X)$ such that

$$F(E) = \int_E g \;\; d\mu, \text{ for all } E \in \Sigma.$$

4. A Banach space X is said to have the *Weak Radon-Nikodym property ($\mathcal{WRNP}$)* [13, page 239] with respect to the finite measure space (Ω, Σ, μ) if for each μ-continuous vector measure $F : \Sigma \to X$ of σ-finite variation, there exists $g \in \mathcal{P}$ such that

$$F(E) = (\mathcal{P}) \int_E g \;\; d\mu, \text{ for all } E \in \Sigma.$$

For more details on vector measures, see [4].

As X is a commutative Banach algebra with identity e of norm 1, by [14, Theorem 18.13], every proper ideal of X is contained in a maximal ideal and every maximal ideal is closed.

Let Δ denote the set of all non-zero multiplicative linear functionals of X. Since X has an identity, we have $\Delta \neq \emptyset$. Moreover, if $\tau \in \triangle$ then $\|\tau\| = 1$. Indeed, by [14, Theorem 18.17], there is a one-to-one correspondence between Δ and the class of maximal ideals, in the sense that every maximal ideal is the kernel of some $\tau \in \Delta$ and conversely, the kernel of every $\tau(\in \Delta)$ is the maximal ideal associated with τ.

It is easy to see that X-valued continuous function are not multipliers of $\mathcal{SHK}$. For example, take

$$f(t) = \sum_{n=1}^{\infty} 2^n c_n \chi_{I_n}(t) e,$$

where Σc_n is a non-absolutely convergent series for which $\sum |c_n|/\sqrt{n}$ does not converge and each $I_n = (2^{-n}, 2^{-n+1})$. Take g to have the value $2(sgn\ c_n)e/\sqrt{n}$ at the midpoint of each I_n, the value zero at the endpoints of each I_n and at 0 and linear on the rest of $[0,1]$. Then it can be shown that $fg \notin \mathcal{SHK}$.

In [16], we proved that if $g : [a,b] \to X$ is of strong bounded variation, then the multiplication operator M_g defined by $M_g(f) = fg$ maps $\mathcal{SHK}$ to $\mathcal{HK}$ and the associated operator T_g defined by $T_g(f) = \int_a^b fg$ maps $\mathcal{SHK}$ to X. Conversely, if X has $\mathcal{RNP}$ and $M : \mathcal{SHK} \to \mathcal{HK}$ is a bounded linear multiplication operator, then there exists a function g of weak bounded variation such that $\tau(M(f)) = \tau(f)\tau(g)$ for all $f \in \mathcal{SHK}$ and every multiplicative linear functional τ of the Banach algebra X. Analogously, if $T : \mathcal{SHK} \to X$ is a bounded linear operator then there exists a function g of weak bounded variation such that $\tau(T(f)) = \int_a^b \tau(fg)$ for all $f \in \mathcal{SHK}$ and every multiplicative linear functional τ of the Banach algebra X.

In section 3, we show that if X satisfies $\mathcal{RNP}$ and $g : [a,b] \to X$ is of strong bounded variation, then the multiplication operator $M_g(f) = fg$ maps $\mathcal{SHK}$ to $\mathcal{SHK}$. This result changes the range space in the results of [16]. In Theorem 7, under the hypothesis of X satisfying $\mathcal{RNP}$, we prove that for $f \in \mathcal{HK}, \tau \in \triangle$ and $g : [a,b] \to X$ of strong bounded variation there exists $h \in \mathcal{SHK}$ such that $\tau(fg) = \tau(h)$. In Theorem 8, X satisfies $\mathcal{WRNP}$ and we show that the function h is in $\mathcal{HKP}$.

Remark. *For each $f \in \mathcal{HK}$, if we define*

$$\|f\|_{\mathcal{HK}} := \sup\left\{ \left\|(\mathcal{HK}) \int_I f\right\|_X : I \in \mathcal{I} \right\},$$

then $\|.\|_{\mathcal{HK}}$ is a semi-norm on $\mathcal{HK}$.

Further, if we define a relation $\sim$ on $\mathcal{HK}$ as $f \sim g$ if $f = g$ a.e., then $\sim$ is an equivalence relation on $\mathcal{HK}$. Therefore $\mathcal{HK}/\sim$ is a normed linear space, which is not a Banach space even for $X = \mathbf{R}$.

In the sequel, we shall need the following results:

Theorem 1. *(Lemma 3.3, [1]) Let X be a Banach space and let $\mu : \mathcal{L} \to X$ be a m-continuous measure of finite variation. If $\phi : \mathcal{I} \to X$ is defined by $\phi(I) = \mu(I)$ for all $I \in \mathcal{I}$, then V_ϕ is finite, $V_\phi \ll \mu$ and $V_\phi(E) \leq |\mu|(E)$, whenever $E \in \mathcal{L}$.*

Theorem 2. *(Theorem 3.6,(i)⇔(vi) [1]) Let X be a Banach space. Then the following are equivalent:*

(i) X has $\mathcal{RNP}$;

(ii) If $V_\phi \ll \mu$, then there exists $f \in \mathcal{SHK}$ such that $\phi(I) = (\mathcal{SHK}) \int_I f \ \ dm$. Here $\phi : \mathcal{I} \to X$ is an additive interval function.

Theorem 3. *(Theorem 4.5,(i)⇔(vii) [2]) Let X be a Banach space. Then the following are equivalent:*

(i) X has $\mathcal{WRNP}$;

(ii) If $V_\phi \ll \mu$, then there exists $f \in \mathcal{HKP}$ such that $\phi(I) = (\mathcal{HKP}) \int_I f \ \ dm$. Here $\phi : \mathcal{I} \to X$ is an additive interval function.

3 Main results

Theorem 4. *Let X be a commutative Banach algebra, with identity of norm one, satisfying $\mathcal{RNP}$. If $\nu : [a, b] \to X$ is a vector measure of strong bounded variation, then for each $f \in \mathcal{SHK}$, the product $f(.)\nu[a, .) \in \mathcal{SHK}$. Moreover, the linear operator $T : \mathcal{SHK} \to X$ defined by $T(f) = (\mathcal{SHK}) \int_{[a,b)} f(t)\nu[a, t)dt$ is $\|.\|_{\mathcal{HK}}$-bounded.*

Proof. We follow the proof of Theorem 3.2 [9]. By continuity of the integral $(\mathcal{SHK}) \int_t^b f(s)ds$ [15, Theorem 7.4.1] and the fact that ν is a vector measure of strong bounded variation, we have that

$$T_\nu(f) = (RS) \int_{[a,b)} \left((\mathcal{SHK}) \int_t^b f(s)ds \right) d\nu(t)$$

is a $\|.\|_{\mathcal{HK}}$-bounded linear operator from $\mathcal{SHK}$ to X. Since $\chi_{[a,t)}(s) = \chi_{(s,b]}(t)$, we have for bounded Borel measurable function f,

$$\begin{aligned}\int_{[a,b)} f(t)\nu[a,t)dt &= \int_{[a,b]} \left[f(t)\int_{[a,b)} \chi_{[a,t)}(s)d\nu(s)\right]dt \\ &= \int_{[a,b)} \left[\int_{[a,b]} f(t)\chi_{(s,b]}(t)dt\right]d\nu(s) \\ &= (RS)\int_{[a,b)} \left((\mathcal{SHK})\int_s^b f(t)dt\right)d\nu(s),\end{aligned}$$

using Fubini's theorem.

Fix $f \in \mathcal{SHK}$ and define $H(I) = T_\nu(f\chi_I), \quad I \in \mathcal{I}$. Note that for bounded measurable function f, the integral $\int_E f(t)\nu[a,t)dt, E \in \mathcal{L}$ is a vector valued measure. By Theorem 1, $V_{\mathcal{HK}}(H) \ll m$. Since X satisfies $\mathcal{RNP}$, using Theorem 2 there exists $h \in \mathcal{SHK}$ such that $H(I) = (\mathcal{SHK})\int_I h(t)dt, \quad I \in \mathcal{I}$. It remains to show that $H'(t) = f(t)\nu[a,t),\ a.e.[m]$.

Let $F(t) = (\mathcal{SHK})\int_a^t f(s)ds,\ t \in [a,b]$. Then $F'(t) = f(t)\ a.e.$ [15, Theorem 7.4.2] so there exists $W \subset [a,b], m(W) = 0$ such that for $t \in [a,b] \setminus W$ and given $\epsilon > 0$ there exists $\delta_1(t) > 0$ such that

$\|f(t) - \frac{F(I)}{m(I)}\| < \frac{\epsilon}{2(\|T_\nu\|+1)}$ for $t \in I \subset (t-\delta_1(t), t+\delta_1(t))\bigcap[a,b]$.

Define $g(t) = \nu[a,t)$. Then g is of strong bounded variation and hence bounded measurable function. Enlarging W, if necessary, we may assume that if $t \notin W$ then $G'(t) = g(t)$ where $G(t) = \int_a^t g(s)ds,\ a \le t < b$.

There exists $\delta_2(t) > 0$ such that

$\|g(t) - \frac{G(I)}{m(I)}\| < \frac{\epsilon}{2(\|f(t)\|+1)}$ for all $t \in I \subset (t-\delta_2(t), t+\delta_2(t))\bigcap[a,b]$.

Hence for each $t \in [a,b] \setminus W$, we have

$$\begin{aligned}\left\|\frac{H(I)}{m(I)} - f(t)g(t)\right\| &= \left\|\frac{T_\nu(f\chi_I)}{m(I)} - f(t)g(t) - \frac{f(t)G(I)}{m(I)} + \frac{f(t)G(I)}{m(I)}\right\| \\ &\le \frac{1}{m(I)}\|T_\nu(f(t)\chi_I) - T_\nu(f\chi_I)\| + \|f(t)\| \quad \left\|g(t) - \frac{G(I)}{m(I)}\right\| \\ &\le \frac{\|T_\nu\|}{m(I)}\|f(t)\chi_I - f\chi_I\|_{\mathcal{SHK}} + \|f(t)\| \quad \left\|g(t) - \frac{G(I)}{m(I)}\right\| \\ &< \epsilon\end{aligned}$$

for $t \in I \subset (t-\delta(t), t+\delta(t))\bigcap[a,b]$, where $\delta(t) = min\{\delta_1(t), \delta_2(t)\}$. This completes the proof. □

Conversely, we have the following results.

Theorem 5. *If the Banach algebra X has $\mathcal{RNP}$ and $T : \mathcal{SHK} \to X$ is a bounded linear operator, then there exists a function g of weak bounded variation such that*

$$\tau(T(f)) = (\mathcal{HK})\int_a^b \tau(f)\tau(g), \text{ for all } f \in \mathcal{SHK} \text{ and } \tau \in \Delta.$$

For the proof, see [16, Theorem 4.1].

Theorem 6. *If the Banach algebra X has $\mathcal{RNP}$ and $M : \mathcal{SHK} \to \mathcal{SHK}$ is a bounded linear multiplication operator, then there exists a function g of weak bounded variation such that*

$$\tau(M(f)) = \tau(fg), \text{ for all } f \in \mathcal{SHK} \text{ and } \tau \in \Delta.$$

In other words, $\tau(M) = \tau(M_g)$.

For the proof of this, see [16, Theorem 4.2].

The following theorem deals with multipliers of $\mathcal{HK}$-integrable functions.

Theorem 7. *Let X be a commutative Banach algebra, with identity of norm one, satisfying $\mathcal{RNP}$. If $\nu : [a,b] \to X$ is a vector measure of strong bounded variation, then for $f \in \mathcal{HK}$ and $\tau \in \triangle$ the product*

$$\tau(f(.)\nu[a,.)) = \tau(h(.)) \ a.e.[m]$$

for some $h \in \mathcal{SHK}$.(The set of measure zero depends on τ.)

Proof. By continuity of the integral $(\mathcal{HK})\int_t^b f(s)ds$ [15, Theorem 7.4.1] and the fact that ν is a vector measure of strong bounded variation, we have that

$$T_\nu(f) = (RS)\int_{[a,b)} \left((\mathcal{HK})\int_t^b f(s)ds\right) d\nu(t)$$

is a $\|.\|_{\mathcal{HK}}$-bounded linear operator from $\mathcal{HK}$ to X. Since $\chi_{[a,t)}(s) = \chi_{(s,b]}(t)$, we have for bounded Borel measurable function f,

$$\begin{aligned}
\int_{[a,b)} f(t)\nu[a,t)dt &= \int_{[a,b]} \left[f(t)\int_{[a,b)} \chi_{[a,t)}(s)d\nu(s)\right]dt \\
&= \int_{[a,b)} \left[\int_{[a,b]} f(t)\chi_{(s,b]}(t)dt\right]d\nu(s) \\
&= (RS)\int_{[a,b)} \left((\mathcal{HK})\int_s^b f(t)dt\right)d\nu(s),
\end{aligned}$$

using Fubini's theorem.

Fix $f \in \mathcal{HK}$ and define $H(I) = T_\nu(f\chi_I),\ I \in \mathcal{I}$. Note that for bounded measurable function f, the integral $\int_E f(t)\nu[a,t)dt, E \in \mathcal{L}$ is a vector valued measure. By Theorem 1, $V_{\mathcal{HK}}(H) \ll m$. Since X satisfies $\mathcal{RNP}$, we have using Theorem 2 that there exists $h \in \mathcal{SHK}$ such that $H(I) = (\mathcal{SHK}) \int_I h(t)dt,\ I \in \mathcal{I}$. It remains to show that for $\tau \in \triangle, \tau(H'(t)) = \tau(f(t)\nu[a,t)),\ a.e.[m]$.
Let $F(t) = (\mathcal{HK}) \int_a^t f(s)ds,\ t \in [a,b]$. Then, as f is the scalar derivative of F, we have for $x^* \in X^*$, $(x^*F)'(t) = x^*(f(t))$ *a.e.* [15, Theorem 7.4.20] so there exists a set W (depending on x^*), $W \subset [a,b], m(W) = 0$ such that for $t \in [a,b] \setminus W$ and given $\epsilon > 0$ there exists $\delta_1(t) > 0$ such that
$\|x^*(f(t)) - \frac{x^*(F(I))}{m(I)}\| < \frac{\epsilon}{2(|\nu|([a,b])+1)}$ for $t \in I \subset (t - \delta_1(t), t + \delta_1(t)) \bigcap [a,b]$.
Define $g(t) = \nu[a,t)$. Then g is of strong bounded variation and hence bounded measurable function. Enlarging W, if necessary, we may assume that if $t \notin W$ then $G'(t) = g(t)$ where $G(t) = \int_a^t g(s)ds, \quad a \le t < b$.
There exists $\delta_2(t) > 0$ such that
$\|g(t) - \frac{G(I)}{m(I)}\| < \frac{\epsilon}{2(\|f(t)\|+1)}$ for all $t \in I \subset (t - \delta_2(t), t + \delta_2(t)) \bigcap [a,b]$.
Hence for $\tau \in \triangle \subseteq X^*$ and $t \in [a,b] \setminus W$ (W depending on τ), we have

$$\begin{aligned}
\left\| \frac{\tau(H(I))}{m(I)} - \tau(f(t)g(t)) \right\| &= \left\| \tau \left[\frac{T_\nu(f\chi_I)}{m(I)} - f(t)g(t) - \frac{f(t)G(I)}{m(I)} + \frac{f(t)G(I)}{m(I)} \right] \right\| \\
&\le \frac{1}{m(I)} \left\| \tau \left[T_\nu(f(t)\chi_I) - T_\nu(f\chi_I) \right] \right\| \\
&\qquad + \|\tau(f(t))\| \left\| \tau\big(g(t) - \frac{G(I)}{m(I)}\big) \right\| \\
&\le \frac{1}{m(I)} \left\| T_{\tau\nu}(\tau(f(t)\chi_I - f\chi_I) \right\| \\
&\qquad + \|\tau\|^2 \|f(t)\| \left\| g(t) - \frac{G(I)}{m(I)} \right\| \\
&\le \|T_{\tau\nu}\| \frac{1}{m(I)} \|\tau(f(t)\chi_I - f\chi_I)\|_{\mathcal{HK}} \\
&\qquad + \|f(t)\| \left\| g(t) - \frac{G(I)}{m(I)} \right\| \\
&< \epsilon
\end{aligned}$$

for $t \in I \subset (t - \delta(t), t + \delta(t)) \bigcap [a,b]$, where $\delta(t) = min\{\delta_1(t), \delta_2(t)\}$, using the fact that $\|\tau\| = 1$. Note that $\|T_{\tau\nu}\| = sup_{\|f\|_{\mathcal{HK}} \le 1} \|T_{\tau\nu}(f)\| \le \|\tau\|\ |\nu|[a,b] = |\nu|[a,b]$. This completes the proof. □

Remark: If in the above theorem $\nu : [a,b] \to X$ is a vector measure of weak bounded variation, then for $f \in \mathcal{HK}$ and $x^* \in X^*$, the product $x^*(f(.)\nu[a,.)) = x^*(h_{x^*})$ $a.e.[m]$ for some $h_{x^*} \in \mathcal{SHK}$. Indeed, the function $g_{x^*}(t) = x^*\nu[a,t)$ is a scalar function of bounded variation on $[a,b]$.
Defining $T_\nu(f) = (RS)\int_{[a,b)}\left((\mathcal{HK})\int_t^b f(s)ds\right)dg_{x^*}(t)$ and proceeding as in the above proof, we get that $(x^*H)'(t) = x^*(f)g_{x^*} = x^*(fg_{x^*}) = x^*(h_{x^*})$ for some $h_{x^*} \in \mathcal{SHK}$. In [5], it is proved that the scalar multipliers of $\mathcal{SHK}(\mathcal{HK})$ functions are functions of essentially bounded variation. Our functions $h(h_{x^*})$ are from $\mathcal{SHK}$ because of the additional condition that X has $\mathcal{RNP}$.

Theorem 8. *Let X be a commutative Banach algebra, with identity of norm one, satisfying $\mathcal{WRNP}$. If $\nu : [a,b] \to X$ is a vector measure of strong bounded variation, then for $f \in \mathcal{HK}$ and $\tau \in \triangle$ the product $\tau(f(.)\nu[a,.)) = \tau(h(.))$ $a.e.[m]$ for some $h \in \mathcal{HKP}$.(The set of measure zero depends on τ.)*

The proof is similar to the above proof with the only difference that we now use Theorem 3 instead of Theorem 2.

Remark: If $X = \mathbf{R}$, then $\mathcal{SHK} = \mathcal{HK} = \mathcal{HKP}$ and our results reduce to real valued multipliers case. Thus Theorems 4, 7 and 8 are generalizations of real multipliers case to vector valued case.

Acknowledgement: The author is grateful to Professor(retired) Ajit Iqbal Singh of University of Delhi for her helpful suggestions and continuous encouragement.

References

[1] B. Bongiorno, L. Di Piazza and K. Musial, *A variational Henstock integral characterization of the Radon-Nikodym property*, Illinois J. Math., **53(1)** (2009), 87–99.

[2] B. Bongiorno, L. Di Piazza and K. Musial, *A characterization of the weak Radon-Nikodym property by finitely additive interval functions*, Bull. Aus. Math. Soc., **80** (2009), 476–485.

[3] S. Cao, *The Henstock integral for Banach valued function*, SEA Bull. Math., **16(1)** (1992), 35–40.

[4] J. Diestel and J. J. Uhl, *Vector Measures*, Amer. Math. Soc. Mathematical Surveys, **15** (1977).

[5] L. Di Piazza and V. Marraffa, *The McShane, PU and Henstock integrals of Banach valued functions*, Czech. Math. Jour., **52(127)** no. 3, (2002) 609–633.

[6] L. Di Piazza, V. Marraffa and K. Musial, *Variational Henstock integrability of Banach space valued functions*, Math. Bohem., **141(2)** (2016), 287–296.

[7] Lee Peng Yee, *Lanzhou Lectures on Henstock integration*, Series in Real Analysis Vol.2, World Scientific Singapore (1989).

[8] Lee Tuo-Yeong, *Henstock-Kurzweil integration on Euclidean spaces*, Ser. Real Anal., **12**, World Scientific Publishing Co., Singapore, (2011)

[9] Lee Tuo-Yeong, *A characterisation of multipliers for the Henstock-Kurzweil integral*, Math. Proc. Camb. Phil. Soc., **138** (2005), 487–492.

[10] V. Marraffa, *A descriptive characterization of the variational Henstock integral*, Matimyas Mathematica, **22(2)** (1999), 73–84.

[11] V. Marraffa, *A characterization of strongly measurable Kurzweil-Henstock integrable functions and weakly continuous operators* , J. Math. Anal. Appl., **340(2)** (2008), 1171–1179.

[12] V. Marraffa, *Strongly measurable Kurzweil-Henstock type integrable functions and series*, Quaestiones Mathematicae, **31(4)** (2008), 379–386.

[13] K. Musial, *Topics in the theory of Pettis integration*, Rend. Instit. Mat. Univ. Trieste, **23** (1991), 177–262.

[14] W. Rudin, *Real and Complex Analysis*, Third edition, McGraw Hill International edition (1987).

[15] Stefań Schwabik and Ye Guoju, *Topics in Banach Space Integration*, Series in Real Analysis, **10**, World Scientific, (2005).

[16] S. P. Singh and S. Bhatnagar, *On vector valued multipliers for the class of strongly $\mathcal{HK}$- integrable functions*, Tatra Mountains Math. Publ., **68** (2017), 69–79.