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Antonio Boccuto,* Dipartimento di Matematica e Informatica, University of Perugia, Via Vanvitelli, 1, 06123 Perugia, Italy.
email: antonio.boccuto@unipg.it, boccuto@yahoo.it

HAHN-BANACH-TYPE THEOREMS AND APPLICATIONS TO OPTIMIZATION FOR PARTIALLY ORDERED VECTOR SPACE-VALUED INVARIANT OPERATORS

Abstract

We prove sandwich, Hahn-Banach, Fenchel duality theorems and a version of the Moreau-Rockafellar formula for invariant partially ordered vector space-valued operators. As consequences and applications, we give some versions of Farkas and Kuhn-Tucker-type optimization results and separation theorems, we prove the equivalence of these results and give a further application to Tarski-type theorems and probability measures defined on suitable product spaces.

1 Introduction.

The Hahn-Banach theorem is one of the most important and studied in Functional Analysis, and is related with results on extensions of measures and

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operators, sandwich theorems and separation of convex sets by means of hyperplanes. It has several applications in various fields of Mathematics, for example Optimization, Convex Analysis, Numerical Analysis, Differential Equations and Calculus of Variations (see also [1, 15, 16, 18, 32, 44, 46]). Among the consequences, we recall some Fenchel-type duality theorems, which deal with the problem of minimizing the dual energy, and is in general easier to handle than the primal energy. These topics have several applications, for instance in formulating algorithms to reconstruct images (see also [2, 6, 11, 12, 22, 23]).

Other applications of Hahn-Banach-type theorems deal with Farkas and Kuhn-Tucker-type theorems, related to optimization problems with suitable constraints (see also [1, 4, 5, 26, 28, 31, 37, 42, 52, 56]).

These topics are very important also in the development of subdifferential calculus and related subjects (see for instance [14, 27, 38, 39, 50]).

For a survey about different versions and applications of theorems of this kind for operators taking values in abstract vector spaces, see e.g. [19, 41].

These topics have several applications also in Measure Theory and Probability, in particular in stochastic processes (see also [20, 33, 47]). In [33] it is dealt with different kinds of probabilistic symmetries, exchangeability and coding results, for random elements invariant under finite or compact groups (see also the reference therein). In many cases it is advisable to deal with ordered vector spaces, since we often consider operators or probability measures which can depend not only on the considered event, but also on the time and on the state of knowledge (see also [9, 10, 13] and the references therein).

Another related field is the study of invariant or equivariant linear functionals, invariant measures and amenable (semi)groups (see also [8, 17, 21, 45, 48, 49]). Recent studies about invariance and equivariance and applications to Machine Learning can be found, for instance, in [24, 30, 35].

Observe that sandwich and extension theorems for Dedekind complete partially ordered vector space-valued operators or measures, invariant with respect to a given semigroup G of transformations, are always valid if and only if G is amenable. Note that every abelian semigroup is amenable, while the group of all permutations $\phi : \mathbb{N} \to \mathbb{N}$ which keep fixed all but a finite number of elements is amenable but not abelian. Moreover, we can see that the group $SO(2, \mathbb{R})$ of all orthogonal matrices of type 2×2 with real entries and whose determinant is equal to one (that is, the group of all rotations of $\mathbb{R}^2$) is abelian, while $SO(n, \mathbb{R})$ is not amenable for every $n \geq 3$ (see also [45]).

In the literature, many studies about these topics have been extended to the context of partially ordered space-valued operators and measures (see also [9, 13, 19, 41] and the references therein).

Some theorems of this kind for invariant partially order vector space-operators were given, for instance, in [7, 8, 21, 29, 48, 49]. In particular, in [7] and [8] some characterizations of amenable (semi)groups were given, in terms of these kinds of theorems, in the context of vector lattice-valued invariant operators and set functions.

In this paper we extend to invariant operators with values in a partially ordered vector space R earlier results proved in [27, 43, 55, 56, 57] in the linear case. We use a Hahn-Banach-type theorem given in [21] for invariant partially ordered vector space-valued operators with respect to amenable semigroups of transformations (here, the role of amenability is essential) and we prove a sandwich theorem in the linear and invariant case. Successively, as consequences, we give a Hahn-Banach-type theorem, a Fenchel-type duality theorem and a version of the Moreau-Rockafellar formula in subdifferential calculus. Moreover, as applications, we present Farkas-type results and a saddle point Kuhn-Tucker-type theorem on convex optimization under suitable given constraints. Furthermore, we prove that our given theorems are equivalent. Note that our results can be viewed as characterizations of the amenability of semigroups, extending previous results proved in [7] and [8], and also as characterizations of the Dedekind completeness of partially ordered vector spaces, extending earlier results given in [27] and [51]. Finally, as a further application, we give a Tarski-type theorem for finitely additive and invariant set functions, and an example of extensions of vector lattice-valued probability measures on suitable product spaces. Note that, in our setting, no topological assumptions are required on the involved space R.

2 Preliminaries.

Let X be a real vector space. An *affine combination* of elements $x_1, x_2, \dots, x_n$ of X is any linear combination of the form $\sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_1,\ \lambda_2, \dots, \lambda_n \in \mathbb{R}$ and $\sum_{i=1}^{n} \lambda_i = 1$. An *affine manifold* of X is a nonempty subset of X, closed under affine combinations.

If $\emptyset \neq Z \subset X$, then the *affine hull of* Z is the smallest affine manifold of X which contains Z, and we denote it by $\text{span}_{\text{aff}}(Z)$.

A point $x_0 \in Z$ is an *algebraic relative interior point of* Z iff for each $x \in \text{span}_{\text{aff}}(Z)$ there exists $\lambda_0 > 0$ with $(1-\lambda)x_0 + \lambda\, x \in Z$ for each $\lambda \in [-\lambda_0, \lambda_0]$. The sets of all algebraic relative interior points of Z is denoted by $int(Z)$.

A nonempty set $Z \subset X$ is said to be *algebraically expanded* iff there exists at least an element $a \in int(Z)$ with $a + \lambda(b-a) \in int(Z)$ for each $b \in Z$ and

$\lambda \in]0,1[$.

Given $Z \subset X$, $Z \neq \emptyset$, the *algebraic hull* Z^a of Z is the set of all points $y \in Z$ such that there exists an element $x \in Z$ with $x + \lambda(y-x) \in Z$ for every $\lambda \in]0,1[$. We say that Z is *algebraically closed* iff $Z = Z^a$.

A nonempty subset D of any real vector space X is said to be *convex* iff $\lambda x_1 + (1-\lambda)x_2 \in D$ for every x_1, $x_2 \in D$ and $\lambda \in [0,1]$.

Given any two real vector spaces X, Y, where Y is partially ordered, and a convex set $D \subset X$, we say that a function $U : X \to Y$ is *convex* on D iff $U(\lambda x_1 + (1-\lambda)x_2) \leq \lambda U(x_1) + (1-\lambda)U(x_2)$ for every x_1, $x_2 \in D$ and $\lambda \in [0,1]$. A function $U : D \to Y$ is said to be *concave* iff $-U$ is convex. We set $D(U) = D$.

Let G be a semigroup, and let $\mathcal{P}(G)$ be the family of all subsets of G. We say that G is *left* (resp. *right*) *amenable*) iff there exists a finitely additive set function $\mu : \mathcal{P}(G) \to [0,1]$ with $\mu(G) = 1$ and $\mu(h^{-1}(E)) = \mu(\{hg : g \in E\})$ (resp. $\mu((E)h^{-1}) = \mu(\{gh : g \in E\}) = \mu(E)$ for every $E \subset G$ and $g \in G$. Such a function μ is called a *left* (resp. *right*) *G-invariant mean.* We say that G is *amenable* iff it is both left and right amenable. In general, left and right amenability are not equivalent, but, if G a group, then they coincide (see also [25]).

Let $G \subset X^X$ be a semigroup of (linear) homomorphisms with $(gh)(x) = g(hx)$ for any g, $h \in G$ and $x \in X$, let R be a Dedekind complete partially ordered vector space, let $R^+ = \{y \in R : y \geq 0\}$ and let $l_b(G,R)$ be the space of all bounded R-valued functions defined on G. Given $f \in l_b(G,R)$ and $h \in G$, defined by ${}_hf(g) = f(hg)$ (resp. $f_h(g) = f(gh)$), $g \in G$. A *left* (resp. *right*)-*G-invariant R-functional* is a linear positive function $m : L^\infty(G) \to R$ such that $m({}_hf) = m(f)$ (resp. $m(f_h) = m(f)$) for all $f \in l_b(G,R)$ and $h \in G$, and $m(\mathbf{y}) = y$ for each $y \in R$, where $\mathbf{y}$ is the constant function which associates the value y to every element $g \in G$.

A set $\emptyset \neq Z \subset X$ is *G-invariant* iff $gz \in Z$ whenever $z \in Z$. A function $L : X \to R$ is *G-subinvariant* (resp. *G-superinvariant, G-invariant*) iff $L(gx) \leq L(x)$ (resp. $L(gx) \geq L(x)$, $L(gx) = L(x)$) for every $g \in G$ and $x \in X$.

We denote by $\mathcal{L}(X,R)$ and $\mathcal{L}(R,R)$ the sets of all linear functions from X to R and from R to R, respectively. We indicate with $\mathcal{L}_{\text{inv}}(X,R)$ the set of all linear G-invariant functions $L \in \mathcal{L}(X,R)$.

A nonempty set $A \subset X$ is called a *cone with vertex* $x_0 \in X$ iff $\lambda(A - x_0) \subset A - x_0$ for every positive real number λ. Sometimes we associate with X a G-invariant cone $X^+ \subset X$ with vertex 0, and the corresponding order on X defined by $x_1 \geq x_2$ if and only if $x_1 - x_2 \in X^+$. In this context we always require that $gx_1 \geq gx_2$ whenever $g \in G$ and $x_1 \geq x_2$, without saying it explicitly. If X has such a cone X^+, then we say that X^+ has *property* $\mathcal{K}$).

A linear function $L : X \to R$ is said to be *positive* iff $L(x) \geq 0$ whenever $x \geq 0$. We denote by $\mathcal{L}_{+,\mathrm{inv}}(X, R)$ the set of all positive functions belonging to $\mathcal{L}_{\mathrm{inv}}(X, R)$.

Given $0 \not\equiv L \in \mathcal{L}(X, R)$, $0 \not\equiv L' \in \mathcal{L}(X, R)$ and u_0 in R, set

$$H = \{(x, y) \in X \times R : L(x) + L'(y) = u_0\}. \tag{1}$$

It is not difficult to check that the set H defined in (1) is empty or an affine manifold of $X \times R$ (see also [43, §1]).

If A, B are two nonempty subsets of $X \times R$ and $H \neq \emptyset$ is as in (1), then we say that H *separates* A and B iff $A \subset H^-$ and $B \subset H^+$, where

$$H^+ = \{(x, y) \in X \times R : L(x) + L'(y) \leq u_0\}, \text{and}$$
$$H^- = \{(x, y) \in X \times R : L(x) + L'(y) \geq u_0\}.$$

The *projection* of $X \times R$ onto X is the function $P_X : X \times R \to X$ defined by $P_X(x, y) = x$ for every $(x, y) \in X \times R \to X$. Moreover, for any nonempty set $A \subset X \times R$, put

$$P_X(A) = \{x \in X : \text{ there exists } y \in R \text{ with } (x, y) \in A\}.$$

We always suppose that G is a right amenable semigroup. Moreover, we often make the following assumption.

$\mathcal{H}$) Suppose that $D(U)$ and $D(V)$ are nonempty convex and G-invariant subsets of X such that

$$0 \in int(D(U) - D(V)) \tag{2}$$

for any two convex and G-subinvariant functions $U : D(U) \to R$ and $V : D(V) \to R$.

Put $P_{U,V} = D(U) \cap D(V)$. Note that $P_{U,V} \neq \emptyset$, thanks to (2).

The *G-invariant conjugate* (or *conjugate*) of U is the R-valued function U^c defined as

$$U^c(L) = \bigvee\{L(x) - U(x) : x \in D(U)\}, \quad L \in D(U^c), \tag{3}$$

where

$$D(U^c) = \{L \in \mathcal{L}_{\mathrm{inv}}(X, R) : \bigvee\{L(x) - U(x) : x \in D(U)\} \text{ exists in } R\}, \tag{4}$$

provided that $D(U^c) \neq \emptyset$. If $x_0 \in D(U)$, then we call *G-invariant subdifferential* (or *subdifferential*) *at* x_0 the set $\partial_{\mathrm{inv}} U(x_0)$ defined as

$$\partial_{\mathrm{inv}} U(x_0) = \{L \in \mathcal{L}_{\mathrm{inv}}(X, R) : L(x) - L(x_0) \leq U(x) - U(x_0) \text{ for any } x \in D(U)\}.$$

Any element $L \in \partial_{\rm inv} U(x_0)$ will be called (G-*invariant*) *subgradient of* U *at* x_0.

Given a nonempty set $A \subset X$ and $x_0 \in X$, we call G-*invariant polar* (or *polar*) *of* A *at* x_0 the set

$$A^*_{\rm inv}(x_0) := \{L \in \mathcal{L}_{\rm inv}(X,R) : L(x) - L(x_0) \leq 0 \text{ for all } x \in A\}.$$

In formulating our version of the duality theorem, we study the following problems (see also [55]).

Problem I) Find $r = \bigwedge\{U(x) + V(x) : x \in P_{U,V}\}$ in R.

Problem II) Find $s = \bigvee\{-U^c(L) - V^c(-L) : L \in D(U^c) \cap D(V^c)\}$ in R, provided that $D(U^c) \cap D(V^c) \neq \emptyset$.

3 The main results.

We begin with the following sandwich theorem in the setting of invariance with respect to amenable semigroups of transformations and partially ordered vector spaces, extending [57, Sandwich Theorem 3.1]. Our technique is based on the existence of linear operators, not necessarily invariant, due to the corresponding classical results, and of suitable invariant partially ordered vector space-valued means, from which it is possible to construct invariant linear functionals.

Theorem 1. *Let* $U : D(U) \to R$ *and* $V : D(V) \to R$ *satisfy assumption* $\mathcal{H}$). *Suppose that* $U(x) + V(x) \geq 0$ *for all* $x \in P_{U,V}$. *Then, there exist* $L \in \mathcal{L}_{\rm inv}(X,R)$ *and* $u_0 \in R$ *with* $L(x) - u_0 \leq U(x)$ *for every* $x \in D(U)$ *and* $L(x) - u_0 \geq -V(x)$ *for each* $x \in D(V)$.

Proof. By [57, Sandwich Theorem 3.1], there exist an element $u_0 \in R$ and a function $L^* \in \mathcal{L}(X,R)$ (not necessarily G-invariant) with $L^*(x) - u_0 \leq U(x)$ for every $x \in D(U)$ and $L^*(x) - u_0 \geq -V(x)$ for each $x \in D(V)$. Pick arbitrarily $x \in X$, and define $f_x \in l_b(G,R)$ by $f_x(g) = L^*(gx)$, $g \in G$. As R is Dedekind complete and G is right amenable, by [21, Théorème 2] there exists a right G-invariant R-functional $m : l_b(G,R) \to R$. Set $L(x) = m(f_x)$, $x \in X$. Since $f_{hx}(g) = L^*(ghx) = f_x(gh) = (f_x)_h(g)$ for any $g \in G$, then $L(hx) = m(f_{hx}) = m((f_x)_h) = m(f_x) = L(x)$ for every $h \in G$, and hence, L is G-invariant. As $D(U)$ and $D(V)$ are G-invariant, U and V are G-subinvariant, $L^*(x) \leq u_0 + U(x)$ for every $x \in D(U)$ and $L^*(x) \geq u_0 - V(x)$ for each $x \in D(V)$, then we obtain

$$f_x(g) = L^*(gx) \leq u_0 + U(gx) \leq u_0 + U(x)$$

for every $x \in D(U)$ and $g \in G$, and hence,

$$L(x) = m(f_x) \le m(\mathbf{u_0} + \mathbf{U}(\mathbf{x})) = m(\mathbf{u_0}) + m(\mathbf{U}(\mathbf{x})) = u_0 + U(x)$$

for any $x \in D(U)$. Analogously it is possible to prove that $L(x) \ge u_0 - V(x)$ for all $x \in D(V)$. Moreover, if $\lambda_1, \lambda_2 \in \mathbb{R}$, $x_1, x_2 \in X$ and $g \in G$, then

$$\begin{aligned} f_{\lambda_1 x_1 + \lambda_2 x_2}(g) &= L^*(g(\lambda_1 x_1 + \lambda_2 x_2)) \\ &= L^*(\lambda_1 g x_1 + \lambda_2 g x_2) \\ &= \lambda_1 L^*(g x_1) + \lambda_2 L^*(g x_2) \\ &= \lambda_1 f_{x_1}(g) + \lambda_2 f_{x_2}(g), \end{aligned}$$

and therefore

$$\begin{aligned} L(\lambda_1 x_1 + \lambda_2 x_2) &= m(f_{\lambda_1 x_1 + \lambda_2 x_2}) \\ &= \lambda_1 m(f_{x_1}) + \lambda_2 m(f_{x_2}) \\ &= \lambda_1 L(x_1) + \lambda_2 L(x_2). \end{aligned}$$

Thus, $L \in \mathcal{L}_{\text{inv}}(X, R)$. This completes the proof. □

Now we present the following Fenchel-type duality theorem, which extends [55, Theorem 2] to invariance.

Theorem 2. *Under the assumption* $\mathcal{H}$), *suppose that*

$$r = \bigwedge \{U(x) + V(x) : x \in P_{U,V}\} \tag{5}$$

exists in R. *Then, Problem* II *has a solution* L_0, *and* $-U^c(L_0) - V^c(-L_0) = r$.

PROOF. Let $\widetilde{U}(x) = U(x) - r$, where r is as in (5). The convexity of $\widetilde{U}$ follows from the convexity of U. Moreover, $P_{\widetilde{U},V} = D(\widetilde{U}) \cap D(V) = D(U) \cap D(V) = P_{U,V}$. For any $x \in P_{U,V}$ it is $r \le U(x) + V(x)$, and hence, $\widetilde{U}(x) + V(x) \ge 0$. Thus, $\widetilde{U}$ and V satisfy the hypotheses of Theorem 1. So, there exist $L_0 \in \mathcal{L}_{\text{inv}}(X, R)$ and $u_0 \in R$ with $L_0(x) - u_0 \le \widetilde{U}(x) = U(x) - r$ for all $x \in D(U)$ and $L_0(x') - u_0 \ge -V(x')$ for all $x' \in D(V)$. From this we deduce

$$L_0(x - x') = L_0(x) - L_0(x') \le u_0 + U(x) - r - u_0 + V(x') \tag{6}$$

and hence,

$$r + L_0(x) - U(x) \le L_0(x') + V(x') \tag{7}$$

for each $x \in D(U)$ and $x' \in D(V)$. Thus,

$$\begin{aligned} r + \bigvee\{L_0(x) - U(x) : x \in D(U)\} &\le \bigwedge\{L_0(x) + V(x) : x \in D(V)\} \\ = -\bigvee\{-L_0(x) - V(x) : x \in D(V)\}, \end{aligned}$$

and hence, $L_0 \in D(U^c) \cap D(V^c)$ and $r \le -U^c(L_0) - V^c(-L_0)$. Furthermore, observe that

$$-U^c(L) - V^c(-L) \le -L(x) + U(x) + L(x) + V(x) = U(x) + V(x)$$

for every $x \in P_{U,V}$ and $L \in D(U^c) \cap D(V^c)$. Taking the infimum as $x \in P_{U,V}$ and the supremum as $L \in D(U^c) \cap D(V^c)$, we obtain

$$r \le -U^c(L_0) - V^c(-L_0) \le s = \bigvee\{-U^c(L) - V^c(-L) : L \in D(U^c) \cap D(V^c)\} \le r.$$

Thus, $r = -U^c(L_0) - V^c(-L_0) = s$, and the supremum in Problem II is a maximum, assumed by L_0. This concludes the proof. □

Remark 3. In general, the converse of Theorem 2 is not true (see also [55, §2]).

Now we characterize the solutions of Problem I in terms of G-invariant subgradients, and extend [55, Theorem 3] to invariance.

Theorem 4. *Under the assumption $\mathcal{H}$), let $x_0 \in P_{U,V}$ be a solution of Problem* I. *Then,*

$$\partial_{\mathrm{inv}} U(x_0) \cap (-\partial_{\mathrm{inv}} V(x_0)) \neq \emptyset.$$

Proof. Let x_0 be as in the hypothesis. By Theorem 2, Problem II has a solution L_0. Thus, we find an $L_0 \in \mathcal{L}_{\mathrm{inv}}(X, R)$ with

$$U(x_0) + V(x_0) = -U^c(L_0) - V^c(-L_0). \tag{8}$$

By definition of the conjugate function, from (8) we get

$$U(x_0) + V(x_0) \le U(x) + V(x') - L_0(x) + L_0(x') \tag{9}$$

for every $x \in D(U)$ and $x' \in D(V)$. From (9) used with $x = x_0$ and $x' = x_0$ we obtain $-L_0 \in \partial_{\mathrm{inv}} V(x_0)$ and $L_0 \in \partial_{\mathrm{inv}} U(x_0)$, respectively.

Conversely, let $L \in \partial_{\mathrm{inv}} U(x_0) \cap (-\partial_{\mathrm{inv}} V(x_0))$. Then, for every $x \in P_{U,V}$ it is

$$U(x) - U(x_0) \ge L(x) - L(x_0) \ge V(x_0) - V(x)$$

and hence, $U(x) + V(x) \ge U(x_0) + V(x_0)$. Thus, x_0 is a solution of Problem I. □

We now turn to the following versions of the Hahn-Banach theorem, extending [57, Theorem 2.1] to invariance.

Theorem 5. *Let $U : D(U) \to R$ be convex and G-subinvariant, let $D(U)$ be G-invariant, assume that $U(0) = 0$ and let $Z \subset X$ be a G-invariant subspace. Suppose that $0 \in int(D(U) - Z)$. Let $T_0 \in \mathcal{L}_{\text{inv}}(Z, R)$ be such that $T_0(z) \le U(z)$ for every $z \in D(U) \cap Z$.*

Then, T_0 has an extension $T \in \mathcal{L}_{\text{inv}}(X, R)$ such that $T(x) \le U(x)$ for each $x \in D(U)$.

Proof. Let Z, T_0, U be as in the hypotheses of Theorem 5. Put $V = -T_0$. Then, $D(V) = D(T_0) = Z$. Since $T_0(z) \le U(z)$ for each $z \in D(U) \cap Z$ and $T_0(0) = U(0) = 0$, we have

$$0 = U(0) + V(0) = \min\{U(x) + V(x) : x \in D(U) \cap Z\}.$$

By Theorem 4 there exists $T \in \partial_{\text{inv}} U(x_0) \cap (-\partial_{\text{inv}} V(x_0))$. It is

$$T(x) = T(x) - T(0) \le U(x) - U(0) = U(x)$$

for all $x \in D(U)$ and

$$-T(z) = -T(z) + T(0) \le -T_0(z) + T_0(0) = -T_0(z), \tag{10}$$

namely

$$T(z) \ge T_0(z) \tag{11}$$

for all $z \in Z$. Taking in (10) $-z$ instead of z, we obtain

$$T(z) = -T(-z) \le -T_0(-z) = T_0(z) \tag{12}$$

for any $z \in Z$. From (11) and (12) it follows that $T(z) = T_0(z)$ for all $z \in Z$. Thus, we get the assertion. □

Theorem 6. *Let U and $D(U)$ be as in Theorem 5. If $0 \in int(D(U))$ and $U(0) = 0$, then there exists $T \in \mathcal{L}_{\text{inv}}(X, R)$ with $T(x) \le U(x)$ for each $x \in D(U)$.*

Proof. It is an immediate consequence of Theorem 5, taking $Z = \{0\}$. □

Now we extend to invariance [57, Corollary 2.6].

Theorem 7. *Let U and $D(U)$ be as in Theorem 5. If $0 \in int(D(U))$ and $U(0) \ge 0$, then there exists $T \in \mathcal{L}_{\text{inv}}(X, R)$ such that $T(x) \le U(x)$ for all $x \in D(U)$.*

Proof. Set $\widetilde{U}(x) = U(x) - U(0)$. Since $\widetilde{U}(0) = 0$, $\widetilde{U}$ satisfies the hypotheses of Theorem 6. So, there exists $T \in \mathcal{L}_{\mathrm{inv}}(X, R)$ with $T(x) \leq \widetilde{U}(x) \leq U(x)$ for every $x \in D(U)$, since $U(0) \geq 0$. This ends the proof. □

Now we give the following version of the Moreau-Rockafellar formula, extending [55, Theorem 4] to G-invariant subdifferentials (see also [27, 38, 39]).

Theorem 8. *Under the assumption $\mathcal{H}$), let $x_0 \in P_{U,V}$ and assume that $\partial_{\mathrm{inv}} U(x_0) \neq \emptyset$ and $\partial_{\mathrm{inv}} V(x_0) \neq \emptyset$. Then,*

$$\partial_{\mathrm{inv}}(U + V)(x_0) = \partial_{\mathrm{inv}}(U)(x_0) + \partial_{\mathrm{inv}}(V)(x_0). \tag{13}$$

Proof. Pick arbitrarily $L' \in \partial_{\mathrm{inv}}(U + V)(x_0)$, and set $V'(x) = V(x) - L'(x)$, $x \in D(V)$. It is not difficult to check that V' is convex. Moreover, by definition of subdifferential, it is

$$L'(x) - L'(x_0) \leq (U + V)(x) - (U + V)(x_0),$$

and hence,

$$U(x_0) + V(x_0) - L'(x_0) \leq U(x) + V(x) - L'(x) = U(x) + V'(x) \tag{14}$$

for every $x \in D(U) \cap D(V)$.

Furthermore, it is

$$(V')^c(-L) = \bigvee \{L'(x) - L(x) - V(x) : x \in D(V)\} = V^c(L' - L)$$

for every $L \in D((V')^c) = D(V^c) - L'$.

Thanks to (14) and Theorem 2, there exists $L_0 \in \mathcal{L}_{\mathrm{inv}}(X, R)$ with

$$U(x_0) + V(x_0) - L'(x_0) = -U^c(L_0) - (V')^c(-L_0) = -U^c(L_0) - V^c(L' - L_0),$$

namely

$$U^c(L_0) + V^c(L' - L_0) = L_0(x_0) - U(x_0) + (L' - L_0)(x_0) - V(x_0). \tag{15}$$

By definition of the conjugate function, we have

$$U^c(L_0) \geq L_0(x_0) - U(x_0),\ V^c(L' - L_0) \geq L'(x_0) - L_0(x_0) - V(x_0). \tag{16}$$

From (15) and (16) it follows that

$$U^c(L_0) = L_0(x_0) - U(x_0),\ V^c(L' - L_0) = L'(x_0) - L_0(x_0) - V(x_0),$$

and hence,

$$L_0(x) - U(x) \leq L_0(x_0) - U(x_0) \text{ for all } x \in D(U),$$

$$L'(x) - L_0(x) - V(x) \leq L'(x_0) - L_0(x_0) - V(x_0) \text{ for all } x \in D(V).$$

Thus, $L_0 \in \partial_{\text{inv}}(U)(x_0)$, $L' - L_0 \in \partial_{\text{inv}}(V)(x_0)$, and hence, $L' \in \partial_{\text{inv}}(U)(x_0) + \partial_{\text{inv}}(V)(x_0)$. By arbitrariness of L', we deduce

$$\partial_{\text{inv}}(U+V)(x_0) \subset \partial_{\text{inv}}(U)(x_0) + \partial_{\text{inv}}(V)(x_0).$$

The proof of the converse inclusion is straightforward. □

A consequence of Theorem 8 is the following:

Theorem 9. *Let A_1, $A_2 \subset X$ be convex G-invariant sets with $0 \in int(A_1 - A_2)$. Then,*

$$(A_1)^*_{\text{inv}}(x_0) + (A_2)^*_{\text{inv}}(x_0) = (A_1 \cap A_2)^*_{\text{inv}}(x_0)$$

for every $x_0 \in A_1 \cap A_2$.

Proof. Let A_1, A_2 be as in the hypothesis, let $A_0 = A_1 \cap A_2$, and let $\mathcal{N}_{A_j}(x) = 0$ for each $x \in A_j$, $j = 0, 1, 2$. It is not difficult to see that

$$\partial_{\text{inv}} \mathcal{N}_{A_j}(x) = (A_j)^*_{\text{inv}}(x) \text{ for each } x \in A_j,\ j = 0, 1, 2, \tag{17}$$

and

$$\mathcal{N}_{A_0}(x) = \mathcal{N}_{A_1}(x) + \mathcal{N}_{A_2}(x) \text{ for all } x \in A_0. \tag{18}$$

From Theorem 8, (17) and (18) we get

$$\begin{aligned}(A_1)^*_{\text{inv}}(x_0) + (A_2)^*_{\text{inv}}(x_0) &= \partial_{\text{inv}} \mathcal{N}_{A_1}(x_0) + \partial_{\text{inv}} \mathcal{N}_{A_2}(x_0) \\ &= \partial_{\text{inv}}(\mathcal{N}_{A_1} + \mathcal{N}_{A_2})(x_0) \\ &= \partial_{\text{inv}} \mathcal{N}_{A_0}(x_0) \\ &= (A_0)^*_{\text{inv}}(x_0).\end{aligned}$$

So, the assertion follows. □

Now we extend [57, Corollary 3.11] to invariance, proving the following Krein-type monotone extension theorem.

Theorem 10. *Let $X^+ \subset X$ be a G-invariant cone satisfying property $\mathcal{K}$), let $Z \subset X$ be a subspace with the order generated by $Z \cap X^+$ such that $0 \in int(X^+ - Z)$, and let $L_0 \in \mathcal{L}_{+,\text{inv}}(Z, R)$. Then, there exists $L \in \mathcal{L}_{+,\text{inv}}(Z, R)$ with $L(z) = L_0(z)$ for all $z \in Z$.*

Proof. Let X^+, Z and L_0 be as in the hypotheses of Theorem 10. By [36, Theorem 7.3.3], there exists a subspace Z' of X such that every $x \in X$ can be uniquely expressed as $x = z + z'$, where $z \in Z$, $z' \in Z'$. The space Z' is called *algebraic complement* of Z.

Let us define $L_1 : X \to R$ by $L_1(x) = -L_0(z)$, $x \in X$. Since $L_0 \in \mathcal{L}_{+,\mathrm{inv}}(Z, R)$, then $L_1 \in (X^+ \cap Z)^*(0)$. By applying Theorem 9 with $A_1 =^+$, $A_2 = Z$, $x_0 = 0$, we find L, $L' \in \mathcal{L}_{\mathrm{inv}}(X, R)$ with $L_1(x) = L(x) + L'(x)$ for all $x \in X$, $L'(x) \leq 0$ for each $x \in X^+$ and $L(z) \leq 0$ for any $z \in Z$. In particular, we get $-L(z) = L(-z) \leq 0$, and hence, $L(z) = 0$, for all $z \in Z$. If $L^* = -L'$, then $L^* \in \mathcal{L}_{+,\mathrm{inv}}(X, R)$ and $L^*(z) = L_0(z)$ for every $z \in Z$. □

4 Applications.

In this section, as consequences and applications of the results previously given, we present some Farkas and Kuhn-Tucker-type theorems for problems of convex optimization for functions, taking values in partially ordered vector spaces. Successively, we show the equivalence of the given theorems. Finally, we give a further application, proving a Tarski-type extension theorem and considering probability measures defined on suitable product spaces.

With the same notations as above, let X^+ have property $\mathcal{K}$), let D_0, D_1, $D_2, \dots, D_n$ be nonempty convex and G-invariant subsets of X, and let $U : D(U) = D_0 \to R$ be convex. For $i = 1, 2, \dots, n$, consider convex functions $U_i : D(U_i) = D_i \to X$. Suppose that $D = \bigcap_{i=0}^{n} D_i \neq \emptyset$. Moreover, assume that U is G-subinvariant and U_i is *G-equivariant*, that is $U_i(gx) = g(U_i(x))$ for all $g \in G$ and $x \in D$.

Put $W = X^n$, $K = (X^+)^n$. The set K induces on W the "componentwise" order, defined by $y = (y_1, y_2, \dots, y_n) \geq y' = (y'_1, y'_2, \dots, y'_n)$ iff $y_i \geq y'_i$ for all $i = 1,\ 2, \dots, n$. For every $g \in G$ and $w \in W$, $w = (y_1, y_2, \dots, y_n)$, put $gw = (gy_1, gy_2, \dots, gy_n)$. Set

$$h(x) = (U_1(x), U_2(x), \dots, U_n(x)), \quad x \in D.$$

We say that h is *G-equivariant* iff $h(gx) = g(hx)$ whenever $g \in G$ and $x \in D$. Note that this property is equivalent to the G-equivariance of the U_i's, $i = 1, 2, \dots, n$.

Now we consider the following minimization problem.

Problem III) Find $x_0 \in D$ such that $U(x_0) = \min\{U(x) : x \in D,\ U_i(x) \leq 0,\ i = 1,\ 2, \dots, n\}$.

The next Farkas-type theorems extend [40, Theorem 1.1] and [56, Theorem 3] to invariance.

Theorem 11. *Assume that* $0 \in int(h(D) + K)$*, and suppose that, for each* $x \in D$,

$$U(x) \geq 0 \text{ if, for all } i = 1, 2, \ldots, n, \text{ it is } U_i(x) \leq 0. \tag{19}$$

Then, there exist $T_i \in \mathcal{L}_{+,inv}(X, R)$*,* $i = 1, 2, \ldots, n$ *with*

$$U(x) + \sum_{i=1}^{n} T_i(U_i(x)) \geq 0 \text{ for each } x \in D. \tag{20}$$

PROOF. Let $\mathcal{Y} = h(D) + K$. We claim that

11.1) the set $\mathcal{Y}$ is convex.

For $i = 1, 2$, let $x^{(i)} \in D$, $y^{(i)} \in R$, be with $y^{(i)} - h(x^{(i)}) \geq 0$. Choose arbitrarily $\lambda \in [0, 1]$, and set $x_\lambda = \lambda\, x^{(1)} + (1-\lambda)\, x^{(2)}$, $y_\lambda = \lambda\, y^{(1)} + (1-\lambda)\, y^{(2)}$. Then, $x_\lambda \in D$ (since D is convex), $y_\lambda \in R$. Moreover, taking into account the convexity of h (which follows from the convexity of the U_i's), we have

$$y_\lambda \geq \lambda\, h(x^{(1)}) + (1-\lambda)\, h(x^{(2)}) \geq h(x_\lambda),$$

getting the claim.

Now we prove that

11.2) the set $\mathcal{Y}$ is G-invariant.

Pick arbitrarily $w \in \mathcal{Y}$ and $g \in G$. We claim that $gw \in \mathcal{Y}$. Indeed, if $w = h(x) + k$, where x and k are suitable elements of D and K, respectively, then $gw = g(h(x)) + gk = h(gx) + gk$, since h is G-equivariant. As D and K are G-invariant, then $gx \in D$ and $gk \in K$ (indeed, the elements of G are increasing homomorphisms). This yields 11.2).

Now, put

$$A = \{(w, y) \in W \times R : \text{ there exists } x_0 \in D \text{ with } w \geq h(x_0) \text{ and } y \geq U(x_0)\};$$
$$B = \bigcup_{\lambda \geq 0} \lambda\, A. \tag{21}$$

We prove that

11.3) The sets A and B defined in (21) are convex, and

$$(w_1 + w_2, y_1 + y_2) \in B \text{ whenever } (w_1, y_1) \in B \text{ and } (w_2, y_2) \in B. \quad (22)$$

Choose arbitrarily (w_1, y_1), $(w_2, y_2) \in A$ and $\lambda \in [0, 1]$, and let $x_i \in D$ satisfy $w_i \geq h(x_i)$ and $y \geq U(x_i)$, $i = 1, 2$. Since h is convex, we have

$$h(\lambda\, x_1 + (1-\lambda)\, x_2) \leq \lambda\, h(x_1) + (1-\lambda)\, h(x_2) \leq \lambda\, w_1 + (1-\lambda)\, w_2,$$

and similarly

$$U(\lambda\, x_1 + (1-\lambda)\, x_2) \leq \lambda\, y_1 + (1-\lambda)\, y_2.$$

Thus, A is convex. From this, since B is a cone with vertex 0, it is not difficult to deduce the convexity of B and formula (22).

Now we construct a convex and G-subinvariant function $p : \mathcal{Y} \to R$, and prove that p satisfies the hypotheses of Theorem 6.

For every $w \in \mathcal{Y}$, set

$$S_w = \{y \in R : (w, y) \in B\}.$$

We claim that

11.4)

$$S_w \neq \emptyset \text{ for every } w \in \mathcal{Y}. \quad (23)$$

Fix arbitrarily $w \in \mathcal{Y}$. Since $0 \in int(\mathcal{Y})$, there exists a positive real number λ_0 with $\lambda\, w \in \mathcal{Y}$ whenever $|\lambda| \leq \lambda_0$. In particular, $\lambda_0\, w \in \mathcal{Y}$, and hence, there exists $x_0 \in D$ with

$$0 \leq \lambda_0 w - h(x_0) = \lambda_0 \Big(w - \frac{1}{\lambda_0} h(x_0)\Big).$$

As $(\lambda_0\, w, U(x_0)) \in A$, then

$$\Big(w, \frac{1}{\lambda_0} U(x_0)\Big) = \frac{1}{\lambda_0} (\lambda_0\, w, U(x_0)) \in B.$$

Thus, taking $y = \dfrac{1}{\lambda_0} h(x_0)$, we get (23).

Now we claim that

11.5)
$$S_{w_1} + S_{w_2} \subset S_{w_1+w_2} \text{ whenever } w_1,\, w_2 \in \mathcal{Y}. \tag{24}$$

Choose arbitrarily $y_1 \in S_{w_1}$, $y_2 \in S_{w_2}$. Then, $(w_1, y_1) \in B$, $(w_2, y_2) \in B$, and hence, $(w_1 + w_2, y_1 + y_2) \in B$, thanks to (22). Therefore, $y_1 + y_2 \in S_{w_1+w_2}$, getting (24).

Now we prove that

11.6) for each $w \in \mathcal{Y}$, S_w is (order) bounded from below.

Pick arbitrarily $y \in S_w$. As $S_{-w} \neq \emptyset$, then there exists $y_0 \in R$ such that $-y_0 \in S_{-w}$. Then,

$$y - y_0 \in S_w + S_{-w} \subset S_0 = \{z \in R : (0, z) \in B\}, \tag{25}$$

thanks to (24). Hence, there exists $x_0 \in D$ such that $h(x_0) \leq 0$, $z \geq U(x_0)$. From this, (19) and (25) we deduce $U(x_0) = 0$, and a fortiori $y - y_0 \geq 0$. So, the element y_0 is a lower (order) bound for the set S_w, getting 11.6).

Note that a similar argument shows that

$$S_0 \subset R^+. \tag{26}$$

Indeed, if $(0, z) \in B$, then either $z = 0$ or there exists $\lambda > 0$ with $(0, \lambda\, z) \in A$. Hence, $\lambda\, z \geq U(x_0) = 0$. This yields (26).

Thus, it makes sense to define a function $p : \mathcal{Y} \to R$, by

$$p(w) = \bigwedge S_w, \quad w \in \mathcal{Y}. \tag{27}$$

Now, we claim that

11.7)
$$p(0) = 0. \tag{28}$$

If $S_0 = \{0\}$, this is straightforward. Otherwise, there exists $y_0 \in S_0$, $y_0 \neq 0$. By (26), y_0 is (strictly) positive. So, $(0, y_0) \in B$, and hence, $(0, \lambda\, y_0) \in B$ for each $\lambda > 0$. Thus, taking into account (26), it is

$$\begin{aligned} 0 \leq p(0) = \bigwedge S_0 = \bigwedge \{y \in R : (0, y) \in B\} \leq \\ \leq \bigwedge \{\lambda\, y_0 : \lambda > 0\} = 0, \end{aligned} \tag{29}$$

getting (28).

Now we prove that

11.8)
$$p(w_1 + w_2) \leq p(w_1) + p(w_2) \text{ for every } w_1, w_2 \in \mathcal{Y}. \tag{30}$$

Fix arbitrarily w_1, $w_2 \in \mathcal{Y}$. From (27) we get $p(w_1 + w_2) \leq y_1 + y_2$ whenever $y_1 \in S_{w_1}$ and $y_2 \in S_{w_2}$, and thus, by arbitrariness of $y_1 \in S_{w_1}$, $p(w_1+w_2) \leq p(w_1)+y_2$ for every $y_2 \in S_{w_2}$. By arbitrariness of $y_2 \in S_{w_2}$, we deduce (30).

Now we claim that

11.9)
$$p(\lambda\, w) = \lambda\, p(w) \quad \text{ for all } w \in \mathcal{Y} \text{ and } \lambda > 0. \tag{31}$$

Indeed, since B is a cone with vertex 0, it is

$$\begin{aligned} p(\lambda\, w) &= \bigwedge S_{\lambda w} \\ &= \bigwedge \{y \in R : (\lambda\, w, y) \in B\} \\ &= \lambda \bigwedge \Big\{ z \in R : \Big(w, \frac{z}{\lambda}\Big) \in B \Big\} \\ &= \lambda \bigwedge \Big\{ v \in R : (w, v) \in B \Big\} \\ &= \lambda \bigwedge S_w = \lambda\, p(w) \end{aligned} \tag{32}$$

for every $w \in \mathcal{Y}$ and $\lambda > 0$.

Thus, from (30) and (31) we deduce that p is convex.

Now we claim that

11.10)
$$11.10)\ S_w \subset S_{gw} \quad \text{ for all } w \in \mathcal{Y}. \tag{33}$$

Indeed, if $y \in S_w$, then $(w, y) \in B$, and hence, there exist a positive real number λ_0 and an element $x_0 \in D$ with

$$\frac{1}{\lambda_0} w \geq h(x_0) \text{ and } \frac{1}{\lambda_0} y \geq U(x_0). \tag{34}$$

Choose arbitrarily $g \in G$. By applying g in (34), since the elements of G are increasing homomorphisms, D is G-invariant, U is G-subinvariant and h is G-equivariant, we obtain

$$\frac{1}{\lambda_0} gw \geq h(gx_0) \text{ and } \frac{1}{\lambda_0} y \geq U(gx_0). \tag{35}$$

Thus, $(gw, y) \in B$, and hence, $y \in S_{gw}$, getting (33).

From (33) we deduce that $p(gw) = \bigwedge S_{gw} \leq \bigwedge S_w = p(w)$ for each $w \in \mathcal{Y}$. Hence, p is G-subinvariant.

By Theorem 6 used with $U = p$, $D(U) = \mathcal{Y}$, $X = W$, there exists a function $T \in \mathcal{L}_{\text{inv}}(W, R)$ with

$$-T(w) \leq p(w) \text{ for every } w \in \mathcal{Y}. \tag{36}$$

Now we prove (20). Pick arbitrarily $x \in D$ and $k \in K$, namely $k = (c_1, c_2, \dots, c_n)$ with $c_i \geq 0$ for each $i = 1, \dots, n$. By definition of A, we get $(h(x) + k, U(x)) \in A \subset B$, and hence, $U(x) \in S_{h(x)+k}$. By definition of $S_{h(x)+k}$, it is $f(x) - p(h(x) + k) \geq 0$, and a fortiori

$$U(x) + T(h(x) + k) \geq 0, \tag{37}$$

thanks to (36). For every $i = 1, 2, \dots, n$ and $w = (w_1, w_2, \dots, w_n)$, let $T_i(w_i) = T(0, 0, \dots, w_i, 0, \dots, 0)$, where w_i is at the i-th place. Note that G-invariance of the T_i's follows from G-invariance of T. By linearity of T, it is

$$T(h(x) + k) = \sum_{i=1}^{n} T_i(U_i(x) + c_i) = \sum_{i=1}^{n} (T_i(U_i(x)) + T_i(c_i)). \tag{38}$$

From (37) and (38) we deduce

$$U(x) + \sum_{i=1}^{n} (T_i(U_i(x)) + T_i(c_i)) \geq 0 \text{ for every } x \in D. \tag{39}$$

From (39), taking $c_i = 0$ for every $i = 1, 2, \dots, n$, we obtain (20).

Finally, we prove that

11.11) T_i is positive, that is $T_i(a_i) \geq 0$ whenever $a_i \in X$, $a_i \geq 0$, for every $i = 1, \dots, n$.

Pick arbitrarily $\overline{x} \in X$, $i \in \{1, \dots, n\}$ and $\nu \in \mathbb{N}$. If in (39) we take $c_i = \nu\, a_i$ and $c_j = 0$ whenever $j \neq i$, then we get

$$\nu(T_i(a_i)) = T_i(\nu\, a_i) \geq -\sum_{i=1}^{n} T_i(U_i(\overline{x})) - U(\overline{x}), \tag{40}$$

that is

$$T_i(a_i) \geq \frac{1}{\nu} \Big(-\sum_{i=1}^{n} T_i(U_i(\overline{x})) - U(\overline{x}) \Big). \tag{41}$$

By arbitrariness of ν, taking into account Dedekind completeness of R, from (41) we get

$$T_i(a_i) \geq \bigvee_{\nu \in \mathbb{N}} \Big[\frac{1}{\nu} \Big(- \sum_{i=1}^{n} T_i(U_i(\overline{x})) - U(\overline{x}) \Big) \Big] = 0,$$

since, thanks to (20), it is

$$\sum_{i=1}^{n} T_i(U_i(\overline{x})) + U(\overline{x}) \geq 0.$$

Thus, we get 11.11). This ends the proof.

□

The next result is a consequence of Theorem 11, and extends [40, Theorem 3.1] and [56, Corollary of Theorem 3] to invariance.

Theorem 12. *Let $Z \subset X$ be a G-invariant subspace, let $L \in \mathcal{L}_{inv}(Z, R)$, let $F_i \in \mathcal{L}(Z, X)$, $i = 1, 2, \ldots, n$ be G-equivariant linear functions, let $b \in R$, and let $a = (a_1, a_2, \ldots, a_n)$ be with $a_i \in X$, $i = 1, 2, \ldots, n$. Assume that*

12.1) $0 \in int(a - F(Z) + K)$.

Moreover, suppose that, for each $x \in Z$, it is

12.2)

$$L(x) \geq b \text{ whenever } F_i(x) \geq a_i \text{ for all } i = 1, 2, \ldots, n. \tag{42}$$

Then, there exist $T_i \in \mathcal{L}_{+,inv}(X, R)$, $i = 1, 2, \ldots, n$ with

$$L(x) = \sum_{i=1}^{n} T_i(F_i(x)) \text{ for each } x \in Z \text{ and } \sum_{i=1}^{n} T_i(a_i) \geq b. \tag{43}$$

Proof. For every $x \in Z$, let

$$U(x) = L(x) - b,\ U_i(x) = a_i - F_i(x),\ i = 1, \ldots, n,\ a = (a_1, a_2, \ldots, a_n),$$
$$F(x) = (F_1(x), F_2(x), \ldots, F_n(x)),$$
$$h(x) = (U_1(x), U_2(x), \ldots, U_n(x)) = a - F(x).$$

By 12.1), $0 \in int(h(Z) + K)$. Moreover, by 12.2), we get $U(x) \geq 0$, for each $x \in Z$ such that $U_i(x) \leq 0$ for all $i = 1, 2, \ldots, n$. By Theorem 11, there exist $T_i \in \mathcal{L}_{+,\mathrm{inv}}(X, R)$, $i = 1, 2, \ldots, n$ with

$$U(x) + \sum_{i=1}^{n} T_i(U_i(x)) \geq 0 \text{ for each } x \in Z,$$

namely

$$L(x) + \sum_{i=1}^{n} T_i(-F_i(x)) \geq b - \sum_{i=1}^{n} T_i(a_i) \text{ for any } x \in Z. \tag{44}$$

If we choose $x = 0$ in (44), then we have

$$b - \sum_{i=1}^{n} T_i(a_i) \leq 0. \tag{45}$$

Pick arbitrarily $x \in Z$ and $\nu \in \mathbb{N}$. It is

$$\nu\Big(L(x) + \sum_{i=1}^{n} T_i(-F_i(x))\Big) = (L(\nu x) + \sum_{i=1}^{n} T_i(-F_i(\nu x)) \geq b - \sum_{i=1}^{n} T_i(a_i).$$

From this we deduce

$$L(x) + \sum_{i=1}^{n} T_i(-F_i(x)) \geq \frac{1}{\nu}\Big(b - \sum_{i=1}^{n} T_i(a_i)\Big) \text{ for any } x \in Z. \tag{46}$$

By arbitrariness of ν and taking into account (45), from (46) we obtain

$$L(x) + \sum_{i=1}^{n} T_i(-F_i(x)) \geq \bigvee_{\nu \in \mathbb{N}} \Big[\frac{1}{\nu}\Big(b - \sum_{i=1}^{n} T_i(a_i)\Big)\Big] = 0 \text{ for every } x \in Z,$$

that is (43). This completes the proof. □

Another consequence of Theorem 11 is the following Kuhn-Tucker-type condition for the existence of saddle points for Problem III, which extends [56, Theorem 5] to invariance.

Theorem 13. *Under the same hypotheses as in Theorem* 11 *, if x_0 is a solution of Problem* III, *then there exist $T_i^0 \in \mathcal{L}_{+,inv}(X, R)$, $i = 1, 2, \ldots, n$ with*

$$U(x_0) + \sum_{i=1}^{n} T_i(U_i(x_0)) \leq U(x_0) + \sum_{i=1}^{n} T_i^0(U_i(x_0)) \leq U(x) + \sum_{i=1}^{n} T_i^0(U_i(x))$$

for every $x \in D$ and $T_i \in \mathcal{L}_{+,inv}(X, R)$.

Proof. Let x_0 be a solution of Problem III. Set $U'(x) = U(x) - U(x_0)$, $x \in D$. It is not difficult to see that U' and U_i, $i = 1, 2, \ldots, n$, satisfy (19), and thus also (20), thanks to Theorem 11. Hence there exist $T_1^0, T_2^0, \dots T_n^0 \in \mathcal{L}_{+,\mathrm{inv}}(X,R)$ with

$$\sum_{i=1}^{n} T_i^0(U_i(x)) + U'(x) \geq 0 \tag{47}$$

for every $x \in D$. Taking $x = x_0$ in (47), we obtain

$$\sum_{i=1}^{n} T_i^0(U_i(x_0)) \geq 0. \tag{48}$$

Since $U_i(x_0) \leq 0$ and T_i^0 are positive for every $i = 1, 2, \dots, n$, from (48) it follows that

$$\sum_{i=1}^{n} T_i^0(U_i(x_0)) = 0. \tag{49}$$

Moreover, for each $T_i \in \mathcal{L}_{+,\mathrm{inv}}(X,R)$, $i = 1, 2, \dots, n$, we get

$$U(x_0) + \sum_{i=1}^{n} T_i(U_i(x_0)) \leq U(x_0) = U(x_0) + \sum_{i=1}^{n} T_i^0(U_i(x_0)). \tag{50}$$

From (47) and (49) it follows that

$$0 \leq \sum_{i=1}^{n} T_i^0(U_i(x)) - \sum_{i=1}^{n} T_i^0(U_i(x_0)) + U(x) - U(x_0),$$

that is

$$U(x_0) + \sum_{i=1}^{n} T_i^0(U_i(x_0)) \leq U(x) + \sum_{i=1}^{n} T_i^0(U_i(x)). \tag{51}$$

Thus, the assertion follows from (50) and (51). □

Remark 14. (a) Observe that, arguing similarly as in the proof of Theorem 11 and [43, Theorem 3 (1)], it is possible to see that, when $\mathcal{Y}$ is also algebraically expanded, Theorem 11 holds even if we require the convexity of B and the hypothesis that $(gx, y) \in B$ whenever $(x, y) \in B$ and $g \in G$, instead of the corresponding properties for the set A. Thus, it is possible to include in our setting even some cases of non-convex optimization (see also [43, Remark]).

(b) Note that, when X^+ is algebraically closed and $int(X^+) \neq \emptyset$, the converse of Theorem 13 holds, and follows directly from its corresponding version without invariance (see also [36, §17.5, (2)], [43, Theorem 3 (2)], [56, Theorem 1 and following Remark]).

The next result is a consequence of Theorem 12.

Theorem 15. *Let $X^+ \subset X$ be a G-invariant cone satisfying property $\mathcal{K}$), let $Z \subset X$ be a G-invariant subspace, let $T_1 \in \mathcal{L}(Z, X)$ be a G-equivariant function and let $T_2 \in \mathcal{L}_{\text{inv}}(Z, R)$. Assume that*

$$0 \in int(-T_1(Z) + X^+), \tag{52}$$

and

$$T_2(x) \geq 0 \text{ whenever } x \text{ belongs to } Z \text{ and } T_1(x) \leq 0. \tag{53}$$

Then there exists $T_0 \in \mathcal{L}_{+,\text{inv}}(X, R)$ with

$$T_0(T_1(x)) + T_2(x) = 0 \text{ for all } x \in Z. \tag{54}$$

Proof. By virtue of (52) and Theorem 12 used with $n = 1$, $F = -T_1$, $L = T_2$, $a = 0$, $b = 0$, there exists $T_0 \in \mathcal{L}_{+,\text{inv}}(X, R)$ such that

$$T_2(x) = T_0(-T_1(x)) \text{ for each } x \in Z,$$

that is (54). This ends the proof. □

Now, using Theorem 6, we prove the existence of affine manifolds, separating two nonempty sets of a product space in the setting of partially ordered vector spaces. We extend [34, Theorem 4.1] and [43, Theorem 1] to invariance.

Theorem 16. *Let A, B be two nonempty subsets of $X \times R$ such that $A - B$ is convex. Assume that*

$$(gx, y) \in A - B \text{ whenever } (x, y) \in A - B \text{ and } g \in G. \tag{55}$$

Suppose that

$$0 \in int(P_X(A - B)) \tag{56}$$

and

$$y_1 \geq y_2 \text{ whenever } (x, y_1) \in A \text{ and } (x, y_2) \in B. \tag{57}$$

Then there exist $L \in \mathcal{L}_{inv}(X, R)$ and $u_0 \in R$ such that the affine manifold

$$H = \{(x, y) \in X \times R : L(x) - y = u_0\} \tag{58}$$

separates A and B.

Proof. We begin with proving that

16.1) the set $P_X(A-B)$ is convex.

Let $x_1,\, x_2 \in P_X(A-B)$ and $\lambda \in [0,1]$. By definition of $P_X(A-B)$, there exist $y_1,\, y_2 \in R$ with (x_1,y_1), $(x_2,y_2) \in A-B$. Since $A-B$ is convex, we get

$$(\lambda\, x_1 + (1-\lambda)\, x_2, \lambda\, y_1 + (1-\lambda)\, y_2) \in A-B. \tag{59}$$

Hence, $\lambda\, x_1 + (1-\lambda)\, x_2 \in P_X(A-B)$, getting 16.1).

Now we claim that

16.2) the set $P_X(A-B)$ is G-invariant.

Let $x \in P_X(A-B)$ and $g \in G$. There exists $y \in R$ such that $(x,y) \in A-B$. By (55), $(gx, gy) \in A-B$, and hence, $gx \in P_X(A-B)$. So, we get the claim.

Now, let

$$C = \bigcup_{\lambda \geq 0} \lambda\,(A-B). \tag{60}$$

Note that

16.3) the set C defined in (60) is convex, and

$$(x_1+x_2, y_1+y_2) \in C \text{ whenever } (x_1,y_1) \in C \text{ and } (x_2,y_2) \in C. \tag{61}$$

The proof of 16.3) is analogous to that of 11.3).

Now we define a convex and G-subinvariant function $p^* : P_X(A-B) \to R$, which fulfils the hypotheses of Theorem 6.

For every $x \in P_X(A-B)$, set

$$E_x = \{y \in R : (x,y) \in C\}.$$

We claim that

$$16.4)\ E_x \neq \emptyset \text{ for every } x \in P_X(A-B). \tag{62}$$

Fix arbitrarily $x \in P_X(A-B)$. As $0 \in int(P_X(A-B))$, there exists $\lambda_0 > 0$ with $\lambda\, x \in P_X(A-B)$ whenever $|\lambda| \leq \lambda_0$. In particular, there exists $y \in R$ with $(\lambda_0\, x, y) \in A-B$. From this it follows that

$$\left(x, \frac{1}{\lambda_0} y\right) = \frac{1}{\lambda_0}(\lambda_0\, x, y) \in C,$$

getting the claim.

Now, put

$$p^*(x) = \bigwedge\{y : y \in E_x\}, \quad x \in P_X(A - B).$$

Proceeding analogously as in the proof of Theorem 11, it is possible to prove that p^* is convex and G-subinvariant, and $p^*(0) = 0$.

By Theorem 6, there exists $T \in \mathcal{L}_{\text{inv}}(X, R)$ with $T(x) \le p^*(x)$ whenever $x \in P_X(A - B)$. From this it follows that

$$T(x_1) - T(x_2) = T(x_1 - x_2) \le p^*(x_1 - x_2) \le y_1 - y_2 \qquad (63)$$

for any $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$. From (63) we obtain $T(x_1) - y_1 \le T(x_2) - y_2$. As R is Dedekind complete, there exists $u_0 \in R$ with

$$\bigvee\{T(x_1) - y_1 : (x_1, y_1) \in A\} \le u_0 \le \bigwedge\{T(x_2) - y_2 : (x_2, y_2) \in B\}.$$

This ends the proof.

□

Now we prove the following equivalence results between our given theorems, extending [27, Theorems 1 and 2] to invariance.

Theorem 17.

17.1) Theorems 1-2,4-7, *and* 16 *are equivalent.*

Moreover, the following implications hold:

(2) $\Longrightarrow$ (8) $\Longrightarrow$ (9) $\Longrightarrow$ (10);

(6) $\Longrightarrow$ (11) $\Longrightarrow$ (13) $\Longrightarrow$ (15) $\Longrightarrow$ (10);

(11) $\Longrightarrow$ (12) $\Longrightarrow$ (15).

17.2) *If int*$(R^+) \neq \emptyset$*, then Theorems* 1-2,4-13,15-16 *are equivalent.*

PROOF. (1) $\Longrightarrow$ (2) See Theorem 2.
(2) $\Longrightarrow$ (4) See Theorem 4.
(4) $\Longrightarrow$ (5) See Theorem 5.
(5) $\Longrightarrow$ (6) See Theorem 6.
(6) $\Longrightarrow$ (7) See Theorem 7.
(2) $\Longrightarrow$ (8) See Theorem 8.
(8) $\Longrightarrow$ (9) See Theorem 9.

(9) $\Longrightarrow$ (10) See Theorem 10.
(6) $\Longrightarrow$ (11) See Theorem 11.
(11) $\Longrightarrow$ (12) See Theorem 12.
(11) $\Longrightarrow$ (13) See Theorem 13.
(12) $\Longrightarrow$ (15) See Theorem 15.
(6) $\Longrightarrow$ (16) See Theorem 16.
(7) $\Longrightarrow$ (1) Set $\mathcal{D} = D(U) - D(V)$, and $\mathcal{E} = \{(x_1 - x_2, U(x_1) + V(x_2) + z) \in X \times R$: there exist $x_1 \in D(U)$, $x_2 \in D(V)$ and $z \in R^+$ such that $x = x_1 - x_2$, $y = U(x_1) + V(x_2) + z\}$. For each $x \in \mathcal{D}$, put $\mathcal{E}_x = \{y \in R : (x, y) \in \mathcal{E}\}$ (see also [57, Sandwich Theorem 3.1]).

We claim that $\mathcal{E}_x \neq \emptyset$ for every $x \in \mathcal{D}$. Indeed, choose arbitrarily $x \in \mathcal{D}$. There are $x_1 \in D(U)$, $x_2 \in D(V)$ with $x = x_1 - x_2$. Taking $z = 0$, we get that $(x_1 - x_2, U(x_1) + V(x_2)) \in \mathcal{E}$, getting the claim.

It is not difficult to check that, since U, V are convex and $D(U)$, $D(V)$ are convex, then $\mathcal{D}$ and $\mathcal{E}$ are convex. Furthermore, note that $\mathcal{E}_0 \subset R^+$: indeed, if $y \in \mathcal{E}_0$, then there exist $x_0 \in P_{U,V}$ and $z \in R^+$ with $y = U(x_0) + V(x_0) + z \geq 0$, since $U(x) + V(x) \geq 0$ for any $x \in P_{U,V}$.

Now we prove that

$$(gx, y) \in \mathcal{E} \text{ whenever } (x, y) \in \mathcal{E} \text{ and } g \in G. \tag{64}$$

Fix arbitrarily $(x, y) \in \mathcal{E}$ and $g \in G$, and let x_1, x_2 and z be as in the definition of $\mathcal{E}$. By G-subinvariance of U and V, there exists $\zeta \in R^+$ such that $y = U(gx_1) + V(gx_2) + \zeta + z = U(x_1) + V(x_2) + z$. Since $g(x_1 - x_2) = gx_1 - gx_2$, we get $(gx, y) \in \mathcal{E}$, that is (64).

Now we define $p : \mathcal{D} \to R$ by $p(x) = \bigwedge \mathcal{E}_x$, $x \in \mathcal{D}$. We show that p is well-defined. Pick arbitrarily $x \in \mathcal{D}$. Since, by hypothesis, $0 \in int(\mathcal{D})$, we find a $\lambda > 0$ with $-\lambda x \in \mathcal{D}$. Pick $y' \in \mathcal{E}_{-\lambda x}$. Since $\mathcal{E}$ is convex, then for every $y \in \mathcal{E}_x$ it is

$$\Big(0, \frac{1}{1+\lambda} y' + \frac{\lambda}{1+\lambda} y\Big) = \frac{1}{1+\lambda}(-\lambda x, y') + \frac{\lambda}{1+\lambda}(x, y) \in \mathcal{E},$$

namely, $\dfrac{1}{1+\lambda} y' + \dfrac{\lambda}{1+\lambda} y \in \mathcal{E}_0$, and hence, $\dfrac{1}{1+\lambda} y' + \dfrac{\lambda}{1+\lambda} y \geq 0$, as $\mathcal{E}_0 \subset R^+$. So, since R is Dedekind complete, $\bigwedge \mathcal{E}_x$ exists in R.

Now we prove that p is convex. If $y_1 \in \mathcal{E}_{x_1}$, $y_2 \in \mathcal{E}_{x_2}$ and $\lambda \in [0, 1]$, then $(x_1, y_1) \in \mathcal{E}$, $(x_2, y_2) \in \mathcal{E}$, and by convexity of $\mathcal{E}$ we get $(\lambda x_1 + (1-\lambda) x_2, \lambda y_1 + (1-\lambda) y_2) \in \mathcal{E}$. Thus, we obtain

$$\lambda y_1 + (1-\lambda) y_2 \in \mathcal{E}_{\lambda x_1 + (1-\lambda) x_2}. \tag{65}$$

From (65) and arbitrariness of y_1 and y_2 we obtain

$$\begin{aligned} p(\lambda x_1 + (1-\lambda)x_2) &= \bigwedge \mathcal{E}_{\lambda x_1 + (1-\lambda)x_2} \\ &\leq \lambda \bigwedge \mathcal{E}_{x_1} + (1-\lambda) \bigwedge \mathcal{E}_{x_2} \\ &= \lambda\, p(x_1) + (1-\lambda)\, p(x_2), \end{aligned}$$

which means that p is convex.

Furthermore we get $p(0) \geq 0$, as $\mathcal{E}_0 \subset R^+$, and $p(gx) = \bigwedge \mathcal{E}_{gx} \leq \bigwedge \mathcal{E}_x = p(x)$ for all $x \in \mathcal{D}$ and $g \in G$, thanks to (64). Thus, p is G-subinvariant.

By Theorem 7, there exists $L \in \mathcal{L}_{\text{inv}}(X, R)$ such that

$$L(x - x') \leq p(x - x') = \bigwedge \mathcal{E}_{x-x'} \leq U(x) + V(x'),$$

and hence,

$$L(x) - U(x) \leq L(x') + V(x') \tag{66}$$

for any $x \in D(U)$ and $x' \in D(V)$. Set

$$u_0 = \bigwedge \{L(x') + V(x') : x' \in D(V)\}. \tag{67}$$

Note that $u_0 \in R$, since R is Dedekind complete. From (66) we get $L(x) - u_0 \leq U(x)$ for each $x \in D(U)$, and from (67) we obtain $L(x') - u_0 \geq -V(x')$ for all $x' \in D(V)$. Thus, we get 1.

(13) $\Longrightarrow$ (15). Set $U = L_1$, $V = L_2$ and $x_0 = 0$. By Theorem 13 used with $n = 1$, $U_1 = T_1$, $U = T_2$ and $x_0 = 0$, we find $T_0 \in \mathcal{L}_{\text{inv}}(X, R)$ with

$$T_0(T_1(x)) + T_2(x) \geq 0 \text{ for each } x \in Z. \tag{68}$$

Changing x with $-x$ in (68), we obtain

$$T_0(T_1(x)) + T_2(x) \leq 0,$$

and hence,

$$T_0(T_1(x)) + T_2(x) = 0 \text{ for all } x \in Z.$$

Thus we get 15.

(15) $\Longrightarrow$ (10). Let $L_1(z) = -z$ and $L_2(z) = L_0(z)$ for all $z \in Z$. From (53) we obtain $L_0(z) \geq 0$ for each $z \in Z \cap X^+$. By Theorem 13, we find a function $L \in \mathcal{L}_{+,\text{inv}}(X, R)$ with $L(-z) + L_0(z) = -L(z) + L_0(z) = 0$ for each $z \in Z$. So, 10 follows.

(16) $\Longrightarrow$ (5). Let Z be any G-invariant subspace of X, let $T_0 \in \mathcal{L}_{\text{inv}}(Z, R)$, let $U : D(U) \to R$ be a convex and G-subinvariant function, assume that $U(0) = 0$ and suppose that $T_0(z) \leq U(z)$ for every $z \in Z$. Let

$$A = \{(x, y) \in X \times R, y \geq U(x)\},$$

$$B = \{(x, y) \in Z \times R, y \leq T_0(x)\}.$$

It is not difficult to see that $A - B$ is a nonempty convex subset of $X \times R$ and satisfies (55), because U is convex and G-subinvariant, and T_0 is linear and G-invariant. Moreover, we have $P_X(A) = D(U)$, $P_X(B) = Z$,

$$P_X(A - B) = P_X(A) - P_X(B) = D(U) - Z.$$

Since, by hypothesis, $0 \in int(D(U) - Z)$, then we get $0 \in int(P_X(A - B))$. Furthermore, if $(x, y_1) \in A$, $(x, y_2) \in B$, then $x \in Z$, and $y_1 \geq U(x) \geq L_0(x) \geq y_2$.

By Theorem 16, there exist $T \in \mathcal{L}_{\text{inv}}(X, R)$ and $u_0 \in R$ with

$$T(x_1) - y_1 \leq u_0 \leq T(x_2) - y_2 \text{ for all } (x_1, y_1) \in A \text{ and } (x_2, y_2) \in B. \tag{69}$$

Choose arbitrarily $z \in Z$. Since $U(0) = 0$, from (69) used with $x_1 = 0$, $x_2 = z$, $y_1 = 0$, $y_2 = T_0(z)$ and with $x_1 = 0$, $x_2 = -z$, $y_1 = 0$, $y_2 = T_0(z)$ we obtain

$$\begin{aligned} &0 \leq u_0 \leq T(z) - T_0(z), \\ &0 \leq u_0 \leq T(-z) - T_0(-z) = T_0(z) - T(z), \end{aligned} \tag{70}$$

respectively. From (70) we deduce that $u_0 = 0$ and $T(z) = T_0(z)$.

Now, pick arbitrarily $x \in X$. From (69) used with $x_1 = x$, $y_1 = U(x)$, $x_2 = 0$, $y_2 = 0$, we obtain $T(x) - U(x) \leq 0$. Thus, we get 5.

Finally, assume that $int(R^+) \neq \emptyset$ and let us prove (10) $\Longrightarrow$ (5). Let $U : D(U) \to R$ be a convex and G-invariant function such that $D(U)$ is G-invariant and $U(0) = 0$. Let

$$C = \{(x, y) \in X \times R, y \geq U(x)\}.$$

As U is convex, G-subinvariant and $U(0) = 0$, it is not difficult to check that C is a convex cone with vertex $(0, 0)$, and that $(gx, y) \in C$ whenever $(x, y) \in C$ and $g \in G$. In the set $X \times R$, we consider the order generated by the cone C. Moreover, since $int(R^+) \neq \emptyset$ and $int(X) = X$, it is $int(C) \neq \emptyset$ and

$$(int(C)) \cap (Z \times R) \supset \{0\} \times int(R^+) \neq \emptyset$$

(see also [27, Theorem 2]). Since $int(Z \times R) = Z \times R$ and $int(C - (Z \times R)) \supset int(C)) - int(Z \times R)$, it follows that $0 \in int(C - (Z \times R))$ (see also [3, I.8.4.a)], [43, Corollary, (4)]).

Define the action of G on $X \times R$ by $g(x, y) = (gx, y)$, $(x, y) \in X \times R$. Let $L_0 \in \mathcal{L}_{\text{inv}}(Z, R)$ be such that $L_0(z) \leq U(z)$ for all $z \in D(U) \cap Z$. Define $L' : Z \times R \to R$ by $L'(z, y) = -L_0(z) + y$, $(z, y) \in Z \times R$. It is not difficult to check that $L'(z, y) \geq 0$ for all $(z, y) \in (Z \times R) \cap (C)$. By Theorem 10, there exists $L'' \in \mathcal{L}_{\text{inv}}(X \times R, R)$ with $L''(z, y) = L'(z, y) = -L_0(z) + y$ for every $(z, y) \in Z \times R$ and $L''(x, y) \geq 0$ for any $(x, y) \in C$. Thus, $L''(0, y) = y$ for all $y \in R$. Set $L(x) = -L''(x, 0)$, $x \in X$. It is easy to see that $L \in \mathcal{L}_{\text{inv}}(X, R)$. We have $L(x) = y - L''(x, y)$ for every $(x, y) \in X \times R$. Thus, when $y = U(x)$, we get $L(x) \leq y = U(x)$. Moreover, for any $z \in Z$ and $y \in R$, it is $L(z) = y - L''(z, y) = y - L'(z, y) = L_0(z)$. So, we get 5. □

Remark 18. Observe that our given results are a characterization of the amenability of G and of the Dedekind completeness of R (see also [7, 8] and [27, 51], respectively).

Now we give some applications of the given results to finitely additive and invariant vector lattice-valued set functions. The classical Vitali example shows that it is not possible to define a σ-additive measure on the family of all subsets of the real line, invariant with respect to the group of all translations, extending the Lebesgue measure (see also [53]). Such a pathology does not exist in the finitely additive case, thanks to the Tarski theorem, whose we give an extension to the setting of partially ordered vector spaces and invariance with respect to any amenable semigroup. The next result is a consequence of Theorem 1.

Theorem 19. *Let $\Omega \neq \emptyset$ be any set, let $\mathcal{P}(\Omega)$ be the family of all subsets of Ω, let $\mathcal{A} \subset \mathcal{P}(\Omega)$ be an algebra, let $G \subset \Omega^\Omega$ be a left (resp. right) amenable semigroup of functions such that $g^{-1}(A) = \{\omega \in \Omega\colon g\omega \in A\}$ (resp. $\{\omega \in \Omega\colon \omega g \in A\}$) $\in \mathcal{A}$ for all $A \in \mathcal{A}$ and $g \in G$, and let $\psi : \mathcal{A} \to R^+$ be a G-invariant finitely additive set function (that is, $\psi(g^{-1}(A)) = \psi(A)$ for every $A \in \mathcal{A}$ and $g \in G$). Then ψ has a G-invariant finitely additive extension $\widetilde{\psi} : \mathcal{P}(\Omega) \to R^+$.*

PROOF. We prove the theorem only in the setting of left amenability, since the case of right amenability is analogous.

Let X be the space of all bounded real-valued functions defined on Ω and let Z be the linear subspace of X generated by all characteristic functions χ_A, as A varies in $\mathcal{A}$ (that is, $\chi_A(\omega) = 1$ if $\omega \in A$ and $\chi_A(\omega) = 0$ if $\omega \in \Omega \setminus A$). For each $f \in X$, $g \in G$ and $\omega \in \Omega$, put $(gf)(\omega) = f(g\omega)$. Note that, for each $g \in G$, g is an increasing linear homomorphism. For every $f \in Z$,

$f = \sum_{i=1}^{n} r_i \chi_{A_i}$ with $A_i \in \mathcal{A}$, $i = 1, 2, \dots, n$, set $\phi(f) = \sum_{i=1}^{n} r_i \psi(A_i)$. It is easy to check that ϕ is well-defined (that is, independent of the representation of f), linear, increasing and G-invariant. Let us define $\phi^+ : X \to R$, $\phi^- : X \to R$ by

$$\begin{aligned} \phi^+(x) &= \bigwedge\Big\{\phi(z) : x \le z, z \in Z\Big\}, \\ \phi^-(x) &= \bigvee\Big\{\phi(z) : x \ge z', z' \in Z\Big\}. \end{aligned} \tag{71}$$

Pick arbitrarily $x \in X$. Thanks to monotonicity of ϕ, for each z, $z' \in Z$ such that $z' \le x \le z$ it is $\phi(z') \le \phi(z)$. Taking the supremum and the infimum in (71), we deduce $\phi^-(x) \le \phi^+(x)$. Moreover, it is not difficult to see that $\phi^+(z) = \phi^-(z) = \phi(z)$ whenever $z \in Z$.

Now we prove that ϕ^+ is convex. Choose arbitrarily x_1, $x_2 \in X$ and $\lambda \in [0, 1]$, and take z_1, $z_2 \in Z$ with $x_i \le z_i$, $i = 1, 2$. Then $\lambda x_1 + (1-\lambda)x_2 \le \lambda z_1 + (1-\lambda)z_2$. As ϕ is linear, we obtain

$$\phi^+(\lambda x_1 + (1-\lambda)x_2) \le \phi(\lambda z_1 + (1-\lambda)z_2) = \lambda\phi(z_1) + (1-\lambda)\phi(z_2).$$

By arbitrariness of z_1 and z_2, we get

$$\phi^+(\lambda x_1 + (1-\lambda)x_2) \le \lambda\phi^+(x_1) + (1-\lambda)\phi^+(x_2).$$

Thus, ϕ^+ is convex. Analogously, it is possible to prove that $-\phi^-$ is convex.

Now we claim that ϕ^+ is G-subinvariant. Choose arbitrarily $x \in X$ and $g \in G$. For each $z \in Z$ such that $x \le z$, it is $gx \le gz$. Thus, we have $\phi^+(gx) \le \phi(gz) = \phi(z)$. Taking the infimum, we deduce $\phi^+(gx) \le \phi(x)$, getting the claim. Similarly, it is possible to check that $-\phi^-$ is G-subinvariant.

By virtue of Theorem 1, there exist $\widetilde{\phi} \in \mathcal{L}_{\rm inv}(X, R)$ and $u_0 \in R$ such that

$$\phi^-(x) \le \widetilde{\phi}(x) - u_0 \le \phi^+(x) \tag{72}$$

for all $x \in X$. As $\phi^+(0) = \phi^-(0) = 0$, by (72) used with $x = 0$ we get $u_0 = \widetilde{\phi}(0) = 0$. Thus, $\widetilde{\phi}(z) = \phi(z)$ for every $z \in Z$. Putting $\widetilde{\psi}(C) = \widetilde{\phi}(\chi_C)$, $C \subset \Omega$, we obtain that $\widetilde{\psi}$ is the requested extension. □

Now we apply Theorem 19 to construct finitely additive and invariant extension of set functions. In the literature, in certain types of problems and investigations (for instance, in stochastic processes), it would be advisable to deal with some kinds of "probabilities", which associate to each event not necessarily a real number, but a real-valued function. Indeed, it is possible to give different values of the knowledge on an event E, depending, for instance, on the time or on some other events associated to E (for example, in the case of conditional probability). More generally, it is advisable to associate to each event an element of a Dedekind complete vector lattice R, because, by virtue of the Maeda-Ogasawara-Vulikh representation Theorem, R can be viewed as a suitable space of continuous extended real-valued functions.

On the other hand, in order to take into account the time, it is advisable also to associate to an event a time value, for example the "current" instant, in which the involved "probabilities" are formulated, taking into account the state of knowledge at this instant. Thus, we can consider as "events" some sets of the type $\prod_{t\in\mathcal{T}} B_t$, where $\mathcal{T}$ is the "time set" and the B_t's are subsets of a fixed abstract nonempty set $\widetilde{R}$. Now we present the following

Example 20. Let R be a Dedekind complete vector lattice, let $\widetilde{R}$ and $\mathcal{T}$ be any two nonempty sets, let $\mathcal{B}$ be an algebra of subsets of $\widetilde{R}$, let $\Omega = \widetilde{R}^{\mathcal{T}}$ and let $\mathcal{C} = \{f \in \Omega$: there exist $t \in \mathcal{T}$, $B \in \mathcal{B}$: $f(t) \in B\}$. Observe that $\mathcal{C}$ is not an algebra: indeed, if C_1, $C_2 \in \mathcal{C}$, $t_1 \neq t_2 \in \mathcal{T}$, $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$, then $C_1 \cup C_2 = \{f \in G : f(t_1) \in B_1\} \cup \{f \in G : f(t_2) \in B_2\} \notin \mathcal{C}$.

It is not difficult to check that the algebra $\mathcal{A}(\mathcal{C})$ generated by $\mathcal{C}$ is the family of all finite (disjoint) unions of sets of the type

$$E = \bigcap_{i\in\Lambda} \{f \in \Omega : f(t_i) \in B_i\}, \tag{73}$$

where Λ is a finite subset of $\mathcal{T}$.

Let $G \subset \mathcal{T}^{\mathcal{T}}$ be any left or right amenable semigroup of functions (in this setting a concrete example, when $\mathcal{T} = \mathbb{R}^n$, is the group generated by a given isometry). If $f \in \widetilde{R}^{\mathcal{T}}$, then we define the action of G on $\widetilde{R}^{\mathcal{T}}$ by setting $(\tau f)(t) = f(\tau(t))$, $t \in \mathcal{T}$. It is not difficult to check that the families $\mathcal{C}$ and $\mathcal{A}(\mathcal{C})$ are G-invariant.

Let $u \in R^+$ be such that $u \neq 0$, and for each $t \in \mathcal{T}$ let $P_t : \mathcal{B} \to R$ be a finitely additive set function with $P_{\tau(t)}(B) = P_t(B)$ whenever $B \in \mathcal{B}$ and $\tau \in G$ such that $P_t(\emptyset) = 0$ and with $P_t(\widetilde{R}) = u$ for any $t \in \mathcal{T}$. Note that $\{P_t(B) : t \in \mathcal{T}, B \in \mathcal{B}\} \subset V[u] = \{x \in R$: there exists a positive real number c with $-cu \leq x \leq cu\}$. Since $V[u]$ is Dedekind complete, by virtue of the Kakutani representation theorem, there exist a compact Hausdorff topological space Ξ and an isomorphism ι from $V[u]$ into $C(\Xi) = \{f \in \mathbb{R}^{\Xi}$: f is continuous$\}$, which maps u into $\mathbf{1}_{\Xi}$, the function defined on Ξ and which associates to each element of Ξ the constant value 1 (see also [54]). Since the P_t's are (order) equibounded, then they can be considered as $V[u]$-valued set functions. For every set E as in (73), where $\Lambda = \{t_1, t_2, \dots, t_q\}$, define

$$P(E) = \iota^{-1}(\iota(P_{t_1}(B_1)) \cdot \iota(P_{t_2}(B_2)) \cdot \ldots \cdot \iota(P_{t_q}(B_q))).$$

By construction, $0 \le P(E) \le u$. Moreover, we get

$$P(\tau^{-1}(E)) = \iota^{-1}(\iota(P_{\tau(t_1)}(B_1)) \cdot \iota(P_{\tau(t_2)}(B_2)) \cdot \ldots \cdot \iota(P_{\tau(t_q)}(B_q))),$$

for every $\tau \in G$. If $\widetilde{E} \in \mathcal{A}(\mathcal{C})$, $\widetilde{E} = \bigcup_{i=1}^{n} B_i$, where B_i is as in (73), $i = 1, 2, \dots, n$, and the B_i's are pairwise disjoint, then set $\widetilde{P}(\widetilde{E}) = \sum_{i=1}^{n} P(B_i)$. It is not difficult to see that $\widetilde{P}$ is a finitely additive G-invariant set function, defined on $\mathcal{A}(\mathcal{C})$ and with values in $V[u]$. By Theorem 19, $\widetilde{P}$ has a finitely additive G-invariant extension $\widehat{P}$, taking values in $V[u]$ and defined on the family of all subsets of $\widetilde{R}^{\mathcal{T}}$, and in particular on the sets of the type $\prod_{t \in \mathcal{T}} B_t$, as B_t varies in the whole of $\mathcal{P}(\widetilde{R})$ for all $t \in \mathcal{T}$ and does not belong necessarily to $\mathcal{B}$. $\square$

5 Conclusions.

We have given some sandwich and Hahn-Banach-type theorems for invariant linear operators with values in a Dedekind complete partially ordered vector space, using special properties of convex functions. As consequences, we have proved a duality Fenchel-type theorem and a version of the Moreau-Rockafellar formula. As applications, we have investigated some convex optimization problems. We have given some conditions for the existence of an optimal solution and have proved separation, Farkas and Kuhn-Tucker-type theorems. We have demonstrated the equivalence of our presented results. Finally, we have given some further applications to Tarski-type extension problems and probability measures on suitable product sets.

Several kinds of minimization problems in convex analysis, subdifferential calculus, probability theory, Calculus of Variations, reconstruction of images, and other branches of Mathematics, can be investigated according to the given approach, considering invariance or equivariance with respect to amenable groups of transformations.

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