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A NOTE ON THE UNIQUENESS PROPERTY FOR BOREL G-MEASURES

Abstract

In terms of a group G of isometries of Euclidean space, it is given a necessary and sufficient condition for the uniqueness of a G-measure on the Borel σ-algebra of this space.

Throughout this paper, $\mathbf{N}$ denotes the set of all natural numbers, for each $n \in \mathbf{N}$ the symbol $\mathbf{R}^n$ denotes the n-dimensional Euclidean space, and G denotes a subgroup of the group of all isometries of $\mathbf{R}^n$. In addition, the symbol l_n stands for the classical n-dimensional Lebesgue measure on $\mathbf{R}^n$ and b_n stands for the restriction of l_n to the Borel σ-algebra $\mathcal{B}(\mathbf{R}^n)$ of $\mathbf{R}^n$. The symbol C_n denotes the closed unit ball in $\mathbf{R}^n$, i.e.,

$$C_n = \{x \in \mathbf{R}^n : ||x|| \leq 1\}.$$

A non-negative functional μ defined on some G-invariant σ-ring of subsets of $\mathbf{R}^n$ is called a G-measure if the following three conditions are satisfied:

(1) $\mu(C_n) = b_n(C_n)$;

(2) μ is countably additive on its domain $\mathrm{dom}(\mu)$;

(3) if $X \in \mathrm{dom}(\mu)$ and $Y \in \mathrm{dom}(\mu)$ are any two G-congruent sets, then $\mu(X) = \mu(Y)$ (the G-invariance of μ).

Clearly, the standard examples of G-measures on $\mathbf{R}^n$ are l_n and b_n.

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Remark 1. *Replacing in the above definition the term "σ-ring" with "ring" and the term "countably additive" with "finitely additive," we come to the notion of a G-volume in $\mathbf{R}^n$ (cf. [3], [10]).*

Remark 2. *Let $\mathcal{M}(G)$ be the class of all G-measures on $\mathbf{R}^n$. Some properties of this class are considered in [5]. The natural question arises whether $\mathcal{M}(G)$ completely characterizes an original group G. In this direction, it was proved that if G_1 and G_2 are two groups of isometries of $\mathbf{R}^n$ such that $G_1 \setminus G_2 \neq \emptyset$, then there exists a G_2-measure on $\mathbf{R}^n$ which is not G_1-invariant. In particular, for any two groups G and H of isometries of $\mathbf{R}^n$, the equality $\mathcal{M}(G) = \mathcal{M}(H)$ implies the equality $G = H$ (for more details, see [5]). The proof of this implication is heavily based on an uncountable form of the Axiom of Choice ($\mathbf{AC}$). Indeed, in Solovay's model [9] of a fragment of set theory with some countable version of $\mathbf{AC}$, all subsets of the real line $\mathbf{R} = \mathbf{R}^1$ are Lebesgue measurable. It follows from this fact that, for the additive group $(\mathbf{Q}, +)$ of all rational numbers and for the additive group $(\mathbf{R}, +)$, the equality $\mathcal{M}((\mathbf{Q}, +)) = \mathcal{M}((\mathbf{R}, +))$ holds true in Solovay's model, but these two groups trivially differ from each other.*

We shall say that μ is a Borel G-measure on $\mathbf{R}^n$ if μ is a G-measure on $\mathbf{R}^n$ and $\mathrm{dom}(\mu) = \mathcal{B}(\mathbf{R}^n)$.

Obviously, b_n is a standard Borel G-measure on $\mathbf{R}^n$. Below we will establish, in terms of a group G, a necessary and sufficient condition for the uniqueness of b_n in the class of all Borel G-measures on $\mathbf{R}^n$. More precisely, the statement we intend to prove is formulated as follows: the measure b_n is a unique Borel G-measure on $\mathbf{R}^n$ if and only if all G-orbits are everywhere dense in $\mathbf{R}^n$. For this purpose, we need several auxiliary results.

Lemma 1. *Let G be a group of isometries of $\mathbf{R}^n$. If at least one of the G-orbits is not everywhere dense in $\mathbf{R}^n$, then there are two distinct Borel G-measures on $\mathbf{R}^n$.*

Proof. As usual, for any point $z \in \mathbf{R}^n$ and any real $r > 0$, we denote by $B(z, r)$ the open ball in $\mathbf{R}^n$ centered in z, with radius r.

Let now a point $x \in \mathbf{R}^n$ be such that its G-orbit $G(x)$ is not everywhere dense in $\mathbf{R}^n$. Then there exists an open ball $B(y, r) \subset \mathbf{R}^n$ satisfying the relation

$$G(x) \cap B(y, r) = \emptyset.$$

This relation implies that

$$(\cup\{g(B(x, r/2)) : g \in G\}) \cap (\cup\{g(B(y, r/2)) : g \in G\}) = \emptyset.$$

We introduce the following three sets:

$A_1 = \cup\{g(B(x, r/2)) : g \in G\}$;
$A_2 = \cup\{g(B(y, r/2)) : g \in G\}$;
$A_3 = \mathbf{R}^n \setminus (A_1 \cup A_2)$.

Observe that A_1, A_2, A_3 are G-invariant subsets of $\mathbf{R}^n$, both A_1 and A_2 are nonempty open sets in $\mathbf{R}^n$, and A_3 is closed in $\mathbf{R}^n$. Further, since

$$\mathbf{R}^n = A_1 \cup A_2 \cup A_3,$$

we have the disjunction

$$b_n(C_n \cap A_1) > 0 \ \vee \ b_n(C_n \cap A_2) > 0 \ \vee \ b_n(C_n \cap A_3) > 0.$$

Consider three possible cases.

1. $b_n(C_n \cap A_1) > 0$. In this case, for each set $X \in \mathcal{B}(\mathbf{R}^n)$, we define

$$\nu(X) = b_n(C_n)\frac{b_n(X \cap A_1)}{b_n(C_n \cap A_1)}.$$

A straightforward verification shows that the functional ν is a Borel G-measure on $\mathbf{R}^n$. At the same time, we have

$$\nu(A_2) = 0, \qquad b_n(A_2) > 0,$$

whence it follows that ν and b_n differ from each other.

2. $b_n(C_n \cap A_2) > 0$. Similarly to the previous case, for any set $X \in \mathcal{B}(\mathbf{R}^n)$, we put

$$\nu(X) = b_n(C_n)\frac{b_n(X \cap A_2)}{b_n(C_n \cap A_2)}.$$

Again, a direct verification shows that the functional ν is a Borel G-measure on $\mathbf{R}^n$. At the same time,

$$\nu(A_1) = 0, \qquad b_n(A_1) > 0,$$

which shows that ν and b_n are two distinct Borel G-measures on $\mathbf{R}^n$.

3. $b_n(C_n \cap A_3) > 0$. In this case, for each set $X \in \mathcal{B}(\mathbf{R}^n)$, we define

$$\nu(X) = b_n(C_n)\frac{b_n(X \cap A_3)}{b_n(C_n \cap A_3)}.$$

Once again, a straightforward verification yields that ν is a Borel G-measure on $\mathbf{R}^n$. At the same time, we have

$$\nu(A_1) = 0, \qquad \nu(A_2) = 0, \qquad b_n(A_1) > 0, \qquad b_n(A_2) > 0,$$

whence it follows that ν and b_n differ from each other.

So, we conclude that if there exists a G-orbit which is not everywhere dense in $\mathbf{R}^n$, then there are at least two distinct Borel G-measures on $\mathbf{R}^n$. This finishes the proof of Lemma 1. □

Remark 3. *As can readily be checked, for a group G of isometries of the space $\mathbf{R}^n$, these two assertions are equivalent:*

(a) there exists at least one point $x \in \mathbf{R}^n$ such that the orbit $G(x)$ is everywhere dense in $\mathbf{R}^n$;

(b) for any point $z \in \mathbf{R}^n$, the orbit $G(z)$ is everywhere dense in $\mathbf{R}^n$.

If (b) holds true, then it makes sense to say that the group G acts almost transitively in $\mathbf{R}^n$.

Below, the group of all isometries of the space $\mathbf{R}^n$ is assumed to be endowed with its standard topology (induced by the topology of Euclidean space of dimension $n^2 + n$).

Lemma 2. *Let G be a group of isometries of $\mathbf{R}^n$ such that all G-orbits are everywhere dense in $\mathbf{R}^n$, let G^* denote the closure of G, and let μ be a Borel G-measure on $\mathbf{R}^n$.*

Then the following two relations are satisfied:

(1) for any compact set K in $\mathbf{R}^n$, one has $\mu(K) < +\infty$ (consequently, μ is a σ-finite measure);

(2) μ is a G^*-invariant measure.

Proof. Since all G-orbits are everywhere dense in $\mathbf{R}^n$, the family of open balls $\{g(\mathrm{int}(C_n)) : g \in G\}$ is a covering of $\mathbf{R}^n$. Therefore, if K is a compact set in $\mathbf{R}^n$, then there exists a finite family $\{g_1, g_2, ..., g_m\} \subset G$ such that

$$K \subset g_1(\mathrm{int}(C_n)) \cup g_2(\mathrm{int}(C_n)) \cup ... \cup g_m(\mathrm{int}(C_n)),$$

which immediately gives us $\mu(K) \leq m b_n(C_n) < +\infty$. This establishes (1) and also implies the equality

$$\mu(K) = \inf\{\mu(U) : K \subset U,\ U \text{ is an open set in } \mathbf{R}^n\},$$

because K is representable in the form $K = \cap\{U_j : j \in \mathbf{N}\}$, where all U_j are open subsets of $\mathbf{R}^n$ and

$$U_0 \supset U_1 \supset ... \supset U_j \supset ..., \qquad \mu(U_0) < +\infty.$$

To show the validity of (2), we use the regularity of μ, i.e., the fact that μ is a Radon measure. So, it suffices to prove that $\mu(h(K)) = \mu(K)$ whenever $h \in G^*$ and K is compact in $\mathbf{R}^n$. Take any real $\varepsilon > 0$. There exists an open set $U \subset \mathbf{R}^n$ such that

$$h(K) \subset U, \quad \mu(U \setminus h(K)) < \varepsilon.$$

Further, since G is everywhere dense in G^*, there exists an element $g \in G$ belonging to an appropriate neighborhood of h and also satisfying $g(K) \subset U$. Therefore, in view of the G-invariance of μ, we may write

$$\mu(K) = \mu(g(K)) \leq \mu(U) \leq \mu(h(K)) + \varepsilon,$$

whence it follows that $\mu(K) \leq \mu(h(K))$. Taking in the last inequality $h^{-1}(K)$ instead of K, we get $\mu(h^{-1}(K)) \leq \mu(K)$ and then easily infer the G^*-invariance of μ. Lemma 2 has thus been proved. □

Lemma 3. *If a group G of isometries of $\mathbf{R}^n$ is such that all G-orbits are everywhere dense in $\mathbf{R}^n$, then the group G^* (the closure of G) acts transitively in $\mathbf{R}^n$.*

Proof. Let 0 denote the neutral element of $\mathbf{R}^n$ and let x be an arbitrary point of $\mathbf{R}^n$. Since the orbit $G(0)$ is everywhere dense in $\mathbf{R}^n$, there exists a sequence $\{g_m : m \in \mathbf{N}\}$ of elements from G such that

$$\lim_{m \to +\infty} g_m(0) = x.$$

It can readily be seen that the family of transformations $\{g_m : m \in \mathbf{N}\}$ is bounded in the group of all isometries of $\mathbf{R}^n$. Therefore, this family contains a convergent subsequence $\{g_{m(i)} : i \in \mathbf{N}\}$ such that

$$\lim_{i \to +\infty} g_{m(i)} = g^* \in G^*.$$

Now, it is clear that $g^*(0) = x$, which completes the proof. □

Before formulating the next auxiliary result, let us recall that a Borel measure on $\mathbf{R}^n$ is said to be G-quasi-invariant if G preserves the class of all μ-measure zero sets.

Obviously, for measures the property of G-quasi-invariance is much weaker than the property of G-invariance.

The next lemma is proved in [6]. However, we enclosed its (highly nontrivial) proof for the reader's convenience.

Lemma 4. *Let G be a closed group of isometries of the space $\mathbf{R}^n$ acting transitively in $\mathbf{R}^n$ and let θ denote the left Haar measure on G. Suppose also that μ is a nonzero σ-finite G-quasi-invariant Borel measure on $\mathbf{R}^n$. Then, for each set $X \in \mathcal{B}(\mathbf{R}^n)$, the equivalence*

$$\mu(X) = 0 \ \Leftrightarrow\ \theta(\{g \in G : g(0) \in X\}) = 0$$

holds true.

Proof. Our argument follows [6] (cf. also Chapter 9 of [7]). First of all, we may assume without loss of generality that μ is a Borel probability G-quasi-invariant measure on $\mathbf{R}^n$. Let us define a surjective continuous mapping $\phi : G \to \mathbf{R}^n$ by the formula

$$\phi(g) = g(0) \qquad (g \in G)$$

and introduce the class of sets

$$\mathcal{S} = \{\phi^{-1}(X) \ : \ X \in \mathcal{B}(\mathbf{R}^n)\}.$$

Clearly, $\mathcal{S}$ is a countably generated σ-subalgebra of the Borel σ-algebra of G. We can also define a probability measure ν on $\mathcal{S}$ by putting

$$\nu(\phi^{-1}(X)) = \mu(X) \qquad (X \in \mathcal{B}(\mathbf{R}^n)).$$

Since the original measure μ is G-quasi-invariant, the measure ν on $\mathcal{S}$ is left G-quasi-invariant. Applying the measure extension theorem from [2], we may extend ν to a Borel probability measure ν' on G. Let us denote by θ' a probability measure equivalent to the Haar measure θ. Further, for each Borel subset Z of G, consider a function $\psi_Z : G \to \mathbf{R}$ defined by the formula

$$\psi_Z(g) = \nu'(gZ) \qquad (g \in G).$$

It is not hard to check that ψ_Z is a Borel function on G integrable with respect to the measure θ'. So we may put

$$\nu''(Z) = \int_G \psi_Z(g) d\theta'(g) = \int_G \nu'(gZ) d\theta'(g).$$

A direct verification shows that ν'' is a left G-quasi-invariant Borel probability measure on G. According to a well-known fact from the Haar measure theory, ν'' and θ are equivalent measures. Hence, by the Radon–Nikodym theorem, there exists a strictly positive Borel function $p : G \to \mathbf{R}$ such that

$$\nu''(Z) = \int_Z p(g) d\theta(g)$$

for each Borel subset Z of G. In view of the definition of ν it is clear that, for any set $X \in \mathcal{B}(\mathbf{R}^n)$, we have

$$\mu(X) = 0 \Leftrightarrow \nu(\phi^{-1}(X)) = 0.$$

At the same time, we may write

$$\nu(\phi^{-1}(X)) = 0 \Leftrightarrow \nu'(\phi^{-1}(X)) = 0 \Leftrightarrow \nu''(\phi^{-1}(X)) = 0.$$

Keeping in mind the strict positivity of p, we obtain the equivalence

$$\mu(X) = 0 \Leftrightarrow \theta(\phi^{-1}(X)) = 0.$$

This completes the proof of Lemma 4. □

We thus see that the family of all μ-measure zero subsets of $\mathbf{R}^n$ is completely determined by θ, so does not depend on the choice of μ satisfying the assumptions of Lemma 4.

Lemma 5. *Let G be a group of isometries of $\mathbf{R}^n$, all G-orbits of which are everywhere dense in $\mathbf{R}^n$, and let μ be a σ-finite G-invariant Borel measure on $\mathbf{R}^n$ absolutely continuous with respect to b_n. Then μ is proportional to b_n, i.e., there exists a real constant $t \geq 0$ such that $\mu = tb_n$.*

PROOF. Since all G-orbits are everywhere dense in $\mathbf{R}^n$, the measure b_n is metrically transitive (ergodic) with respect to G, i.e., for any Borel set $X \subset \mathbf{R}^n$ with $b_n(X) > 0$, there exists a countable family $\{g_i : i \in I\} \subset G$ such that

$$b_n(\mathbf{R}^n \setminus \cup\{g_i(X) : i \in I\}) = 0.$$

This fact readily follows from the classical Lebesgue theorem on the existence of density points in X. Now, it suffices to apply one general theorem from the theory of invariant measures, stating that if a σ-finite invariant measure is absolutely continuous with respect to a σ-finite metrically transitive measure, then these two measures are proportional (see, e.g., [5] for a proof of the above-mentioned general statement). □

We now are ready to establish the following result.

Theorem 1. *For a group G of isometries of $\mathbf{R}^n$, these two assertions are equivalent:*

(1) all G-orbits are everywhere dense in $\mathbf{R}^n$;

(2) any Borel G-measure on $\mathbf{R}^n$ is identical with b_n.

Proof. The implication (2) $\Rightarrow$ (1) immediately follows from Lemma 1. Let us show the validity of the converse implication (1) $\Rightarrow$ (2). Suppose (1) and let μ be any Borel G-measure on the space $\mathbf{R}^n$. Denote by G^* the closure of G. According to Lemma 3, the group G^* acts transitively in $\mathbf{R}^n$ and, according to Lemma 2, the measure μ is G^*-invariant. Further, by virtue of Lemma 4, μ and b_n are equivalent measures and, in particular, μ is absolutely continuous with respect to b_n. So, we may apply Lemma 5 and infer that μ is proportional to b_n. Finally, taking into account that

$$\mu(C_n) = b_n(C_n),$$

we conclude that μ and b_n coincide with each other. □

Remark 4. *The proof of the above theorem is not quite elementary in the sense that it uses the notion of a Haar measure on a closed (in general, non-commutative) group of isometries of* $\mathbf{R}^n$*, so the presented argument leaves the framework of classical real analysis. In this connection, it would be interesting to give an elementary proof of Theorem 1 without appealing to profound properties of the Haar measure.*

Remark 5. *The assertion of Theorem 1 fails to be true if we somehow weaken the definition of a* G*-measure on* $\mathbf{R}^n$*. For instance, consider a Borel measure* μ *on* $\mathbf{R}^2$ *satisfying the following condition:*

(*) $\mu(B) = b_2(C_2) = \pi$ *for every closed disc* $B \subset \mathbf{R}^2$ *of radius* 1*.*

Then we cannot assert, in general, that μ *is identical with* b_2*. Indeed, as shown in [1], there are two real constants* $\alpha \neq 0$ *and* $\beta \neq 0$ *such that*

$$\int\int_B \sin(\alpha x + \beta y)dxdy = 0$$

for any disc $B \subset \mathbf{R}^2$ *congruent to* C_2*. Consequently, if* μ *is defined by*

$$\mu(Z) = b_2(Z) + \int\int_Z \sin(\alpha x + \beta y)dxdy$$

for each set $Z \in \mathcal{B}(\mathbf{R}^2)$*, then* μ *satisfies (*) but differs from* b_2 *(see also [4] for some related interesting results).*

Remark 6. *Let* G *be again a group of isometries of* $\mathbf{R}^n$*, all* G*-orbits of which are everywhere dense in* $\mathbf{R}^n$*. In general, the standard Borel measure* b_n *does not possess the uniqueness property with respect to the class of all* σ*-finite*

G-invariant Borel measures on $\mathbf{R}^n$ (here the uniqueness means the proportionality of measures). For example, define the subgroup H of $\mathbf{R}^2$ by the equality

$$H = \mathbf{R} \times \mathbf{Q}.$$

Clearly, H is a Borel uncountable everywhere dense subgroup of $\mathbf{R}^2$. For any Borel set $X \subset \mathbf{R}^2$, put

$$\nu(X) = \sum \{b_1(X \cap (\mathbf{R} \times \{q\})) : q \in \mathbf{Q}\}.$$

It is not difficult to see that:

(a) *ν is a Borel σ-finite H-invariant measure on $\mathbf{R}^2$;*

(b) *$\nu(C_2) = +\infty$.*

Actually, for the uniqueness of b_n with respect to the class of all σ-finite G-invariant Borel measures on $\mathbf{R}^n$, a much stronger assumption on G is needed. One of the sufficient conditions is formulated as follows: for each point $x \in \mathbf{R}^n$, the G-orbit $G(x)$ is everywhere dense in $\mathbf{R}^n$, and the group G is thick in its closure G^*, which means that $\theta_*(G^* \setminus G) = 0$, where θ_* denotes the inner measure canonically associated with the left Haar measure θ on G^* (cf. [6]; see also Chapter 9 of [7]). It is still unknown whether the above sufficient condition is also necessary. In this context, let us notice that some necessary and sufficient conditions for the uniqueness property of l_n with respect to the class of all σ-finite G-invariant measures on $\mathbf{R}^n$ are presented in Chapter 9 of [7].

At the end of this note, we would like to give another, slightly stronger formulation of Theorem 1. For this purpose, we need one well-known statement from classical descriptive set theory.

Lemma 6. *Let E be a Polish space and let $\{Z_m : m \in \mathbf{N}\}$ be a family of Borel subsets of E. Then the following two assertions are equivalent:*

(1) *the family $\{Z_m : m \in \mathbf{N}\}$ separates points in E, i.e., for any two distinct points $x \in E$ and $y \in E$, there exists $m \in \mathbf{N}$ such that $\mathrm{card}(Z_m \cap \{x, y\}) = 1$;*

(2) *the σ-algebra generated by $\{Z_m : m \in \mathbf{N}\}$ is identical with the Borel σ-algebra of E.*

PROOF. The implication (2) $\Rightarrow$ (1) is almost trivial, so we restrict our attention to the implication (1) $\Rightarrow$ (2). Let $\{Z_m : m \in \mathbf{N}\}$ separate points of E.

Consider Marczewski's characteristic function χ indiced by $\{Z_m : m \in \mathbf{N}\}$. As known, χ acts from E into Cantor's discontinuum $\{0,1\}^{\mathbf{N}}$ and is defined as follows: for each $z \in E$ one has

$$\chi(z) = \{i_z(m) : m \in \mathbf{N}\},$$

where $i_z(m) = 1$ if $z \in Z_m$, and $i_z(m) = 0$ if $z \notin Z_m$. By virtue of (1), this χ turns out to be an injective Borel mapping, so the χ-images of all Borel subsets of E are Borel sets in $\{0,1\}^{\mathbf{N}}$ and, actually, χ is a Borel isomorphism between E and $\chi(E)$ (see, for instance, [8]). Taking into account the fact that

$$\chi(Z_m) = \{t \in \{0,1\}^{\mathbf{N}} : t_m = 1\} \cap \chi(E) \qquad (m \in \mathbf{N}),$$

one easily concludes that the σ-algebra generated by $\{Z_m : m \in \mathbf{N}\}$ coincides with the Borel σ-algebra of E. □

Theorem 2. *For a group G of isometries of $\mathbf{R}^n$, the following two assertions are equivalent:*

(1) all G-orbits are everywhere dense in $\mathbf{R}^n$;

(2) any G-measure μ is an extension of the measure b_n.

Proof. By virtue of Lemma 1 and Theorem 1, it suffices to demonstrate that if (1) is valid and μ is a G-measure on $\mathbf{R}^n$, then $\mathrm{dom}(\mu)$ entirely contains the Borel σ-algebra $\mathcal{B}(\mathbf{R}^n)$. For this purpose, denote again by 0 the neutral element of the additive group $\mathbf{R}^n$ and observe that there exists a countable family $\{g_i : i \in I\}$ of elements of G such that the set $\{g_i(0) : i \in I\}$ is everywhere dense in $\mathbf{R}^n$. From this fact it is not difficult to deduce that the countable family of sets $\{g_i(C_n) : i \in I\}$ separates the points in $\mathbf{R}^n$. Consequently, according to Lemma 6, the σ-ring generated by the family $\{g_i(C_n) : i \in I\}$ coincides with the Borel σ-algebra $\mathcal{B}(\mathbf{R}^n)$. Thus, the inclusion $\mathcal{B}(\mathbf{R}^n) \subset \mathrm{dom}(\mu)$ holds true. In fact, a more simple geometric argument also leads to the required result. Namely, for any real $\varepsilon > 0$, there exist two indices $i \in I$ and $j \in I$ such that the set $g_i(C_n) \cap g_j(C_n)$ has nonempty interior and its diameter is strictly less than ε. From this circumstance it is not hard to infer that every open subset of $\mathbf{R}^n$ belongs to the σ-algebra generated by the family $\{g_i(C_n) : i \in I\}$, whence the inclusion $\mathcal{B}(\mathbf{R}^n) \subset \mathrm{dom}(\mu)$ trivially follows. □

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