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LOCAL DIMENSIONS OF OVERLAPPING SELF-SIMILAR MEASURES

Abstract

We show that any equicontractive, self-similar measure arising from the IFS of contractions (S_j), with self-similar set $[0,1]$, admits an isolated point in its set of local dimensions provided the images of $S_j(0,1)$ (suitably) overlap and the minimal probability is associated with one (resp., both) of the endpoint contractions. Examples include m-fold convolution products of Bernoulli convolutions or Cantor measures with contraction factor exceeding $1/(m+1)$ in the biased case and $1/m$ in the unbiased case. We also obtain upper and lower bounds on the set of local dimensions for various Bernoulli convolutions.

1   Introduction

Consider the iterated function system (IFS) consisting of the contractions of $S_j : [0,1] \to [0,1]$, with common contraction factor ρ, and probabilities p_j, $j = 0, ..., m \geq 1$. By the equicontractive, self-similar measure associated with this IFS we mean the unique Borel probability measure μ satisfying

$$\mu = \sum_{j=0}^{m} p_j \cdot \mu \circ S_j^{-1}. \tag{1}$$

This measure is supported on the associated self-similar set and is well known to be either purely absolutely continuous with respect to Lebesgue measure or purely singular. When $S_0(x) = \rho x$, $S_1(x) = \rho x + 1 - \rho$ and $m = 1$, the associated self-similar measures are known as Cantor measures or Bernoulli convolutions, and are sometimes referred to as unbiased if $p_0 = p_1$, or biased if $p_0 \neq p_1$.

Our interest is in the local behaviour of these measures. The *local dimension* of a measure μ at a point x in the support of μ is defined as

$$\dim_{\text{loc}} \mu(x) = \lim_{\epsilon \to 0} \frac{\log(\mu([x - \epsilon, x + \epsilon]))}{\log \epsilon}. \tag{2}$$

In the case that the IFS satisfies the open set condition, it is well known that the set of local dimensions of the associated self-similar measure is a closed interval and there are formulas for the endpoints of the interval which depend on the contraction factors and the probabilities. See [5] for more details. For measures that do not satisfy the open set condition the situation is much less well understood. In [14], Hu and Lau discovered that the 3-fold convolution of the unbiased middle-third Cantor measure admits an isolated point in its set of local dimensions. This was later found to be true for certain other equicontractive self-similar Cantor-like measures arising from IFS which have enough 'overlap', such as the m-fold convolution product of the unbiased Cantor measure with contraction factor $1/m$, [2, 20]. These measures all had the so-called 'finite type' property, a separation condition permitting overlaps, but stronger than the weak separation condition.

The Bernoulli convolutions with contraction factor the inverse of a Pisot number[1] also have the finite type property. These are particularly interesting being the only known singular Bernoulli convolutions, see [3, 19, 21]. There is a long history of studying the dimensionality properties of these measures, c.f., [17, 22, 23] and the many references cited therein for historical information. In [6, 7], Feng conducted a study of mainly unbiased Bernoulli convolutions with contraction factor the inverse of a simple Pisot number and proved that for this class of measures the set of local dimensions is an interval. In contrast, in [11] it was shown that all biased Bernoulli convolutions with these contraction factors admit an isolated point.

In this paper, we show that *any* equicontractive, self-similar measure will admit an isolated point in its set of local dimensions provided the images of $[0, 1]$ under the contractions strictly overlap and p_0 is the unique minimal probability (Theorem 4). We also prove there is an isolated point if $p_0 = p_m$

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[1]A Pisot number is a real algebraic integer, greater than 1, such that all of its Galois conjugates are strictly less than 1 in absolute value.

are the unique minimal probabilities and there is 'sufficient' overlap (Theorem 7). In particular, we prove that if μ is any Bernoulli convolution or Cantor measure with contraction factor $\rho > 1/(m+1)$ in the biased case and $\rho > (\sqrt{m^2+4}-m)/2$ in the unbiased case, then the m-fold convolution of μ with itself has an isolated point in its set of local dimensions, improving upon the examples given in [2, 20].

In all these cases, the isolated point is the local dimension at 0. This local dimension at $x = 0$ is easy to compute and is the maximum local dimension. A challenging problem is to find sharp bounds for the set of local dimensions at other x. Upper bounds have recently been found in [1] for special classes of examples of these measures, including the biased Bernoulli convolutions. Our arguments of Section 3 also give upper bounds. In Section 4.1 we discuss other computational techniques that allow us to prove even better upper bounds for the local dimensions. A variation of these techniques are used in Section 4.2 to find lower bounds for local dimension in the case where the self-similar measure satisfies the asymptotically weak separation condition. These techniques are applied to various Bernoulli convolutions.

2   Terminology and Basic Properties

Throughout the paper we study the IFS (S_j, p_j) consisting of the contractions

$$S_j(x) = \rho x + d_j \text{ for } 0 = d_0 < d_1 < \cdots < d_m = 1 - \rho, \ j = 0, ..., m, \ m \geq 1 \quad (3)$$

and probabilities $p_j > 0$, $\sum_{j=0}^{m} p_j = 1$, and the associated self-similar measure μ satisfying (1). We refer to ρ as the contraction factor of the IFS or the self-similar measure. When $m = 1$ the associated self-similar measure is a Cantor measure (when $\rho < 1/2$) or Bernoulli convolution (when $\rho > 1/2$). If, in addition, $p_0 = p_1 = 1/2$ (the unbiased case) we often denote the Cantor measure or Bernoulli convolution by μ_ρ.

We generally assume that $d_i - d_{i-1} \leq \rho$ from which it follows that the associated self-similar set (and hence support of μ) is $[0, 1]$.

The notion of local dimension of a measure was stated in (2). Of course, the limit need not exist and when we replace the limit by lim sup or lim inf, then this gives the *upper and lower local dimensions of μ at x* denoted by $\overline{\dim}_{\text{loc}}\mu(x)$ and $\underline{\dim}_{\text{loc}}\mu(x)$ respectively:

$$\overline{\dim}_{\text{loc}}\mu(x) = \limsup_{\epsilon \to 0} \frac{\log(\mu([x - \epsilon, x + \epsilon]))}{\log \epsilon},$$

$$\underline{\dim}_{\text{loc}}\mu(x) = \liminf_{\epsilon \to 0} \frac{\log(\mu([x - \epsilon, x + \epsilon]))}{\log \epsilon}.$$

Given $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{A}^n$ where $\mathcal{A} = \{0, 1, \ldots, m\}$, we denote by S_σ the concatenation $S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}$ and put $p_\sigma = p_{\sigma_1} p_{\sigma_2} \ldots p_{\sigma_n}$. For $\sigma = (\sigma_1, \sigma_2, \sigma_3, \ldots) \in \mathcal{A}^{\mathbb{N}}$ we define $S_\sigma(0) = \lim_{n \to \infty} S_{\sigma_1 \sigma_2 \ldots \sigma_n}(0)$. We let

$$\mathcal{E}(x) = \{\sigma \in \mathcal{A}^{\mathbb{N}} : S_\sigma(0) = x\}.$$

If $\sigma = (\sigma_i) \in \mathcal{E}(x)$ is a presentation of x, then $x = \sum \sigma_i \rho^{i-1}(1 - \rho)$, thus the set $\mathcal{E}(x)$ can be thought of as the set of beta-expansions of x with digit set $\mathcal{A}$, as first introduced in [16, 18].

We also set

$$\begin{aligned}
\mathcal{E}_n(x) &= \{\sigma \in \mathcal{A}^n : \text{there exists } \tau \in \mathcal{A}^{\mathbb{N}} \text{ such that } S_{\sigma\tau}(0) = x\} \\
&= \{\sigma \in \mathcal{A}^n : x \in S_\sigma([0, 1])\}
\end{aligned} \tag{4}$$

and

$$\mathcal{N}_n(x) = \sum_{\sigma \in \mathcal{E}_n(x)} p_\sigma . \tag{5}$$

Much is known about $\mathcal{E}_n(x)$ and $\mathcal{N}_n(x)$, especially when μ_ρ is a unbiased Bernoulli convolution, see [9, 15] for recent results. In particular, note that if $\sigma \in \mathcal{E}_n(x)$, then $S_\sigma([0, 1]) \subset [x - \rho^n, x + \rho^n]$. Thus $\mu([x - \rho^n, x + \rho^n]) \geq \mathcal{N}_n(x)$ from which the following statement is immediate.

Lemma 1. *Assume μ is an equicontractive self-similar measure with contraction factor ρ. For all $x \in \mathrm{supp}\mu$ we have*

$$\overline{\dim}_{\mathrm{loc}}\mu(x) \leq \limsup_{n \to \infty} \frac{\log(\mathcal{N}_n(x))}{n \log \rho}.$$

Remark 2. Suppose μ is any equicontractive self-similar measure with contraction factor ρ and that $|S_0(0) - S_1(0)| = a$. One can easily check that if $\sigma \in \mathcal{A}^n$ and $[0, a\rho^n) \cap S_\sigma[0, 1]$ is not empty, then $\sigma = (0)^n$. Thus $\mu([0, a\rho^n]) \leq p_0^n$. Clearly, $\mu([0, \rho^n]) \geq p_0^n$, hence if J is chosen so $\rho^J \leq a$, then

$$p_0^{n+J} \leq \mu([0, \rho^{n+J}]) \leq \mu([0, a\rho^n]) \leq p_0^n.$$

It follows easily from this that $\dim_{\mathrm{loc}}\mu(0) = \log p_0 / \log \rho$. A similar statement holds for $\dim_{\mathrm{loc}}\mu(1)$.

Together with an older result of Erdös, we can quickly deduce that unbiased Bernoulli convolutions with large enough contraction factors have isolated points in their set of local dimensions. This is essentially proved in [4], but we include a sketch here for completeness.

Proposition 3. *Let μ_ρ be the unbiased Bernoulli convolution with contraction factor $\rho > (\sqrt{5} - 1)/2$. Then*

$$\sup_{x \in (0,1)} \overline{\dim}_{\mathrm{loc}}\mu_\rho(x) < \dim_{\mathrm{loc}} \mu_\rho(0) = \dim_{\mathrm{loc}} \mu_\rho(1).$$

PROOF. In the proof of Theorem 3 of [4] it is shown that if k is chosen such that $1 < \rho^2 + \rho^3 + \cdots + \rho^k$, then for any $x \in (0,1)$, we have $\#\mathcal{E}_n(x) \geq c(x)2^{n/k}$ for some $c(x) > 0$ and independent of n. Thus $\mathcal{N}_n(x) \geq c(x)2^{-n(1-1/k)}$. Appealing to the previous lemma it follows that for all $x \neq 0, 1$,

$$\overline{\dim}_{\mathrm{loc}}\mu_\rho(x) \leq \limsup_{n\to\infty} \frac{\log(c(x)2^{-n(1-1/k)})}{n \log \rho} \leq \left(1 - \frac{1}{k}\right) \frac{\log 2}{|\log \rho|}.$$

The conclusion of the proposition holds since $\dim_{\mathrm{loc}} \mu_\rho(0) = \dim_{\mathrm{loc}} \mu_\rho(1) = \log 2/ |\log \rho|$. $\qquad\square$

3   Isolated points in the set of local dimensions

We will say the IFS of (3) has *strict overlap* if $S_j(1) > S_{j+1}(0)$ for each $j = 0, ..., m - 1$. Equivalently, $S_j(0,1) \cap S_{j+1}(0,1) \neq \emptyset$ for $j = 0, \ldots, m - 1$. An example is the IFS generating the Bernoulli convolution with contraction factor $\rho > 1/2$.

Theorem 4. *Suppose μ is an equicontractive, self-similar measure associated with the IFS (S_j, p_j) of (3) that has the strict overlap property. If $p_0 < p_j$ for all $j \neq 0$, then*

$$\sup_{x \neq 0} \overline{\dim}_{\mathrm{loc}}\mu(x) < \dim_{\mathrm{loc}} \mu(0)$$

and thus $\dim_{\mathrm{loc}} \mu(0)$ is an isolated point in the set $\{\dim_{\mathrm{loc}} \mu(x) : x \in \mathrm{supp}\mu\}$.

PROOF. The strict overlap property is equivalent to the inequalities $d_{j-1} + \rho > d_j$ for all $j = 1, \ldots, m$, thus we can choose $0 < \xi < \rho$ so $d_{j-1} + \rho > d_j + \xi$ for all $j = 1, \ldots, m$. Choose an integer $J > 0$ such that $\rho^J < \xi$.

We claim any $x \neq 0$ has a presentation (a_k) where the density of indices k with $a_k \neq 0$ exceeds $1/J$. Assume $p_i = \min_{j\neq 0} p_j$. It follows from the claim that if $x \neq 0$ and n is large, then $\mathcal{N}_n(x) \geq \left(p_i^{1/J}p_0^{(J-1)/J}\right)^n$, and hence by Lemma 1

$$\overline{\dim}_{\mathrm{loc}}\mu(x) \leq \frac{\log p_i + (J - 1) \log p_0}{J \log \rho} < \frac{\log p_0}{\log \rho} = \dim_{\mathrm{loc}} \mu(0),$$

proving the result.

To prove the claim, we will give an iterative algorithm for producing such a presentation. This algorithm is essentially the lazy expansion of x with respect to the alphabet $\mathcal{A} = \{0, 1, ..., m\}$. To begin, if $x \in [0, d_1 + \xi] \subseteq S_0[0, 1]$ choose $a_1 = 0$; if $x \in (d_j + \xi, d_{j+1} + \xi] \subseteq S_j[0, 1]$ for $j = 1, \ldots, m - 1$ take $a_1 = j$; and if $x \in (d_{m-1} + \xi, 1] \subseteq S_m[0, 1]$ take $a_1 = m$.

Assuming $a_1, \ldots, a_N$ have been chosen, set $\sigma = (a_1, \ldots, a_N)$. Then $x \in S_\sigma[0, 1]$. We put $a_{N+1} = 0$ if $x \in S_\sigma[0, d_1 + \xi]$, $a_{N+1} = j$ if $x \in S_\sigma(d_j + \xi, d_{j+1} + \xi]$ and $a_{N+1} = m$ if $x \in S_\sigma(d_{m-1} + \xi, 1]$.

Suppose $x \neq 0$ has presentation (a_k) under this algorithm. As $x \neq 0$, there is some index n such that $a_n \neq 0$, say $a_n = j$ for $j \neq 0$. We will see that it is not possible for all of $a_{n+1}, \ldots, a_{n+J} = 0$. Without loss of generality $j \neq m$. (The arguments are similar when $j = m$ and will be left for the reader.) Put $\sigma = (a_1, \ldots, a_{n-1})$. Then $x \in S_\sigma(d_j + \xi, d_{j+1} + \xi]$, hence $x - S_\sigma(d_j) \geq \xi \rho^{|\sigma|}$. If all $a_{n+j} = 0$ for $j = 1, \ldots, J$, then

$$x \in S_{\sigma j \underbrace{0 \ldots 0}_{J-1}}[0, d_1 + \xi] \subseteq S_{\sigma j \underbrace{0 \ldots 0}_{J-1}}[0, \rho].$$

Thus

$$x \leq \sup S_{\sigma j \underbrace{0 \ldots 0}_{J-1}}[0, \rho] = S_\sigma(d_j) + \rho^{J+|\sigma|}$$
$$< S_\sigma(d_j) + \xi \rho^{|\sigma|},$$

which is a contradiction. $\qquad\square$

Note that the proof actually establishes that

$$\sup_{x \neq 0} \overline{\dim}_{\mathrm{loc}} \mu(x) \leq \frac{\log(\min_{j \neq 0} p_j) + \left(\frac{\log \xi}{\log \rho} - 1\right) \log p_0}{\log \xi}$$

where $\xi = \min_{j=1,\ldots,m}(d_{j-1} + \rho - d_j)$. The corollary below is a special case of this.

Corollary 5. *If μ_ρ is a biased Bernoulli convolution with $\rho > (\sqrt{5} - 1)/2$ and $p_0 < p_1$, then*

$$\sup_{x \neq 0} \overline{\dim}_{\mathrm{loc}} \mu_\rho(x) \leq \frac{2/3 \log p_0 + 1/3 \log(1 - p_0)}{\log \rho}.$$

PROOF. In the notation of the proof of Theorem 4 we can choose any $\xi < 2\rho - 1$. As $\rho^3 = 2\rho - 1$ when $\rho = (\sqrt{5} - 1)/2$, it follows that we can choose $\xi > \rho^3$. $\qquad\square$

Remark 6. The m-fold convolution of a measure can be defined inductively as $\mu^m = \mu^{m-1} * \mu$. For the purposes of this paper, we typically rescale the convolution so that μ^m still has support in $[0, 1]$. This will not affect dimensionality results. When μ arises from an equicontractive IFS, then μ^m is again a self-similar measure. For example, if μ is the biased Bernoulli convolution or Cantor measure with contraction factor ρ and probabilities $p, 1 - p$, then μ^m is the self-similar measure associated with the IFS $S_j(x) = \rho x + j(1 - \rho)/m$ and probabilities $p_j = \binom{m}{j} p^j (1 - p)^{m-j}$ for $j = 0, ..., m$. It is easy to check that this IFS has the strict overlapping property if $\rho > 1/(m + 1)$ and thus the set of local dimensions will have an isolated point if $p \neq 1 - p$.

If a stricter overlapping property is satisfied, more can be proven.

Theorem 7. *Suppose μ is an equicontractive, self-similar measure associated with the IFS (S_j, p_j) of (3) that has the strict overlap property. Suppose $m \geq 2$ and $p_0 = p_m < p_j$ for all $j \neq 0, m$. In addition, assume that $S_{m-1}(1) > S_m S_1(0)$ and $S_1(0) < S_0 S_{m-1}(1)$. Then*

$$\sup_{x \neq 0,1} \overline{\dim}_{\mathrm{loc}} \mu(x) < \dim_{\mathrm{loc}} \mu(0)$$

and thus $\dim_{\mathrm{loc}} \mu(0) = \dim_{\mathrm{loc}} \mu(1)$ is an isolated point in the set of local dimensions of μ.

PROOF. The additional overlapping condition, $S_{m-1}(1) > S_m S_1(0)$ and $S_1(0) < S_0 S_{m-1}(1)$, is equivalent to the two inequalities $\rho(d_{m-1} + \rho) > d_1$ and $d_m + \rho d_1 < d_{m-1} + \rho$. Thus we can choose $\xi > 0$ such that for each j,

1. $d_j + \rho - \xi > d_{j+1} + \xi$

2. $\rho(d_{m-1} + \rho - \xi) > d_1 + \xi$

3. $d_m + \rho(d_1 + \xi) < d_{m-1} + \rho - \xi$.

Let
$$L = [0, d_1 + \xi), R = (d_{m-1} + \rho - \xi, 1],$$
$$M_j = [d_j + \xi, d_j + \rho - \xi] \text{ for } j = 1, \ldots, m - 1$$

and $M = \bigcup\limits_{j=1}^{m-1} M_j$. Then $[0, 1] = L \cup M \cup R$. One should observe that properties (1)-(3) above ensure that $\bigcup\limits_{\sigma} S_\sigma(M) = (0, 1)$ where the union is taken over all words σ on the alphabet $\{0, 1, \ldots, m\}$.

As in the previous proof, we claim that if $\rho^J < \xi$, then any $x \neq 0, 1$ has a presentation (a_i) where the density of indices i with $a_i \neq 0, m$ is at least $1/J$.

We use the following algorithm to produce the presentation: Take $a_1 = j$ if $x \in M_j$ (if there is a non-unique choice, choose either index). If $x \notin \bigcup_j M_j$, then either $x \in L$ or R and we take $a_1 = 0$ or m respectively. Now assume $a_1, \ldots, a_{n-1}$ have been determined and put $\sigma = (a_1, \ldots, a_{n-1})$, so $x \in S_\sigma[0, 1]$. If $x \in S_\sigma(M_j)$ take $a_n = j$, while if $x \in S_\sigma(L)$ or $S_\sigma(R)$ take $a_n = 0, m$ respectively.

If $x \neq 0, m$ then there must be an index n where $a_n = j \in \{1, \ldots, m-1\}$. We claim that $a_{n+k} \neq 0, m$ for some $k < J$. That is, we cannot have all of $a_{n+1}, a_{n+2}, \ldots, a_{n+J} \in \{0, m\}$.

To prove this claim, put $\sigma = (a_1, \ldots, a_{n-1})$. As $a_n = j$, we have

$$x \in S_\sigma(M_j) \subseteq S_{\sigma j}[0, 1] = S_{\sigma j}(L) \cup S_{\sigma j}(M) \cup S_{\sigma j}(R).$$

If $x \in S_{\sigma j}(M)$, then $x \in S_{\sigma j}(M_k)$ for some $k \neq 0, m$ and we are done since that ensures $a_{n+1} \neq 0, m$. So we can assume $x \in S_{\sigma j}(L)$ or $S_{\sigma j}(R)$.

We will assume that $x \in S_{\sigma j}(L)$; the latter case is similar. Hence $a_{n+1} = 0$. Upon rescaling we can assume

$$x \in M_j = [d_j + \xi, d_j + \rho - \xi] \subseteq [d_j, d_j + \rho] = S_j[0, 1]$$

and $x \in S_j(L)$ where

$$
\begin{aligned}
S_j(L) &= [d_j, d_j + \rho(d_1 + \xi)) \subseteq [d_j, d_j + \rho^2] \\
&= S_j[0, \rho] = S_{j0}[0, 1] = S_{j0}(L \cup M \cup R).
\end{aligned}
$$

But

$$\inf S_{j0}(R) = S_{j0}(d_{m-1} + \rho - \xi) = d_j + \rho^2(d_{m-1} + \rho - \xi).$$

Property (2) thus implies that $\sup S_j(L) < \inf S_{j0}(R)$ and hence we must actually have $x \in S_{j0}(L \cup M)$. Thus $a_{n+2} \neq m$. If $a_{n+2} \neq 0$ we are done and otherwise $x \in S_{j0}(L)$. We repeat the argument. Since $S_{j\underbrace{0 \ldots 0}_{J-1}}(L)$ is an interval of length at most ρ^J with left endpoint d_j and $x \geq d_j + \xi$, we see that if $\rho^J < \xi$ we cannot have $x \in S_{j\underbrace{0 \ldots 0}_{J-1}}(L)$. This completes the proof. $\qquad \square$

Corollary 8. *Suppose μ is the self-similar measure associated with the IFS*

$$S_j(x) = \rho x + \frac{j}{m}(1 - \rho),$$

with $j = 0, ..., m$, $m \geq 2$, and probabilities p_j that satisfy $p_0 = p_m < \min_{j \neq 0,k} p_j$. If

$$\rho > \frac{\sqrt{m^2 + 4} - m}{2},$$

then the set of local dimensions of μ has an isolated point.

PROOF. A routine calculation shows that all three overlap requirements of Theorem 7, $d_j < d_{j-1} + \rho$, $\rho(d_{m-1} + \rho) > d_1$ and $d_m + \rho d_1 < d_{m-1} + \rho$, are satisfied for such ρ. □

We remark that $(\sqrt{m^2 + 4} - m)/2 < 1/m$, so this improves upon the fact that the local dimension of the m-fold convolution of the uniform Cantor measure on the Cantor set with ratio $1/m$ has an isolated point at 0, as shown in [2, 20]. Note that when $\rho = 1/(m + 1)$, the IFS satisfies the open set condition and hence there is no isolated point.

4   Computational techniques to find upper and lower bounds

4.1   Upper bounds

Proposition 3 and Theorem 4 give upper bounds on the local dimension of the Bernoulli convolution μ_ρ at x for any $x \in (0, 1)$ in the unbiased ($\rho > (\sqrt{5} - 1)/2$) and biased ($\rho > 1/2$) cases respectively. Upper bounds are also given in [1]. In all of these cases, these bounds can be used to show that the local dimension at $x = 0$ is an isolated point within the set of all possible local dimensions. However, while sufficient to demonstrate a gap in the set of local dimensions, these bounds are not tight. This section will discuss computational techniques that can be used to improve the upper bounds for the set of local dimensions for $x \in (0, 1)$. We do this for both the unbiased and biased Bernoulli convolutions.

In the case of the unbiased Bernoulli convolution, we see that $\dim_{\text{loc}} \mu_\rho(0) = \log 2/|\log \rho|$. This is given by the blue curve in Figure 1. Further, as shown in Proposition 3 (and [4]), taking k such that $1 < \rho^2 + \cdots + \rho^k$, gives

$$\overline{\dim}_{\text{loc}}\mu_\rho(x) \leq \left(1 - \frac{1}{k}\right) \frac{\log 2}{|\log \rho|} \text{ for } x \in (0, 1).$$

This formula is shown by the red curve in Figure 1. The black curve in Figure 1 shows the tighter upper bound given by an application of Theorem 9 below.

Now consider the case of the biased Bernoulli convolutions. Assume $p_0 < p_1$. It is easy to see that $\dim_{\text{loc}} \mu_\rho(0) = \log p_0/\log \rho$. This is given by the blue
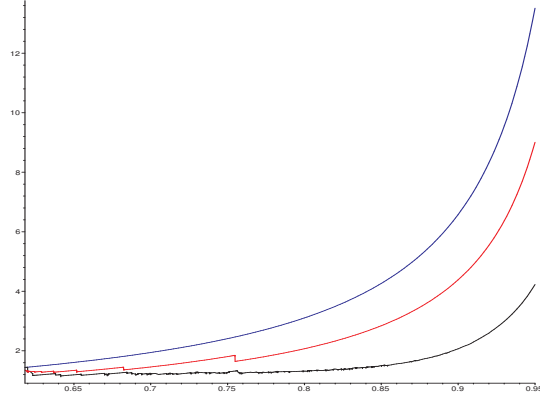
Figure 1: Upper bounds for local dimensions of unbiased Bernoulli convolutions

curve in Figure 2. Both Theorem 4 and Baker in [1], show there exists some k such that

$$\overline{\dim}_{\mathrm{loc}}\mu_\rho(x) \le \frac{\log p_0 + (k-1)p_1}{k \log \rho}.$$

The choice of k varies in the two approaches and depends on ρ. When $\rho < \frac{\sqrt{5}-1}{2}$, the k found by Theorem 4 results in a tighter upper bound than that found in [1], while for $\rho > \frac{\sqrt{5}-1}{2}$, the converse is true. The green curve in Figure 2 shows the upper bound given by Theorem 4, while the red curve shows the bound found in [1]. The black curve is, again, the upper bound found using Theorem 9.

Let μ be a self-similar measure with support $[0,1]$. Let I be a subset of $[0,1]$. We generalize equation (4) to give

$$
\begin{aligned}
\mathcal{E}_n(x, I) &= \{\sigma \in \mathcal{A}^n : \exists\, \tau \in \mathcal{A}^{\mathbb{N}} \text{ such that } S_{\sigma\tau}(0) = x \text{ and } S_\tau(0) \in I\} \\
&= \{\sigma \in \mathcal{A}^n : x \in S_\sigma(I)\}.
\end{aligned}
$$

We similarly generalize equation (5) to give

$$\mathcal{N}_n(x, I) = \sum_{\sigma \in \mathcal{E}_n(x, I)} p_\sigma.$$

We easily see that $\mathcal{N}_n(x, I) \le \mathcal{N}_n(x)$ for all x, I and n.
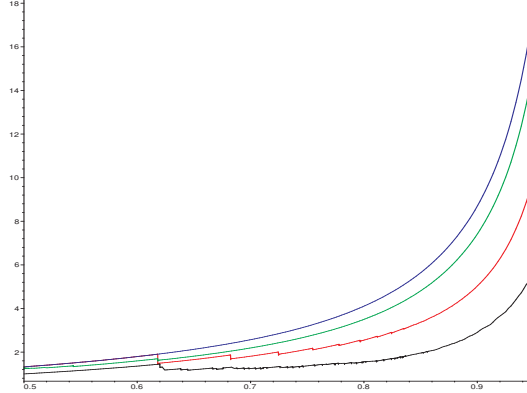
Figure 2: Upper bounds for local dimensions of biased ($p_0 = 0.4, p_1 = 0.6$) Bernoulli convolutions

Theorem 9. *Let μ be a self-similar measure with support $[0, 1]$. Let $I \subset [0, 1]$ be an open interval such that for all $x \in (0, 1)$ there exists a word σ with $x \in S_\sigma(I)$. Let $k = \min_{x \in I} \mathcal{N}_n(x, I)$. Then*

$$\overline{\dim}_{\text{loc}}\mu(x) \leq \frac{\log k}{n \log \rho}.$$

For a given measure μ, if we can find an interval I that satisfies the requirements of the theorem and can compute k, then we will have a computational method to find an upper bound for $\overline{\dim}_{\text{loc}}\mu(x)$ for $x \in (0, 1)$.

PROOF. First, assume that $x \in I$. We will further assume that $k > 0$, otherwise the bound is trivial. Hence there exists at least one σ such that $x \in S_\sigma(I)$. This gives us that

$$
\begin{aligned}
\mathcal{E}_{2n}(x, I) &= \{\sigma \in \mathcal{A}^{2n} : x \in S_\sigma(I)\} \\
&= \{\sigma_1, \sigma_2 \in \mathcal{A}^n : x \in S_{\sigma_1\sigma_2}(I)\} \\
&\supseteq \{\sigma_1, \sigma_2 \in \mathcal{A}^n : x \in S_{\sigma_1\sigma_2}(I), x \in S_{\sigma_1}(I)\} \\
&= \{\sigma_1, \sigma_2 \in \mathcal{A}^n : x \in S_{\sigma_1}(I), S_{\sigma_1}^{-1}(x) \in S_{\sigma_2}(I)\} \\
&= \{\sigma_1, \sigma_2 : \sigma_1 \in \mathcal{E}_n(x, I), \sigma_2 \in \mathcal{E}_n(S_{\sigma_1}^{-1}(x), I)\}.
\end{aligned}
$$

Note that if $x \in S_{\sigma_1}(I)$, then $S_{\sigma_1}^{-1}(x)$ is well defined and in I.

From this it follows that

$$\mathcal{N}_{2n}(x) \geq \mathcal{N}_{2n}(x, I) \geq \sum_{\sigma_1 \in \mathcal{E}_n(x,I)} \sum_{\sigma_2 \in \mathcal{E}_n(S_{\sigma_1}^{-1}(x),I)} p_{\sigma_1} p_{\sigma_2} \geq k^2.$$

Similarly $\mathcal{N}_{mn}(x) \geq k^m$. Furthermore for $mn \geq N \geq (m-1)n$ we have $\mathcal{N}_N(x) \geq k^m$. Taking limits gives

$$\overline{\dim}_{\mathrm{loc}}\mu(x) \leq \limsup_{N \to \infty} \frac{\log \mathcal{N}_N(x)}{\log \rho^N} \leq \lim_{m \to \infty} \frac{\log k^m}{\log \rho^{n(m-1)}} \leq \frac{\log k}{n \log \rho}.$$

The case for $x \in (0,1) \setminus I$ is similar. We know there is some σ, say of length t, such that $x \in S_\sigma(I)$. As $S_\sigma^{-1}(x) \in I$, we have that $\mathcal{N}_{nm}(S_\sigma^{-1}(x), I) \geq k^m$ as above. Thus $\mathcal{N}_{mn+t}(x, I) \geq p_\sigma k^m$, and the result follows as before, taking limits. $\qquad\square$

Next, we will demonstrate how the theorem can be implemented to produce upper bounds on local dimensions by means of an example. Consider the unbiased Bernoulli convolution with contraction factor $\rho = 0.8$ and let $I = (0.3, 0.7)$. One can check that $\bigcup_{|\sigma|=1} S_\sigma(I) = (0.24, 0.76)$ and in general that $\bigcup_{|\sigma|=n} S_\sigma(I) = (0.3(0.8)^n, 1 - 0.3(0.8)^n)$. It follows that the hypothesis of the theorem is satisfied. There are 16 images of $S_\sigma(I)$ for $|\sigma| = 4$; these are given in Table 1. One can readily check that for each $x \in (0.3, 0.7)$ there are at least 3 words σ such that $x \in S_\sigma(I)$. Thus $\mathcal{N}_4(x, (0.3, 0.7)) \geq 3/16$. Hence $k \geq 3/16$ and $\overline{\dim}_{\mathrm{loc}}\mu_\rho(x) \leq \frac{\log(3/16)}{4\log(0.8)} \sim 1.876$.

These calculations while exact, are also locally constant. It can be shown that there is a neighbourhood around $\rho = 0.8$ and around the endpoint 0.3 and 0.7 such that $k \geq 3/16$ within this neighbourhood. This comment is true in general, not just for the special case of $\rho = 0.8$, $I = [0.3, 0.7]$ and $n = 4$.

This process can be generalized to other Bernoulli convolutions and automated. Consider, first, the interval $I = (a, 1-a)$ where $\rho a + 1 - \rho < 1/2$ and $\rho(1-a) > 1/2$. (For example, take any a satisfying $0 < a < 1 - 1/(2\rho)$.) Then $S_0(I) \cup S_1(I) = (\rho a, 1 - \rho a)$ and, more generally, $\bigcup_{|\sigma|=n} S_\sigma(I) = (\rho^n a, 1 - \rho^n a)$. It is clear that the hypothesis of the theorem is satisfied for such I.

Consider, next, an interval $I = (b, 1-b)$ where b may be larger than $1 - 1/(2\rho)$. If there exists a choice of a with $0 < a < 1 - 1/(2\rho)$ and integer n such that $(a, 1-a) \subseteq \bigcup_{|\sigma|=n} S_\sigma(I)$, then again I will satisfy the hypothesis of the theorem.

For the purposes of the graphs, we considered the intervals $(\frac{1}{2}(1-1/(2\rho)), 1 - \frac{1}{2}(1-1/(2\rho))) = (\frac{1}{2} - \frac{1}{4\rho}, \frac{1}{2} + \frac{1}{4\rho})$, $(0.1, 0.9)$, $(0.2, 0.8)$ and $(0.3, 0.7)$. The first always satisfies the conditions of the theorem and we compute the associated

σ	$S_\sigma([0.3, 0.7]$
0000	[.12288, .28672]
0001	[.22528, .38912]
0010	[.25088, .41472]
0100	[.28288, .44672]
1000	[.32288, .48672]
0011	[.35328, .51712]
0101	[.38528, .54912]
0110	[.41088, .57472]
1001	[.42528, .58912]
1010	[.45088, .61472]
1100	[.48288, .64672]
0111	[.51328, .67712]
1011	[.55328, .71712]
1101	[.58528, .74912]
1110	[.61088, .77472]
1111	[.71328, .87712]

Table 1: Images of $S_\sigma([0.3, 0.7])$ for $|\sigma| = 4$

k. The other three may or may not depending on whether we can find a choice
of $n \leq 10$ as above where we view these intervals as the choice $(b, 1 - b)$ and
understand the first interval as $(a, 1 - a)$. If we can quickly find a suitable n,
we compute the associated k. Otherwise we ignore the interval. We take the
minimum k resulting from these choices of intervals.

A similar method can be used for any self-similar measure with non-trivial
overlaps, with the details being left to the reader.

4.2   Lower bounds

In Section 4.1, we showed how one could use computational techniques to find
upper bounds for the local dimension for $\mu_\rho(x)$ for any $x \in (0, 1)$. Similar
techniques can be used to find lower bounds for the range of possible local
dimensions assuming the IFS satisfies a suitable separation condition.

The IFS (or any associated self-similar measure) with contraction factor
ρ is said to satisfy the *weak separation condition* (wsc) if there is a constant
$c > 0$ such that whenever $\sigma, \tau \in \mathcal{A}^n$, then either

$$S_\sigma(0) = S_\tau(0) \text{ or } |S_\sigma(0) - S_\tau(0)| \geq c\rho^n. \tag{6}$$

The following definition is equivalent to that of [8].

Definition 10. A equicontractive IFS with ratio of contraction ρ satisfies the *asymptotically weak separation condition* (asymptotically wsc) if there exists a sequence $f(n)$ such that $\log f(n)/n \to 0$ as $n \to \infty$, and such that for each $n \in \mathbb{N}$ and each $x \in [0,1]$ we have

$$\#\{S_\sigma[0,1] : \sigma \in \mathcal{A}^n, S_\sigma[0,1] \cap (x - \rho^n, x + \rho^n) \neq \emptyset\} \leq f(n) \qquad (7)$$

The weak separation condition implies the asymptotically wsc, with the latter being strictly weaker. Indeed, as observed in [8], a Bernoulli convolution with contraction factor the reciprocal of a Salem number[2] in $(1,2)$ satisfies the asymptotically wsc, but not the wsc. It is widely believed that if there are any contraction factors which give rise to purely singular Bernoulli convolutions other than reciprocals of Pisot numbers, then the prime candidates would be reciprocals of Salem numbers.

It is worth noting that if $\rho \in (1/2, 1)$ is transcendental then $S_\sigma(0) \neq S_\tau(0)$ for all $\sigma \neq \tau$ and hence all images on the left hand side of (7) are unique. Hence if ρ is transcendental then the IFS canot satisfy the asympototically weak separation condition.

We can obtain lower bounds in the spirit of Lemma 1 under the assumption of the asymptotically wsc.

Lemma 11. *Assume μ is an equicontractive, self-similar measure with contraction factor ρ that satisfies the asymptotically weak separation condition. Then for all $x \in \mathrm{supp}\mu$ we have*

$$\underline{\dim}_{\mathrm{loc}}\mu(x) \geq \liminf_{n \to \infty} \frac{\log(\sup_y \mathcal{N}_n(y))}{n \log \rho}.$$

PROOF. Suppose $f(n)$ is a sequence with $(\log f(n))/n \to 0$ and satisfying (7).

It is convenient to put

$$H_n = \{\sigma \in \mathcal{A}^n : S_\sigma[0,1] \cap [x - \rho^n, x + \rho^n] \neq \emptyset\}.$$

With this notation, we have $\mu[x - \rho^n, x + \rho^n] \leq \sum_{\sigma \in H_n} p_\sigma$ for all n. Let

$$\mathcal{H}_n = \{S_\sigma(0) : \sigma \in H_n\}.$$

Note that $\mathcal{H}_n$ may contain fewer elements than H_n as we may have $S_\sigma(0) = S_\tau(0)$ for $\sigma, \tau \in H_n$. Indeed, the asymptotically wsc guarantees $\#\mathcal{H}_n \leq f(n)$. Since $\sum_{\sigma \in \mathcal{A}^n : S_\sigma(0) = y} p_\sigma \leq \mathcal{N}_n(y)$, we see that

$$\mu[x - \rho^n, x + \rho^n] \leq \sum_{y \in \mathcal{H}_n} \mathcal{N}_n(y) \leq \#\mathcal{H}_n \sup_y \mathcal{N}_n(y) \leq f(n) \sup_y \mathcal{N}_n(y).$$

---

[2]A Salem number is a real algebraic integer, greater than 1, such that all of its Galois conjugates ≤ 1 in absolute value and at least one conjugate is 1 in absolute value.

σ	$S_\sigma([0,1]$
0000	$[.0000, .4096]$
0001	$[.1024, .5120]$
0010	$[.1280, .5376]$
0100	$[.1600, .5696]$
1000	$[.2000, .6096]$
0011	$[.2304, .6400]$
0101	$[.2624, .6720]$
0110	$[.2880, .6976]$
1001	$[.3024, .7120]$
1010	$[.3280, .7376]$
1100	$[.3600, .7696]$
0111	$[.3904, .8000]$
1011	$[.4304, .8400]$
1101	$[.4624, .8720]$
1110	$[.4880, .8976]$
1111	$[.5904, 1.000]$

Table 2: Images of $S_\sigma([0,1])$ for $|\sigma| = 4$

Thus

$$\underline{\dim}_{\text{loc}}\mu(x) = \liminf_{n\to\infty} \frac{\log(\mu[x - \rho^n, x + \rho^n])}{\log(2\rho^n)}$$
$$\geq \liminf_{n\to\infty} \frac{\log(\sup_y \mathcal{N}_n(y)) + \log(f(n))}{n\log\rho + \log 2}$$
$$= \liminf_{n\to\infty} \frac{\log\sup_y(\mathcal{N}_n(y))}{n\log\rho}.$$

$\square$

Note that $\sup_{y\in[0,1]} \mathcal{N}_{n+m}(y) \geq \sup_{y\in[0,1]} \mathcal{N}_n(y) \times \sup_{y\in[0,1]} \mathcal{N}_m(y)$. Thus good bounds on $\sup_{y\in[0,1]} \mathcal{N}_n(y)$ will result in good lower bounds for $\underline{\dim}_{\text{loc}}\mu(x)$.

We will consider the same example as in the previous subsection. Let $\rho = 0.8$ and $n = 4$. We will again assume that $p_0 = p_1 = 1/2$, the unbiased case. When considering upper bounds, we wished to use some $I \subsetneq [0,1]$. In this case we will use $I = [0,1]$. There are 16 images of $S_\sigma([0,1])$ for $|\sigma| = 4$. These are listed in Table 2 in increasing order of the left endpoint.

It is easy to compute that $\sup_{y\in[0,1]} \mathcal{N}_4(y) = \mathcal{N}_4(0.5) = 14/2^4$. From this we conclude that $\underline{\dim}_{\text{loc}}\mu(x) \geq \frac{\log 14/2^4}{4\log(0.8)} \sim 0.5984102692$. Because we are using

Range of ρ	Lower bound for $\underline{\dim}_{\mathrm{loc}}\mu(x)$
$[0.50, 0.55]$	0.792021
$[0.55, 0.60]$	0.825663
$[0.60, 0.65]$	0.840348
$[0.65, 0.70]$	0.824701
$[0.70, 0.75]$	0.750984
$[0.75, 0.80]$	0.635012
$[0.80, 0.851]$	0.416226

Table 3: Lower bound for local dimensions

$[0, 1]$ exactly, we need to worry about situations where $S_\sigma(0) = S_\tau(1)$ for some $|\sigma| = |\tau|$. Such cases are known as *transition points.* In these cases the value of $\sup_y \mathcal{N}_n(y)$ may change as ρ is increased or decreased slightly. Such transition points occur when ρ satisfies very precise algebraic conditions and are easily enumerated for each n. A discussion of how to find and properly compute these transition points for a fixed n is discussed in [12, 13].

In Figure 3 we indicate the lower bounds using n up to 10. The points on this graph indicate the lower bounds at transition points, whereas the lines indicate regions between transition points when $\sup_{y \in [0,1]} \mathcal{N}_n(y)$ is constant. We have computed the lower bounds for $\sup_{y \in [0,1]} \mathcal{N}_n(y)$ at all transition points. The following theorem illustrates the kind of information this approach will yield.

Theorem 12. *Suppose the unbiased Bernoulli convolution μ_ρ satisfies the asymptotically weak separation condition. Then for all $x \in [0, 1]$ we have lower bounds as described in Table 3. More precise information can be found in Figure 3.*

It is worth noting that Theorem 12 applies only to those ρ for which the measure satisfies the asymptotically wsc, whereas this is not clearly indicated in Figure 3. Figure 3 should be interpreted as: If μ_ρ satisfies the asymtotically wsc, then the value on the graph is a lower bound of the set $\{\dim_{\mathrm{loc}} \mu(x) : x \in [0, 1]\}$. If, instead, μ_ρ does not satisfy the asymptotically wsc, then the corresponding value on the graph has no meaning.

The smallest known Salem number is approximately 1.176280, the root of $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$. This has an approximate reciprocal of 0.850137. The smallest Pisot number is approximately 1.324718, the root of $x^3 - x - 1$, with approximate reciprocal of 0.754877. As the only known Bernoulli convolutions satisfying the asymptotically wsc are those where ρ is the reciprocal of a Pisot or Salem number, Table 3 was restricted to $\rho \le 0.851$.
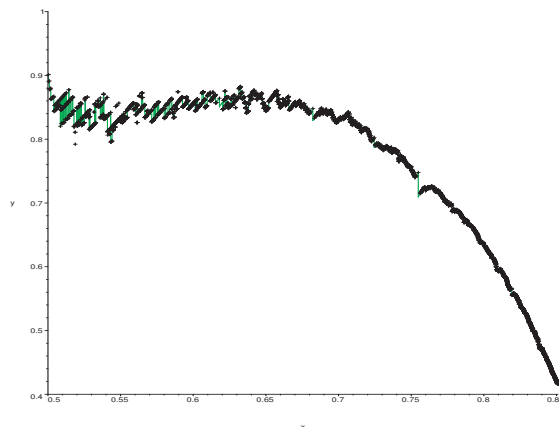
Figure 3: Lower bounds for local dimensions of unbiased Bernoulli convolutions

Remark 13. After this work was completed, Tom Kempton (private communication) observed that the requirement of asymptotically weak separation condition is not needed in the statement or proof of Lemma 11. Instead we note that if $|\sigma| = n$ is such that $S_\sigma([0,1]) \cap [x - \rho^n, x + \rho^n]$ then $\sigma \in \mathcal{E}_n(x) \cup \mathcal{E}_n(x - \rho^n) \cup \mathcal{E}_n(x + \rho^n)$. This gives that $\mu[x - \rho^n, x + \rho^n] \leq 3 \sup_y \mathcal{N}_n(y)$ from which the result follows.

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