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MYCIELSKI-REGULARITY OF GIBBS MEASURES ON COOKIE-CUTTER SETS

Abstract

It has been shown that all Radon probability measures on $\mathbb{R}$ are Mycielski-regular, as well as Lebesgue measure on the unit cube and certain self-similar measures. In this paper, these results are extended to Gibbs measures on cookie-cutter sets.

1 Introduction

Let μ be a Radon probability measure on the Euclidean space $\mathbb{R}^d$ for $d \geq 1$, and $f : \mathbb{R}^d \to \mathbb{R}$ a measurable function. Given a sequence (x_n) in $(\mathbb{R}^d)^{\mathbb{N}}$, for any $x \in \mathbb{R}^d$ define $f_n(x) = f(x_k)$, where x_k is the first among $x_0, \ldots, x_{n-1}$ that minimizes the distance from x to x_k, $0 \leq k \leq n-1$. The measures for which the sequence $(f_n)_{n=1}^{\infty}$ converges in measure to f for almost every sequence $(x_0, x_1, \ldots)$ are called Mycielski-regular. The question was first posed by Mycielski as to which measures have this property [6]. In [1], self-similar measures with probabilities r_i^s (where s is the Hausdorff measure) are shown to be Mycielski-Regular. The method used to prove this result - the method of Voronoi tessellations - was first used by Fremlin [4] to show that all Radon probability measures are Mycielski-Regular when $d = 1$ and also for Lebesgue measure on the unit cube.

Since Fremlin has proved it in this way for all Radon probability measures in the case $d = 1$, it gives some hope that the result can be generalized. On the other hand, it could be that this property belongs to all Radon probability measures in this space because of the structure of $\mathbb{R}$. Two questions emerge. First, can the result be generalized to $\mathbb{R}^d$ for $d \geq 2$? Second, how might this

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be done? Might the method of Voronoi tessellations be used to prove that this property belongs to all Radon probability measures in higher dimensions? It is known that for certain measures on higher dimensional spaces, such as Lebesgue measure on the unit cube or self-similar measures on bounded subsets of $\mathbb{R}^d$, the same method works. This paper considers the method of Voronoi tesellations with respect to other measures. In particular, it is shown that certain Gibbs measures have this property. This is significant in the sense that such Gibbs measures are a generalization of the self-similar measures studied in [1].

We begin with some basic definitions. In this paper, $X \subseteq \mathbb{R}^d$, ρ is the Euclidean metric on X, Ω is the infinite product space $X^{\mathbb{N}}$, and λ is the infinite product measure $\mu^{\mathbb{N}}$ with domain $\mathcal{B}(\Omega)$. The following definitions are from Fremlin [4].

Definition 1.1. *Let (X, ρ) be a metric space. Let $\omega = (x_k)_{k=0}^{\infty}$ be an infinite sequence in $X^{\mathbb{N}}$ and let $\omega[n] = \{x_0, \ldots, x_{n-1}\}$. Suppose that $z \in \omega[n]$. Define the Voronoi tile $V(\omega\restriction n, z)$ by*

$$V(\omega\restriction n, z) = \{x \in X : \rho(x,z) = \rho(x,\omega[n]) \text{ and if } i < j < n \text{ and } z = x_j \neq x_i, \text{ then } \rho(x,z) < \rho(x,x_i)\}. \tag{1}$$

We call the collection of such $V(\omega\restriction n, z)$ the Voronoi tessellation defined by $\omega[n]$.

Definition 1.2. *Let $f : X \to \mathbb{R}$, and $\omega[n]$ as above, and write $x_i = x(i)$. Let $k(\omega[n], x)$ be the least i such that $\rho(x,\omega[n]) = \rho(x, x(i))$, so that $x \in V(\omega\restriction n, x(k(\omega[n], x)))$. Define $F(\omega\restriction n, f)(x) = f(x(k(\omega[n], x)))$.*

The previous definition is another way of defining the function f_n. This will be used in the discussion to follow on the conditions for a measure to be Mycielski-Regular.

2 Conditions for a measure to be Mycielski-Regular

Here we follow David Fremlin's development of the conditions for a measure to be Mycielski-Regular[4]. Following Fremlin, we define a functional $\theta : \Sigma \to [0,1]$ such that for any measurable E,

$$\limsup_{n\to\infty} \int F(\omega\restriction n, \mathbb{1}_E) d\mu = \theta(E) \tag{2}$$

for λ-almost every $\omega \in \Omega$. Fremlin has shown that θ has the following properties:

(i) θ is a unital submeasure.
(ii) $\theta(H) \leq \mu(H)$ for every closed $H \subseteq X$, and $\theta(G) \geq \mu(G)$ for every open $G \subseteq X$.
(iii) If a measurable set E is such that $\mu(\partial E) = 0$, then $\theta(E) = \mu(E)$, where ∂E is the topological boundary of the set E.

To show these three properties, we note first that this function is measurable with respect to $\mathcal{B}(\Omega)$. To see this, write

$$F(\omega \restriction n, f)(x) = \sum_{i=1}^{n} f(\omega(i)) \mathbb{1}_{V(\omega \restriction n, \omega(i))}(x). \tag{3}$$

If $f = \mathbb{1}_E$ for $E \in \Sigma$, then for all $x \in X$, $F(\omega \restriction n, \mathbb{1}_E)(x) \leq 1$, for every $\omega \in \Omega$. Hence,

$$\int_{\Omega} F(\omega \restriction n, \mathbb{1}_E)) d\lambda \quad = \quad \int_{\Omega \times X} F(\omega \restriction n, \mathbb{1}_E)) d(\lambda \times \mu) < \infty. \tag{4}$$

So $F(\omega \restriction n, \mathbb{1}_E)) \in \mathcal{L}_1(\Omega \times X, \mathcal{B}(\Omega) \otimes \Sigma, \lambda \times \mu)$. It follows by Fubini's theorem [5] that the function

$$\omega \mapsto \int_X F(\omega \restriction n, \mathbb{1}_E)) d\mu \tag{5}$$

is in $\mathcal{L}_1(X^{\mathbb{N}}, \mathcal{B}(\Omega), \lambda)$ and, in particular, is λ-measurable.

At first sight, it appears that θ depends both on $E \subseteq X$ and $\omega \in \Omega$. We will show that if ω and $\omega^{'} \in \Omega$ are eventually equal, then $\lim_{n\to\infty}(F(\omega \restriction n, f)(x) - F(\omega^{'} \restriction n, f)(x)) = 0$ for almost every $x \in X$, and so

$$\lim_{n\to\infty} \left(\int F(\omega \restriction n, f) - \int F(\omega^{'} \restriction n, f) \right) = 0. \tag{6}$$

Hence the function $h : \omega \in \Omega \mapsto \limsup_{n\to\infty} \int F(\omega \restriction n, f) d\mu$ is measurable and is constant on all sequences that are eventually equal. By the Zero-One Law [5], the set $\{\omega \in \Omega : h(\omega) > \alpha\}$ has measure 0 or 1 for every $\alpha \in \mathbb{R}$, and so there is an α such that $h(\omega) = \alpha$ for almost every ω. To show (6), we enlist the aid of the following two propositions:

Proposition 2.1. *Let (X, ρ) be a separable metric space and let μ be a topological probability measure. If X_0 is the support of μ, then for every $k \in \mathbb{N}$, $X_0 = \overline{\omega[\mathbb{N} \setminus k]}$ for λ-a.e. ω, where $\omega[\mathbb{N} \setminus k] = \{x_k, x_{k+1}, \ldots\}$.*

Proof. If X is separable metric, then any subspace is separable metric; in particular, it holds for X_0. Let $\mathcal{U}$ be a countable base for X_0. Since X_0 is the support of μ, if $U \in \mathcal{U}$, then $\mu(U) > 0$, and so $\lambda(\{\omega : \omega[\mathbb{N} \setminus k] \cap U \neq \varnothing\}) = 1$, and since

$$\bigcap_{U \in \mathcal{U} \setminus \{\varnothing\}} \{\omega : U \cap \omega[\mathbb{N} \setminus k] \neq \varnothing\} \subseteq \{\omega : X_0 \subseteq \overline{\omega[\mathbb{N} \setminus k]}\}, \tag{7}$$

it follows that $\lambda(\{\omega : X_0 \subseteq \overline{\omega[\mathbb{N} \setminus k]}\}) = 1$ as well. □

Proposition 2.2. *Let (X, ρ) be a separable metric space and let μ be a topological probability measure such that μ has no atoms. There exists $\Omega_0 \subseteq \Omega$ with $\lambda(\Omega_0) = 1$, such that if ω,$\omega^{'} \in \Omega_0$ are eventually equal, then for μ-a.e. $x \in X$, there is an $n \in \mathbb{N}$ such that $F(\omega \restriction m, f)(x) = F(\omega^{'} \restriction m, f))(x)$ for every $m \geq n$ and for every f defined on X.*

PROOF. Let ω,$\omega^{'} \in \Omega_0$ such that $\omega(m) = \omega^{'}(m)$ for every $m \geq l$. Let $X_0 = \overline{\omega[\mathbb{N} \setminus l]} \setminus I$ and $I = \omega[l] \cup \omega^{'}[l]$. Then $\mu(X_0) = 1$ since $\mu(I) = 0$. Now if $x \in X_0$ then there exists $n \geq l$ such that $\rho(x, \omega[n \setminus l]) < \rho(x, I)$, and the same is true for all $m \geq n$. So for any $m \geq n$, $k(\omega \restriction m, x) = k(\omega^{'} \restriction m, x)$, and hence that $F(\omega \restriction m, f)(x) = F(\omega^{'} \restriction m, f)(x)$. □

It thus happens that the functional θ is constant on measurable sets. We now establish the three properties mentioned above. First, θ is a unital submeasure. By "unital" is meant that $\theta : \Sigma \to [0, 1]$, which is clear. That it is a submeasure is also easy to see. By "submeasure" is meant that θ has the following three properties:

(i) $\theta(A \cup B) \leq \theta(A) + \theta(B)$ for all $A, B \in \Sigma$,
(ii) $\theta(A) \leq \theta(B)$ if $A \subseteq B$, and
(iii) $\theta(\varnothing) = 0$.

These properties follow because we can write

$$\theta(E) = \limsup_{n\to\infty} \int F(\omega \restriction n, \mathbb{1}_E) d\mu \tag{8}$$

$$= \limsup_{n\to\infty} \int \sum_{i=1}^{n} \mathbb{1}_E(\omega(i)) \mathbb{1}_{V(\omega \restriction n, \omega(i))}(x) d\mu \tag{9}$$

$$= \limsup_{n\to\infty} \sum_{i=1}^{n} \int \mathbb{1}_E(\omega(i)) \mathbb{1}_{V(\omega \restriction n, \omega(i))}(x) d\mu, \tag{10}$$

and because of the properties of the characteristic function. Thus, $\mathbb{1}_{A \cup B}(x) \leq \mathbb{1}_A(x) + \mathbb{1}_B(x)$, and if $A \subseteq B$, then $\mathbb{1}_A \leq \mathbb{1}_B$, and $\mathbb{1}_\varnothing \equiv 0$.

To show that $\theta(H) \leq \mu(H)$ for every closed $H \subseteq X$, we first need the following lemma:

Lemma 2.3. *Let f be a real-valued continuous function defined on X. Then for almost every $\omega \in \Omega$ and for every $x \in supp(\mu) = X_0$, $F(\omega \restriction n, f)(x)$ converges to $f(x)$ as $n \to \infty$.*

PROOF. Let $\epsilon > 0$. By the continuity of f, there exists a $\delta > 0$ such that if $\rho(x,y) < \delta$ then $|f(x) - f(y)| < \epsilon$. Further, as $n \to \infty$, for every $x \in X_0$, we have that $\rho(x, \omega[n]) \to 0$. So there is an $n_0 \in \mathbb{N}$, such that if $n \geq n_0$, then $\rho(x, \omega[n]) < \delta$. So,

$$\begin{aligned} |F(\omega \restriction n, f)(x) - f(x)| &= |\sum_{i=1}^{n} f(\omega(i)) \mathbb{1}_{V(\omega \restriction n, \omega(i))}(x) - f(x)| && (11) \\ &= |f(\omega(j)) - f(x)| && (12) \end{aligned}$$

for the $1 \leq j \leq n$ such that $x \in V(\omega \restriction n, \omega(j))$. As $n \to \infty$, $\rho(\omega(j), x) < \delta$, and so we get that $|f(\omega(j)) - f(x)| < \epsilon$. $\square$

Now let $\epsilon > 0$, and let $H \subseteq X$ be closed. There is a continuous function f, such that $\mathbb{1}_H \leq f$ and $\int f d\mu < \mu(H) + \epsilon$. Further, since $\lim_{n\to\infty} F(\omega \restriction n, f)(x) = f(x)$ for almost every x, then we have that

$$\begin{aligned} \theta(H) &= \limsup_{n\to\infty} \int F(\omega \restriction n, \mathbb{1}_H) d\mu && (13) \\ &= \int f(x) d\mu \qquad \text{(by Lemma 2.3)} && (14) \\ &< \mu(H) + \epsilon, && (15) \end{aligned}$$

and hence we have that $\theta(H) \leq \mu(H)$.

On the other hand, if $G \subseteq X$ is open, then $\theta(G) = 1 - \theta(X \setminus G) \geq 1 - \mu(X \setminus G) = \mu(G)$.

Finally, we show that if a measurable set E is such that $\mu(\partial E) = 0$, then $\theta(E) = \mu(E)$, where ∂E is the topological boundary of the set E. This follows from

$$\mu(E) = \mu(\overline{E}) = \mu(\operatorname{int} E) \leq \theta(\operatorname{int} E) \leq \theta(E) \leq \theta(\overline{E}) \leq \mu(\overline{E}) = \mu(E). \quad (16)$$

This lays the groundwork for the following two fundamental theorems, and which provide the key to proving which measures are in fact Mycieski-regular:

Theorem 2.4. *Let (X, ρ) be a separable metric space, μ a topological probability measure on X and $\theta : \Sigma \to [0,1]$ the functional defined above. Then the following are equivalent:*

(i) μ is Mycielski-regular;
(ii) θ is absolutely continuous with respect to μ;
(iii) $\theta = \mu$.

PROOF. It is clear that (i) $\implies$ (iii) $\implies$ (ii). It is thus sufficient to show that (ii) $\implies$ (i). Suppose then that θ is absolutely continuous with respect to μ. Let $f : X \to \mathbb{R}$ be measurable, and for each $k \in \mathbb{N}$, let $\delta_k > 0$ be such that $\theta(E) \leq 2^{-k}$ whenever $\mu(E) \leq \delta_k$. By Lusin's theorem [5], there exists a continuous function, call it $g_k : X \to \mathbb{R}$, and a set $E_k = \{x \in X : g_k(x) \neq f(x)\}$, such that $\mu(E_k) \leq \min\{2^{-k}, \delta_k\}$. Note that $\{x \in X : F(\omega \restriction n, f)(x) \neq F(\omega \restriction n, g_k)(x)\} \subseteq \{x \in X : F(\omega \restriction n, \mathbb{1}_{E_k}) = 1\}$ for every $\omega \in \Omega$. Define $W_k \subseteq \Omega$ such that $\omega \in W_k$ if and only if $\lim_{n\to\infty} F(\omega \restriction n, g_k)(x) = g_k(x)$ for almost every x and $\limsup_{n\to\infty} \int F(\omega \restriction n, \mathbb{1}_{E_k}) d\mu \leq 2^{-k}$. Then $\lambda(W_k) = 1$. Let $W = \cap_{k\in\mathbb{N}} W_k$. For any ω, we have that

$$\begin{aligned} \min\{|F(\omega \restriction n, f) - F(\omega \restriction n, g_k)|, \mathbb{1}_X\} &\leq \min\{F(\omega \restriction n, |f - g_k|), \mathbb{1}_X\} &(17)\\ &\leq F(\omega \restriction n, \mathbb{1}_{E_k}). &(18)\end{aligned}$$

Hence,

$$\begin{aligned} \min &\{|F(\omega \restriction n, f) - f|, \mathbb{1}_X\} \leq \min\{|F(\omega \restriction n, f) - F(\omega \restriction n, g_k)|, \mathbb{1}_X\} &(19)\\ + &\min\{|F(\omega \restriction n, g_k) - g_k|, \mathbb{1}_X\} + \min\{|g_k - f|, \mathbb{1}_X\} &(20)\\ \leq &F(\omega \restriction n, \mathbb{1}_{E_k}) + \min\{|F(\omega \restriction n, g_k) - g_k|, \mathbb{1}_X\} + \mathbb{1}_{E_k}. &(21)\end{aligned}$$

Thus we have that if $\omega \in W$, then

$$\begin{aligned} \overline{\lim_{n\to\infty}} &\int \min\{|F(\omega \restriction n, f) - f|, \mathbb{1}_X\} d\mu \leq \overline{\lim} \int F(\omega \restriction n, \mathbb{1}_{E_k}) d\mu &(22)\\ + &\overline{\lim} \int \min\{|F(\omega \restriction n, g_k) - g_k|, \mathbb{1}_X\} d\mu &(23)\\ + &\overline{\lim} \int \mathbb{1}_{E_k} d\mu &(24)\\ \leq &2^{-k} + 0 + 2^{-k+1}. &(25)\end{aligned}$$

Since this is true for every $k \in \mathbb{N}$, it follows that $F(\omega \restriction n, f)$ converges to f in measure. And since f is arbitrary, we have that μ is Mycieski-regular. □

We now give a sufficient condition for a measure to be Mycielski-regular in terms of its tessellations. We need the following definition:

Definition 2.5. *Let X, ρ, μ, Ω, and λ be as defined above. We say that μ has moderated Voronoi tessellations if for every $\epsilon > 0$ there exists $M \geq 0$ such that*

$$\sum_{n=1}^{\infty} \lambda\{\omega : \mu\left(\bigcup\{V^{'}(\omega{\restriction}n, z) : z \in \omega[n], \mu(V^{'}(\omega{\restriction}n, z)) \geq M/n\}\right) \geq \epsilon\} < \infty, \tag{26}$$

where each $V^{'}(\omega{\restriction}n, z)$ is the punctured Voronoi tile $V(\omega{\restriction}n, z) \setminus \{z\}$.

Note that if μ has moderated Voronoi tessellations for M then μ has moderated Voronoi tessellations for all $M^{'} \geq M$. The reason for this is as follows: call $A(n, M, \epsilon) = \{\omega : \mu\left(\bigcup\{V^{'}(\omega{\restriction}n, z) : z \in \omega[n], \mu(V^{'}(\omega{\restriction}n, z)) \geq M/n\}\right) \geq \epsilon\}$, and $B(n, M) = \bigcup\{V^{'}(\omega{\restriction}n, z) : z \in \omega[n], \mu(V^{'}(\omega{\restriction}n, z)) \geq M/n\}$. If $V^{'}(\omega{\restriction}n, z)) \in B(n, M^{'})$, then $V^{'}(\omega{\restriction}n, z)) \in B(n, M)$ so that $B(n, M^{'}) \subseteq B(n, M)$. Thus, if $\mu(B(n, M^{'})) \geq \epsilon$ then $\mu(B(n, M)) \geq \epsilon$. So if $\omega \in A(n, M^{'}, \epsilon)$ then $\omega \in A(n, M, \epsilon)$. Hence, if $\sum A(n, M, \epsilon) < \infty$ then $\sum A(n, M^{'}, \epsilon) < \infty$. We now have the proper background to state the following theorem.

Theorem 2.6. *Let (X, ρ) be a separable metric space, μ a topological probability measure on X which has moderated Voronoi tessellations. Then μ is Mycielski-regular.*

PROOF. Let θ be the submeasure introduced above. We will show that θ is absolutely continuous with respect to μ and therefore by the previous theorem, it will follow that μ is Mycielski-regular.

Let $\epsilon > 0$, and let $M \geq 0$ such that

$$\sum_{n=1}^{\infty} \lambda\{\omega : \mu\left(\bigcup\{V(\omega{\restriction}n, z) : z \in \omega[n], \mu(V(\omega{\restriction}n, z)) \geq M/n\}\right) \geq \epsilon/3\} < \infty. \tag{27}$$

Let

$$\Omega_1 = \{\omega : \mu\left(\bigcup\{V(\omega{\restriction}n, z) : z \in \omega[n], \mu(V(\omega{\restriction}n, z)) \geq M/n\}\right) < \epsilon/3 \text{ for all but finitely } n\}. \tag{28}$$

It follows that $\lambda(\Omega_1) = 1$.

Now, let $\delta > 0$ such that $2M\delta \leq \epsilon/3$, $\delta \leq \epsilon/3$, and $\delta \leq 1/2$. Suppose that $\mu(E) \leq \delta$. Let

$$\Omega_2 = \{\omega : \text{Card}\{n : \text{Card}\{i : i < n, \omega(i) \in E\} > 2\delta n\} < \infty\}. \tag{29}$$

It follows by the Strong Law of Large Numbers [5] that $\lambda(\Omega_2) = 1$. Let $\omega \in \Omega_1 \cap \Omega_2$. Let n be such that

$$\mu\left(\bigcup\{V(\omega\restriction n, z) : z \in \omega[n], \mu(V(\omega\restriction n, z)) \geq M/n\}\right) \leq \epsilon/3, \tag{30}$$

and

$$\text{Card}\{i : i < n, \omega(i) \in E\} \leq 2\delta n. \tag{31}$$

Set $I = E \cap \omega[n]$, and $J = \{z : z \in \omega[n], \mu(V(\omega\restriction n, z)) \geq M/n\}$. Thus,

$$\begin{aligned}
\int F(\omega\restriction n, \mathbb{1}_E) d\mu &= \sum_{z\in I} \mu(V(\omega\restriction n, z)) && (32)\\
&= \sum_{z\in I\cap J} \mu(V(\omega\restriction n, z)) + \sum_{z\in I\setminus J} \mu(V(\omega\restriction n, z)) && (33)\\
&\leq \epsilon/3 + \text{Card}(I \setminus J) \cdot M/n && (34)\\
&\leq \epsilon/3 + M \cdot \text{Card}(I)/n && (35)\\
&\leq \epsilon/3 + 2M\delta \leq \epsilon. && (36)
\end{aligned}$$

As this is true for all but finitely n, it follows that $\theta(E) \leq \epsilon$, and thus that θ is absolutely continuous with respect to μ. □

3 Gibbs Measures

As in the case of the measures studied in [1], we are interested in measures that concentrate their mass on fractal sets which are constructed via a set of contractions. Let $(F_1, \ldots, F_l)$ be a conformal iterated function system satisfying the Hölder condition (for a complete definition, see [7]) on a closed and bounded subset X of $\mathbb{R}^d$. Let C be the unique, nonempty compact subset of X satisfying

$$C = \bigcup_{i=1}^{l} F_i(C). \tag{37}$$

We look at the measure defined on sets which can be thought of as non-linear analogues of Cantor sets, sometimes called cookie-cutter sets. These are sets which are approximately self-similar in the sense that small parts of the set can be mapped by a transformation onto a large part of the set without too much distortion. Gibbs measure is a measure which has its support on a cookie-cutter set and is a generalization of a self-similar measure. We will show that certain Gibbs measures are Mycielski-Regular. The proof for such measures would also include the self-similar measures as a special case.

A cookie-cutter set can be thought of in two related ways: as the repeller of a dynamical system or as the attractor of an iterated function system. Following Falconer [3], let f be an expanding function (with $|f'(x)| > 1$) defined on a finite collection of disjoint subsets $X_1, X_2, \ldots, X_l$ of X. We assume that the function f is of class C^2 and maps each X_j bijectively onto X, for each j. Consider the set of all points x that remain in $\cup_{k=1}^{l} X_k$ for all iterates under f. That is, consider the set

$$C = \{x \in X : f^k(x) \in \bigcup_{i=1}^{l} X_i \text{ for all } k \geq 0\} = \bigcap_{k=0}^{\infty} f^{-k}(X). \tag{38}$$

Associated with the function f is the conformal iterated function system, consisting of conformal mappings F_i for $i = 1 \ldots l$, with $F_i(x) = f^{-1}(x) \cap X_i$. Let $I_k = \{1, \ldots, l\}^k$, and let $\sigma = i_1 i_2 \ldots i_k$, with $i_m \in \{1, \ldots, l\}$ for $1 \leq m \leq k$. Define $X_\sigma = F_{i_1} \circ \cdots \circ F_{i_k}(X) = f_\sigma^{-k}(X)$. Thus, for $\sigma \in I_k$, X_σ is a set that f^k maps bijectively onto X. Moreover, there exist real numbers γ and η between 0 and 1 such that

$$\eta \leq |F_i(x)| \leq \gamma \tag{39}$$

for each i, $1 \leq i \leq l$ and for all $x \in X$.

We want to define a measure which concentrates its mass on the cookie-cutter set C. To do so, we need a Lipschitz function $\phi : \cup_{i=1}^{l} X_i \to \mathbb{R}$, and a function

$$S_k\phi(x) = \sum_{j=0}^{k-1} \phi(f^j(x)). \tag{40}$$

The following theorem is proved in [3]; it establishes the existence of a Gibbs measure with support in the set C:

Theorem 3.1. *For all k and $\sigma \in I_k$, let $x_\sigma \in X_\sigma$. Then the limit*

$$P(\phi) = \lim_{k \to \infty} \frac{1}{k} \log \sum_{\sigma \in I_k} \exp S_k\phi(x_\sigma) \tag{41}$$

exists and does not depend on the $x_\sigma \in X_\sigma$ chosen. Furthermore, there exists a Borel probability measure μ supported on C and a number $a > 0$ such that for all k and all $\sigma \in I_k$,

$$a^{-1} \leq \frac{\mu(X_\sigma)}{\exp(-kP(\phi) + S_k\phi(x))} \leq a \tag{42}$$

for all $x \in X_\sigma$.

Let $\phi(x) = -s\log|f'(x)|$ and let $s \in \mathbb{R}$ be such that $P(\phi) = 0$. Then it can be shown (see the proof of Theorem 5.3 in [3]) that there exists $b \in \mathbb{R}$ such that

$$b^{-1}|X_\sigma|^s \leq \mu(X_\sigma) \leq b|X_\sigma|^s \tag{43}$$

for all $\sigma \in I_k$. It should be pointed out that the existence of these bounds depends upon the fact that the IFS is conformal.

As Falconer points out, the pressure formula (41) generalizes the dimension formula for self-similar sets to a non-linear setting [3]. Hence, the proof for the Gibbs measure associated with this choice of s also proves the result for self-similar measures.

We are now in a position to show that the Gibbs measure associated with the function $\phi(x) = -s\log|f'(x)|$ is Mycielski-Regular. This will be done by showing that it has moderated Voronoi Tessellations.

Theorem 3.2. *Let $\phi(x) = -s\log|f'(x)|$ and let $s \in \mathbb{R}$ be such that $P(\phi) = 0$. Then the associated Gibbs measure μ is Mycielski-Regular.*

Before presenting the proof of the main result, a couple of lemmas are inserted to which reference will be made in the course of the proof. The first result is obvious, but it is included here to keep down the clutter in the following proof. A very similar version to the second lemma is also stated and proved in [2].

Lemma 3.3. *Let $\{x_k\}$ and $\{y_k\}$ be sequences of positive real numbers with $x_k, y_k \to \infty$ as $k \to \infty$. Then there exist real numbers $\alpha > 1$ and $0 < \beta < 1$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $\dfrac{\lceil x_k \rceil}{\lfloor y_k \rfloor} \leq \dfrac{\alpha x_k}{\beta y_k}$.*

Lemma 3.4. *Let $\{V_i\}$ be a collection of disjoint open sets of $\mathbb{R}^n$ such that each V_i contains a ball of radius $c_1 r$ and is contained in a ball of radius $c_2 r$. Then any ball B of radius $2c_2 r$ intersects, at most, $(4c_2)^n c_1^{-n}$ of the sets $\overline{V_i}$.*

PROOF. If $\overline{V_i}$ meets B, then $\overline{V_i}$ is contained in a ball concentric with B and of radius $4c_2 r$. If Γ of these sets meets B, then we use the fact that each of the sets also contains a ball of radius $c_1 r$ to obtain an upper bound for the number of sets meeting B. Summing up over the volumes of the interior balls, we obtain $\Gamma(c_1 r)^n \leq (4c_2 r)^n$, which gives $\Gamma \leq (4c_2)^n c_1^{-n}$. □

We now proceed with the proof of our main result.

PROOF. Let $\epsilon > 0$ and let

$$M = \left\lceil 2\ln\left(\frac{8e\alpha b^2}{\beta\eta^s\epsilon}\right)\right\rceil, \tag{44}$$

where α and β are chosen as in Lemma 3.3 (in particular, we will want to take $x_k = b(\eta\zeta)^{-s}$ and $y_k = \frac{\epsilon}{2b\zeta^s}$). Note that for ϵ small enough, $M > 1$. For large $n \in \mathbb{N}$, choose $k(n) \in \mathbb{N}$ that satisfies the following:

$$\frac{bM}{2(\eta\zeta)^s} \leq n \leq \frac{bM}{(\eta\zeta)^s} \leq 2^{\epsilon/4b\zeta^s}, \tag{45}$$

where $\zeta = \gamma^{k(n)}$ (here γ is defined by (39)). Also note that in the following pages, $k(n)$ will be denoted by k.

For the moment, fix n satisfying (45), and let $z \in C$. Define $\mathcal{S}_k$ to be the set of sequences which are truncated at the first j such that $\eta\zeta \leq |X_{i_1 \ldots i_j}| \leq \zeta$. Let $\mathcal{V}_k = \{X_\sigma : \sigma \in \mathcal{S}_k\}$ and let $\mathcal{K}$ be the set of members of $\mathcal{V}_k$ meeting $B(z,\zeta)$. By Lemma 3.4, if B is a ball of radius ζ, then there are a finite number (call it Γ) of members of $\mathcal{V}_k$ that intersect B and this number is independent of the level k.

Let $V \subseteq X$ be a convex set. Suppose that $y \in (V \cap C) \setminus \cup\mathcal{K}$ such that $y \in X_\sigma$ for some $X_\sigma \in \mathcal{V}_k$. Since $y \notin \cup\mathcal{K}$, it follows that $\rho(y,z) > \zeta$. Since $|X_\sigma| \leq \zeta$, it follows that $X_\sigma \subseteq intB(y, \rho(y,z))$. Define

$$V_z = V \bigcap \bigcup_{w \in (V\cap C)\setminus \cup\mathcal{K}} \{X_\sigma \in \mathcal{V}_k : X_\sigma \subseteq intB(w, \rho(w,z))\}. \tag{46}$$

From above, it follows that $y \in V_z$. Accordingly, $(V \cap C) \setminus V_z \subseteq \cup\mathcal{K}$ and is covered by at most Γ members of $\mathcal{V}_k$, and so (we can drop the C since μ is a measure with support in C)

$$\mu(V \setminus V_z) \leq \Gamma b|X_\sigma|^s \leq \Gamma b\zeta^s = \Gamma'\zeta^s, \tag{47}$$

where $\Gamma' = \Gamma b$.

Now let

$$H_n(\omega) = \bigcup \{V(\omega\restriction n, z) : z \in \omega[n], \mu(V(\omega\restriction n, z)) \geq 2\Gamma'\zeta^s\}, \tag{48}$$

and

$$\mathcal{K}_\omega = \{X_\sigma \in \mathcal{V}_k : X_\sigma \cap \omega[n] = \varnothing\}. \tag{49}$$

Note that the set $V(\omega\restriction n, z)$ is a convex set. Let this set play the role of the set V above. Thus, if $V(\omega\restriction n, z) \subseteq H_n(\omega)$, then $\mu(V(\omega\restriction n, z)) \leq 2\mu(V(\omega\restriction n, z) \cap \cup\mathcal{K}_\omega)$. This is because

$$\begin{aligned} \mu(V(\omega\restriction n, z)) &= \mu(V(\omega\restriction n, z) \setminus \cup\mathcal{K}_\omega) + \mu(V(\omega\restriction n, z) \cap \cup\mathcal{K}_\omega) && (50)\\ &\leq \mu(V(\omega\restriction n, z) \cap \cup\mathcal{K}_\omega) + \mu(V(\omega\restriction n, z) \cap \cup\mathcal{K}_\omega) && (51)\\ &= 2\mu(V(\omega\restriction n, z) \cap \cup\mathcal{K}_\omega), && (52) \end{aligned}$$

where the inequality in the second line comes from the fact that $\mu(V(\omega \upharpoonright n, z) \backslash \cup \mathcal{K}_\omega) \leq \Gamma' \zeta^s$, whereas from the definition of $H_n(\omega)$ we have that $\mu(V(\omega \upharpoonright n, z)) \geq 2\Gamma' \zeta^s$. (If $y \in V(\omega \upharpoonright n, z)$ with $y \neq z$, then $intB(y, \rho(y, z)) \cap \omega[n] = \varnothing$, and if $X_\sigma \subseteq intB(y, \rho(y, z))$ then $X_\sigma \in \mathcal{K}_\omega$.)

Thus,

$$\begin{aligned} \mu(H_n(\omega)) &\leq 2\mu(H_n(\omega) \cap \cup \mathcal{K}_\omega) && (53) \\ &\leq 2\mu(\cup \mathcal{K}_\omega) && (54) \\ &\leq 2b\zeta^s \mathrm{Card} \mathcal{K}_\omega. && (55) \end{aligned}$$

Therefore, if $\mu(H_n(\omega)) \geq \epsilon$, then $\mathrm{Card}(\mathcal{K}_\omega) \geq \dfrac{\epsilon}{2b\zeta^s} \geq m$, where $m = \left\lfloor \dfrac{\epsilon}{2b\zeta^s} \right\rfloor$. It follows that

$$\{\omega \in \Omega : \mu(H_n(\omega)) \geq \epsilon\} \subseteq \{\omega \in \Omega : \mathrm{Card}(\mathcal{K}_\omega) \geq m\}. \quad (56)$$

Define $[\mathcal{V}_k]^m = \{\mathcal{K} \subseteq \mathcal{V}_k : \mathrm{Card}(\mathcal{K}) = m\}$. Then,

$$\begin{aligned} &\lambda(\{\omega \in \Omega : \mu(H_n(\omega)) \geq \epsilon\}) \leq \lambda(\{\omega \in \Omega : \mathrm{Card}(\mathcal{K}_\omega) \geq m\}) && (57) \\ \leq &\sum_{\mathcal{K} \in [\mathcal{V}_k]^m} \lambda(\{\omega : \omega[n] \text{ does not meet } \cup \mathcal{K}\}). && (58) \end{aligned}$$

Also, note that $1 \geq \mu(\cup \mathcal{V}_k) \geq \mathrm{Card}(\mathcal{V}_k) b^{-1} \eta^s \zeta^s$, so that

$$\mathrm{Card}(\mathcal{V}_k) \leq b(\eta\zeta)^{-s}. \quad (59)$$

Then,

$$\begin{aligned} \sum_{\mathcal{K} \in [\mathcal{V}_k]^m} \lambda(\{\omega : \omega[n] \cap \cup \mathcal{K} = \varnothing\}) &\leq \binom{\lceil b(\eta\zeta)^{-s} \rceil}{m} \left(1 - \frac{m(\eta\zeta)^s}{b}\right)^n && (60) \\ &\leq \frac{(\lceil b(\eta\zeta)^{-s} \rceil)^m}{m!} \left(1 - \frac{m(\eta\zeta)^s}{b}\right)^n && (61) \\ &\leq \frac{(\lceil b(\eta\zeta)^{-s} \rceil)^m}{m!} \left(1 - \frac{m(\eta\zeta)^s}{b}\right)^{\frac{(bM(\eta\zeta)^{-s})}{2}} && (62) \\ &\leq \frac{e^m (\lceil b(\eta\zeta)^{-s} \rceil)^m}{m^m} \left(1 - \frac{m(\eta\zeta)^s}{b}\right)^{\frac{(bM(\eta\zeta)^{-s})}{2}} && (63) \end{aligned}$$

$$\leq \frac{e^m \left(\lceil b(\eta\zeta)^{-s}\rceil\right)^m}{m^m} \left(\frac{1}{e}\right)^{Mm/2} \tag{64}$$

$$\leq \left(\frac{e \left(\lceil b(\eta\zeta)^{-s}\rceil\right)}{me^{M/2}}\right)^m \tag{65}$$

$$\leq \left(\frac{1}{2}\right)^m \quad \text{(by Lemma 3.3 and choice of } M) \tag{66}$$

$$= \left(\frac{1}{2}\right)^{\left\lfloor \frac{\epsilon}{2b\zeta^s} \right\rfloor}. \tag{67}$$

By (45), it follows that for n sufficiently large,

$$\left(\frac{1}{2}\right)^{\left\lfloor \frac{\epsilon}{2b\zeta^s} \right\rfloor} \leq \frac{1}{n^2}. \tag{68}$$

Let n_0 be such that for all $n \geq n_0$, (45) holds. Then,

$$\sum_{n=1}^{\infty} \lambda(\{\omega \in \Omega : \mu(H_n(\omega)) \geq \epsilon\}) \leq n_0 + \sum_{n=n_0}^{\infty} \lambda(\{\omega \in \Omega : \mu(H_n(\omega)) \geq \epsilon\}) \tag{69}$$

$$\leq n_0 + \sum_{n=n_0}^{\infty} \frac{1}{n^2} < \infty. \tag{70}$$

□

4 Concluding Remarks

The arguments used here show both the utility and short-comings of the method of moderated Voronoi tessellations. Its utility lies in its ability to show Mycielski-Regularity for numerous measures. It works especially well if the sets are disjoint at any level k in the construction of the set on which the measure concentrates its mass. If the sets are not disjoint, trying to find an upper bound on the number Γ of sets becomes problematic - an upper bound which is crucial in the proof. Moreover, we first set out to show that Gibbs measures were Mycielski-Regular for any $s \in \mathbb{R}$; however, so far we have been unable to do this. In the end, this is still a particular case of the more general problem: are all Radon probability measures Mycielski-Regular? We think so, but it remains to be proven.

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