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SOME APPLICATIONS OF ORDER-EMBEDDINGS OF COUNTABLE ORDINALS INTO THE REAL LINE

Abstract

It is a well-known fact that an ordinal α can be embedded into the real line $\mathbb{R}$ in an order-preserving manner if and only if α is countable. However, it would seem that outside of set theory, this fact has not yet found any concrete applications. The goal of this paper is to present some applications. More precisely, we show how two classical results, one in point-set topology and the other in real analysis, can be proven by defining specific order-embeddings of countable ordinals into $\mathbb{R}$.

1 Introduction.

Let α be an ordinal. An order-embedding of α into the real line $\mathbb{R}$ is defined as an injective function $\iota : \alpha \hookrightarrow \mathbb{R}$ such that $\iota(\beta) < \iota(\gamma)$ for all ordinals β, γ that satisfy $\beta < \gamma < \alpha$.

As the set of rational numbers is countable and dense in $\mathbb{R}$, it is evident that if an order-embedding of an ordinal α into $\mathbb{R}$ exists, then α is countable. Conversely, if α is a countable ordinal, then there is an order-embedding of α into $\mathbb{R}$ (cf. Theorem 34 in Section 5.3 of [2]).

Before conceiving the idea for this paper, I was unaware of any applications of the order-embedding result above outside of set theory. While re-visiting certain classical theorems in point-set topology and real analysis, I discovered that one could devise alternative proofs using transfinite-recursion arguments.

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These arguments depend subtly upon order-embeddings of countable ordinals into $\mathbb{R}$, tailored to suit the specific needs of the situation at hand. The goal of this paper, then, is to explain how this idea works.

In Section 2, we offer an alternative proof of the classical result in topology that the Sorgenfrey line is hereditarily Lindelöf.

In Section 3, we generalize a result of C. E. Aull on symmetric derivatives using order-embeddings of countable ordinals into $\mathbb{R}$. Aull's paper [1] begins with the *Quasi-Rolle's Theorem*, which yields the *Quasi-Mean Value Theorem.* The following fact is then proven as a corollary: *If the symmetric derivative of a real-valued function f defined on an open interval exists and is non-negative, then f is non-decreasing.*

Our argument proceeds backward, in the sense that we first prove this fact as a special case of a more general result, and then derive the Quasi-Rolle's Theorem and the Quasi-Mean Value Theorem as corollaries.

Throughout this paper, ω_1 denotes the first uncountable ordinal.

2 The Sorgenfrey Line

Definition 1. A topological space (X, τ) is said to be *Lindelöf* if and only if every τ-open cover for X has a countable sub-cover. It is said to be *hereditarily Lindelöf* if and only if every subspace is Lindelöf.

Definition 2. The *Sorgenfrey line* $\mathbb{R}_l$ is the topological space $(\mathbb{R}, \mathcal{S})$, where $\mathcal{S}$ is the topology on $\mathbb{R}$ with $\mathcal{B} := \{[a, b) \mid a, b \in \mathbb{R} \text{ and } a < b\}$ as a base.

The Sorgenfrey line is a famous example in topology of a first-countable and separable topological space that is not second-countable. Another well-known property is that it is hereditarily Lindelöf, and it would be enlightening to see how one may prove this using order-embeddings of countable ordinals into $\mathbb{R}$.

We begin with two lemmas that belong to topological folklore.

Lemma 1. *A topological space is hereditarily Lindelöf if and only if each of its open subspaces is Lindelöf.*

Lemma 2. *Each element of $\mathcal{S}$ is a countable <u>disjoint</u> union of elements of $\mathcal{B}$.*

Lemma 1 and Lemma 2 say that to prove that $\mathbb{R}_l$ is hereditarily Lindelöf, it suffices to show that if $a, b \in \mathbb{R}$ satisfy $a < b$ and $\mathcal{U}$ is an $\mathcal{S}$-open cover for $[a, b)$, then there exists a countable sub-cover of $\mathcal{U}$ for $[a, b)$.

Theorem 1. *The Sorgenfrey line is hereditarily Lindelöf.*

Proof. Pick $a, b \in \mathbb{R}$ satisfying $a < b$, and let $\mathcal{U}$ be an $\mathcal{S}$-open cover for $[a, b)$. Then by the Axiom of Choice, there exists a function $f : [a, b) \to \mathcal{U}$ such that $x \in f(x)$ for each $x \in [a, b)$.

Define a (unique but f-dependent) non-decreasing function $\iota : \omega_1 \to [a, b]$ via transfinite recursion as follows:

- Let $\iota(0) := a$.
- If $\alpha < \omega_1$ and $\iota(\alpha)$ is defined, then let

$$\iota(\alpha + 1) := \begin{cases} \sup(\{x \in \mathbb{R} \mid x \leq b \text{ and } [\iota(\alpha), x) \subseteq f(\iota(\alpha))\}), & \text{if } \iota(\alpha) < b; \\ b, & \text{if } \iota(\alpha) = b. \end{cases}$$

- If $\lambda < \omega_1$ is a limit and $\iota(\alpha)$ is defined for all $\alpha < \lambda$, then let $\iota(\lambda) := \sup_{\alpha < \lambda} \iota(\alpha)$.

It is clear from the definition of ι that there exists an $\alpha < \omega_1$ such that $\iota(\alpha) = b$, otherwise ι would be an order-embedding of ω_1 into $\mathbb{R}$, which contradicts the order-embedding result. Therefore, $\{f(\iota(\beta)) \mid \beta < \alpha\}$ is a countable sub-cover of $\mathcal{U}$ for $[a, b)$. □

An advantage of this approach is that it offers a very efficient construction of countable sub-covers for elements of $\mathcal{B}$.

3 Generalized Symmetric Derivatives

Definition 3. Let I be a non-degenerate interval of $\mathbb{R}$, and let $f : I \to \mathbb{R}$. The *symmetric derivative* of f, denoted by f^*, is the function with domain

$$D := \left\{ x \in I^\circ \;\middle|\; \lim_{h \to 0^+} \frac{f(x+h) - f(x-h)}{2h} \text{ exists} \right\}$$

such that

$$\forall x \in D : \qquad f^*(x) := \lim_{h \to 0^+} \frac{f(x+h) - f(x-h)}{2h}.$$

Note: I° denotes the interior of I, which is non-empty as I is non-degenerate.

Let $a, b \in \mathbb{R}$ satisfy $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the following results were proven by C. E. Aull in [1].

Theorem 2 (The Quasi-Rolle's Theorem)**.** *If $f(a) = f(b)$ and f^* is defined everywhere on (a, b), then there exist points $x_1, x_2 \in (a, b)$ such that*

$$f^*(x_1) \leq 0 \leq f^*(x_2).$$

From this, it is relatively easy to deduce the Quasi-Mean Value Theorem.

Theorem 3 (The Quasi-Mean Value Theorem)**.** *If f^* is defined everywhere on (a,b), then there exist points $x_1, x_2 \in (a,b)$ such that*

$$f^*(x_1) \leq \frac{f(b) - f(a)}{b - a} \leq f^*(x_2).$$

This readily yields the following corollary.

Corollary 1. *If f^* is defined and non-negative everywhere on (a,b), then f is non-decreasing on $[a,b]$.*

By defining an order-embedding of a countable limit ordinal into $\mathbb{R}$ that possesses very specific properties, we can actually establish the corollary from first principles and *then* obtain from it Theorem 2 and Theorem 3.

In fact, we can prove something more, but first, we require a definition of the generalized symmetric derivative.

Definition 4. Let I be a non-degenerate interval of $\mathbb{R}$, and let $f : I \to \mathbb{R}$. Given $p, q \in \mathbb{R}_{>0}$, we define the (p,q)-*generalized symmetric derivative* of f, denoted by $f_{p,q}$, as the function with domain

$$D := \left\{ x \in I^\circ \;\middle|\; \lim_{h \to 0^+} \frac{f(x + qh) - f(x - ph)}{(p + q)h} \text{ exists} \right\}$$

such that

$$\forall x \in D : \qquad f_{p,q}(x) := \lim_{h \to 0^+} \frac{f(x + qh) - f(x - ph)}{(p + q)h}.$$

Fix $p, q \in \mathbb{R}_{>0}$. As before, let $a, b \in \mathbb{R}$ satisfy $a < b$, and let $f : [a,b] \to \mathbb{R}$ be a continuous function.

Proposition 1. *If $f_{p,q}$ is defined and non-negative everywhere on (a,b), then f is non-decreasing on $[a,b]$.*

The proof that we are about to furnish is interesting because it summons a property of the ordinals that is seldom applied to results in real analysis, namely, that there does not exist an infinite descending chain of ordinals.

We first give some motivation for the proof in the case $p = q = 1$, which is exactly Corollary 1. Let f^* be defined and non-negative everywhere on (a,b). Fix $\epsilon > 0$. Then for each $x \in (a,b)$, there exist points $s, t \in (a,b)$, symmetric about x and satisfying $s < t$, such that

$$\frac{f(t) - f(s)}{t - s} \geq -\epsilon, \qquad \text{or equivalently,} \qquad f(t) - f(s) \geq -\epsilon(t - s).$$

If one can "link" an increasing sequence $(x_n)_{n=1}^N$ of points in (a,b) so that

$$\forall n \in \mathbb{N}_{\leq N-1}: \qquad f(x_{n+1}) - f(x_n) \geq -\epsilon(x_{n+1} - x_n),$$

then a telescoping sum yields

$$f(x_N) - f(x_1) \geq -\epsilon(x_N - x_1).$$

Let $a', b' \in (a,b)$ satisfy $a' < b'$. If x_1 can be chosen arbitrarily close to a' and x_N arbitrarily close to b', then

$$f(b') - f(a') \geq -\epsilon(b' - a').$$

As ϵ is arbitrary, it follows that $f(b') - f(a') \geq 0$, so f is non-decreasing.

We will now give a rigorous realization of this idea in the general case.

Proof. Fix $\epsilon, \delta > 0$ arbitrarily. According to the hypotheses, there exists a function $N : (a,b) \to \mathbb{N}$ such that for each $x \in (a,b)$, we have

$$N(x) = \min\left(\left\{n \in \mathbb{N} \;\middle|\; \frac{1}{2^n} < \delta \text{ and } \frac{f(x+qh) - f(x-ph)}{(p+q)h} > -\epsilon \right.\right.$$
$$\left.\left.\text{for all } h \in \left(0, \frac{1}{2^n}\right)\right\}\right).$$

Let $a', b' \in (a,b)$ satisfy $a' < b'$. It suffices by the continuity of f to show that $f(a') \leq f(b')$.

Construct a function $\iota : \omega_1 \to [a', b']$ via transfinite recursion as follows:

- Let $\iota(0) := a' < b'$.
- Let $\iota(1) := a' + \dfrac{1}{2^n} < b'$, where $n := \min\left(\left\{m \in \mathbb{N} \;\middle|\; \dfrac{1}{2^m} < b' - a', \delta\right\}\right)$.
- Suppose that $k \in [1, \omega)$ and $\iota(k-1), \iota(k)$ are defined.
  - If $\iota(k-1) < \iota(k) < b'$, then let $\iota(k+1) := \iota(k) + \dfrac{1}{2^n} < b'$, where
  $$n := \min\left(\left\{m \in \mathbb{N} \;\middle|\; \frac{1}{2^m} < \min\left(b' - \iota(k), \frac{q}{2^{N(\iota(k))}}, \frac{q}{p}[\iota(k) - \iota(k-1)]\right)\right\}\right). \tag{1}$$
  - Otherwise, let $\iota(k+1) := b'$.

- For a limit $\lambda < \omega_1$, if $\iota(\alpha)$ is defined for all $\alpha < \lambda$, then let $\iota(\lambda) := \sup_{\alpha<\lambda} \iota(\alpha)$.
  - If $\iota(\alpha) < \iota(\lambda) < b'$ for all $\alpha < \lambda$, then, with γ_λ denoting the ordinal
  $$\min\left(\left\{\beta \in [0,\lambda) \;\middle|\; 0 < \iota(\lambda) - \iota(\beta+1) < \min\left(\frac{p}{q}[b' - \iota(\lambda)], \frac{p}{2^{N(\iota(\lambda))}}\right)\right\}\right),$$
  let
  $$\iota(\lambda+1) := \left(1 + \frac{q}{p}\right)\iota(\lambda) - \frac{q}{p}\iota(\gamma_\lambda + 1) < b', \tag{2}$$
  $$\iota(\lambda+2) := \iota(\lambda+1) + \frac{1}{2^n} < b', \tag{3}$$
  where
  $$\begin{aligned} n := \min\Bigg(\Bigg\{m \in \mathbb{N} \;\Bigg|\; \frac{1}{2^m} < \min\Bigg(& b' - \iota(\lambda+1), \frac{q}{2^{N(\iota(\lambda+1))}}, \\ & \frac{q}{p}[\iota(\lambda+1) - \iota(\lambda)], \frac{q^2}{p^2}[\iota(\gamma_\lambda+2) - \iota(\gamma_\lambda+1)]\Bigg)\Bigg\}\Bigg). \end{aligned} \tag{4}$$
  - Otherwise, let $\iota(\lambda+1), \iota(\lambda+2) := b'$.
- Suppose that $\lambda < \omega_1$ is a limit, $k \in [2,\omega)$ and $\iota(\lambda+k-1), \iota(\lambda+k)$ are defined.
  - If $\iota(\lambda+k-1) < \iota(\lambda+k) < b'$, then let $\iota(\lambda+k+1) := \iota(\lambda+k) + \dfrac{1}{2^n} < b'$, where
  $$\begin{aligned} n := \min\Bigg(\Bigg\{m \in \mathbb{N} \;\Bigg|\; \frac{1}{2^m} < \min\Bigg(& b' - \iota(\lambda+k), \frac{q}{2^{N(\iota(\lambda+k))}}, \\ & \frac{q}{p}[\iota(\lambda+k) - \iota(\lambda+k-1)]\Bigg)\Bigg\}\Bigg). \end{aligned} \tag{5}$$
  - Otherwise, let $\iota(\lambda+k+1) := b'$.

There exists an ordinal $\alpha < \omega_1$ such that $\iota(\alpha) = b'$, otherwise ι would be an order-embedding of ω_1 into $\mathbb{R}$, which contradicts the order-embedding result. Let Λ denote the smallest such α. Then Λ is clearly a limit ordinal, and $\iota|_\Lambda$ is an order-embedding of Λ into $[a', b']$.

Given $x \in \mathbb{R}$, define a "skewed-reflection" map $R^{p,q}_x : [x,\infty) \to (-\infty, x]$ by
$$\forall y \in \mathbb{R}: \qquad R^{p,q}_x(y) := x - \frac{p}{q}(y - x).$$
For $p = q = 1$, $R^{p,q}_x$ is ordinary reflection from the right of x to the left of x.

Claim 1. For each $\alpha < \Lambda$, we have

$$s \in (\iota(\alpha+1), \iota(\alpha+2)) \quad \Longrightarrow \quad R^{p,q}_{\iota(\alpha+1)}(s) \in (\iota(\alpha), \iota(\alpha+1)).$$

Proof of Claim 1. Firstly, the map $R^{p,q}_{\iota(\alpha+1)}$ behaves so that

$$\begin{aligned} & \iota(\alpha+1) < s < \iota(\alpha+2) \\ \Longrightarrow\quad & R^{p,q}_{\iota(\alpha+1)}(\iota(\alpha+2)) < R^{p,q}_{\iota(\alpha+1)}(s) < \iota(\alpha+1) \\ \Longleftrightarrow\quad & \iota(\alpha+1) - \frac{p}{q}[\iota(\alpha+2) - \iota(\alpha+1)] < R^{p,q}_{\iota(\alpha+1)}(s) < \iota(\alpha+1). \end{aligned} \tag{6}$$

Secondly, (1), (4) and (5) ensure that

$$\iota(\alpha+2) - \iota(\alpha+1) < \frac{q}{p}[\iota(\alpha+1) - \iota(\alpha)]. \tag{7}$$

After rearranging terms in (7) and then applying (6), we obtain

$$\iota(\alpha) < \iota(\alpha+1) - \frac{p}{q}[\iota(\alpha+2) - \iota(\alpha+1)] < R^{p,q}_{\iota(\alpha+1)}(s).$$

This concludes the proof of the claim. □

Claim 2. For each limit $\lambda < \Lambda$, we have

$$s \in (\iota(\lambda+1), \iota(\lambda+2)) \quad \Longrightarrow \quad R^{p,q}_{\iota(\lambda)}\left(R^{p,q}_{\iota(\lambda+1)}(s)\right) \in (\iota(\gamma_\lambda+1), \iota(\gamma_\lambda+2)).$$

Proof of Claim 2. Firstly, the maps $R^{p,q}_{\iota(\lambda)}$ and $R^{p,q}_{\iota(\lambda+1)}$ behave so that

$$\begin{aligned} & \iota(\lambda+1) < s < \iota(\lambda+2) \\ \Longrightarrow\quad & R^{p,q}_{\iota(\lambda+1)}(\iota(\lambda+2)) < R^{p,q}_{\iota(\lambda+1)}(s) < \iota(\lambda+1) \\ \Longleftrightarrow\quad & \iota(\lambda+1) - \frac{p}{q}[\iota(\lambda+2) - \iota(\lambda+1)] < R^{p,q}_{\iota(\lambda+1)}(s) < \iota(\lambda+1) \\ \Longrightarrow\quad & \underbrace{\iota(\lambda) < \iota(\lambda+1) - \frac{p}{q}[\iota(\lambda+2) - \iota(\lambda+1)]}_{\text{As a consequence of (3) and (4).}} < R^{p,q}_{\iota(\lambda+1)}(s) < \iota(\lambda+1) \end{aligned}$$

$$\Longrightarrow$$

$$\begin{cases} R^{p,q}_{\iota(\lambda)}(\iota(\lambda+1)) < R^{p,q}_{\iota(\lambda)}\left(R^{p,q}_{\iota(\lambda+1)}(s)\right) \quad \text{and} \\ R^{p,q}_{\iota(\lambda)}\left(R^{p,q}_{\iota(\lambda+1)}(s)\right) < R^{p,q}_{\iota(\lambda)}\left(\iota(\lambda+1) - \dfrac{p}{q}[\iota(\lambda+2) - \iota(\lambda+1)]\right) \end{cases}$$

$$\Longleftrightarrow$$

$$\begin{cases} \iota(\lambda) - \dfrac{p}{q}[\iota(\lambda+1) - \iota(\lambda)] < R^{p,q}_{\iota(\lambda)}\left(R^{p,q}_{\iota(\lambda+1)}(s)\right) \quad \text{and} \\ R^{p,q}_{\iota(\lambda)}\left(R^{p,q}_{\iota(\lambda+1)}(s)\right) < \iota(\lambda) - \dfrac{p}{q}\left[\iota(\lambda+1) - \dfrac{p}{q}[\iota(\lambda+2) - \iota(\lambda+1)] - \iota(\lambda)\right]. \end{cases}$$

Secondly, (2) says that

$$\begin{aligned} \iota(\lambda) - \frac{p}{q}[\iota(\lambda+1) - \iota(\lambda)] &= \iota(\lambda) - \frac{p}{q}\left[\left(1 + \frac{q}{p}\right)\iota(\lambda) - \frac{q}{p}\iota(\gamma_\lambda + 1) - \iota(\lambda)\right] \\ &= \iota(\gamma_\lambda + 1). \end{aligned}$$

Thirdly, observe that

$$\begin{aligned} & \iota(\lambda) - \frac{p}{q}\left[\iota(\lambda+1) - \frac{p}{q}[\iota(\lambda+2) - \iota(\lambda+1)] - \iota(\lambda)\right] \\ = \; & \iota(\lambda) - \frac{p}{q}[\iota(\lambda+1) - \iota(\lambda)] + \frac{p^2}{q^2}[\iota(\lambda+2) - \iota(\lambda+1)] \\ = \; & \iota(\gamma_\lambda + 1) + \frac{p^2}{q^2}[\iota(\lambda+2) - \iota(\lambda+1)] \qquad \text{(By (2).)} \\ < \; & \iota(\gamma_\lambda + 1) + \frac{p^2}{q^2} \cdot \frac{q^2}{p^2}[\iota(\gamma_\lambda + 2) - \iota(\gamma_\lambda + 1)] \qquad \text{(By (4).)} \\ = \; & \iota(\gamma_\lambda + 2). \end{aligned}$$

Therefore, $R^{p,q}_{\iota(\lambda)}\left(R^{p,q}_{\iota(\lambda+1)}(s)\right) \in (\iota(\gamma_\lambda + 1), \iota(\gamma_\lambda + 2))$. □

If s is a point in $\mathbb{R}$ such that $s \in (\iota(\alpha), \iota(\alpha+1))$ for some $\alpha < \Lambda$, then we call s a *legitimate point* and the map $R^{p,q}_{\iota(\alpha)}$ a *legitimate reflection* of s.

By Claim 1 and Claim 2, applying successive legitimate reflections to any legitimate point $s \in (\iota(\alpha+1), \iota(\alpha+2))$, with $\alpha < \Lambda$, yields a legitimate point. As an infinite descending chain of ordinals does not exist, a terminal point $r \in (\iota(0), \iota(1)) \subseteq (a', a' + \delta)$ will be attained after applying a finite number of successive legitimate reflections.

Now, let $(x_k)_{k=1}^N$ denote the maximal finite sequence obtainable from s by applying successive legitimate reflections, so that $x_1 = s$ and $x_N = r$. Suppose $k \in \mathbb{N}_{\leq N-1}$, and $\beta \in (0, \Lambda)$ satisfies $x_k \in (\iota(\beta), \iota(\beta+1))$. Then as $x_{k+1} := R^{p,q}_{\iota(\beta)}(x_k)$,

$$h := \frac{\iota(\beta) - x_{k+1}}{p} = \frac{x_k - \iota(\beta)}{q} < \frac{\iota(\beta+1) - \iota(\beta)}{q}.$$

In the case that β is a limit ordinal, it is a consequence of (2) that

$$h < \frac{\iota(\beta+1) - \iota(\beta)}{q} = \frac{1}{q} \cdot \frac{q}{p}[\iota(\beta) - \iota(\gamma_\beta + 1)] < \frac{1}{q} \cdot \frac{q}{p} \cdot \frac{p}{2^{N(\iota(\beta))}} = \frac{1}{2^{N(\iota(\beta))}}.$$

If β is a successor, then

$$h < \frac{\iota(\beta+1) - \iota(\beta)}{q} < \frac{1}{q} \cdot \frac{q}{2^{N(\iota(\beta))}} = \frac{1}{2^{N(\iota(\beta))}}.$$

Hence, in all cases, $h < \frac{1}{2^{N(\iota(\beta))}}$, so by the definition of the function N,

$$\frac{f(x_k) - f(x_{k+1})}{x_k - x_{k+1}} = \frac{f(\iota(\beta) + qh) - f(\iota(\beta) - ph)}{(p+q)h} \geq -\epsilon,$$

or equivalently,

$$f(x_k) - f(x_{k+1}) \geq -\epsilon(x_k - x_{k+1}).$$

As a consequence,

$$f(s) - f(r) = \sum_{k=1}^{N-1} [f(x_k) - f(x_{k+1})] \geq -\epsilon \sum_{k=1}^{N-1} (x_k - x_{k+1}) = -\epsilon(s - r). \quad (8)$$

Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers decreasing to 0. This yields a corresponding sequence $(\iota_n : \omega_1 \to [a', b'])_{n\in\mathbb{N}}$ of functions, given by the recursive construction earlier on, and a corresponding sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of ordinals $< \omega_1$ such that Λ_n for each $n \in \mathbb{N}$ is the largest limit Λ for which $\iota_n|_\Lambda$ is an order-embedding of Λ into $[a', b']$.

Define a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of ordinals $< \omega_1$ by

$$\forall n \in \mathbb{N} : \qquad \alpha_n := \min\left(\left\{\alpha \in [0, \Lambda_n) \;\middle|\; b' - \frac{1}{2^n} < \iota_n(\alpha) < b'\right\}\right) + 1.$$

Then define a sequence $(s_n)_{n\in\mathbb{N}}$ in (a', b') of legitimate points by

$$\forall n \in \mathbb{N} : \qquad s_n := \frac{\iota_n(\alpha_n) + \iota_n(\alpha_n + 1)}{2},$$

which produces, by successive legitimate reflections, a corresponding sequence $(r_n)_{n\in\mathbb{N}}$ of terminal points, so that $r_n \in (a', a' + \delta_n)$ for each $n \in \mathbb{N}$. Invoking (8), we get

$$\forall n \in \mathbb{N}: \qquad f(s_n) - f(r_n) \geq -\epsilon(s_n - r_n).$$

However, $\lim_{n\to\infty} r_n = a'$ and $\lim_{n\to\infty} s_n = b'$, so by the continuity of f,

$$f(b') - f(a') \geq -\epsilon(b' - a').$$

Therefore, as ϵ is arbitrary, we conclude that $f(b') - f(a') \geq 0$. □

Notice that we have not used the Axiom of Choice anywhere in the proof. We can now prove the generalized Quasi-Mean Value Theorem, of which the generalized Quasi-Rolle's Theorem is just a special case.

Corollary 2. *If $f_{p,q}$ is defined everywhere on (a, b), then there exist points $x_1, x_2 \in (a, b)$ such that*

$$f_{p,q}(x_1) \leq \frac{f(b) - f(a)}{b - a} \leq f_{p,q}(x_2).$$

PROOF. Define a continuous function $g : [a, b] \to \mathbb{R}$ by

$$\forall x \in [a, b]: \qquad g(x) := f(x) - \left[\frac{f(b) - f(a)}{b - a} \cdot (x - a) + f(a)\right].$$

Then $g_{p,q}$ is defined on (a, b); indeed,

$$\forall x \in (a, b): \qquad g_{p,q}(x) = f_{p,q}(x) - \frac{f(b) - f(a)}{b - a}.$$

Suppose that $f_{p,q}(x) > \dfrac{f(b) - f(a)}{b - a}$ for all $x \in (a, b)$. Then $g_{p,q}$ is positive on (a, b), so by Proposition 1, we find that g is non-decreasing on $[a, b]$. However, $g(a) = 0 = g(b)$, which yields $g(x) = 0$ and therefore $f_{p,q}(x) = \dfrac{f(b) - f(a)}{b - a}$, for all $x \in (a, b)$. Hence, our earlier hypothesis is contradicted.

Similarly, assuming that $f_{p,q}(x) < \dfrac{f(b) - f(a)}{b - a}$ for all $x \in (a, b)$ produces a contradiction. The proof is therefore complete. □

4 Conclusion

Although the embedding technique presented in this paper may not necessarily be the most elegant approach toward solving those problems to which it may be adapted, it supports the idea that order-embeddings of countable ordinals into $\mathbb{R}$ can play an important role in areas of mathematics outside of set theory. We hope that more substantial applications of the technique will be found.

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