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FRACTIONAL HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE MODULUS OF THE MIXED DERIVATIVES ARE CO-ORDINATED EXTENDED (s_1, m_1)-(s_2, m_2)-PREINVEX

Abstract

In this paper, we establish a new fractional identity involving a functions of two independent variables, and then we derive some fractional Hermite-Hadamard type integral inequalities for functions whose modulus of the mixed derivatives are co-ordinated extended (s_1, m_1)-(s_2, m_2)-preinvex.

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1 Introduction

One of the most well-known inequalities in mathematics for convex functions is the so called Hermite-Hadamard's integral inequality, which can be stated as follows: for each convex function f on the finite interval $[a,b]$ we have

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int_a^b f(x)\,dx \leq \tfrac{f(a)+f(b)}{2}. \tag{1}$$

If the function f is concave, then (1) holds in the reverse direction (see [14]).

In [5], Dragomir established the two-dimensional analogue of (1) given by

$$\begin{aligned} f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right) \leq& \tfrac{1}{2}\left(\tfrac{1}{b-a}\int_a^b f(x,\tfrac{c+d}{2})\,dx + \tfrac{1}{d-c}\int_c^d f(\tfrac{a+b}{2},y)\,dy\right) \\ \leq& \tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\ \leq& \tfrac{1}{4}\left(\tfrac{1}{b-a}\int_a^b f(x,c)\,dx + \tfrac{1}{b-a}\int_a^b f(x,d)\,dx \right. \\ &\left. + \tfrac{1}{d-c}\int_c^d f(a,y)\,dy + \tfrac{1}{d-c}\int_c^d f(b,y)\,dy\right) \\ \leq& \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{2}$$

The inequalities (1) and (2) have attracted many researchers. Various generalizations, refinements, extensions and variants of (1)-(2) have appeared in the literature, see [1, 2, 8–11, 17] and references therein.

In recent years, many efforts have been made by many mathematicians to generalize the classical convexity. Hanson [6] introduced a new class of generalized convex functions, called invex functions. Many authors studied their properties and applications in mathematical programming and optimizations (for instance, see [4, 12, 13, 15, 18, 20].)

Sarikaya [16] gave the following fractional Hermite-Hadamard for co-ordinated convex functions.

Theorem 1. *Let $f:\Delta\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\Big|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\big(J^{\alpha,\beta}_{a^{+},c^{+}}f(b,d)+J^{\alpha,\beta}_{a^{+},d^{-}}f(b,c)\big)$$
$$+J^{\alpha,\beta}_{b^{-},c^{+}}f(a,d)+J^{\alpha,\beta}_{b^{-},d^{-}}f(a,c)-C\Big|$$
$$\le \tfrac{(b-a)(d-c)}{4(\alpha+1)(\beta+1)}\left(\left|\tfrac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\tfrac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\tfrac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\tfrac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right),$$

where

$$C=\tfrac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left(J^{\beta}_{c^{+}}f(a,d)+J^{\beta}_{c^{+}}f(b,d)+J^{\beta}_{d^{-}}f(a,c)+J^{\beta}_{d^{-}}f(b,c)\right)$$
$$+\tfrac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left(J^{\alpha}_{a^{+}}f(b,c)+J^{\alpha}_{a^{+}}f(b,d)+J^{\alpha}_{b^{-}}f(a,c)+J^{\alpha}_{b^{-}}f(a,d)\right). \quad (3)$$

Theorem 2. *Let $f:\Delta\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is a convex function on the co-ordinates on Δ, where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then one has the inequalities*

$$\left|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\left(J^{\alpha,\beta}_{a^{+},c^{+}}f(b,d)+J^{\alpha,\beta}_{a^{+},d^{-}}f(b,c)\right.\right.$$
$$\left.\left.+J^{\alpha,\beta}_{b^{-},c^{+}}f(a,d)+J^{\alpha,\beta}_{b^{-},d^{-}}f(a,c)-C\right)\right|$$
$$\le \tfrac{(b-a)(d-c)}{4^{\frac{1}{q}}((\alpha p+1)(\beta p+1))^{\frac{1}{p}}}$$
$$\times\left(\left|\tfrac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\tfrac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\tfrac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\tfrac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right)^{\frac{1}{q}}$$

where C is defined by (3).

Motivated by the above results, in this paper we introduce the concept of extended (s_1,m_1)-(s_2,m_2)-preinvex functions. Also we establish a new fractional identity involving a functions of two independent variables, and by using this identity we derive some fractional Hermite-Hadamard type integral inequalities for functions whose modulus of the mixed derivatives lies in this class of functions.

2 Preliminaries

In this section, we recall some definitions of classical and generalized convexity, as well as definitions concerning some fractional operators. These latter can be seen as notations which we will use in the later section. In what follows we assume that $\Delta := [a,b] \times [c,d]$ is a bidimensional interval in $\mathbb{R}^2$ with $a < b$ and $c < d$ and $\Delta_0 := [0,b] \times [0,d]$ with $0 < b$ and $0 < d$.

We note that Dragomir is the first to have introduced the concept of convexity for two-variable functions. The formal writing of co-ordinated convex functions is due to Latif et. al. and is given by the following definition.

Definition 3. *[8] A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ, if the inequality*

$$f\big(tx + (1-t)\,u, \lambda y + (1-\lambda)\,v\big) \leq t\lambda f(x,y) + t\,(1-\lambda)\,f(x,v) \\ + (1-t)\,\lambda f(u,y) + (1-t)\,(1-\lambda)\,f(u,v)$$

holds for all $t, \lambda \in [0,1]$ and $(x,y), (x,v), (u,y), (u,v) \in \Delta$.

Among the generalizations of convex functions, we mention the s-convexity, which was introduced in [2] by Alomari and Darus.

Definition 4. *[2] A nonnegative function $f : \Delta \subset [0,\infty)^2 \to \mathbb{R}$ is said to be s-convex in the second sense on the co-ordinates on Δ for some fixed $s \in (0,1]$, if the following inequality*

$$f(\lambda x + (1-\lambda)z, ty + (1-t)w) \leq \lambda^s t^s f(x,y) + \lambda^s (1-t)^s f(x,w) \\ +(1-\lambda)^s t^s f(z,y) + (1-\lambda)^s (1-t)^s f(z,w)$$

holds, for all $(x,y), (z,w), (x,w), (z,y) \in \Delta$ and $\lambda, t \in [0,1]$.

Recently, Bai and Qi have generalized the above notions by introducing the concept of co-ordinated (s_1,m_1)-(s_2,m_2)-convex functions by the following definition.

Definition 5. *[3] A function $f : \Delta_0 \to \mathbb{R}$ is said to be (s_1,m_1)-(s_2,m_2)-convex on the co-ordinates on Δ_0, if the following inequality*

$$f\,(\lambda x + m_1\,(1-\lambda)\,u, ty + m_2\,(1-t)\,v) \leq \lambda^{s_1} t^{s_2} f(x,y) + m_2 \lambda^{s_1} (1-t^{s_2}) f(x,v) \\ +m_1(1-\lambda^{s_1}) t^{s_2} f(u,y) + m_1 m_2 (1-\lambda^{s_1})(1-t^{s_2}) f(u,v)$$

holds for all $t, \alpha \in [0,1], s_1, m_1, s_2, m_2 \in (0,1]$ and $(x,u), (y,v) \in \Delta_0$.

Xi et al. gave an extension of Definition 5 as follows.

Definition 6. *[19] A function $f : \Delta_0 \to \mathbb{R}$ is said to be co-ordinated extended (s_1,m_1)-(s_2,m_2)-convex on Δ_0, if the following inequality*

$$f\,(\lambda x + m_1\,(1-\lambda)\,u, ty + m_2\,(1-t)\,v) \leq \lambda^{s_1} t^{s_2} f(x,y)$$

$$+m_2\lambda^{s_1}(1-t)^{s_2} f(x,v) + m_1(1-\lambda)^{s_1} t^{s_2} f(u,y)$$
$$+m_1 m_2 (1-\lambda)^{s_1}(1-t)^{s_2} f(u,v)$$

holds for all $t, \alpha \in [0,1], s_1, m_1, s_2, m_2 \in (0,1]$ *and* $(x,u),(y,v) \in \Delta_0$.

Without giving too much detail, we list some definitions concerning the generalized convex functions and certain definitions of fractional integration in the Riemann-Liouville sense.

Definition 7. *[10] Let* K_1, K_2 *be nonempty subsets of* $\mathbb{R}^n$, $(u,v) \in K_1 \times K_2$. *We say* $K_1 \times K_2$ *is invex at* (u,v) *with respect to* η_1 *and* η_2, *if for each* $(x,y) \in K_1 \times K_2$ *and* $t, s \in [0,1]$, *we have*

$$(u + t\eta_1(x,u), v + s\eta_2(y,v)) \in K_1 \times K_2.$$

$K_1 \times K_2$ is said to be an invex set with respect to η_1 and η_2 if $K_1 \times K_2$ is invex at each $(u,v) \in K_1 \times K_2$.

In what follows we assume that $K_1 \times K_2$ be an invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$.

Definition 8. *[9] A function* $f : K_1 \times K_2 \to \mathbb{R}$ *is said to be preinvex on the co-ordinates, if the inequality*

$$f(u + \lambda\eta_1(x,u), v + t\eta_2(y,v)) \le (1-\lambda)(1-t)f(u,v) + (1-\lambda)tf(u,y)$$
$$+(1-t)\lambda f(x,v) + \lambda t f(x,y)$$

holds for all $t, \lambda \in [0,1]$ *and* $(x,y),(x,v),(u,y),(u,v) \in K_1 \times K_2$.

Definition 9. *[11] A nonnegative function* $f : K_1 \times K_2 \subset [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ *is said to be s-preinvex in the second sense on co-ordinates for some fixed* $s \in (0,1]$, *if the inequality*

$$f(u + \lambda\eta_1(x,u), v + t\eta_2(y,v)) \le (1-\lambda)^s(1-t)^s f(u,v) + (1-\lambda)^s t^s f(u,y)$$
$$+(1-t)^s\lambda^s f(x,v) + \lambda^s t^s f(x,y)$$

holds for all $t, \lambda \in [0,1]$ *and* $(x,y),(x,v),(u,y),(u,v) \in K_1 \times K_2$.

Definition 10. *[7] Let* $f \in L_1[a,b]$. *The Riemann-Liouville integrals* $J^{\alpha}_{a^+}f$ *and* $J^{\alpha}_{b^-}f$ *of order* $\alpha > 0$ *with* $a \ge 0$ *are defined by*

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt, \qquad x > a,$$
$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt, \qquad b > x,$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\,dt$ *is the Gamma function and* $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.

Definition 11. *[7] Let $f \in L([a,b]\times[c,d])$. The Riemann–Liouville integrals $J^{\alpha,\beta}_{a^+,c^+}$, $J^{\alpha,\beta}_{a^+,d^-}$, $J^{\alpha,\beta}_{b^-,c^+}$, and $J^{\alpha,\beta}_{b^-,d^-}$ of order $\alpha,\beta>0$ with $a,c\geq 0$, $a<b$, and $a<d$ are defined by*

$$J^{\alpha,\beta}_{a^+,c^+} f(b,d) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (b-x)^{\alpha-1} (d-y)^{\beta-1} f(x,y)\, dy\, dx, \quad (4)$$

$$J^{\alpha,\beta}_{a^+,d^-} f(b,c) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (b-x)^{\alpha-1} (y-c)^{\beta-1} f(x,y)\, dy\, dx, \quad (5)$$

$$J^{\alpha,\beta}_{b^-,c^+} f(a,d) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (x-a)^{\alpha-1} (d-y)^{\beta-1} f(x,y)\, dy\, dx, \quad (6)$$

and

$$J^{\alpha,\beta}_{b^-,d^-} f(a,c) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (x-a)^{\alpha-1} (y-c)^{\beta-1} f(x,y)\, dy\, dx, \quad (7)$$

where Γ is the Gamma function, and

$$J^{0,0}_{a^+,c^+} f(b,d) = J^{0,0}_{a^+,d^-} f(b,c) = J^{0,0}_{b^-,c^+} f(a,d) = J^{0,0}_{b^-,d^-} f(a,c) = f(x,y).$$

Definition 12. *[16] Let $f \in L([a,b]\times[c,d])$. The Riemann–Liouville integrals $J^{\alpha}_{b^-} f(a,c)$, $J^{\alpha}_{a^+} f(b,c)$, $J^{\beta}_{d^-} f(a,c)$, and $J^{\alpha}_{c^+} f(a,d)$ of order $\alpha,\beta>0$ with $a,c\geq 0$, $a<b$, and $c<d$ are defined by*

$$J^{\alpha}_{b^-} f(a,c) = \frac{1}{\Gamma(\alpha)} \int_a^b (x-a)^{\alpha-1} f(x,c)\, dx, \quad (8)$$

$$J^{\alpha}_{a^+} f(b,c) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} f(x,c)\, dx, \quad (9)$$

$$J^{\beta}_{d^-} f(a,c) = \frac{1}{\Gamma(\beta)} \int_c^d (y-c)^{\beta-1} f(a,y)\, dy, \quad (10)$$

and

$$J^{\alpha}_{c^+} f(a,d) = \frac{1}{\Gamma(\beta)} \int_c^d (d-y)^{\beta-1} f(a,y)\, dy, \quad (11)$$

where Γ is the Gamma function.

We also recall that beta function is defined by

$$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

3 Main results

In what follows we assume that $K=[a,a+\eta_1(b,a)]\times[c,c+\eta_2(d,c)]$ be an invex subset of $[0,+\infty)\times[0,+\infty)$ with respect to η_1,η_2, where $\eta_1,\eta_2:\mathbb{R}^2\to\mathbb{R}$ are two bifunctions such that $\eta_1(b,a)>0$ and $\eta_2(d,c)>0$.

Definition 13. *A nonnegative function $f:K_1\times K_2\subset[0,+\infty)\times[0,+\infty)\to\mathbb{R}$ is said to be extended (s_1,m_1)-(s_2,m_2)-preinvex function on the co-ordinates for some fixed $s_1,s_2\in[-1,1]$ and $m_1,m_2\in(0,1]$, if the inequality*

$$
\begin{aligned}
&f(u+\lambda\eta_1(x,u),v+t\eta_2(y,v))\\
&\quad\le(1-\lambda)^{s_1}(1-t)^{s_2}f(u,v)+m_2(1-\lambda)^{s_1}t^{s_2}f(u,\tfrac{y}{m_2})\\
&\qquad+m_1\lambda^{s_1}(1-t)^{s_2}f(\tfrac{x}{m_1},v)+m_1m_2\lambda^{s_1}t^{s_2}f(\tfrac{x}{m_1},\tfrac{y}{m_2})
\end{aligned}
$$

holds for all $t,\lambda\in[0,1]$ and $(x,y),(x,v),(u,y),(u,v)\in K_1\times K_2$.

Remark 14. *Obviously, Definition 13, recapture Definitions 3 to 6 and Definitions 8 and 7 for well-chosen values of s_1,m_1,s_2,m_2,η_1 and η_2.*

In order to prove our results we need the following lemma.

Lemma 15. *Let $f:K\to\mathbb{R}$ be a partially differentiable function on K. If $\frac{\partial^2 f}{\partial t\partial s}\in L(K)$, then the following equality holds*

$$
\begin{aligned}
&L(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J)\\
&\quad=\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1(t^\alpha-(1-t)^\alpha)(s^\beta-(1-s)^\beta)\\
&\qquad\times\tfrac{\partial^2 f}{\partial t\partial s}(a+t\eta_1(b,a),c+s\eta_2(d,c))\,ds\,dt,
\end{aligned}\tag{12}
$$

where

$$
\begin{aligned}
&L(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J)\\
&\quad=\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\\
&\qquad+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\
&\qquad+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\
&\qquad+J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))),
\end{aligned}\tag{13}
$$

with

$$
A=\tfrac{\Gamma(\alpha+1)}{4(\eta_1(b,a))^\alpha}(J^\alpha_{(a+\eta_1(b,a))^-}f(a,c+\eta_2(d,c))+J^\alpha_{(a+\eta_1(b,a))^-}f(a,c)
$$

$$+ J^{\alpha}_{a^+} f(a+\eta_1(b,a), c+\eta_2(d,c)) + J^{\alpha}_{a^+} f(a+\eta_1(b,a), c))$$
$$+ \tfrac{\Gamma(\beta+1)}{4(\eta_2(d,c))^{\beta}} (J^{\beta}_{(c+\eta_2(d,c))^-} f(a+\eta_1(b,a), c) + J^{\beta}_{(c+\eta_2(d,c))^-} f(a,c)$$
$$+ J^{\beta}_{c^+} f(a+\eta_1(b,a), c+\eta_2(d,c)) + J^{\beta}_{c^+} f(a, c+\eta_2(d,c))) \tag{14}$$

PROOF. By integration by parts, we get

$$\begin{aligned}
&\int_0^1 \int_0^1 (t^{\alpha} - (1-t)^{\alpha})(s^{\beta} - (1-s)^{\beta}) \tfrac{\partial^2 f}{\partial t \partial s}(a + t\eta_1(b,a), c + s\eta_2(d,c))\, ds\, dt \\
&\quad = \int_0^1 (t^{\alpha} - (1-t)^{\alpha}) \\
&\qquad \times (\int_0^1 (s^{\beta} - (1-s)^{\beta}) \tfrac{\partial^2 f}{\partial t \partial s}(a + t\eta_1(b,a), c + s\eta_2(d,c))\, ds)\, dt \\
&\quad = \int_0^1 (t^{\alpha} - (1-t)^{\alpha})(\tfrac{1}{\eta_2(d,c)} \tfrac{\partial f}{\partial t}(a + t\eta_1(b,a), c + \eta_2(d,c)) \\
&\qquad + \tfrac{1}{\eta_2(d,c)} \tfrac{\partial f}{\partial t}(a + t\eta_1(b,a), c) \\
&\qquad - \tfrac{\beta}{\eta_2(d,c)} \int_0^1 (s^{\beta-1} + (1-s)^{\beta-1}) \tfrac{\partial f}{\partial t}(a + t\eta_1(b,a), c + s\eta_2(d,c))\, ds)\, dt \\
&\quad = \tfrac{1}{\eta_2(d,c)} \int_0^1 (t^{\alpha} - (1-t)^{\alpha}) \tfrac{\partial f}{\partial t}(a + t\eta_1(b,a), c + \eta_2(d,c))\, dt \\
&\qquad + \tfrac{1}{\eta_2(d,c)} \int_0^1 (t^{\alpha} - (1-t)^{\alpha}) \tfrac{\partial f}{\partial t}(a + t\eta_1(b,a), c)\, dt \\
&\qquad - \tfrac{\beta}{\eta_2(d,c)} \int_0^1 \int_0^1 (s^{\beta-1} + (1-s)^{\beta-1})(t^{\alpha} - (1-t)^{\alpha}) \\
&\qquad \times \tfrac{\partial f}{\partial t}(a + t\eta_1(b,a), c + s\eta_2(d,c))\, dt\, ds \\
&\quad = \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} f(a + \eta_1(b,a), c + \eta_2(d,c)) + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} f(a, c + \eta_2(d,c)) \\
&\qquad - \tfrac{\alpha}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 (t^{\alpha-1} + (1-t)^{\alpha-1}) f(a + t\eta_1(b,a), c + \eta_2(d,c))\, dt \\
&\qquad + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} f(a + \eta_1(b,a), c) + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} f(a,c) \\
&\qquad - \tfrac{\alpha}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 (t^{\alpha-1} + (1-t)^{\alpha-1}) f(a + t\eta_1(b,a), c)\, dt \\
&\qquad - \tfrac{\beta}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 (s^{\beta-1} + (1-s)^{\beta-1}) f(a + \eta_1(b,a), c + s\eta_2(d,c))\, ds \\
&\qquad - \tfrac{\beta}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 (s^{\beta-1} + (1-s)^{\beta-1}) f(a, c + s\eta_2(d,c))\, ds
\end{aligned}$$

$$
\begin{aligned}
&+ \tfrac{\alpha\beta}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 \int_0^1 (s^{\beta-1} + (1-s)^{\beta-1})(t^{\alpha-1} + (1-t)^{\alpha-1}) \\
&\times f(a + t\eta_1(b,a), c + s\eta_2(d,c))\, ds\, dt \\
= {}& \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} f(a + \eta_1(b,a), c + \eta_2(d,c)) + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} f(a, c + \eta_2(d,c)) \\
&- \tfrac{\alpha}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 (t^{\alpha-1} + (1-t)^{\alpha-1}) f(a + t\eta_1(b,a), c + \eta_2(d,c))\, dt \\
&+ \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} f(a + \eta_1(b,a), c) + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} f(a,c) \\
&- \tfrac{\alpha}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 (t^{\alpha-1} + (1-t)^{\alpha-1}) f(a + t\eta_1(b,a), c)\, dt \\
&- \tfrac{\beta}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 (s^{\beta-1} + (1-s)^{\beta-1}) f(a + \eta_1(b,a), c + s\eta_2(d,c))\, ds \\
&- \tfrac{\beta}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 (s^{\beta-1} + (1-s)^{\beta-1}) f(a, c + s\eta_2(d,c))\, ds \\
&+ \tfrac{\alpha\beta}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 \int_0^1 (s^{\beta-1} + (1-s)^{\beta-1})(t^{\alpha-1} + (1-t)^{\alpha-1}) \\
&\times f(a + t\eta_1(b,a), c + s\eta_2(d,c))\, ds\, dt \\
= {}& \tfrac{f(a,c) + f(a, c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a+\eta_1(b,a), c+\eta_2(d,c))}{\eta_1(b,a)\eta_2(d,c)} \\
&- \tfrac{\alpha}{\eta_1(b,a)\eta_2(d,c)} \Big(\int_0^1 t^{\alpha-1} f(a + t\eta_1(b,a), c + \eta_2(d,c))\, dt \\
&+ \int_0^1 (1-t)^{\alpha-1} f(a + t\eta_1(b,a), c + \eta_2(d,c))\, dt \\
&+ \int_0^1 t^{\alpha-1} f(a + t\eta_1(b,a), c)\, dt \\
&+ \int_0^1 (1-t)^{\alpha-1} f(a + t\eta_1(b,a), c)\, dt \Big) \\
&- \tfrac{\beta}{\eta_1(b,a)\eta_2(d,c)} \Big(\int_0^1 s^{\beta-1} f(a + \eta_1(b,a), c + s\eta_2(d,c))\, ds \\
&+ \int_0^1 (1-s)^{\beta-1} f(a + \eta_1(b,a), c + s\eta_2(d,c))\, ds \\
&+ \int_0^1 s^{\beta-1} f(a, c + s\eta_2(d,c))\, ds \\
&+ \int_0^1 (1-s)^{\beta-1} f(a, c + s\eta_2(d,c))\, ds \Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \tfrac{\alpha\beta}{\eta_1(b,a)\eta_2(d,c)}\Big(\int_0^1\int_0^1 s^{\beta-1}t^{\alpha-1}f(a+t\eta_1(b,a),c+s\eta_2(d,c))\,ds\,dt\\
&+ \int_0^1\int_0^1 s^{\beta-1}(1-t)^{\alpha-1}f(a+t\eta_1(b,a),c+s\eta_2(d,c))\,ds\,dt\\
&+ \int_0^1\int_0^1 (1-s)^{\beta-1}t^{\alpha-1}f(a+t\eta_1(b,a),c+s\eta_2(d,c))\,ds\,dt\\
&+ \int_0^1\int_0^1 (1-s)^{\beta-1}(1-t)^{\alpha-1}f(a+t\eta_1(b,a),c+s\eta_2(d,c))\,ds\,dt\Big).
\end{aligned} \tag{15}
$$

For $x = a + t\eta_1(b,a)$ and $y = c + s\eta_2(d,c)$, (15) becomes

$$
\begin{aligned}
&\int_0^1\int_0^1 (t^\alpha-(1-t)^\alpha)(s^\beta-(1-s)^\beta)\tfrac{\partial^2 f}{\partial t\partial s}(a+t\eta_1(b,a),c+s\eta_2(d,c))\,ds\,dt\\
&= \tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{\eta_1(b,a)\eta_2(d,c)}\\
&- \tfrac{\alpha}{(\eta_1(b,a))^{\alpha+1}\eta_2(d,c)}\int_a^{a+\eta_1(b,a)} (x-a)^{\alpha-1}f(x,c+\eta_2(d,c))\,dx\\
&- \tfrac{\alpha}{(\eta_1(b,a))^{\alpha+1}\eta_2(d,c)}\int_a^{a+\eta_1(b,a)} (a+\eta_1(b,a)-x)^{\alpha-1}f(x,c+\eta_2(d,c))\,dx\\
&- \tfrac{\alpha}{(\eta_1(b,a))^{\alpha+1}\eta_2(d,c)}\int_a^{a+\eta_1(b,a)} (x-a)^{\alpha-1}f(x,c)\,dx\\
&- \tfrac{\alpha}{(\eta_1(b,a))^{\alpha+1}\eta_2(d,c)}\int_a^{a+\eta_1(b,a)} (a+\eta_1(b,a)-x)^{\alpha-1}f(x,c)\,dx\\
&- \tfrac{\beta}{\eta_1(b,a)(\eta_2(d,c))^{\beta+1}}\Big(\int_c^{c+\eta_2(d,c)} (y-c)^{\beta-1}f(a+\eta_1(b,a),y)\,dy\\
&+ \int_c^{c+\eta_2(d,c)} (c+\eta_2(d,c)-y)^{\beta-1}f(a+\eta_1(b,a),y)\,dy\\
&+ \int_c^{c+\eta_2(d,c)} (y-c)^{\beta-1}f(a,y)\,dy\\
&+ \int_c^{c+\eta_2(d,c)} (c+\eta_2(d,c)-y)^{\beta-1}f(a,y)\,dy\Big)\\
&+ \tfrac{\alpha\beta}{(\eta_1(b,a))^{\alpha+1}(\eta_2(d,c))^{\beta+1}}\\
&\times \Big(\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} (x-a)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dy\,dx
\end{aligned}
$$

$$+\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}(a+\eta_1(b,a)-x)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dy\,dx$$
$$+\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}(x-a)^{\alpha-1}(c+\eta_2(d,c)-y)^{\beta-1}f(x,y)\,dy\,dx$$
$$+\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}(a+\eta_1(b,a)-x)^{\alpha-1}(c+\eta_2(d,c)-y)^{\beta-1}$$
$$\times f(x,y)\,dy\,dx\Big). \tag{16}$$

Multiplying both sides of (16) by $\frac{\eta_1(b,a)\eta_2(d,c)}{4}$, we get the desired result. □

Having established a partial result given by the above identity, our first result is to establish an estimate of the fractional trapezoid inequality for functions whose modulus of the mixed derivatives lies in the class of functions given in Definition 13. Some special cases are also discussed.

Theorem 16. *Let $f : K \to \mathbb{R}$ be a partially differentiable function on K. If $|\frac{\partial^2 f}{\partial t\partial\lambda}|$ is co-ordinated extended (s_1, m_1)-(s_2, m_2)-preinvex function on K with respect to η_1 and η_2 for some fixed $(s_1, m_1), (s_2, m_2) \in [-1, 1]\times(0, 1]$, then the following fractional inequality holds*

$$\begin{aligned}
&|L(a, b, c, d, \alpha, \beta, \eta_1, \eta_2, A, J)|\\
&\quad\le \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\Big(B(\alpha+1, s_1+1)+\tfrac{1}{\alpha+s_1+1}\Big)\Big(B(\beta+1, s_2+1)+\tfrac{1}{\beta+s_2+1}\Big)\\
&\qquad\times\Big(|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)|+m_2|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,\tfrac{d}{m_2})|+m_1|\tfrac{\partial^2 f}{\partial t\partial\lambda}(\tfrac{b}{m_1},c)|\\
&\qquad+m_1m_2|\tfrac{\partial^2 f}{\partial t\partial\lambda}(\tfrac{b}{m_1},\tfrac{d}{m_2})|\Big),
\end{aligned}$$

where L and A are defined as in (13) *and* (14) *respectively, and $B(\cdot,\cdot)$ is the beta function.*

Proof. From Lemma 15, and properties of modulus we have

$$\begin{aligned}
&|L(a, b, c, d, \alpha, \beta, \eta_1, \eta_2, A, J)|\\
&\quad\le \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1 |t^\alpha-(1-t)^\alpha||\lambda^\beta-(1-\lambda)^\beta|\\
&\qquad\times|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a), c+\lambda\eta_2(d,c))|\,d\lambda\,dt\\
&\quad\le \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1 (t^\alpha+(1-t)^\alpha)(\lambda^\beta+(1-\lambda)^\beta)
\end{aligned}$$

$$\times \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\right| d\lambda\, dt.$$

Using (s_1,m_1)-(s_2,m_2)-preinvexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$, we get

$$\begin{aligned}
&|L(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J)| \\
&\quad\le \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1 (t^\alpha+(1-t)^\alpha)(\lambda^\beta+(1-\lambda)^\beta) \\
&\qquad\times\Big[(1-t)^{s_1}(1-\lambda)^{s_2}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| + m_2(1-t)^{s_1}\lambda^{s_2}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,\tfrac{d}{m_2})\right| \\
&\qquad + m_1 t^{s_1}(1-\lambda)^{s_2}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(\tfrac{b}{m_1},c)\right| + m_1m_2t^{s_1}\lambda^{s_2}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(\tfrac{b}{m_1},\tfrac{d}{m_2})\right|\Big]\, d\lambda\, dt \\
&\quad= \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\Big(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\int_0^1\int_0^1 (t^\alpha(1-t)^{s_1}+(1-t)^{\alpha+s_1}) \\
&\qquad\times(\lambda^\beta(1-\lambda)^{s_2}+(1-\lambda)^{\beta+s_2})\, d\lambda\, dt \\
&\qquad + m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,\tfrac{d}{m_2})\right|\int_0^1\int_0^1 (t^\alpha(1-t)^{s_1}+(1-t)^{\alpha+s_1}) \\
&\qquad\times(\lambda^{\beta+s_2}+\lambda^{s_2}(1-\lambda)^\beta)\, d\lambda\, dt \\
&\qquad + m_1\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(\tfrac{b}{m_1},c)\right|\int_0^1\int_0^1 (t^{\alpha+s_1}+t^{s_1}(1-t)^\alpha) \\
&\qquad\times(\lambda^\beta(1-\lambda)^{s_2}+(1-\lambda)^{\beta+s_2})\, d\lambda\, dt \\
&\qquad + m_1m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(\tfrac{b}{m_1},\tfrac{d}{m_2})\right| \\
&\qquad\times\int_0^1\int_0^1 (t^{\alpha+s_1}+t^{s_1}(1-t)^\alpha)(\lambda^{\beta+s_2}+\lambda^{s_2}(1-\lambda)^\beta)\, d\lambda\, dt\Big) \\
&\quad= \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}(B(\alpha+1,s_1+1)+\tfrac{1}{\alpha+s_1+1})(B(\beta+1,s_2+1)+\tfrac{1}{\beta+s_2+1}) \\
&\qquad\times(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| + m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,\tfrac{d}{m_2})\right| + m_1\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(\tfrac{b}{m_1},c)\right| \\
&\qquad + m_1m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(\tfrac{b}{m_1},\tfrac{d}{m_2})\right|).
\end{aligned}$$

The proof is completed. □

Corollary 17. *In Theorem 16 if we put $m_1 = m_2 = 1$, we get*

$$\begin{aligned}
&\left|L\big(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J\big)\right| \\
&\quad\le \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(B(\alpha+1,s_1+1)+\tfrac{1}{\alpha+s_1+1}\right)\left(B(\beta+1,s_2+1)+\tfrac{1}{\beta+s_2+1}\right) \\
&\qquad\times\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right| + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right| + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right).
\end{aligned}$$

And if we take $s_1 = s_2 = 1$, we get

$$\left|L\big(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J\big)\right| \leq \tfrac{\eta_1(b,a)\eta_2(d,c)}{4(\alpha+1)(\beta+1)} \left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| + m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\tfrac{d}{m_2}\right)\right| + m_1\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(\tfrac{b}{m_1},c\right)\right| + m_1m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(\tfrac{b}{m_1},\tfrac{d}{m_2}\right)\right|\right).$$

In the case where $m_1 = m_2 = s_1 = s_2 = 1$, we obtain

$$\left|L\big(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J\big)\right| \leq \tfrac{\eta_1(b,a)\eta_2(d,c)}{4(\alpha+1)(\beta+1)} \times \left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right| + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right| + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right).$$

Corollary 18. *In Theorem 16 if we choose $\eta_1(b,a) = \eta_2(b,a) = b-a$, we obtain*

$$\begin{aligned}&\Big|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - C + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\\ &\quad\times\left(J^{\alpha,\beta}_{b^-,d^-}f(a,c) + J^{\alpha,\beta}_{a^+,d^-}f(b,c) + J^{\alpha,\beta}_{(b)^-,c^+}f(a,d) + J^{\alpha,\beta}_{a^+,c^+}f(b,d)\right)\Big|\\ &\leq \tfrac{(b-a)(d-c)}{4}\left(B(\alpha+1,s_1+1) + \tfrac{1}{\alpha+s_1+1}\right)\left(B(\beta+1,s_2+1) + \tfrac{1}{\beta+s_2+1}\right)\\ &\quad\times\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| + m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\tfrac{d}{m_2}\right)\right| + m_1\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(\tfrac{b}{m_1},c\right)\right|\right.\\ &\quad\left.+ m_1m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(\tfrac{b}{m_1},\tfrac{d}{m_2}\right)\right|\right).\end{aligned}$$

Moreover if we put $m_1 = m_2 = 1$, we get

$$\begin{aligned}&\Big|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - C + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\\ &\quad\times\left(J^{\alpha,\beta}_{b^-,d^-}f(a,c) + J^{\alpha,\beta}_{a^+,d^-}f(b,c) + J^{\alpha,\beta}_{(b)^-,c^+}f(a,d) + J^{\alpha,\beta}_{a^+,c^+}f(b,d)\right)\Big|\\ &\leq \tfrac{(b-a)(d-c)}{4}\left(B(\alpha+1,s_1+1) + \tfrac{1}{\alpha+s_1+1}\right)\left(B(\beta+1,s_2+1) + \tfrac{1}{\beta+s_2+1}\right)\\ &\quad\times\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right| + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right| + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right).\end{aligned}$$

And if we take $s_1 = s_2 = 1$, we get

$$\begin{aligned}&\Big|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - C + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\\ &\quad\times\left(J^{\alpha,\beta}_{b^-,d^-}f(a,c) + J^{\alpha,\beta}_{a^+,d^-}f(b,c) + J^{\alpha,\beta}_{(b)^-,c^+}f(a,d) + J^{\alpha,\beta}_{a^+,c^+}f(b,d)\right)\Big|\\ &\leq \tfrac{(b-a)(d-c)}{4(\alpha+1)(\beta+1)}\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| + m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\tfrac{d}{m_2}\right)\right| + m_1\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(\tfrac{b}{m_1},c\right)\right|\right.\end{aligned}$$

$$+ m_1 m_2 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, \frac{d}{m_2}\right)\right|\Big),$$

where C is defined by (3).

Remark 19. *Corollary 18 will be reduced to Theorem 5 from [16] if we take $m_1 = m_2 = s_1 = s_2 = 1$.*

Corollary 20. *In Theorem 16 if we put $\alpha = \beta = 1$, we get*

$$\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - D$$
$$+ \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\, dy\, dx\Big|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4(s_1+1)(s_2+1)} \left(\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right| + m_2 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(a, \frac{d}{m_2}\right)\right|\right.$$
$$\left. + m_1 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, c\right)\right| + m_1 m_2 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, \frac{d}{m_2}\right)\right|\right).$$

Moreover if we put $m_1 = m_2 = 1$, we get

$$\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - D$$
$$+ \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\, dy\, dx\Big|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4(s_1+1)(s_2+1)} \left(\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right| + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right| + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right| + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|\right).$$

And if we take $s_1 = s_2 = 1$, we get

$$\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - D$$
$$+ \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\, dy\, dx\Big|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{16} \left(\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right| + m_2 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(a, \frac{d}{m_2}\right)\right|\right.$$
$$\left. + m_1 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, c\right)\right| + m_1 m_2 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, \frac{d}{m_2}\right)\right|\right).$$

In the case where $m_1 = m_2 = s_1 = s_2 = 1$, we obtain

$$\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - D$$
$$+ \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\, dy\, dx\Big|$$

$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{16}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right),$$

where

$$D=\frac{1}{2\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}\left(f(x,c)+f(x,c+\eta_2(d,c))\right)dx + \frac{1}{2\eta_2(d,c)}\int_c^{c+\eta_2(d,c)}\left(f(a+\eta_1(b,a),y)+f(a,y)\right)dy. \tag{17}$$

Corollary 21. *In Theorem 16 if we choose $\eta_1(b,a)=\eta_2(b,a)=b-a$ and $\alpha=\beta=1$, we get*

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{2(b-a)}\int_a^b\left(f(x,c)+f(x,d)\right)dx\right.\\&\quad\left.-\frac{1}{2(d-c)}\int_c^d\left(f(b,y)+f(a,y)\right)dy+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx\right|\\&\leq\frac{(b-a)(d-c)}{4(s_1+1)(s_2+1)}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|\right.\\&\quad\left.+m_1\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},c\right)\right|+m_1m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|\right).\end{aligned}$$

Moreover if we put $m_1=m_2=1$, we get

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{2(b-a)}\int_a^b\left(f(x,c)+f(x,d)\right)dx\right.\\&\quad\left.-\frac{1}{2(d-c)}\int_c^d\left(f(b,y)+f(a,y)\right)dy+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx\right|\\&\leq\frac{(b-a)(d-c)}{4(s_1+1)(s_2+1)}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right).\end{aligned}$$

And if we take $s_1=s_2=1$, we get

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{2(b-a)}\int_a^b\left(f(x,c)+f(x,d)\right)dx\right.\\&\quad\left.-\frac{1}{2(d-c)}\int_c^d\left(f(b,y)+f(a,y)\right)dy+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx\right|\\&\leq\frac{(b-a)(d-c)}{16}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|\right.\end{aligned}$$

$$+ m_1 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, c\right)\right| + m_1 m_2 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, \frac{d}{m_2}\right)\right|\right).$$

In the case where $m_1 = m_2 = s_1 = s_2 = 1$, *we obtain*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{2(b-a)} \int_a^b (f(x,c) + f(x,d))\, dx \right.$$
$$\left. - \frac{1}{2(d-c)} \int_c^d (f(b,y) + f(a,y))\, dy + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy\, dx\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\left(\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right| + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right| + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right| + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|\right).$$

Now, by using Hölder's inequality and power mean inequality we establish the estimate of the fractional trapezoid inequality in the case where a certain power of the modulus of the mixed derivatives are co-ordinated extended (s_1, m_1)-(s_2, m_2)-preinvex, and at each times the special cases will be discussed.

Theorem 22. *Let* $f : K \to \mathbb{R}$ *be a partially differentiable function on* K, *and let* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q$ *is co-ordinated extended* (s_1, m_1)-(s_2, m_2)*-preinvex function on* K *with respect to* η_1 *and* η_2 *for some fixed* $(s_1, m_1), (s_2, m_2) \in [-1, 1] \times (0, 1]$, *then the following fractional inequality holds*

$$\left|L\big(a, b, c, d, \alpha, \beta, \eta_1, \eta_2, A, J\big)\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \times \frac{1}{(s_1+1)^{\frac{1}{q}}(s_2+1)^{\frac{1}{q}}}\left(\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q + m_2\left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(a, \frac{d}{m_2}\right)\right|^q\right.$$
$$\left. + m_1 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, c\right)\right|^q + m_1 m_2 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, \frac{d}{m_2}\right)\right|^q\right)^{\frac{1}{q}},$$

where L *and* A *are defined as in* (13) *and* (14) *respectively.*

PROOF. From Lemma 15, properties of modulus, and Hölder inequality, we have

$$\left|L\big(a, b, c, d, \alpha, \beta, \eta_1, \eta_2, A, J\big)\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\left(\int_0^1 \int_0^1 t^{\alpha p} \lambda^{\beta p}\, d\lambda\, dt\right)^{\frac{1}{p}} + \left(\int_0^1 \int_0^1 t^{\alpha p} (1-\lambda)^{\beta p}\, d\lambda\, dt\right)^{\frac{1}{p}}\right.$$
$$\left. + \left(\int_0^1 \int_0^1 (1-t)^{p\alpha} \lambda^{p\beta}\, d\lambda\, dt\right)^{\frac{1}{p}} + \left(\int_0^1 \int_0^1 (1-t)^{p\alpha} (1-\lambda)^{p\beta}\, d\lambda\, dt\right)^{\frac{1}{p}}\right)$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a),c+\lambda\eta_2(d,c)\right)\right|^q d\lambda\,dt\right)^{\frac{1}{q}}$$
$$=\frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a),c+\lambda\eta_2(d,c)\right)\right|^q d\lambda\,dt\right)^{\frac{1}{q}}. \tag{18}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated extended (s_1,m_1)-(s_2,m_2)-preinvex, from (18) we deduce

$$\begin{aligned}
&|L(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J)|\\
&\le\frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\\
&\quad\times\left(\int_0^1\int_0^1\left((1-t)^{s_1}(1-\lambda)^{s_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right.\right.\\
&\quad+m_2(1-t)^{s_1}\lambda^{s_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q+m_1t^{s_1}(1-\lambda)^{s_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q\\
&\quad\left.\left.+m_1m_2t^{s_1}\lambda^{s_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right)d\lambda\,dt\right)^{\frac{1}{q}}\\
&=\frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\times\frac{1}{(s_1+1)^{\frac{1}{q}}(s_2+1)^{\frac{1}{q}}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q\right.\\
&\quad\left.+m_1\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q+m_1m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

which is the desired result. □

Corollary 23. *In Theorem 22 if we put $m_1=m_2=1$, we get*

$$\begin{aligned}
\left|L\big(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J\big)\right|&\le\frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\times\frac{1}{(s_1+1)^{\frac{1}{q}}(s_2+1)^{\frac{1}{q}}}\\
&\quad\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

And if we take $s_1=s_2=1$, we get

$$|L(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J)|\le\frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}$$
$$\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q+m_1\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q+m_1m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q}{4}\right)^{\frac{1}{q}}$$

In the case where $m_1 = m_2 = s_1 = s_2 = 1$*, we obtain*

$$\left|L\big(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J\big)\right| \leq \frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \times \left(\frac{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}$$

Corollary 24. *In Theorem 22 if we choose* $\eta_1(b,a) = \eta_2(b,a) = b-a$*, we obtain*

$$\begin{aligned}&\Big|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - C + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \\ &\quad\times \left(J^{\alpha,\beta}_{b^-,d^-} f(a,c) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{(b)^-,c^+} f(a,d) + J^{\alpha,\beta}_{a^+,c^+} f(b,d)\right)\Big| \\ &\leq \tfrac{(b-a)(d-c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \times \tfrac{1}{(s_1+1)^{\frac{1}{q}}(s_2+1)^{\frac{1}{q}}} \left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\tfrac{d}{m_2}\right)\right|^q\right. \\ &\quad\left.+ m_1\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(\tfrac{b}{m_1},c\right)\right|^q + m_1 m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(\tfrac{b}{m_1},\tfrac{d}{m_2}\right)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Moreover if we put $m_1 = m_2 = 1$*, we get*

$$\begin{aligned}&\Big|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - C + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \\ &\quad\times \left(J^{\alpha,\beta}_{b^-,d^-} f(a,c) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{(b)^-,c^+} f(a,d) + J^{\alpha,\beta}_{a^+,c^+} f(b,d)\right)\Big| \\ &\leq \tfrac{(b-a)(d-c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \times \tfrac{1}{(s_1+1)^{\frac{1}{q}}(s_2+1)^{\frac{1}{q}}} \left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right. \\ &\quad\left.+ \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

And if we take $s_1 = s_2 = 1$*, we get*

$$\begin{aligned}&\Big|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - C + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \\ &\quad\times \left(J^{\alpha,\beta}_{b^-,d^-} f(a,c) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{(b)^-,c^+} f(a,d) + J^{\alpha,\beta}_{a^+,c^+} f(b,d)\right)\Big| \\ &\leq \tfrac{(b-a)(d-c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \times \left(\tfrac{1}{4}\right)^{\frac{1}{q}} \left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\tfrac{d}{m_2}\right)\right|^q\right. \\ &\quad\left.+ m_1\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(\tfrac{b}{m_1},c\right)\right|^q + m_1 m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(\tfrac{b}{m_1},\tfrac{d}{m_2}\right)\right|^q\right)^{\frac{1}{q}},\end{aligned}$$

where C *is defined by* (3).

Remark 25. *Corollary 24 will be reduced to Theorem 6 from [16] if we take* $m_1 = m_2 = s_1 = s_2 = 1$.

Corollary 26. *In Theorem 22 if we put* $\alpha = \beta = 1$, *we get*

$$\Bigg|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - D + \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dy\,dx\Bigg|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{(p+1)^{\frac{2}{p}}} \times \frac{1}{(s_1+1)^{\frac{1}{q}}(s_2+1)^{\frac{1}{q}}}\Bigg(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q + m_1\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q + m_1 m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\Bigg)^{\frac{1}{q}}.$$

Moreover if we put $m_1 = m_2 = 1$, *we get*

$$\Bigg|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - D + \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dy\,dx\Bigg|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{(p+1)^{\frac{2}{p}}} \times \frac{1}{(s_1+1)^{\frac{1}{q}}(s_2+1)^{\frac{1}{q}}}\Bigg(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\Bigg)^{\frac{1}{q}}.$$

And if we take $s_1 = s_2 = 1$, *we get*

$$\Bigg|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - D + \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dy\,dx\Bigg|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{(p+1)^{\frac{2}{p}}} \times \left(\frac{1}{4}\right)^{\frac{1}{q}}\Bigg(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q + m_1\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q + m_1 m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\Bigg)^{\frac{1}{q}}.$$

In the case where $m_1 = m_2 = s_1 = s_2 = 1$, *we obtain*

$$\Bigg|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - D + \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dy\,dx\Bigg|$$

$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{(p+1)^{\frac{2}{p}}} \times \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|^q}{4} \right)^{\frac{1}{q}},$$

where D is defined by (17).

Corollary 27. *In Theorem 22 if we choose $\eta_1(b,a) = \eta_2(b,a) = b - a$ and $\alpha = \beta = 1$, we get*

$$\Bigg| \tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \tfrac{1}{2(b-a)} \int_a^b (f(x,c) + f(x,d))\, dx$$
$$- \tfrac{1}{2(d-c)} \int_c^d (f(b,y) + f(a,y))\, dy + \tfrac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy\, dx \Bigg|$$
$$\leq \tfrac{(b-a)(d-c)}{(p+1)^{\frac{2}{p}}} \times \tfrac{1}{(s_1+1)^{\frac{1}{q}}(s_2+1)^{\frac{1}{q}}} \left(\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q + m_2 \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}\left(a, \tfrac{d}{m_2}\right)\right|^q \right.$$
$$\left. + m_1 \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}\left(\tfrac{b}{m_1}, c\right)\right|^q + m_1 m_2 \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}\left(\tfrac{b}{m_1}, \tfrac{d}{m_2}\right)\right|^q \right)^{\frac{1}{q}}.$$

Moreover if we put $m_1 = m_2 = 1$, we get

$$\Bigg| \tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \tfrac{1}{2(b-a)} \int_a^b (f(x,c) + f(x,d))\, dx$$
$$- \tfrac{1}{2(d-c)} \int_c^d (f(b,y) + f(a,y))\, dy + \tfrac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy\, dx \Bigg|$$
$$\leq \tfrac{(b-a)(d-c)}{(p+1)^{\frac{2}{p}}} \times \tfrac{1}{(s_1+1)^{\frac{1}{q}}(s_2+1)^{\frac{1}{q}}} \left(\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q + \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right|^q \right.$$
$$\left. + \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|^q + \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|^q \right)^{\frac{1}{q}}.$$

And if we take $s_1 = s_2 = 1$, we get

$$\Bigg| \tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \tfrac{1}{2(b-a)} \int_a^b (f(x,c) + f(x,d))\, dx$$
$$- \tfrac{1}{2(d-c)} \int_c^d (f(b,y) + f(a,y))\, dy + \tfrac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy\, dx \Bigg|$$
$$\leq \tfrac{(b-a)(d-c)}{(p+1)^{\frac{2}{p}}} \times \left(\tfrac{1}{4}\right)^{\frac{1}{q}} \left(\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q + m_2 \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}\left(a, \tfrac{d}{m_2}\right)\right|^q \right.$$

$$+ m_1 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, c\right)\right|^q + m_1 m_2 \left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, \frac{d}{m_2}\right)\right|^q\right)^{\frac{1}{q}}.$$

In the case where $m_1 = m_2 = s_1 = s_2 = 1$*, we obtain*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{2(b-a)}\int_a^b \left(f(x,c)+f(x,d)\right) dx\right.$$
$$\left. - \frac{1}{2(d-c)}\int_c^d \left(f(b,y)+f(a,y)\right) dy + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\, dy\, dx\right|$$
$$\leq \frac{(b-a)(d-c)}{(p+1)^{\frac{2}{p}}} \times \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}.$$

Theorem 28. *Let* $f : K \to \mathbb{R}$ *be a partially differentiable function on* K *and let* $p \geq 1$*. If* $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q$ *is co-ordinated extended*(s_1, m_1)*-*(s_2, m_2)*-preinvex function on* K *with respect to* η_1 *and* η_2 *for some fixed* $(s_1, m_1), (s_2, m_2) \in [-1, 1] \times (0, 1]$*, then the following fractional inequality holds*

$$|L(a, b, c, d, \alpha, \beta, \eta_1, \eta_2, A, J)|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{((\alpha+1)(\beta+1))^{1-\frac{1}{q}}}$$
$$\times \left(B(\alpha+1, s_1+1) + \frac{1}{\alpha+s_1+1}\right)^{\frac{1}{q}} \left(B(\beta+1, s_2+1) + \frac{1}{\beta+s_2+1}\right)^{\frac{1}{q}}$$
$$\times \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q + m_2\left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(a, \frac{d}{m_2}\right)\right|^q + m_1\left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, c\right)\right|^q + m_1 m_2\left|\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, \frac{d}{m_2}\right)\right|^q}{4}\right)^{\frac{1}{q}}$$

where L *and* A *are defined as in* (13) *and* (14) *respectively, and* $B(\cdot,\cdot)$ *is the beta function.*

Proof. From Lemma 15, properties of modulus, and power mean inequality, we have

$$|L(a, b, c, d, \alpha, \beta, \eta_1, \eta_2, A, J)|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1 \left(t^\alpha + (1-t)^\alpha\right)\left(\lambda^\beta + (1-\lambda)^\beta\right) d\lambda\, dt\right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1\int_0^1 \left(t^\alpha + (1-t)^\alpha\right)\left(\lambda^\beta + (1-\lambda)^\beta\right)\right.$$

$$
\begin{aligned}
&\times \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda\, dt\Big)^{\frac{1}{q}} \\
= & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4^{\frac{1}{q}}(\alpha+1)^{1-\frac{1}{q}}(\beta+1)^{1-\frac{1}{q}}} \\
&\times \left(\int_0^1\int_0^1 \left(t^\alpha+(1-t)^\alpha\right)\left(\lambda^\beta+(1-\lambda)^\beta\right)\right. \\
&\times \left.\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda\, dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q$ is co-ordinated extended (s_1, m_1)-(s_2, m_2)-preinvex it yields

$$
\begin{aligned}
&|L(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J)| \\
&\leq \tfrac{\eta_1(b,a)\eta_2(d,c)}{4^{\frac{1}{q}}(\alpha+1)^{1-\frac{1}{q}}(\beta+1)^{1-\frac{1}{q}}}\left(\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q \int_0^1\int_0^1 \left(t^\alpha(1-t)^{s_1}+(1-t)^{\alpha+s_1}\right)\right. \\
&\quad\times \left(\lambda^\beta(1-\lambda)^{s_2}+(1-\lambda)^{\beta+s_2}\right) d\lambda\, dt \\
&\quad+ m_2\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}\left(a,\tfrac{d}{m_2}\right)\right|^q \int_0^1\int_0^1 \left(t^\alpha(1-t)^{s_1}+(1-t)^{\alpha+s_1}\right) \\
&\quad\times \left(\lambda^{\beta+s_2}+\lambda^{s_2}(1-\lambda)^\beta\right) d\lambda\, dt \\
&\quad+ m_1\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}\left(\tfrac{b}{m_1},c\right)\right|^q \int_0^1\int_0^1 \left(t^{\alpha+s_1}+t^{s_1}(1-t)^\alpha\right) \\
&\quad\times \left(\lambda^\beta(1-\lambda)^{s_2}+(1-\lambda)^{\beta+s_2}\right) d\lambda\, dt \\
&\quad+ m_1 m_2\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}\left(\tfrac{b}{m_1},\tfrac{d}{m_2}\right)\right|^q \int_0^1\int_0^1 \left(t^{\alpha+s_1}+t^{s_1}(1-t)^\alpha\right) \\
&\quad\times \left.\left(\lambda^{\beta+s_2}+\lambda^{s_2}(1-\lambda)^\beta\right) d\lambda\, dt\right)^{\frac{1}{q}} \\
&= \tfrac{\eta_1(b,a)\eta_2(d,c)}{4^{\frac{1}{q}}(\alpha+1)^{1-\frac{1}{q}}(\beta+1)^{1-\frac{1}{q}}} \\
&\quad\times \left(B(\alpha+1,s_1+1)+\tfrac{1}{\alpha+s_1+1}\right)^{\frac{1}{q}}\left(B(\beta+1,s_2+1)+\tfrac{1}{\beta+s_2+1}\right)^{\frac{1}{q}} \\
&\quad\times \left(\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q + m_2\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}\left(a,\tfrac{d}{m_2}\right)\right|^q + m_1\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}\left(\tfrac{b}{m_1},c\right)\right|^q\right. \\
&\quad+ \left. m_1 m_2\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}\left(\tfrac{b}{m_1},\tfrac{d}{m_2}\right)\right|^q\right)^{\frac{1}{q}},
\end{aligned}
$$

which is the desired result. □

Corollary 29. *In Theorem 28 if we put $m_1 = m_2 = 1$, we get*

$$\begin{aligned}
&\left|L\left(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J\right)\right| \leq \frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha+1)^{1-\frac{1}{q}}(\beta+1)^{1-\frac{1}{q}}} \\
&\times\left(B\left(\alpha+1,s_1+1\right)+\frac{1}{\alpha+s_1+1}\right)^{\frac{1}{q}}\left(B\left(\beta+1,s_2+1\right)+\frac{1}{\beta+s_2+1}\right)^{\frac{1}{q}} \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

And if we take $s_1 = s_2 = 1$, we get

$$\begin{aligned}
&\left|L\left(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J\right)\right| \\
&\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4^{\frac{1}{q}}(\alpha+1)(\beta+1)}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,c\right)\right|^q+m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q\right. \\
&\left.+m_1\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q+m_1m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

In the case where $m_1 = m_2 = s_1 = s_2 = 1$, we obtain

$$\begin{aligned}
&\left|L\left(a,b,c,d,\alpha,\beta,\eta_1,\eta_2,A,J\right)\right| \leq \frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha+1)(\beta+1)} \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 30. *In Theorem 28 if we choose $\eta_1(b,a) = \eta_2(b,a) = b - a$, we obtain*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-C+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right. \\
&\left.\times\left(J^{\alpha,\beta}_{b^-,d^-}f\left(a,c\right)+J^{\alpha,\beta}_{a^+,d^-}f\left(b,c\right)+J^{\alpha,\beta}_{(b)^-,c^+}f\left(a,d\right)+J^{\alpha,\beta}_{a^+,c^+}f\left(b,d\right)\right)\right| \\
&\leq \frac{(b-a)(d-c)}{4^{\frac{1}{q}}(\alpha+1)^{1-\frac{1}{q}}(\beta+1)^{1-\frac{1}{q}}}\left(B\left(\alpha+1,s_1+1\right)+\frac{1}{\alpha+s_1+1}\right)^{\frac{1}{q}} \\
&\times\left(B\left(\beta+1,s_2+1\right)+\frac{1}{\beta+s_2+1}\right)^{\frac{1}{q}} \\
&\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,c\right)\right|^q+m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q+m_1\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q\right. \\
&\left.+m_1m_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Moreover if we put $m_1 = m_2 = 1$, we get

$$\begin{aligned}&\Big|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - C + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\ &\quad\times \left(J^{\alpha,\beta}_{b^-,d^-} f(a,c) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{(b)^-,c^+} f(a,d) + J^{\alpha,\beta}_{a^+,c^+} f(b,d)\right)\Big|\\ &\le \tfrac{(b-a)(d-c)}{4^{\frac{1}{q}}(\alpha+1)^{1-\frac{1}{q}}(\beta+1)^{1-\frac{1}{q}}}\\ &\quad\times \left(B(\alpha+1, s_1+1) + \tfrac{1}{\alpha+s_1+1}\right)^{\frac{1}{q}} \left(B(\beta+1, s_2+1) + \tfrac{1}{\beta+s_2+1}\right)^{\frac{1}{q}}\\ &\quad\times \left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q + \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

And if we take $s_1 = s_2 = 1$, we get

$$\begin{aligned}&\Big|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - C + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\ &\quad\times \left(J^{\alpha,\beta}_{b^-,d^-} f(a,c) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{(b)^-,c^+} f(a,d) + J^{\alpha,\beta}_{a^+,c^+} f(b,d)\right)\Big|\\ &\le \tfrac{(b-a)(d-c)}{4^{\frac{1}{q}}(\alpha+1)(\beta+1)} \times \left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\tfrac{d}{m_2}\right)\right|^q\right.\\ &\quad\left. + m_1\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(\tfrac{b}{m_1},c\right)\right|^q + m_1 m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(\tfrac{b}{m_1},\tfrac{d}{m_2}\right)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

And if we take $m_1 = m_2 = s_1 = s_2 = 1$, we get

$$\begin{aligned}&\Big|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - C + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\ &\quad\times \left(J^{\alpha,\beta}_{b^-,d^-} f(a,c) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{(b)^-,c^+} f(a,d) + J^{\alpha,\beta}_{a^+,c^+} f(b,d)\right)\Big|\\ &\le \tfrac{(b-a)(d-c)}{(\alpha+1)(\beta+1)}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q}{4}\right)^{\frac{1}{q}},\end{aligned}$$

where C is defined by (3).

Corollary 31. *In Theorem 28 if we put $\alpha = \beta = 1$, we get*

$$\begin{aligned}&\Big|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - D\\ &\qquad + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dy\,dx\Big|\\ &\le \tfrac{\eta_1(b,a)\eta_2(d,c)}{4(s_1+1)^{\frac{1}{q}}(s_2+1)^{\frac{1}{q}}}\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + m_2\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\tfrac{d}{m_2}\right)\right|^q\right.\end{aligned}$$

$$+ m_1 \left| \frac{\partial^2 f}{\partial t \partial \lambda} \left(\frac{b}{m_1}, c \right) \right|^q + m_1 m_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \Bigg)^{\frac{1}{q}}.$$

Moreover if we put $m_1 = m_2 = 1$, *we get*

$$\Bigg| \frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - D$$
$$+ \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\, dy\, dx \Bigg|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4} \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|^q}{(s_1+1)(s_2+1)} \right)^{\frac{1}{q}}.$$

And if we take $s_1 = s_2 = 1$, *we get*

$$\Bigg| \frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - D$$
$$+ \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\, dy\, dx \Bigg|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4^{1+\frac{1}{q}}} \Bigg(\left| \frac{\partial^2 f}{\partial t \partial \lambda}(a,c) \right|^q + m_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda} \left(a, \frac{d}{m_2} \right) \right|^q$$
$$+ m_1 \left| \frac{\partial^2 f}{\partial t \partial \lambda} \left(\frac{b}{m_1}, c \right) \right|^q + m_1 m_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \Bigg)^{\frac{1}{q}}.$$

In the case where $m_1 = m_2 = s_1 = s_2 = 1$, *we obtain*

$$\Bigg| \frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - D$$
$$+ \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\, dy\, dx \Bigg|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4} \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|^q}{4} \right)^{\frac{1}{q}},$$

where D *is defined by* (17).

Corollary 32. *In Theorem 28 if we choose* $\eta_1(b,a) = \eta_2(b,a) = b - a$ *and* $\alpha = \beta = 1$, *we get*

$$\Bigg| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{2(b-a)} \int_a^b (f(x,c) + f(x,d))\, dx$$

$$- \frac{1}{2(d-c)} \int_c^d (f(b,y) + f(a,y))\, dy + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy\, dx \Bigg|$$
$$\leq \frac{(b-a)(d-c)}{4(s_1+1)^{\frac{1}{q}}(s_2+1)^{\frac{1}{q}}} \left(\left| \frac{\partial^2 f}{\partial t \partial \lambda}(a,c) \right|^q + m_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(a, \frac{d}{m_2}\right) \right|^q \right.$$
$$\left. + m_1 \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, c\right) \right|^q + m_1 m_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, \frac{d}{m_2}\right) \right|^q \right)^{\frac{1}{q}}.$$

Moreover if we put $m_1 = m_2 = 1$, we get

$$\Bigg| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{2(b-a)} \int_a^b (f(x,c) + f(x,d))\, dx$$
$$- \frac{1}{2(d-c)} \int_c^d (f(b,y) + f(a,y))\, dy + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy\, dx \Bigg|$$
$$\leq \frac{(b-a)(d-c)}{4} \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|^q}{(s_1+1)(s_2+1)} \right)^{\frac{1}{q}}.$$

And if we take $s_1 = s_2 = 1$, we get

$$\Bigg| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{2(b-a)} \int_a^b (f(x,c) + f(x,d))\, dx$$
$$- \frac{1}{2(d-c)} \int_c^d (f(b,y) + f(a,y))\, dy + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy\, dx \Bigg|$$
$$\leq \frac{(b-a)(d-c)}{4^{(1+\frac{1}{q})}} \left(\left| \frac{\partial^2 f}{\partial t \partial \lambda}(a,c) \right|^q + m_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(a, \frac{d}{m_2}\right) \right|^q \right.$$
$$\left. + m_1 \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, c\right) \right|^q + m_1 m_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{b}{m_1}, \frac{d}{m_2}\right) \right|^q \right)^{\frac{1}{q}}.$$

In the case where $m_1 = m_2 = s_1 = s_2 = 1$, we obtain

$$\Bigg| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{2(b-a)} \int_a^b (f(x,c) + f(x,d))\, dx$$
$$- \frac{1}{2(d-c)} \int_c^d (f(b,y) + f(a,y))\, dy + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy\, dx \Bigg|$$
$$\leq \frac{(b-a)(d-c)}{4} \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|^q}{4} \right)^{\frac{1}{q}}.$$

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