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RIEMANN SUMMABILITY OF TRIGONOMETRIC SERIES AND RIEMANN DERIVATIVES OF REAL FUNCTIONS

Abstract

A relation between Riemann summability and Riemann derivative is established and necessary and sufficient conditions for Riemann summability of trigonometric series are obtained

1 Introduction

If $a_r, b_r \to 0$ as $r \to \infty$ then the term by term twice integrated series of the trigonometric series

$$\frac{1}{2}a_0 + \sum_{r=1}^{\infty}(a_r \cos rx + b_r \sin rx) \tag{1}$$

converges to a continuous function F where

$$F(x) = \frac{1}{2}a_0\frac{x^2}{2} - \sum_{r=1}^{\infty}\frac{1}{r^2}(a_r \cos rx + b_r \sin rx)$$

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and it is easy to verify that for $h \neq 0$

$$\frac{F(x+h)-2F(x)+F(x-h)}{4h^2} = \frac{1}{2}a_0 + \sum_{r=1}^{\infty}(a_r \cos rx + b_r \sin rx)\left(\frac{\sin rh}{rh}\right)^2.$$

Letting $h \to 0$, if the limit exists finitely,

$$\lim_{h\to 0}\frac{F(x+h)-2F(x)+F(x-h)}{4h^2} = \lim_{h\to 0}\left[\frac{1}{2}a_0 + \sum_{r=1}^{\infty}(a_r \cos rx + b_r \sin rx)\left(\frac{\sin rh}{rh}\right)^2\right]. \tag{2}$$

In this case the series (1) is called Riemann summable of order 2 at x and the right hand side of (2) is called its $(R, 2)$ sum at x while the left hand side of (2) is called the symmetric Riemann derivative of F at x of order 2. This is the method of Riemann for studying the behaviour of trigonometric series. Hardy in his book ([6] p-53), remarked that Riemann's methods are fundamental in the theory of trigonometric series. He pointed out two results of Riemann, one on $(R, 2)$-summability and the other on (R_2)-summability which played an important role to study the behaviour of trigonometric series ((R_2)-summability is defined in Section 2 below). Zygmund [33] devoted an entire chapter of his book on Riemann theory of trigonometric series calling these two results as "Riemann's first theorem" and "Riemann's second theorem" respectively (see [33]; Vol I, pp 319-320). Lots of research papers are published thereafter investigating the nature of convergence and various types of summability of the series (1) of which Cesaro summability had played a major role. Higher order Riemann derivatives and their generalizations are introduced. To study Cesaro summabilty de la Vallée-Poussin (d.I.V.P.) derivative (also called generalized symmetric derivative) are introduced. The problem of convergence of the series (1) is old and consequently many papers appeared on Cesaro summabilty (for example see [11, 20, 32, 33]) and also on Riemann summability (for example see [7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 19, 25, 26, 27, 31]). Also many papers appeared on Riemann derivative (see [1, 2, 3, 4, 5, 17, 18, 22, 23, 24]; see also [21] for other references). The present authors have not seen in the literature the relation between Riemann summability and Riemann derivative of order $k > 2$.

In the present paper, we established a connection between Riemann derivative and Riemann summability of any order and generalized the two classical results of Riemann summability cited in the begining.

2 Definitions and notations

Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a positive integer k. The symmetric difference of f at a point x of order k is defined by

$$\Delta_k^s(f;x,2h) = \sum_{j=0}^{k}(-1)^{k-j}\left(\begin{array}{c} k \\ j \end{array}\right) f(x+2jh-kh). \tag{3}$$

It can be verified that the difference operator (3) satisfies the relation

$$\Delta_1^s(f;x,2h) = f(x+h) - f(x-h)$$

$$\Delta_{k+1}^s(f;x,2h) = \Delta_1^s[\Delta_k^s(f;x,2h)] = \Delta_k^s(f;x+h,2h) - \Delta_k^s(f;x-h,2h). \tag{4}$$

(see [4] and [17]).

Definition 2.1. (Riemann derivative) The symmetric Riemann derivative of f at x of order k, denoted by $RD_k^s f(x)$, is defined by

$$RD_k^s f(x) = \lim_{h\to 0+} \frac{\Delta_k^s(f;x,2h)}{(2h)^k}, \tag{5}$$

provided the limit exists.

If $RD_k^s f(x)$ exists then it does not imply that the previous derivative $RD_i^s f(x)$ exists for $i = 1, 2, \ldots.., k-1$.

However, if the ordinary k-*th* derivative $f^{(k)}$ exists then $RD_k^s f(x)$ exists and they are equal.

Definition 2.2. (Riemann smoothness) If

$$\lim_{h\to 0+} \frac{\Delta_k^s(f;x,2h)}{(2h)^k}\, h = 0 \tag{6}$$

then f is said to be Riemann smooth at x of order k.
Clearly, if $RD_k^s f(x)$ exists finitely then f is Riemann smooth at x of order k. For $k = 2$ this is the definition of usual smoothness.

Definition 2.3. (de la Vallée-Poussin derivative) If there is a polynomial $P(t) = P_x(t)$ of degree at most k such that

$$\frac{1}{2}[f(x+t) + (-1)^k f(x-t)] = P(t) + o(t^k) \quad \text{as} \quad t \to 0+, \tag{7}$$

then f is said to have symmetric d.I.V.P. derivative at x of order k and if $\frac{a_k}{k!}$ is the coefficient of t^k in $P(t)$ then a_k is called the symmetric d.l.V.P. derivative

of f at x of order k and is denoted by $f_{(k)}^{(s)}(x)$.
It can be shown that if $f_{(k)}^{(s)}(x)$ exists then $f_{(i)}^{(s)}(x)$ exists for $i = k-2, k-4, \ldots, 2$ or 1 according as k is even or odd, but not necessarily for $i = k-1, k-3, \ldots$. Also it can be shown that if $f_{(k)}^{(s)}(x)$ exists then $RD_k^s f(x)$ also exists and they are equal (see [21]; p 178). The converse is not true. So the Riemann derivative is more general than the d.l.V.P. derivative. It may be noted that $RD_k^s f(x) = f_{(k)}^{(s)}(x)$ for $k = 1$ and 2.

Definition 2.4. (Riemann summability) A series $\sum_{r=0}^{\infty} C_r$ of constant terms is said to be Riemann summable of order k, or (R, k) summable to s if

$$\lim_{h \to 0} \sum_{r=0}^{\infty} C_r \left(\frac{\sin rh}{rh} \right)^k = s. \tag{8}$$

A sequence $\{s_r\}$ of constant terms is said to be (R, k) summable to s if

$$\lim_{h \to 0} \sum_{r=0}^{\infty} s_r \left[\left(\frac{\sin rh}{rh} \right)^k - \left(\frac{\sin(r+1)h}{(r+1)h} \right)^k \right] = s. \tag{9}$$

Clearly (9) agrees with (8). In fact, if s_r is the r-th partial sum of $\sum_{i=0}^{\infty} C_i$ then by summing by parts the left hand side of (8) we get the left hand side of (9). It may be noted that if a series is (R, k) summable then it may not be (R, k') summable for $k' > k$. See [14, 15]

Definition 2.5. (Modified Riemann summability) A sequence $\{s_r\}$ of constant terms is said to be modified Riemann summable of order k, or (R_k) summable to s if

$$\lim_{h \to 0} \sum_{r=0}^{\infty} (s_r - s) \left(\frac{\sin rh}{rh} \right)^k h = 0. \tag{10}$$

A series $\sum_{r=0}^{\infty} C_r$ of constant terms is said to be (R_k) summable to s if the sequence $\{s_r\}$ of its partial sums $s_r = \sum_{i=0}^{r} C_i$ is (R_k) summable to s.

The methods (R, k) and (R_k) are distinct. Marcinkiewicz [16] proved that the methods $(R, 2)$ and (R_2) are incomparable.

3 Preliminaries

Since a summability method is a generalization of convergence, it is desirable that whenever a series converges to s, it should be summable to s by that summability method. If this is so then that summability method is called regular. Let

$$M = (\alpha_{nr}) = \begin{pmatrix} \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0r} & \cdots \\ \alpha_{10} & \alpha_{11} & \cdots & \alpha_{1r} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n0} & \alpha_{n1} & \cdots & \alpha_{nr} & \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

be a doubly infinite matrix. Then the matrix M is called regular if the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_{nr} = 0$ for $r = 0, 1, 2, ...$

(ii) The sequence $\{Q_n\}$ is bounded where $Q_n = \sum_{r=0}^{\infty} |\alpha_{nr}|$

(iii) $\lim_{n\to\infty} \sum_{r=0}^{\infty} \alpha_{nr} = 1.$

The matrix M induces a summability method. For, given a sequence $\{s_r\}$, or a series whose partial sums are s_r, we can find another sequence $\{\sigma_n\}$ such that $\sigma_n = \sum_{r=0}^{\infty} \alpha_{nr} s_r$ and if $\{\sigma_n\}$ converges to σ then $\{s_r\}$ is called summable to σ by the method M.

Theorem 3.1. *If a matrix $M = (\alpha_{nr})$ is regular then the summability method induced by M is regular.*

This is proved in ([33], Vol I, p74).

Theorem 3.2. *Let $k > 1$ and let $u(h) = \left(\frac{\sin h}{h}\right)^k$. For every r, $r = 0, 1, 2, ...$ and any sequence $\{h_n\}$ such that $h_n \to 0$ as $n \to \infty$, let $\alpha_{nr} = u(rh_n) - u((r+1)h_n)$. Then the matrix (α_{nr}) is regular.*

PROOF. We may suppose that $h_n > 0$ for all n. We have

$$\begin{aligned} \lim_{n\to\infty} \alpha_{nr} &= \lim_{n\to\infty} [u(rh_n) - u((r+1)h_n)] \\ &= \lim_{n\to\infty} \left[\left(\frac{\sin rh_n}{rh_n}\right)^k - \left(\frac{\sin(r+1)h_n}{(r+1)h_n}\right)^k \right] = 0 \end{aligned}$$

for $r = 0, 1, 2, \ldots$(assuming $\dfrac{\sin 0}{0} = 1$). So (α_{nr}) satisfies (i). Also writing Q_n as in (ii).

$$
\begin{aligned}
Q_n = \sum_{r=0}^{\infty} \Big| u(rh_n) - u((r+1)h_n) \Big| &= \sum_{r=0}^{\infty} \left| \int_{rh_n}^{(r+1)h_n} u'(t)dt \right| \\
&\le \sum_{r=0}^{\infty} \int_{rh_n}^{(r+1)h_n} |u'(t)|dt = \int_0^{\infty} |u'(t)|dt \\
&= \int_0^{\infty} \left| k \left(\frac{\sin t}{t}\right)^{k-1} \frac{t\cos t - \sin t}{t^2} \right| dt
\end{aligned}
$$

$$
= \int_0^{\frac{\pi}{2}} \left| k \left(\frac{\sin t}{t}\right)^{k-1} \frac{t\cos t - \sin t}{t^2} \right| dt + \int_{\frac{\pi}{2}}^{\infty} \left| k \left(\frac{\sin t}{t}\right)^{k-1} \frac{t\cos t - \sin t}{t^2} \right| dt.
$$

The first integral is finite. The second integral is

$$
\left| k \int_{\frac{\pi}{2}}^{\infty} (\sin t)^{k-1} \frac{t\cos t - \sin t}{t^{k+1}} \right| dt \le k \int_{\frac{\pi}{2}}^{\infty} \left| \frac{t\cos t - \sin t}{t^{k+1}} \right| dt.
$$

Since $\left| \dfrac{t\cos t - \sin t}{t^{k+1}} \right|$ is $O(t^{-k})$ as $t \to \infty$ and since $k > 1$, the second integral is also finite. These integrals are also independent of n. This being true for all n, the sequence $\{Q_n\}$ is bounded and hence $\{\alpha_{nr}\}$ satisfies (ii).

Finally, $\lim\limits_{n\to\infty} \sum\limits_{r=0}^{\infty} \alpha_{nr} = \lim\limits_{n\to\infty} \sum\limits_{r=0}^{\infty} [u(rh_n) - u((r+1)h_n)] = 1$. So (α_{nr}) satisfies (iii). Therefore (α_{nr}) is regular. □

4 Series and sequences of constant terms

Theorem 4.1. *If* $\sum\limits_{r=0}^{\infty} C_r$ *is a series of constant terms which converges to* s *then this series is* (R, k) *summable to* s *for all* $k > 1$.

PROOF. Let $S_r = \sum\limits_{i=0}^{r} C_i$ and $u(h) = \left(\dfrac{\sin h}{h}\right)^k$. Applying summation by parts we have for every sequence $\{h_n\}$ such that $h_n \to 0+$ as $n \to \infty$,

$$
C_0 + \sum_{r=1}^{\infty} C_r \left(\frac{\sin rh_n}{rh_n}\right)^k = \sum_{r=0}^{\infty} S_r [u(rh_n) - u((r+1)h_n)]. \qquad (11)
$$

Writing $\alpha_{nr} = [u(rh_n) - u((r+1)h_n]$ and applying Theorem 3.2, (α_{nr}) is regular. Let $\sigma_n = \sum_{r=0}^{\infty} S_r \alpha_{nr}$. Since $S_r \to s$, by Theorem 3.1, $\sigma_{\mathrm{n}} \to s$ as $n \to \infty$. Hence the left hand side of (11) tends to s as $h_n \to 0+$. Since $\{h_n\}$ is an arbitrary sequence $\lim_{h\to 0+} \left[C_0 + \sum_{r=1}^{\infty} C_r \left(\frac{\sin rh}{rh} \right)^k \right] = s$. This shows that this is also true for $h \to 0-$. Hence $\sum_{r=0}^{\infty} C_r$ is (R,k) summable to s. □

Theorem 4.2. *If $\{s_r\}$ is a sequence of constant terms which converges to* 0 *then this sequence is (R_k) summable to* 0 *for all $k > 1$.*

PROOF. Consider any sequence $\{h_n\}$ such that $h_n \to 0+$ as $n \to \infty$. Let $\alpha_{nr} = \left(\frac{\sin rh_n}{rh_n} \right)^k h_n$. Then $\lim_{n\to\infty} \alpha_{nr} = 0$ for $r = 0, 1, 2, \ldots$, showing that (α_{nr}) satisfies (i). To show that it satisfies (ii) note that

$$Q_n = \sum_{r=0}^{\infty} |\alpha_{nr}| = h_n + \sum_{r=1}^{\infty} \left| \left(\frac{\sin rh_n}{rh_n} \right)^k \right| h_n.$$

We may suppose that $h_n \leq 1$ for all n. Let N be the integer such that $\frac{1}{h_n} < N \leq 1 + \frac{1}{h_n}$ for a fixed h_n. Then $1 < Nh_n \leq h_n + 1$. So

$$\begin{aligned} Q_n = h_n + \sum_{r=1}^{\infty} \left| \left(\frac{\sin rh_n}{rh_n} \right)^k \right| h_n &\leq h_n + \sum_{r=1}^{N} h_n + \sum_{r=N+1}^{\infty} \frac{1}{r^k h_n^{k-1}} \\ = h_n(1+N) + \frac{1}{h_n^{k-1}} \sum_{r=N+1}^{\infty} \frac{1}{r^k} &\leq 3 + \frac{1}{h_n^{k-1}} \int_N^{\infty} \frac{1}{x^k} dx \\ = 3 + \frac{1}{(k-1)(Nh_n)^{k-1}} &< 3 + \frac{1}{k-1}. \end{aligned}$$

This being true for all n, we conclude that (α_{nr}) satisfies (ii). Since $s_r \to 0$ as $r \to \infty$, taking $s = 0$ the condition (iii) is not needed in the proof of Theorem 3.1. So by Theorem 3.1, $\sigma_n = \sum_{r=0}^{\infty} s_r \alpha_{nr} = \sum_{r=0}^{\infty} s_r \left(\frac{\sin rh_n}{rh_n} \right)^k h_n \to 0$ as $n \to \infty$. If the sequence $\{h_n\}$ is such that $h_n \to 0-$ as $n \to \infty$ then putting $t_n = -h_n$ and applying the above argument for the sequence $\{t_n\}$ the above relation is true. This shows that the sequence $\{s_r\}$ is (R_k) summable to 0. □

Corollary 4.3. *If $\{s_r\}$ is a sequence of constant terms which converges to s then this sequence is (R_k) summable to s for all $k > 1$.*

Considering the sequence $\{s_r - s\}$ the result follows from Definition 2.5 and Theorem 4.2.

5 Trigonometric Series

Consider a trigonometric series

$$\frac{1}{2}a_0 + \sum_{r=1}^{\infty}(a_r \cos rx + b_r \sin rx). \tag{12}$$

Note that the series (12) may or may not be convergent and that even if it converges everywhere it may not be a Fourier series of any function. For convenience we write

$$A_0(x) = \frac{1}{2}a_0, A_r(x) = a_r \cos rx + b_r \sin rx, B_r(x) = a_r \sin rx - b_r \cos rx \tag{13}$$

and so the series (12) henceforth will be written as $\sum_{r=0}^{\infty} A_r(x)$.

Theorem 5.1. *If the series $\sum_{r=0}^{\infty} A_r(x)$ converges to s at a point x_0 then the series $\sum_{r=0}^{\infty} A_r(x_0)$ is (R, k) summable to s for all $k > 1$.*

Proof. Since $\sum_{r=0}^{\infty} A_r(x_0)$ is a series of constant terms the result follows from Theorem 4.1. □

Putting $k = 2$ Theorem 5.1 is Riemann's first theorem (see [33], Vol I, p 319).

Theorem 5.2. *If the sequence $\{A_r(x)\}$ converges to 0 at a point x_0 then $\{A_r(x_0)\}$ is (R_k) summable to 0 for all $k > 1$.*

The result follows from Theorem 4.2.

Theorem 5.3. *If $a_r \to 0, b_r \to 0$ as $r \to \infty$ then the sequence $\{A_r(x)\}$ is (R_k) summable to 0 uniformly for all x and all $k > 1$.*

PROOF. Since $|A_r(x)| \leq |a_r| + |b_r|, \{A_r(x)\}$ converges to 0 uniformly for all x and so the result follows from Theorem 5.2. □

For $k = 2$ Theorem 5.3 is Riemann's second theorem (see [33], Vol I, p320)

6 Relation between Riemann derivative and Riemann summability

The following theorem establishes a relation between Riemann derivative and Riemann summability.

Theorem 6.1. *Let k be a positive integer. If the series obtained by k times term-by-term integration of (12) converges uniformly and absolutely to a function F then for $h \neq 0$*

$$\frac{\Delta_k^s(F;x,2h)}{(2h)^k} = A_0 + \sum_{r=1}^{\infty} A_r(x) \left(\frac{\sin rh}{rh} \right)^k . \tag{14}$$

PROOF. Since the k times integrated series of (12) converges uniformly and absolutely to F, the k times term-by-term integrated series of $\sum_{r=1}^{\infty} A_r(x)$ converges uniformly and absolutely to a function G such that $G(x) = F(x) - A_0 \frac{x^k}{k!}$. Writing $H(x) = A_0 \frac{x^k}{k!}$ we have $F = G + H$. So we have

$$\frac{\Delta_k^s(F;x,2h)}{(2h)^k} = \frac{\Delta_k^s(H;x,2h)}{(2h)^k} + \frac{\Delta_k^s(G;x,2h)}{(2h)^k}. \tag{15}$$

To calculate $\Delta_k^s(H;x,2h) = \frac{A_0}{k!}\Delta_k^s(x^k;x,2h)$ we use the relation (see [21], p 177)

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^p = \begin{cases} 0 \text{ if } p = 0, 1, 2, \dots k-1 \\ k! \text{ if } p = k. \end{cases} \tag{16}$$

We have by using (3)

$$\Delta_k^s(x^k;x,2h) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x + 2jh - kh)^k$$

$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{i=0}^{k} \binom{k}{i} x^{k-i} (2j-k)^i h^i$$

$$= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{i=1}^{k}\binom{k}{i}x^{k-i}(2j-k)^i h^i$$

$$= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{i=1}^{k}\binom{k}{i}x^{k-i}h^i\sum_{l=0}^{i}\binom{i}{l}(2j)^l(-k)^{i-l}$$

$$= \sum_{i=1}^{k}\binom{k}{i}x^{k-i}h^i\sum_{l=0}^{i}\binom{i}{l}(2)^l(-k)^{i-l}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(j)^l$$

$= (2h)^k k!$, using (16).

Hence

$$\frac{\Delta_k^s(H;x,2h)}{(2h)^k} = \frac{A_0}{k!}\frac{\Delta_k^s(x^k;x,2h)}{(2h)^k} = A_0. \tag{17}$$

Now we shall prove that

$$\frac{\Delta_k^s(G;x,2h)}{(2h)^k} = \sum_{r=1}^{\infty}A_r(x)\left(\frac{\sin rh}{rh}\right)^k. \tag{18}$$

Let $k=1$. Then G is the sum of the series obtained by term-by- term once integration of $\sum_{r=1}^{\infty}A_r(x)$ and so by (13) $G(x)=\sum_{r=1}^{\infty}\frac{1}{r}B_r(x)$. Since

$$\frac{\Delta_1^s(\frac{1}{r}B_r;x,2h)}{2h} = \frac{B_r(x+h)-B_r(x-h)}{2rh} = \frac{2A_r(x)\sin rh}{2rh} = A_r(x)\left(\frac{\sin rh}{rh}\right),$$

we have

$$\frac{\Delta_1^s(G;x,2h)}{2h} = \sum_{r=1}^{\infty}\frac{\Delta_1^s(\frac{1}{r}B_r;x,2h)}{2h} = \sum_{r=1}^{\infty}A_r(x)\left(\frac{\sin rh}{rh}\right).$$

So (18) is true for $k=1$. Suppose that (18) is true for $k=m$. Let $k=m+1$. Since $\sum_{r=1}^{\infty}\frac{1}{r}B_r(x)$ is the once integrated series of $\sum_{r=1}^{\infty}A_r(x)$, the $(m+1)$ times integrated series of $\sum_{r=1}^{\infty}A_r(x)$ is the same as the m times integrated series of $\sum_{r=1}^{\infty}\frac{1}{r}B_r(x)$ and the m times integrated series of $\sum_{r=1}^{\infty}\frac{1}{r}B_r(x)$

converges uniformly and absolutely to G. Since the result is true for $k = m$, from (18) we have,

$$\frac{\Delta_m^s(G;x,2h)}{(2h)^m} = \sum_{r=1}^{\infty} \frac{1}{r} B_r(x) \left(\frac{\sin rh}{rh}\right)^m . \tag{19}$$

Hence using the relation (4) we get from (19)

$$\frac{\Delta_{m+1}^s(G;x,2h)}{(2h)^{m+1}} = \frac{1}{2h} \Delta_1^s \left(\frac{\Delta_m^s(G;x,2h)}{(2h)^m}\right)$$

$$= \frac{1}{2h}\left[\frac{1}{(2h)^m}\Delta_m^s(G;x+h,2h) - \frac{1}{(2h)^m}\Delta_m^s(G;x-h,2h)\right]$$

$$= \frac{1}{2h}\sum_{r=1}^{\infty}\frac{1}{r}\left[B_r(x+h) - B_r(x-h)\right]\left(\frac{\sin rh}{rh}\right)^m$$

$$= \sum_{r=1}^{\infty} A_r(x) \frac{\sin rh}{rh}\left(\frac{\sin rh}{rh}\right)^m = \sum_{r=1}^{\infty} A_r(x)\left(\frac{\sin rh}{rh}\right)^{m+1}$$

which shows that (18) is true for $k = m + 1$. Therefore (18) is true for all k. So from (15), (17) and (18) we get

$$\frac{\Delta_k^s(F;x,2h)}{(2h)^k} = A_0 + \sum_{r=1}^{\infty} A_r(x)\left(\frac{\sin rh}{rh}\right)^k ,$$

completing the proof. □

Corollary 6.2. *Under the hypothesis of Theorem 6.1, we have for $h \neq 0$*

$$\frac{\Delta_k^s(F;x,2h)}{(2h)^k} h = A_0 h + \sum_{r=1}^{\infty} A_r(x)\left(\frac{\sin rh}{rh}\right)^k h. \tag{20}$$

Theorem 6.3. *Under the hypothesis of Theorem 6.1 the series (12) is (R, k) summable at x_0 to s if and only if the k-th symmetric Riemann derivative of F at x_0, $RD_k^s F(x_0)$ exists finitely and $s = RD_k^s F(x_0)$.*

The proof follows from Theorem 6.1 by taking limit in (14) as $h \to 0$.

Theorem 6.4. *Under the hypothesis of Theorem 6.1 the sequence $\{A_r(x)\}$ of the terms of the series (12) is (R_k) summable at x_0 to 0 if and only if the function F is Riemann smooth at x_0 of order k.*

The proof follows from Corollary 6.2 by taking limit in (20) as $h \to 0$.

Remark 6.5. Suppose $a_r = O(r^{k-2})$ and $b_r = O(r^{k-2})$ where a_r and b_r are as in (12). So there is $M > 0$ such that for all r, $\frac{|a_r|}{r^{k-2}} \le M$ and $\frac{|b_r|}{r^{k-2}} \le M$ which give $\frac{|a_r|}{r^k} \le \frac{M}{r^2}$ and $\frac{|b_r|}{r^k} \le \frac{M}{r^2}$. Therefore the series obtained by k times term-by-term integration of (12) converges uniformly and absolutely to a function F. So Theorem 6.3 gives necessary and sufficient condition of (R, k) summability of the series (12) and Theorem 6.4 gives necessary and sufficient condition of (R_k) summability of the terms of the series (12). Now let us consider $k = 2$. If the series $\sum_{r=1}^{\infty} A_r(x)$ converges at a point x_0 to sum s then $\sum_{r=1}^{\infty} A_r(x_0)$ is a series of constant terms which converges to s. So by Theorem 4.1 we get Riemann's first Theorem (see [33], Vol I, p 319). Again if $a_r, b_r \to 0$ as $r \to \infty$ then since $|A_r(x)| \le |a_r| + |b_r|, \{A_r(x)\}$ converges uniformly to 0 for all x as $r \to \infty$. For a fixed x, $\{A_r(x)\}$ is a sequence of constant terms which converges to 0 and so by Theorem 4.2, $\{A_r(x)\}$ is (R_2)- summable to 0. Considering all x, since $\{A_r(x)\}$ converges uniformly to 0, it is uniformly (R_2) summable to 0. Also the twice integrated series of (12) converges uniformly and absolutely to a function F and so by Corollary 6.2 and Theorem 6.4 we get Riemann's second Theorem (see [33], Vol I, p320).

7 Relation between Riemann summabilty and Cesaro summability

Cesaro summability (for the definition of Cesaro summability see ([6], p46) or ([33], Vol I, p76) and Riemann summability have interesting relation in the sense that if the strength of one is increased it surpasses the other which is seen in the following two theorems.

Theorem 7.1. *If a series $\sum C_r$ is $(C, k - \delta)$ summable to s, where $\delta > 0$, then it is $(R, k + 1)$ summable to s.*

This is proved by Verblunsky [27]

Theorem 7.2. *If a series $\sum C_r$ is (R, k) summable to s then it is $(C, k + \delta)$ summable to s for $k = 1$ and $k = 2$ where $\delta > 0$.*

This is proved by Kuttner [13].

It follows from Theorem 6.1 that Riemann summability of trigonometric series is associated with symmetric Riemann derivative. The following theorem shows that Cesaro summability is associated with symmetric d.l..V. P derivative which is to some extent equivalent to symmetric Cesaro derivative. (see [21], p40 and p106).

Theorem 7.3. *If the series (12) is* (C,α) *summable to* s *at a point* x*, where* $\alpha \geq 0$ *and if the series obtained by* k *times term by term integration of (12) converges uniformly and absolutely to a function* F *where* $k > \alpha + 1$ *then* $F^s_{(k)}(x)$ *exists and equals* s*.*

This follows from a theorem of Wolf [32] (see also [33], Vol.II, p 66). Now we have

Theorem 7.4. *If the series (12) is* (C,α) *summable at a point* x *to* s*, where* $\alpha \geq 0$*, and if the series obtained by* k *times term by term integration of (12) converges uniformly and absolutely to a function* F *where* $k > \alpha + 1$ *then (12) is* (R,k) *summable at* x *to* s*.*

Proof. Since the existence of $F^s_{(k)}(x)$ implies the existence of $RD^s_k F(x)$ and $F^s_k(x) = RD^s_k F(x)$ (see [21], p 178) the proof follows from Theorem 7.3 and Theorem 6.3. □

8 Concluding remarks

Theorem 4.1 and Theorem 4.2 and consequently Theorem 5.1 and Theorem 5.2 show that the methods (R,k) and (R_k) are regular for $k > 1$. The regularity of (R,k) and (R_k) methods for $k \geq 2$ are mentioned in [9, 10] and so these are known, but we could not locate the proofs of these and therefore we give the proofs of these two results for completeness. For $k = 1$ the situation is different. If the series (12) converges at a point x_0 then the once integrated series of (12) need not converge in a neighbourhood of x_0 and so (R,1) summability of (12) need further conditions. This summability method is called Lebesgue summability or summable L. Zygmund proved that if a_n and b_n are $O(\frac{1}{n})$ then (12) is convergent at x_0 to $s(x_0)$ if and only if it is summable L at x_0 to $s(x_0)$ (see [33], Vol I, pp 321-323]). Moricz [19] relaxed this condition on a_n and b_n and proved under this weaker condition that if (12) converges at x_0 to $s(x_0)$ then (12) is summable L at x_0 to $s(x_0)$. Vindas [31] extended this result of Moricz for several other summability methods. It is remarked in [9, 10] that methods $(R,1)$ and (R_1) are not regular. Also it is remarked in [31] that summability L is somehow complicated and it is not regular and that if (12) converges at x_0 then it is not necessarily summable L at x_0 (see [31], p76).

To prove Theorem T' in [8] Hardy and Littlewood proved that if $\phi(n)$ is any positive function of n tending steadily to infinity with n it is possible to find a convergent series $\sum a_n$ for which

$$a_n = O\left(\frac{\phi(n)}{n}\right) \quad \texttt{and} \quad \limsup_{h\to 0} \sum a_n \frac{\sin nh}{nh} = \infty$$

(See [8]; proof of Theorem T', pp 255-261. See also [26]; p 389). In the proof the authors took any function $\phi(n) < \sqrt{n}$. So the following theorem came out:

Theorem 8.1. *(Hardy and Littlewood). There is a convergent series $\sum a_n$ of constant terms such that*

$$a_n = O\left(\frac{1}{2\sqrt{n}}\right) \tag{21}$$

and

$$\limsup_{h\to 0} \sum a_n \frac{\sin nh}{nh} = \infty. \tag{22}$$

This theorem shows that the summability method (R, k) is not regular for $k = 1$. So, for $k = 1$ Theorem 4.1 and Theorem 4.2 do not necessarily hold. For trigonometric series we get:

Theorem 8.2. *There is a trigonometric series $\sum A_r(x)$ which converges at a point x_0 but*

$$\limsup_{h\to 0} \sum A_r(x_0) \frac{\sin rh}{rh} = \infty \tag{23}$$

and hence $\sum A_r(x_0)$ is not $(R, 1)$ summable.

Proof. Without loss of generality we can take $x_0 = 0$. Let us take the series $\sum a_r$ of Theorem 8.1 and consider the series $\sum A_r(x)$ where $A_r(x) = a_r \cos rx$. Then $\sum A_r(x)$ converges at $x = 0$. So by taking $x_0 = 0$ and applying (22) the relation (23) holds for $x_0 = 0$, proving the theorem. □

Theorem 8.2 shows that Theorem 5.1 and Theorem 5.2 do not hold for $k = 1$.
Consider the trigonometric series $\sum A_r(x) = \sum a_r \cos rx$ where a_r are as in Theorem 8.1. Since a_r satisfies (21) the once integrated series $\sum a_r \frac{\sin rx}{r}$ of $\sum A_r(x)$ converges uniformly and absolutely to a function, say $L(x)$ and hence by Theorem 6.1

$$\frac{\Delta_1^s(L; x, 2h)}{2h} = \sum a_r \cos rx \left(\frac{\sin rh}{rh}\right)$$

which gives by (22)

$$\limsup_{h\to 0} \frac{\Delta_1^s(L;0,2h)}{2h} = \infty. \tag{24}$$

However in (24) approximate limit exists and

$$\lim_{h\to 0} ap \frac{\Delta_1^s(L;0,2h)}{2h} = \lim_{h\to 0} ap \sum a_r \frac{\sin rh}{rh} = \sum a_r.$$

For, Rajchmann and Zygmund (see [29], Lemma 24) proved that if a series $\sum C_r$ of constant terms converges to s then

$$\lim_{h\to 0} ap \sum C_r \frac{\sin rh}{rh} = s.$$

This result shows that if a series is convergent and if it is not $(R,1)$-summable then it has a property which may be called approximately $(R,1)$-summable.

We need the following theorem of Hardy.

Theorem 8.3. *The series $\sum n^{-b} e^{Ain^a}$ where $A > 0, 0 < a < 1, b = \beta + i\gamma$, is (C,k) summable for $k > -1$ if and only if $(k+1)a + \beta > 1$.*

This is proved in ([6]; p 141, Theorem 84).

Theorem 8.4. *There exists a series which is not convergent, but is $(R,2)$ summable.*

Proof. Putting $A = 1, a = \frac{1}{2} = b$ the series in Theorem 8.3 becomes $\sum n^{-\frac{1}{2}} e^{in^{\frac{1}{2}}}$. Since in this case $(k+1)a+\beta = (k+1)\frac{1}{2} + \frac{1}{2} = \frac{k}{2} + 1 > 1$ if and only if $k > 0$, by Theorem 8.3 the series $\sum n^{-\frac{1}{2}} e^{in^{\frac{1}{2}}}$ is (C,k) summable if and only if $k > 0$. So, $\sum n^{-\frac{1}{2}} e^{in^{\frac{1}{2}}}$ is not convergent but is $(C, \frac{1}{2})$ summable. Again by Theorem 7.1 $(C, \frac{1}{2})$ summability implies $(R,2)$ summability. Hence the series $\sum n^{-\frac{1}{2}} e^{in^{\frac{1}{2}}}$ is $(R,2)$ summable. This completes the proof. □

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