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THE BAIRE CLASSIFICATION OF STRONGLY SEPARATELY CONTINUOUS FUNCTIONS ON ℓ_∞

Abstract

We prove that for any $\alpha \in [0, \omega_1)$ there exists a strongly separately continuous function $f : \ell_\infty \to [0, 1]$ such that f belongs to the $(\alpha+1)$'th /$(\alpha+2)$'th/ Baire class and does not belong to the α'th Baire class if α is finite /infinite/.

1 Introduction

The notion of real-valued strongly separately continuous function defined on $\mathbb{R}^n$ was introduced and studied by Dzagnidze in his paper [2]. He proved that the class of all strongly separately continuous real-valued functions on $\mathbb{R}^n$ coincides with the class of all continuous functions. Later, Činčura, Šalát and Visnyai [1] considered strongly separately continuous functions defined on the Hilbert space ℓ_2 of sequences $x = (x_n)_{n=1}^\infty$ of real numbers with $\sum\limits_{n=1}^{\infty} x_n^2 < +\infty$ and showed that there are essential differences between some properties of strongly separately continuous functions defined on ℓ_2 and the corresponding

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properties of functions on $\mathbb{R}^n$. In particular, they noticed that there exists a strongly separately continuous function $f : \ell_2 \to \mathbb{R}$ which does not belong to the first Baire class. Extending these results, Visnyai [8] constructed a strongly separately continuous function $f : \ell_2 \to \mathbb{R}$ of the third Baire class which is not quasi-continuous at every point of ℓ_2. It was shown recently in [6] that for every $2 \le \alpha < \omega$ there exists a strongly separately continuous function $f : \ell_p \to \mathbb{R}$ which belongs the α'th Baire class and does not belong to the β'th Baire class on ℓ_p for $\beta < \alpha$, where $p \in [1, +\infty)$.

The aim of this paper is to generalize results from [6] to the case of $p = +\infty$. We develop arguments from [3] and prove that for any $\alpha \in [0, \omega_1)$ there exists a strongly separately continuous function $f : \ell_\infty \to [0, 1]$ such that f belongs to the $(\alpha+1)$'th /$(\alpha+2)$'th/ Baire class and does not belong to the α'th Baire class if α is finite /infinite/.

2 Definitions and notations

Let ℓ_∞ be the Banach space of all bounded sequences of reals with the norm

$$\|x\|_\infty = \sup_{k\in\omega} |x_k|$$

for all $x = (x_k)_{k\in\omega} \in \ell_\infty$. For $x, y \in \ell_\infty$ we denote $d_\infty(x, y) = \|x - y\|_\infty$. If $x \in \ell_\infty$ and $\delta > 0$, then

$$B_\infty(x, \delta) = \{y \in \ell_\infty : \|x - y\|_\infty < \delta\}.$$

Definition 2.1. Let $x^0 = (x^0_k)_{k\in\omega} \in \ell_\infty$ and $(Y, |\cdot - \cdot|)$ be a metric space. A function $f : \ell_\infty \to Y$ is said to be *strongly separately continuous at x^0 with respect to the k-th variable* if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x = (x_k)_{k\in\omega} \in B_\infty(x^0, \delta)$$
$$|f(x_1, \dots, x_k, \dots) - f(x_1, \dots, x_{k-1}, x^0_k, x_{k+1}, \dots)| < \varepsilon. \tag{1}$$

If f is strongly separately continuous at x^0 with respect to each variable, then f is said to be *strongly separately continuous at x^0*. Moreover, f is *strongly separately continuous on ℓ_∞* if it is strongly separately continuous at each point of ℓ_∞.

Strongly separately continuous functions we will also call *ssc functions* for short.

Definition 2.2. A subset $A \subseteq X$ of a Cartesian product $X = \prod_{k=1}^\infty X_k$ of sets $X_1, X_2, \dots$ is called *$\mathcal{S}$-open* [4], if

$$\sigma_1(a) = \{(x_k)_{k=1}^\infty \in X : |\{k : x_k \ne a_k\}| \le 1\} \subseteq A$$

for all $a = (a_k)_{k=1}^{\infty} \in A$.

We put

$$\sigma(a) = \{(x_k)_{k=1}^{\infty} \in X : |\{k : x_k \neq a_k\}| \leq \aleph_0\}$$

and observe that the set $\sigma(a)$ is $\mathcal{S}$-open.

If $x \in \ell_\infty$ and $N \subseteq \omega$, then we put

$$\pi_N(x) = (x_k)_{k \in N}.$$

In the case $N = \{n\}$, we write $\pi_n(x)$ instead of $\pi_{\{n\}}(x)$.

3 Main result

Define a function $(\alpha)^\bullet$ as the following

$$(\alpha)^\bullet = \begin{cases} \alpha, & \alpha \in [0, \omega), \\ \alpha + 1, & \alpha \in [\omega, \omega_1). \end{cases} \tag{2}$$

Theorem 3.1. *For any $\alpha \in [0, \omega_1)$ there exists a strongly separately continuous function $f : \ell_\infty \to [0, 1]$ which belongs to the $(\alpha + 1)^\bullet$'th Baire class and does not belong to the α'th Baire class on ℓ_∞.*

PROOF. We define transfinite sequences $(A_\alpha)_{1 \leq \alpha < \omega_1}$ and $(B_\alpha)_{1 \leq \alpha < \omega_1}$ of subsets of ℓ_∞ inductively and in the following way. Put

$$A_1 = \{(x_n)_{n=1}^{\infty} \in \ell_\infty : \exists m \ \forall n \geq m \ \ x_n = 0\} \quad \text{and} \quad B_1 = \ell_\infty \setminus A_1.$$

Let $(T_n : n \in \omega)$ be a partition of ω onto infinite sets $T_n = \{t_{n0}, t_{n1}, \dots\}$, where $(t_{nm})_{m \in \omega}$ is a strictly increasing sequence of numbers $t_{nm} \in \omega$. We put

$$\ell_\infty^{T_n} = \{(x_{t_{nm}}) \in \ell_\infty : t_{nm} \in T_n \ \ \forall m \in \omega\}.$$

For every $n \in \omega$ we denote by A_1^n /B_1^n/ the copy of the set A_1 /B_1/, which is contained in the space $\ell_\infty^{T_n}$. Assume that for some $\alpha > 1$ we have already defined sequences $(A_\beta)_{1 \leq \beta < \alpha}$ and $(B_\beta)_{1 \leq \beta < \alpha}$ (and their copies $(A_\beta^n)_{1 \leq \beta < \alpha}$ and $(B_\beta^n)_{1 \leq \beta < \alpha}$ in $\ell_\infty^{T_n}$) of subsets of ℓ_∞. Now we put

$$A_\alpha = \begin{cases} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \pi_{T_n}^{-1}(B_\beta^n), & \alpha = \beta + 1, \\ \bigcup_{n=1}^{\infty} \pi_{T_n}^{-1}(A_{\beta_n}^n), & \alpha = \sup \beta_n, \end{cases}$$

and

$$B_\alpha = \ell_\infty \setminus A_\alpha.$$

CLAIM 1. *For every $\alpha \in [1, \omega_1)$ the following statements are true:*

1. *the sets A_α and B_α are $\mathcal{S}$-open in ℓ_∞;*
2. *for any $y = (y_n)_{n=1}^\infty \in \ell_\infty$ with $y_n \neq 0$ for all $n \in \omega$ we have*

$$x = (x_n)_{n\in\omega} \in A_\alpha \;\Leftrightarrow\; z = (x_n \cdot y_n)_{n\in\omega} \in A_\alpha.$$

Proof of Claim 1. (1). Evidently, A_1 and B_1 are $\mathcal{S}$-open. Assume that for some $\alpha < \omega_1$ the claim is valid for all $\beta < \alpha$. Let $\alpha = \beta + 1$ be an isolated ordinal. Take any $x \in A_\alpha$ and $y \in \sigma_1(x)$. Then there exists $m \in \mathbb{N}$ such that $\pi_{T_n}(x) \in B_\beta^n$ for all $n \geq m$. Since $\pi_{T_n}(y) \in \sigma_1(\pi_{T_n}(x))$ and B_β^n is $\mathcal{S}$-open, $\pi_{T_n}(y) \in B_\beta^n$. Therefore, $y \in A_\alpha$. We argue similarly in the case where α is a limit ordinal.

(2). We fix $y = (y_n)_{n=1}^\infty \in \ell_\infty$ such that $y_n \neq 0$ for all $n \in \mathbb{N}$. The statement is true for $\alpha = 1$, since $A_1 = \sigma(0)$. Assume that for some $\alpha < \omega_1$ the property is valid for all $\beta < \alpha$. Let $\alpha = \beta + 1$ for some β. The inductive assumption implies that

$$\begin{array}{ccc} x \in A_\alpha & \Longleftrightarrow & \exists m \in \mathbb{N}\ \forall n \geq m\ \ \pi_{T_n}(x) \in B_\beta^n \\ & & \Updownarrow \\ z \in A_\alpha & \Longleftrightarrow & \exists m \in \mathbb{N}\ \forall n \geq m\ \ \pi_{T_n}(z) \in B_\beta^n \end{array}$$

We argue similarly in the case of limit α. □

Consider the equivalent metric

$$d(x, y) = \min\{d_\infty(x, y), 1\}$$

on the space ℓ_∞.

Claim 2. *For every $\alpha \in [1, \omega_1)$ the following condition holds:*

($*$) *for every set $C \subseteq (\ell_\infty, d)$ of the additive /multiplicative/ class α there exists a contracting mapping $f : (\ell_\infty, d) \to (\ell_\infty, d)$ with the Lipschitz constant $L = \frac{1}{2}$ such that*

$$C = f^{-1}(A_\alpha) \quad /C = f^{-1}(B_\alpha)/, \tag{3}$$

$$|\pi_n(f(x))| < 1 \quad \forall x \in \ell_\infty\ \forall n \in \omega. \tag{4}$$

Proof of Claim 2. We will argue by the induction on α. Let C be an arbitrary F_σ-subset of (ℓ_∞, d). Then there exists an increasing sequence $(C_n)_{n\in\omega}$ of of closed subsets of (ℓ_∞, d) such that $C = \bigcup_{n\in\omega} C_n$. Consider a map $f : \ell_\infty \to \ell_\infty$, defined by the rule

$$f(x) = \big(\tfrac{1}{2}d(x, C_1), \ldots, \tfrac{1}{2}d(x, C_n), \ldots\big)$$

for all $x \in \ell_\infty$.

We show that $C = f^{-1}(A_1)$. Take $x \in C$ and choose $m \in \omega$ such that $x \in C_n$ for all $n \geq m$. Then $d(x, C_n) = 0$ and $\pi_n(f(x)) = 0$ for all $n \geq m$. Hence, x belongs to the right-hand side of the equality. Now we prove the inverse inclusion. Let $x \in f^{-1}(A_1)$. Then there exists $m \in \omega$ such that $\pi_n(f(x)) = 0$ for all $n \geq m$. Consequently, $d(x, C_n) = 0$ for all $n \geq m$. Since C_n is closed, $x \in C_n$ for all $n \geq m$. Therefore, $x \in \bigcup_{n\in\omega} C_n = C$.

Since

$$d(f(x), f(y)) \leq d_\infty(f(x), f(y)) = \sup_{n\in\omega} |\tfrac{1}{2}d(x, C_n) - \tfrac{1}{2}d(y, C_n)| \leq \tfrac{1}{2}d(x, y)$$

for all $x, y \in \ell_\infty$, the mapping f is contracting with the Lipschitz constant $L = \frac{1}{2}$. Moreover,

$$|\pi_n(f(x))| = \tfrac{1}{2}d(x, C_n) < 1$$

for every $n \in \omega$.

Assume that for some $\alpha < \omega_1$ the condition $(*)$ is valid for all $\beta < \alpha$. Let $C \subseteq (\ell_\infty, d)$ be any set of the α'th additive class. Take an increasing sequence of sets C_n such that $C = \bigcup_{n\in\omega} C_n$, where every C_n belongs to the multiplicative class β if $\alpha = \beta + 1$, and in the case $\alpha = \sup \beta_n$ we can assume that C_n belongs to the additive class β_n for every $n \in \omega$. By the inductive assumption there exists a sequence $(f_n)_{n\in\omega}$ of contracting maps $f_n : (\ell_\infty, d) \to (\ell_\infty, d)$ with the Lipschitz constant $L = \frac{1}{2}$ such that

$$C_n = \begin{cases} f_n^{-1}(B_\beta), & \alpha = \beta + 1, \\ f_n^{-1}(A_{\beta_n}), & \alpha = \sup \beta_n, \end{cases} \tag{5}$$

$$|\pi_m(f_n(x))| < 1 \quad \forall x \in \ell_\infty \ \ \forall n, m \in \omega. \tag{6}$$

For every $k \in \omega$ we choose a unique pair $(n(k), m(k)) \in \omega^2$ such that

$$k = t_{n(k)m(k)} \in T_{n(k)}.$$

For all $x \in \ell_\infty$ and $n, m \in \omega$ we put $f_{nm}(x) = \pi_m(f_n(x))$ and consider a map $f : \ell_\infty \to \ell_\infty$, defined by the rule

$$f(x) = \left(\tfrac{1}{2}f_{n(1)m(1)}(x), \dots, \tfrac{1}{2}f_{n(k)m(k)}(x), \dots\right)$$

for all $x \in \ell_\infty$. The inequalities

$$|f_{nm}(x) - f_{nm}(y)| = |\pi_m(f_n(x)) - \pi_m(f_n(y))| \leq$$
$$\leq \sup_{m\in\omega} |\pi_m(f_n(x)) - \pi_m(f_n(y))| = d_\infty(f_n(x), f_n(y))$$

and

$$|f_{nm}(x) - f_{nm}(y)| \leq 2$$

imply that

$$\frac{1}{2}|f_{nm}(x) - f_{nm}(y)| \leq d(f_n(x), f_n(y)) \leq \frac{1}{2}d(x, y)$$

for all $x, y \in \ell_\infty$ and $n, m \in \omega$. Then

$$d(f(x), f(y)) \leq d_\infty(f(x), f(y)) =$$
$$= \sup_{k\in\omega} |\tfrac{1}{2}(f_{n(k)m(k)}(x) - f_{n(k)m(k)}(y))| \leq \frac{1}{2}d(x, y)$$

for all $x, y \in \ell_\infty$. Therefore, $f : (\ell_\infty, d) \to (\ell_\infty, d)$ is a Lipschitz map with the constant $L = \frac{1}{2}$.

It remains to show that $C = f^{-1}(A_\alpha)$. Assume that $\alpha = \beta + 1$ (we argue similarly if α is limit). Let us observe that $x \in C$ if and only if there exists $m \in \omega$ such that $f_n(x) \in B_\beta$ for all $n \geq m$. Since

$$\pi_{T_n}(f(x)) = \Big(\frac{1}{2}\pi_k(f_n(x))\Big)_{k\in T_n},$$

we have

$$f_n(x) \in B_\beta \iff \pi_{T_n}(f(x)) \in B^n_\beta.$$

by statement (2) of Claim 1. Therefore, $C = f^{-1}(A_\alpha)$. □

CLAIM 3. *For every $\alpha \in [1, \omega_1)$ the set A_α belongs to the additive class α and does not belong to the multiplicative class α in ℓ_∞.*

Proof of Claim 3. If $\alpha = 1$, then

$$A_1 = \bigcup_{n\in\omega} \{x \in \ell_\infty : |\{k \in \omega : x_k \neq 0\}| \leq n\}$$

is an F_σ-subset of ℓ_∞, since every set $\{x \in \ell_\infty : |k \in \omega : x_k \neq 0| \leq n\}$ is closed. Consequently, B_1 is G_δ-subset of ℓ_∞. Suppose that for some $\alpha \geq 1$ the set A_β /B_β/ belongs to the additive /multiplicative/ class β in ℓ_∞ for every $\beta < \alpha$. Since every projection $\pi_{T_n} : \ell_\infty \to \ell_\infty^{T_n}$ is continuous, the set A_α belongs to the additive class α in ℓ_∞ and the set B_β belongs to the multiplicative class α in ℓ_∞.

Fix $\alpha \in [1, \omega_1)$. In order to show that A_α does not belong to the α'th multiplicative class we assume the contrary. Claim 2 implies that there exists a contraction $f : (\ell_\infty, d) \to (\ell_\infty, d)$ such that $A_\alpha = f^{-1}(B_\alpha)$. By the Contraction Map Principle, there would be a fixed point for f, which implies a contradiction. □

Now we are ready to construct a function f from the statement of the theorem. Let $\alpha \in [0, \omega_1)$ be fixed. If $\alpha = 0$, then we put $A = c$, where c is the subspace of ℓ_∞ consisting of all convergent sequences of real numbers. If $\alpha > 0$, then previous steps imply the existence of an $\mathcal{S}$-open set $A \subseteq \ell_\infty$ such that A belongs to the $(\alpha)^\bullet$'th additive class and does not belong to the $(\alpha)^\bullet$'th multiplicative class. In any case for every $x \in \ell_\infty$ we put

$$f(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

We prove that $f : \ell_\infty \to [0,1]$ is strongly separately continuous. Fix $\varepsilon > 0$, $k \in \omega$ and $x = (x_n)_{n\in\omega} \in \ell_\infty$. We put $\delta = 1$ and notice that for all $y \in B_\infty(x, \delta)$ we have

$$y = (y_1, \dots, y_k, \dots) \in A \iff z = (y_1, \dots, y_{k-1}, x_k, y_{k+1}, \dots) \in A,$$

since A is $\mathcal{S}$-open. Therefore,

$$|f(y) - f(z)| = 0$$

for all $y \in B_\infty(x, \delta)$ and $z = (y_1, \dots, y_{k-1}, x_k, y_{k+1}, \dots)$. Hence, f is strongly separately continuous at x with respect to the k'th variable.

Notice that both A and $X \setminus A$ are of the $(\alpha + 1)^\bullet$'th additive class, that is, A is ambiguous set of the $(\alpha + 1)^\bullet$'th class in ℓ_∞. It is well-known that the characteristic function of any ambiguous set of the class ξ in any metric space belongs to the ξ'th Baire class [7, §31] for any $\xi \in [1, \omega_1)$. Therefore, $f \in \mathrm{B}_{(\alpha+1)^\bullet}(\ell_\infty, [0,1])$.

If $\alpha = 0$, then f is discontinuous exactly on A and hence $f \notin \mathrm{B}_0(\ell_\infty, [0,1])$.

In case $\alpha > 0$ we assume that $f \in \mathrm{B}_\alpha(\ell_\infty, [0,1])$. Then f belongs to the $(\alpha)^\bullet$'th Borel class. Therefore, $A = f^{-1}(1)$ is the set of the $(\alpha)^\bullet$'th multiplicative class in ℓ_∞, which contradicts to the choice of A. □

Remark 3.2. The existence of an ssc function $f : \ell_\infty \to [0,1]$ which is not Baire measurable was proved in [5]. The Baire classification of ssc functions defined on $\mathbb{R}^\omega$ was studied in [4].

Theorem 3.1 suggests the following question.

Question 3.3. *Does there exist a strongly separately continuous function f : $\ell_\infty \to [0,1]$ such that $f \in \mathrm{B}_{\omega+1} \setminus \mathrm{B}_\omega$?*

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