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APPROXIMATIONS BY DIFFERENCES OF LOWER SEMICONTINUOUS AND FINELY CONTINUOUS FUNCTIONS

Abstract

A classical theorem of W.Sierpiński, S. Mazurkiewicz and S.Kempisty says that the class of all differences of lower semicontinuous functions is uniformly dense in the space of all Baire–one functions. We show a generalization of this result to the case when finely continuous functions of either density topologies or both linear and nonlinear potential theory are involved. Moreover, we examine which topological properties play a crucial role when deriving approximation theorems in more general situations.

1 Introduction

Let $\mathcal{K}$ be a convex cone of functions. In many situations, it is important to approximate a given function by differences of two functions from $\mathcal{K}$. The typical situation arises in potential theory: an easy consequence of the Stone-Weierstrass theorem is that any continuous function on the boundary of any

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bounded domain in $\mathbb{R}^n$ can be uniformly approximated by differences of continuous potentials. Notice also approximation theorems involving differences of functions from cones of convex or superharmonic functions. Another result says that Baire–one functions on the real line can be uniformly approximated by differences of lower semicontinuous functions.

In this paper we concetrate mainly on topological methods allowing an approximation of functions from certain classes of functions.

2 Approximation theorem

In J. Lukeš et al. [17, Corollary 3.36], the following result on approximation was presented (for the convenience of the reader we will give another complete proof of this assertion in the Appendix):

Theorem 1. *Let τ be a fine topology on a metric space (P, ρ) having the Luzin–Menshov property. Then any real ρ–Baire–one and τ–continuous function on P can be uniformly approximated by the differences of two positive τ–continuous and lower ρ–semicontinuous functions.*

In what follows, we assume always that the topology τ is *finer* than the metric topology ρ and the prefices are related to these topologies. When there is no prefix denoting topological properties we always have in mind the original topology of the space. A function f is *positive* if $f \geq 0$.

Remind that *Baire–one functions* are defined as pointwise limits of sequences of continuous functions.

The topology τ on P is said to have the *Luzin–Menshov property* (with respect to ρ) if for any τ–open set $G_\tau \subset P$ and for any set $A \subset P$ which is of type F_σ in ρ, $A \subset G_\tau$, there exists a positive τ–continuous and upper ρ–semicontinuous function f such that $A \subset \{x \in P : f(x) > 0\} \subset G_\tau$. Note that by passing to $\max\{f, 1\}$ we may require f to be bounded.

Observe that the Luzin–Menshov property is a special case of *binormality.* A set X on which two topologies τ and ρ are given is said to be a *binormal* topological space if it satisfies the following binormality condition: Whenever A and B are disjoint subsets of X, A τ–closed, B ρ–closed, there exist disjoint sets G_A and G_B, G_A ρ–open, G_B τ–open such that $A \subset G_A$ and $B \subset G_B$.

In [17, Theorem 3.11] it is shown that our definition of the Luzin–Menshov property expresses nothing else than the metric space P equipped with topologies τ and ρ is binormal.

3 Approximation in Euclidean spaces

It is not difficult to show that the discrete topology on $\mathbb{R}^n$ has the Luzin–Menshov property (with respect to the Euclidean one). Hence Theorem 1 leads to the following proposition which goes back to S. Mazurkiewicz [18] and W. Sierpiński [21] (they proved the result for bounded functions) and to S. Kempisty [13] (who dropped the condition of boundedness).

Proposition 2. *Any Baire–one function on $\mathbb{R}^n$ can be uniformly approximated by differences of two positive lower semicontinuous functions.*

The assertion that there exist bounded Baire–one functions which cannot be uniformly approximated by differences of bounded lower semicontinuous functions belongs to mathematical folklore (see, for example, a paper [22] where C. T. Tucker presents a construction of such a bounded Baire–one function).

An another example appeared in a recent paper by E. Omasta [19]. He constructed a bounded Baire–one function which is even a difference of two lower semicontinuous functions though it cannot be uniformly approximated by differences of two bounded lower semicontinuous functions. So, we have the following assertion (in the following, $\|\cdot\|$ stands for the supremum norm).

Proposition 3. *There exists a bounded Baire–one function h on $\mathbb{R}$ which is a difference of two lower semicontinuous functions such that if f and g are positive lower semicontinuous functions and $\|h-(f-g)\|<1$, then f and g are unbounded.*

In [19], E. Omasta presented also a proof of the following generalization of Proposition 3.

Proposition 4. *Let τ be a fine topology on $\mathbb{R}$ with respect to a metric topology ρ on $\mathbb{R}$ having the Luzin–Menshov property. If any countable subset of $\mathbb{R}$ is τ–closed, then there exists a bounded ρ–Baire–one and τ–continuous function h on $\mathbb{R}$ such that if f and g are lower ρ–semicontinuous functions and $\|h-(f-g)\|<1$, then f and g are unbounded.*

Remark 5. According to Proposition 3, the class of bounded functions which can be uniformly approximated by two *bounded* lower semicontinuous functions is a proper subset of a class of bounded Baire–one functions. This class of functions is of self interest and has been studied by R. Haydon et al. [7] (as the class of $B_{1/2}$–functions) on compact metric spaces with applications to the theory of Banach spaces.

Another applications of $B_{1/2}$–theory appeared recently in a paper [19] by E. Omasta and in a forthcoming paper [20] by P. Pošta. It concerns the Dirichlet solution of the classical Dirichlet problem for a continuous boundary condition on a bounded open subset U of $\mathbb{R}^n$. It is known that this solution need not be continuous on $\overline{U}$ (the closure of U) but it is always Baire–one. It can be proved that the Dirichlet solution can be uniformly approximated by differences of two functions which are sums of potentials continuous on a bounded open ball containing $\overline{U}$ and harmonic on U, thus it is even a $B_{1/2}$–function.

The $B_{1/2}$–class (as B_1^1) also appeared in the study of A. S. Kechris and A. Louveau [8].

4 Density topologies

The *ordinary density topology* on $\mathbb{R}^n$ is defined as the coarsest topology making all approximately continuous functions continuous or, equivalently, it is formed by a collection of all Lebesgue measurable sets on $\mathbb{R}^n$ having any of its points as a point of ordinary density. The ordinary density topology has the Luzin–Menshov property (the proof can be found in C. Goffman et al. [6]; cf. also J. Lukeš et al. [17, Section 6.D]). Since any approximately continuous function on $\mathbb{R}^n$ is a Baire–one function we get the following proposition immediately from Theorem 1 (see also S. Vaněček [23] for bounded functions).

Proposition 6. *Any approximately continuous function on $\mathbb{R}^n$ can be uniformly approximated by differences of two positive approximately continuous and lower semicontinuous functions.*

Remarks 7. (a) In [23], S. Vaněček showed that there exists a bounded approximately continuous function h on $\mathbb{R}$ such that if f, g are positive lower semicontinuous functions and $\|h - (f - g)\| < 1$ then f, g are unbounded. It is easy to see that Vaněček's result follows from Proposition 4.

(b) The density topology is an important example of a more general concept of abstract density topologies. Given a measure space $(X, \mathcal{S}, \mu)$ where μ is a σ–finite and complete measure on a σ–algebra $\mathcal{S}$, a topology τ is said to be an *abstract density topology* on $(X, \mathcal{S}, \mu)$ if the class of μ–null sets coincides with the class of τ–closed and τ–nowhere dense sets and, moreover, a set M belongs to $\mathcal{S}$ if and only M has the τ–Baire property.

5 Porous and $\mathcal{I}$–density topologies

Replacing a measure space $(X, \mathcal{S}, \mu)$ in the definition of an abstract density topology by its topological counterpart we are led to the next topological definition. A topology τ is a *categorial density topology* on a topological space (X, ρ) if ρ–first category subsets of X coincide with τ–closed and τ–nowhere dense sets and, moreover, a set A has the ρ–Baire property if and only if A has the τ–Baire property. We remark that the usual definition of abstract density topologies is different from ours. Moreover, it is a particular case of the general definition using a notion of lower density operators.

For this reason, consider now a more general approach to abstract density topologies. Let $\mathcal{S}$ be a σ–algebra on a set X and $\mathcal{J}$ a proper σ–ideal $\mathcal{J} \subset \mathcal{S}$. A mapping $L : \mathcal{S} \to \mathcal{S}$ is a *lower density operator* if

(a) $L(\emptyset) = \emptyset$ and $L(X) = X$,

(b) $L(A \cap B) = L(A) \cap L(B)$,

(c) $A \sim B$ implies $L(A) = L(B)$,

(d) $L(A) \sim A$

for any $A, B \in \mathcal{S}$. Here $A \sim B$ means that the symmetric difference $A \Delta B \in \mathcal{J}$.

A topology τ on X is an *abstract density topology* on $(X, \mathcal{S}, \mathcal{J})$ if there exists a lower density operator $L : \mathcal{S} \to \mathcal{S}$ such that

$$\tau = \{A \in \mathcal{S} : A \subset L(A)\}\,.$$

We obtain an interesting particular case when (X, ρ) is a topological space, $\mathcal{S}$ is the family of all sets with the Baire property on X and $\mathcal{J}$ is the σ–ideal of all meager subsets of X. The corresponding abstract density topology τ on $(X, \mathcal{S}, \mathcal{J})$ can be characterized by topological conditions as a definition (cf. the beginning of this section) which yields an intrinsic characterization of these abstract density topologies (see J. Lukeš et al. [17, Section 6.E]).

An important example of a categorial density topology (labelled as the $\mathcal{I}$–density topology) starting a study of these topologies was investigated by W. Wilczyński in [24]. A lot of interesting papers appeared since that time. Let us mention, for example, recent papers of J. Hejduk [11] or W. Wojdowski [25].

Wilczyński's definition of the $\mathcal{I}$–density topology uses the algebraic structure of the real line. We briefly describe his topology on the real line. We denote by $\mathcal{S}$ the σ–algebra of all subsets of $\mathbb{R}$ having the Baire property and by $\mathcal{I}$ the σ–ideal of meager subsets of $\mathbb{R}$. We say that $x \in \mathbb{R}$ is a $\mathcal{I}$–density point of a set $A \subset \mathbb{R}$ if

$$\lim_{n\to\infty} \chi_{n(A-x)\cap[-1,1]} = \chi_{[-1,1]}$$

with respect to $\mathcal{I}$. (Here χ_M denotes the characteristic function of M and the convergence means that every subsequence of the sequence $\chi_{n(A-x)\cap[-1,1]}$ contains a subsequence converging to $\chi_{[-1,1]}$ except for a meager set.)

A set $M \in \mathcal{S}$ is said to be $\mathcal{I}$–density open if every point of M is a $\mathcal{I}$–density point of M. The collection of all I–density open sets forms a topology on $\mathbb{R}$. It is labelled as the *$\mathcal{I}$–density topology*.

Another approach to the $\mathcal{I}$–density topology (even in a metric case) is due to L. Zajíček [26]. Sometimes fine topologies can be defined also using the so–called the notion of *thinness*. Let us illustrate this on the case of Zajíček's porous topologies.

Let (P, ρ) be a metric space and let $U(x, r)$ stand for the open ball centered at x and radius of r. Given $M \subset P$, $x \in P$ and $R > 0$ define

$$\gamma(x, M, R) := \sup\{r > 0 : \text{there exists } y \in P \text{ such that } U(y, r) \subset U(x, R) \setminus M\}.$$

A set M is *porous* at x if $\limsup_{R\to 0_+} \frac{1}{R}\, \gamma(x, M, R) > 0$. A set G is said to be *porously thin* at x if $G \cup M$ is porous at x whenever M is porous at x. The collection of all sets which are porously thin at a point forms an ideal. Hence, the family

$$\{G \subset P : P \setminus G \text{ is porously thin at any point of } G\}$$

forms a new topology on P which is finer than the original one. It is called the *porous* topology on P and labelled as the p–topology.

Asume now that (P, ρ) is a metric Baire space. Then we can introduce a $*$–modification p^* of a porous topology p on P defining

$$p^* := \{G \setminus N : G \in p \text{ and } N \text{ is } \rho\text{–meager}\}\,.$$

It can be shown that p^* is a categorial density topology on P (cf., for example, Zajíček's paper [26]).

It was proved by V. Kelar in [12, Theorem 1a] that the porous topology p has the Luzin–Menshov property (with respect to ρ). Since any fine topology with the Luzin–Menchoff property is completely regular (see J. Lukeš et al. [17, Corollary 3.13]), any porous topology p is completely regular. In contrast, the p^*–topology is not even regular (cf. K. Ciesielski et al. [4, Theorem 2.6.2]).

Nevertheless, due to Theorem 4 and Proposition 9 in Zajíček's paper [26] we have the following proposition.

Proposition 8. *Let (P, ρ) be a metric Baire space. Then p^*–continuous and p–continuous functions on P coincide and any p–continuous function (hence any p^*–continuous function) on P is Baire–one.*

Using Proposition 8 and Theorem 1 we can state the following approximation statement.

Proposition 9. *Let (P, ρ) be a metric Baire space. Any p–continuous function on (P, ρ) can be uniformly approximated by differences of two positive p–continuous and lower ρ–semicontinuous functions.*

Any p^*–continuous function on (P, ρ) can be uniformly approximated by differences of two positive p^*–continuous and lower ρ–semicontinuous functions.

For the proof of the next theorem see papers [26] and [27] of L. Zajíček.

Theorem 10. *Zajíček's p^*–topology on $\mathbb{R}$ coincides with Wilczyński's $\mathcal{I}$–density topology.*

6 Fine topology in linear potential theory

There is another example of an interesting topology on $\mathbb{R}^n$, $n \geq 2$, defined in classical potential theory of harmonic functions. Recall that the *classical fine topology* on $\mathbb{R}^n$ is the coarsest topology making all Newtonian ($n \geq 3$) or logarithmic ($n = 2$) potentials continuous. Equivalently, the classical fine topology is generated by the family of all positive superharmonic functions on $\mathbb{R}^n$. Since any Newtonian ($n \geq 3$) or logarithmic ($n = 2$) potential on $\mathbb{R}^n$ is approximately continuous (cf. J. Lukeš et al. [17, Theorem 10.5 and Exercise 10.A.3]), the classical fine topology is coarser than the ordinary density topology on $\mathbb{R}^n$.

In what follows, we will present proofs in a more general setting. Namely, we consider from now a $\mathfrak{P}$–harmonic space $(X, \mathcal{H})$ in the sense of C. Constantinescu and A. Cornea or, even more generally, a balayage space $(X, \mathcal{W})$ of J. Bliedtner and W. Hansen. Here, X is supposed to be a locally compact space with a countable base.

We define the *fine topology* on X as the topology generated by the class of all superharmonic functions. This topology has the Luzin–Menshov property (with respect to the original topology of X). The first proof of this assertions appeared in J. Lukeš [15] under an additional assumption on a harmonic space. The crucial point was to have for a given polar set a superharmonic function (even a potential) having a value $+\infty$ exactly at this polar set. This assertion was proved in the classical case of harmonic functions by G. C. Evans [5] provided the polar set is closed, and by G. Choquet in [3] for the G_δ-case. This theorem still holds in our general setting of harmonic or balayage spaces and for the completeness we sketch its proof.

Theorem 11 (Evans-Choquet theorem)**.** *If P is a polar G_δ–subset of X, there exists a superharmonic function s on X such that*

$$P = \{x \in X : s(x) = +\infty\}.$$

PROOF. The proof of the Evans-Choquet theorem can run as follows. If K is a compact subset of X which is disjoint with P we construct a positive hyperharmonic function u such that $u \le 1$ on K and $u = \infty$ on P. To this end let $X \setminus K = \bigcup_n H_n$ where $\{H_n\}$ is a sequence of compact subsets of X, $H_n \subset \operatorname{int} H_{n+1}$. Then

$$\mathcal{F} := \{u : u \text{ is a positive hyperharmonic function on } X, u \ge 1 \text{ on } H_n \cap P\}$$

is a Perron family on $X \setminus H_{n+1}$. An appeal on Dini's theorem yields a function $u_n \in \mathcal{F}$ such that $u_n \le 2^{-n}$ on K. With $\{u_n\}$ chosen in this manner we set $u = \sum_n u_n$.

Now let $X \setminus P = \bigcup_n K_n$ where K_n are compact. There exists a sequence of positive hyperharmonic functions $\{v_n\}$ such that $v_n = \infty$ on P and $v_n \le 2^{-n}$ on K_n. The function $s := \sum_n \min(1, v_n)$ has all properties desired. □

The complete proof of the Luzin–Menshov property of the fine topology in abstract harmonic spaces appeared in J. Lukeš et al. [17]. Here we present a simplified proof of a similar result even for the case of balayage spaces of W. Hansen and J. Bliedtner.

Theorem 12 (Luzin–Menshov property — a linear case)**.** *Let $(X, \mathcal{W})$ be a balayage space where X is locally compact with a countable base. Let F be a finely closed subset of X and K a closed subset of X disjoint from F. Then there exists a positive finely continuous and upper semicontinuous function φ on X such that $\varphi = 0$ on F and $\varphi > 0$ on K.*

PROOF. Since every closed subset of X is σ–compact, there exists an increasing sequence $\{K_n\}$ of compact subsets of X such that $\bigcup K_n = K$. Let p be a strict potential on X. Then $\widehat{R}_p^F = R_p^F < p$ on $X \setminus F$. According to J. Bliedtner and W. Hansen [2, Lemma VI.2.5], there exists a sequence $\{U_n\}$ of open sets such that $F \subset U_n \subset X$ and $\widehat{R}_p^{U_n} \le R_p^{U_n} \le R_p^F + \frac{1}{n}$ on K_n. Set

$$\varphi := \sum_{n=1}^{\infty} \frac{1}{2^n} (p - \widehat{R}_p^{U_n}) \, .$$

Then $\varphi \ge 0$ is a finely continuous and upper semicontinuous function on X, $\varphi = 0$ on F. Fix $x \in K$. There exists an integer n such that $x \in K_n$ and $R_p^F(x) + \frac{1}{n} < p(x)$. Hence $R_p^{U_n}(x) < p(x)$ which yields $\varphi(x) > 0$. □

Remark 13. An answer to an interesting question concerning situations when finely continuous functions are Baire–one can be found out in J. Lukeš and

J. Malý [16] by reading Section 6.1. In particular, in [16] it is shown that finely continuous functions are Baire–one in the case of porous topologies as well in the cases of both linear and nonlinear potential theories. Moreover, in [16] references and historical remarks are presented.

Having this in mind, using our approximation Theorem 1 we get the following assertion.

Proposition 14. *Any finely continuous function on X can be uniformly approximated by differences of two positive finely continuous and lower semicontinuous functions.*

7 Nonlinear potential theory and the p–fine topology

Another example of a fine topology on $\mathbb{R}^n$, $n \geq 2$, arises from a nonlinear potential theory. Let $p \in (1, n]$. Consider the following *p–Laplace equation*

$$\operatorname{div}\left(|\nabla u|^{p-2}\, \nabla u\right) = 0 \tag{1}$$

where solutions must be understood in a weak sense. When $p = 2$ this equation reduces to the classical Laplace equation.

Let Ω be an open subset of $\mathbb{R}^n$. A function $h : \Omega \to \mathbb{R}$ is said to be *p–harmonic* in Ω if it is a continuous weak solution of (1) in Ω. A function $u : \Omega \to (-\infty, +\infty]$ is *p–superharmonic* in Ω if u is lower semicontinuous (with respect to the Euclidean topology), $u \not\equiv \infty$ in each component of Ω and for each D relatively compact open subset of Ω and each p–harmonic function h on D and continuous on $\overline{D}$, the inequality $u \geq h$ on ∂D (the boundary of D) implies $u \geq h$ in D.

The *p–fine topology* is generated by the collection of all p–superharmonic functions on $\mathbb{R}^n$, it forms with the Euclidean topology a binormal topological space and it is strictly finer than the Euclidean topology on $\mathbb{R}^n$ for $p \in (1, n]$ (cf. J. Heinonen et al. [9, Theorem 3.4] and J. Heinonen et al. [10, Lemma 12.1 and following remarks]).

Here we give a proof of the Luzin–Menshov property of the p–fine topology (with respect to the Euclidean one). The proof is completely different from that one given by J. Heinonen et al. [9, Theorem 3.4] and it is based on the idea used in J. Lukeš [15].

We shall need the following analogy of the Evans-Choquet theorem.

Theorem 15 (Evans-Choquet property)**.** *Let $p \in (1, n]$ and suppose that P is a G_δ-subset of $\mathbb{R}^n$ of Sobolev p–capacity zero. Then there exists a p–superharmonic function s on $\mathbb{R}^n$ such that $P = \{x \in \mathbb{R}^n : s(x) = \infty\}$.*

PROOF. See T. Kilpeläinen [14, Theorem 1.3] (for the case of p–laplacian, put simply $A(x,\xi) = |\xi|^{p-2}\xi$). □

Theorem 16 (Luzin–Menshov property - nonlinear case)**.** *Let $p \in (1, n]$. Then the p–fine topology on $\mathbb{R}^n$ has the Luzin–Menshov property.*

PROOF. Assume first that a closed set $F \subset \mathbb{R}^n$ and a p–finely closed set $Q \subset \mathbb{R}^n \setminus F$ are given. Denote

$$bQ := \{x \in \mathbb{R}^n : \int_0^1 [r^{1-n} C_{1,p}(Q \cap U(x,r))]^{\frac{1}{p-1}}\, dr = +\infty\}$$

the set of all points where Q is not p–thin (here $C_{1,p}(M)$ is a $(1,p)$–capacity of a set M, see D. R. Adams and L. I. Hedberg [1, Definition 2.2.6]). Since the function

$$x \mapsto \int_0^1 [r^{1-n} C_{1,p}(Q \cap U(x,r))]^{\frac{1}{p-1}}\, dr$$

is p–finely continuous and lower semicontinuous, we see that bQ is a zero set of a positive p–finely continuous and upper semicontinuous function φ. Now, the p–fine topology has the Kellog property: The set $Q \setminus bQ$ is of $(1,p)$–capacity zero (cf. D. R. Adams and L. I. Hedberg [1, Corollary 6.3.17]). There exists a G_δ–set P of $(1,p)$–capacity zero such that $Q \setminus bQ \subset P$. We may assume that $P \cap F = \emptyset$. By J. Heinonen et al. [10, Corollary 2.39], P is also of a Sobolev p–capacity zero. An appeal to Theorem 15 yields the existence of a p–superharmonic function s such that $P = \{x \in \mathbb{R}^n : s(x) = +\infty\}$.

Hence, $P = \{x \in \mathbb{R}^n : \frac{\pi}{2} - \operatorname{arctg} s(x) = 0\}$ is a zero set of a positive p–finely continuous and upper semicontinuous function. We see that there exists a p–finely continuous and upper semicontinuous function f such that $0 \le f \le 1$ and $F \subset \{x \in \mathbb{R}^n : f(x) > 0\} \subset \mathbb{R}^n \setminus Q$.

For a general case, let $A \subset G_p$, A be of type F_σ and G_p p–finely open. Write $A = \bigcup_k F_k$ where $\{F_k\}$ is an increasing sequence of closed sets. For each k there exists a p–finely continuous and upper semicontinuous function f_k such that $0 \le f_k \le 1$ and $F_k \subset \{x \in \mathbb{R}^n : f_k(x) > 0\} \subset G_p$. A function $f := \sum_k 2^{-k} f_k$ is a positive p–finely continuous and upper semicontinuous function for which the inclusions $A \subset \{x \in \mathbb{R}^n : f(x) > 0\} \subset G_p$ hold. □

We can now formulate the following proposition.

Proposition 17. *Any p–finely continuous function on $\mathbb{R}^n$ can be uniformly approximated by differences of two positive p–finely continuous and lower semicontinuous functions.*

Remark 18. The result presented above can be generalized to the case of the $\mathcal{A}$-fine topology on unweighted $\mathbb{R}^n$ that stems from the second order quasilinear elliptic equations (see, for example, J. Heinonen et al. [10]).

We note that recently a nonlinear potential theory on metric spaces has been developed. We do not know whether or under what conditions the Luzin–Menshov property holds for the corresponding fine and metric topologies.

8 Appendix

Let $\alpha, \beta \in \mathbb{R}$ and $f : P \to \mathbb{R}$. Throughout the proof, the symbol $[f < \alpha]$ stands for $\{x \in P : f(x) < \alpha\}$ and the symbols $[f > \alpha]$, $[\alpha < f < \beta]$ are defined analogically.

PROOF OF THEOREM 1. Assume f is a real ρ–Baire–one and τ–continuous function on P, $M > 1$ and $k \in \mathbb{Z}$. Recall that Baire-one functions on metric spaces are characterized as those functions for which all level sets $[f > \alpha]$, $[f < \alpha]$ are of type F_σ. By the assumptions on f, the set $[a < f(x) < b]$ is τ-open and simultaneously of type F_σ in ρ for every pair of real numbers $a < b$. Since τ has the Luzin–Menshov property there exists a positive bounded τ–continuous and upper ρ–semicontinuous function w_k such that

$$\left[\frac{k-1}{M} < f < \frac{k+1}{M}\right] \subset [w_k > 0] \subset \left[\frac{k-2}{M} < f < \frac{k+2}{M}\right].$$

By multiplying w_k by an appropriate constant we can assume that

$$0 \le w_k \le \frac{1}{4} e^{-k^2}.$$

Set

$$u(x) := \sum_{k \in \mathbb{Z}} w_k(x) \quad \text{and} \quad v(x) := \sum_{k \in \mathbb{Z}} e^{\frac{k}{M}} w_k(x).$$

The sums are finite for every $x \in P$ since $w_k(x)$ can be strictly positive only for $Mf(x) - 2 < k < Mf(x) + 2$.

Thus u, v are τ–continuous and upper ρ–semicontinuous and $0 < u(x) < 1$, $0 < v(x) < 1$ for every $x \in P$. Moreover

$$\mathrm{e}^{f(x)-\frac{2}{M}} \sum_{k \in \mathbb{Z}} w_k(x) < \sum_{k \in \mathbb{Z}} \mathrm{e}^{\frac{k}{M}} w_k(x) < \mathrm{e}^{f(x)+\frac{2}{M}} \sum_{k \in \mathbb{Z}} w_k(x)$$

and so

$$\mathrm{e}^{f(x)-\frac{2}{M}} < \frac{v(x)}{u(x)} < \mathrm{e}^{f(x)+\frac{2}{M}}$$

for any $x \in P$. The functions

$$s(x) := -\log u(x) \quad \text{and} \quad t(x) := -\log v(x)$$

are positive τ–continuous and lower ρ–semicontinuous and moreover

$$|f(x) - (s(x) - t(x))| < \frac{2}{M}$$

for every $x \in P$. Since M was arbitrary, the assertion easily follows. □

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