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FOURIER METHOD REVISED TO SOLVE PARTIAL DIFFERENTIAL EQUATIONS AND PROVE UNIQUENESS AT ONE STROKE

Abstract

We present a novel application of Fourier analysis for solving PDEs which is much faster than the usual separation of variables method and, moreover, it implies uniqueness of the obtained solution at the same time.

1 Introduction

The heat conduction problem

$$u_t - au_{xx} = 0, \quad 0 \le x \le L, \ t > 0, \tag{1}$$

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{2}$$

$$u(0,t) = u(L,t) = 0, \quad t \ge 0, \tag{3}$$

where $a > 0$, $L > 0$ and $f : [0, L] \longrightarrow \mathbb{R}$ are given, is a mathematical model for the evolution of temperatures in a thin rod of length L. Equation (1) is the well–known heat equation in dimension one. The initial condition (2) means

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that the point x of the rod has temperature $f(x)$ at the initial time $t = 0$. Finally, the boundary conditions (3) mean that the temperature is kept equal to zero at every moment.

The standard Fourier method of separation of variables for solving (1)–(2)–(3) starts by searching for solutions of the form

$$u(x,t) = X(x)T(t),$$

which, upon substitution in (1) and (3), leads to an eigenvalue problem. Namely, we have to find all possible constants $\lambda \in \mathbb{R}$ so that the boundary value problem

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0, \tag{4}$$

and the equation $T' = \lambda\, k\, T$, have non-zero solutions. After finding those adequate eigenvalues λ and corrresponding eigenfunctions (non–zero solutions), one gets a formal series solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-an^2\pi^2\, t/L^2} \sin \frac{n\pi x}{L}. \tag{5}$$

The coefficients a_n must be chosen so that (2) holds, i.e.

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L},$$

so the a_n's must be the coefficients of the sine Fourier series of f (which coincides with f under suitable assumptions).

Once the formal solution (5) is computed, one has to prove that it makes sense, i.e., the series converges, the series can be differentiated term by term and the series really solves the problem.

However, this procedure does not imply uniqueness, and we have to prove that (5) is the unique solution (1)–(2)–(3) by means of other techniques, usually energy methods or as a consequence of maximum principles. Readers are referred to [1, 2] for details.

Here we present a kind of inverse Fourier method, starting with the sine Fourier series of an arbitrary solution and ending up with (5), thus computing the solution and proving its uniqueness at once. This method is valid for many other equations or boundary conditions.

2 A fast Fourier method implying uniqueness

Assume that

$$f \in PC^1([0,L]), \;\; f(0) = f(L) = 0,$$

where $PC^1([a,b])$ denotes the set of all functions which are continuous on $[a,b]$ and have a bounded continuous derivative on $[a,b]\setminus A$, for some empty or finite set A. This condition shall be used to prove convergence of a certain Fourier series and it can be relaxed.

Let $u = u(x,t) \in \mathcal{C}([0,L]\times[0,\infty)) \cap \mathcal{C}^{2,1}_{x,t}([0,L]\times(0,\infty))$ be a solution of (1)–(2)–(3). Here, the notation means that u is continuous on $[0,L]\times[0,\infty)$ and its derivatives u_{xx} and u_t exist and are continuous on $[0,L]\times(0,\infty)$. Derivatives with respect to x at the endpoints $x=0$ and $x=L$ are just the corresponding side derivatives.

We shall prove that u is necessarily given by (5), obtaining in this way the solution formula and its uniqueness.

For each fixed $t\geq 0$ we expand the one–variable function $x\mapsto u(x,t)$ as a sine Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi x}{L}, \quad 0\leq x\leq L, \tag{6}$$

where, by definition of the n–th sine Fourier coefficient,

$$T_n(t) = \frac{2}{L}\int_0^L u(x,t)\sin\frac{n\pi x}{L}\,dx, \quad n=1,2,\dots \tag{7}$$

The identity in (6) is guaranteed by standard results on uniform convergence of Fourier series: note that the odd extension of $u(\cdot,t)$ to the interval $[-L,L]$ belongs to $PC^1([-L,L])$ and assumes the same value at the endpoints thanks to (3) and thanks to the assumptions on f for the case $t=0$.

Computing $T_n(t)$ explicitly is all that remains to do.

First, we deduce from (7) and the assumptions on u that $T_n \in \mathcal{C}([0,\infty)) \cap \mathcal{C}^1(0,\infty)$ and

$$T_n(0) = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n=1,2,\dots \tag{8}$$

Second, we differentiate with respect to t in (7) and, since $u_t = au_{xx}$ for $t>0$, we get

$$T_n'(t) = a\frac{2}{L}\int_0^L u_{xx}(x,t)\sin\frac{n\pi x}{L}\,dx, \quad t>0.$$

Now we integrate by parts twice to obtain

$$T_n'(t) = -a\frac{n^2\pi^2}{L^2}T_n(t), \quad t>0, \tag{9}$$

which, along with (8), implies that

$$T_n(t) = a_n e^{-an^2\pi^2 t/L^2} \quad \text{where} \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{10}$$

The computation of the solution, and the proof of its uniqueness in the set $\mathcal{C}([0,L]\times[0,\infty)) \cap \mathcal{C}^{2,1}([0,L]\times(0,\infty))$, are finished.

Remark. *Standard arguments show that the assumptions on f ensure that (5) belongs to $\mathcal{C}([0,L]\times[0,\infty)) \cap \mathcal{C}^{\infty}([0,L]\times(0,\infty))$ and solves (1)–(2)–(3). One also has to check this when employing the usual version of Fourier method.*

The previous arguments can be simplified a little if we are only interested on proving uniqueness of solution to (1)–(2)–(3). Indeed, to do so assume that u_1 and u_2 are two solutions of (1)–(2)–(3) in the class $\mathcal{C}([0,L]\times[0,\infty)) \cap \mathcal{C}^{2,1}_{x,t}([0,L]\times(0,\infty))$, then

$$u = u_1 - u_2 \in \mathcal{C}([0,L]\times[0,\infty)) \cap \mathcal{C}^{2,1}_{x,t}([0,L]\times(0,\infty)),$$

and u solves (1), (3) and (2) with $f(x)=0$ for all $x\in[0,L]$. Then u can be expressed as in (6) and the T_n's are solutions of

$$T_n'(t) = -a\frac{n^2\pi^2}{L^2}T_n(t), \quad t>0, \quad T_n(0)=0,$$

hence $T_n(t)=0$ for all n and t. Therefore, $u=0$ or, equivalently, $u_1=u_2$.

Remark. *Other boundary conditions can be considered and the method still works with some modifications. See next section.*

3 A big family of problems suitable for the fast Fourier method

Let α_1, α, $\beta \in \mathcal{C}([0,L])$, $F = F(x,t) \in \mathcal{C}([0,L]\times(0,\infty))$ and consider the Neumann problem

$$u_{tt} + \alpha_1(t)\,u_t + \alpha(t)u - \beta(t)\,u_{xx} = F(x,t), \quad 0\le x\le L,\ t>0, \tag{11}$$
$$u(x,0) = f(x), \quad 0\le x\le L, \tag{12}$$
$$u_t(x,0) = g(x), \quad 0\le x\le L, \tag{13}$$
$$u_x(0,t) = u_x(L,t) = 0, \quad t\ge 0. \tag{14}$$

Two conditions at $t = 0$ must be imposed in this case because (11) is a second order equation also in the t variable. Observe that the PDE includes many important ones, such as the wave or the Laplace equations. We assume that

$$f,\, g \in PC^1([0, L]), \quad f'(0) = f'(L) = 0.$$

We want to compute the solution and prove its uniqueness in the set of functions

$$X = \mathcal{C}^{0,1}_{x,t}([0, L] \times [0, \infty)) \cap \mathcal{C}^2([0, L] \times (0, \infty)).$$

Let $u \in X$ be a solution of (11)–(12)–(13)–(14). For each $t \geq 0$ we expand $u(\cdot, t)$ as a cosine Fourier series (this is the proper type of series for the Neumann problem because every term satisfies the Neumann conditions):

$$u(x,t) = \frac{T_0(t)}{2} + \sum_{n=1}^{\infty} T_n(t) \cos\frac{n\pi x}{L}, \quad 0 \leq x \leq L, \tag{15}$$

where

$$T_n(t) = \frac{2}{L}\int_0^L u(x,t)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, 2, \ldots \tag{16}$$

For each $n = 0, 1, 2, \ldots$ the function T_n is continuously differentiable on $[0, \infty)$, twice continuously differentiable on $(0, \infty)$, and satisfies the initial conditions

$$T_n(0) = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad T_n'(0) = \frac{2}{L}\int_0^L g(x)\cos\frac{n\pi x}{L}\,dx. \tag{17}$$

Differentiating with respect to t and using (11) we get

$$\begin{aligned} T_n''(t) + \alpha_1(t)\,T_n'(t) + \alpha(t)\,T_n(t) &= \beta(t)\frac{2}{L}\int_0^L u_{xx}(x,t)\cos\frac{n\pi x}{L}\,dx \\ &\quad + \frac{2}{L}\int_0^L F(x,t)\cos\frac{n\pi x}{L}\,dx, \end{aligned}$$

and integrating by parts twice we deduce that T_n solves

$$T_n'' + \alpha_1(t)\,T_n' + \left(\alpha(t) + \beta(t)\frac{n^2\pi^2}{L^2}\right) T_n = \gamma_n(t), \quad t > 0, \tag{18}$$

where

$$\gamma_n(t) = \frac{2}{L}\int_0^L F(x,t)\cos\frac{n\pi x}{L}\,dx.$$

Equation (18) is linear with continuous coefficients, so the initial value problem (18)–(17) has a unique solution. This proves that there is only one possible choice for T_n, thus proving that (11)–(12)–(13)–(14) has at most one solution $u \in X$ given by (15) with the T_n's as the unique solutions of (17)–(18).

4 Other boundary conditions

Mixed or periodic problems can be treated in a similar way.

Consider equation (11) along with the initial conditions (12)–(13) and mixed boundary data

$$u(0,t) = u_x(L,t) = 0, \quad t > 0. \tag{19}$$

This problem can be reduced to a Dirichlet problem on the double interval $[0, 2L]$ by means of the following idea: extend the definitions of $F(\cdot, t)$, f and g to the interval $[0, 2L]$ as even functions with respect to the line $x = L$, namely,

$$F(x,t) = F(2L - x, t),\ f(x) = f(2L - x),\ g(x) = g(2L - x) \quad \text{for } x \in (L, 2L].$$

Now, use the method described in section 1 to solve the Dirichlet problem

$$u_{tt} + \alpha_1(t)\, u_t + \alpha(t) u - \beta(t)\, u_{xx} = F(x,t), \quad 0 \le x \le 2L,\ t > 0, \tag{20}$$

$$u(x,0) = f(x), \quad 0 \le x \le 2L, \tag{21}$$

$$u_t(x,0) = g(x), \quad 0 \le x \le 2L, \tag{22}$$

$$u(0,t) = u(2L,t) = 0, \quad t \ge 0. \tag{23}$$

The obtained solution can be proven to be a solution to (11)–(12)–(13)–(19).

The case of reversed mixed conditions

$$u_x(0,t) = u(L,t) = 0, \quad t > 0, \tag{24}$$

can be reduced to (19) by the change of variable $v(x,t) = u(L - x, t)$.

Finally, to solve and prove uniqueness to the problem (11)–(12)–(13) with periodic conditions

$$u(0,t) = u(L,t),\ u_x(0,t) = u_x(L,t), \quad t > 0, \tag{25}$$

it suffices to expand any solution $u = u(x,t)$ as a complete Fourier series with respect to x, i.e.

$$u(x,t) = T_0(t) + \sum_{n=1}^{\infty} \left(T_n(t) \cos \frac{n\pi x}{L} + \tilde{T}_n(t) \sin \frac{n\pi x}{L} \right),$$

and then repeat the arguments in the previous sections with the obvious modifications.

5 Problems in higher dimensions

To show that our method works in higher dimensions we consider the following problem with the heat equation:

$$u_t - k\,(u_{xx} + u_{yy}) = 0, \quad 0 \le x \le L,\ 0 \le y \le R,\ t > 0, \tag{26}$$
$$u(x, y, 0) = f(x, y), \quad 0 \le x \le L,\ 0 \le y \le R, \tag{27}$$
$$u(0, y, t) = u(L, y, t) = 0, \quad 0 \le y \le R,\ t \ge 0, \tag{28}$$
$$u(x, 0, t) = u(x, R, t) = 0, \quad 0 \le x \le L,\ t \ge 0, \tag{29}$$

where $k > 0$, $L > 0$, $R > 0$ and, for simplicity, $f \in \mathcal{C}^1([0, L] \times [0, R])$ and $f = 0$ on the boundary of the rectangle $[0, L] \times [0, R]$.

We can compute and prove uniqueness of a solution of (26)–(27)–(28)–(29) in the set

$$X = \mathcal{C}([0, L] \times [0, R] \times [0, \infty)) \cap \mathcal{C}^{2,2,1}_{x,y,t}([0, L] \times [0, R] \times (0, \infty)),$$

by applying the previous method twice. Specifically, we assume a solution $u \in X$ and we expand it as a sine Fourier series with respect to y:

$$u(x, y, t) = \sum_{n=1}^{\infty} U_n(x, t) \sin \frac{n\pi y}{R},$$

where

$$U_n(x, t) = \frac{2}{R} \int_0^R u(x, y, t) \sin \frac{n\pi y}{R}\, dy, \quad n = 1, 2, \ldots$$

For each fixed $n = 1, 2, \ldots$ we have

$$U_n(x, 0) = \frac{2}{R} \int_0^R f(x, y) \sin \frac{n\pi y}{R}\, dy = F_n(x),$$

and we compute

$$\begin{aligned} \frac{\partial}{\partial t} U_n(x, t) - k \frac{\partial^2}{\partial x^2} U_n(x, t) &= k \frac{2}{R} \int_0^R u_{yy}(x, y, t) \sin \frac{n\pi y}{R}\, dy \\ &= -k \frac{n^2\pi^2}{R^2} U_n(x, t). \end{aligned}$$

Hence, for each $n = 1, 2, \ldots$, U_n is a solution of

$$\begin{aligned} v_t - k\, v_{xx} + k \frac{n^2\pi^2}{R^2} v &= 0, \quad 0 \le x \le L,\ t > 0, \\ v(x, 0) &= F_n(x), \quad 0 \le x \le L, \\ v(0, t) = v(L, t) &= 0, \quad t \ge 0. \end{aligned}$$

Arguing as in our first section, this problem can be proven to have a unique solution

$$U_n(x,t) = \sum_{m=1}^{\infty} T_{mn}(t) \sin \frac{m\pi x}{L},$$

where T_{mn} is the unique solution of the initial value problem

$$T' + k\left(\frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{R^2}\right)T = 0, \quad T(0) = a_{mn},$$

and

$$\begin{aligned} a_{mn} &= \frac{2}{L}\int_0^L F_n(x) \sin \frac{m\pi x}{L}\, dx \\ &= \frac{4}{LR}\int_0^L \int_0^R f(x,y) \sin \frac{n\pi y}{R} \sin \frac{m\pi x}{L}\, dy dx. \end{aligned}$$

We have proven that if $u \in X$ solves (26)–(27)–(28)–(29) then it can only be

$$\begin{aligned} u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{mn}(t) \sin \frac{m\pi x}{L} \sin \frac{m\pi y}{R} \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \mathrm{e}^{-k\pi^2((m/L)^2+(n/R)^2)t} \sin \frac{m\pi x}{L} \sin \frac{m\pi y}{R}. \end{aligned}$$

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