RESEARCH

Real Analysis Exchange
Vol. 43(1), 2017, pp. 137–154
DOI: 10.14321/realanalexch.41.1.0137

Rachel L. Bayless, Department of Mathematics, Agnes Scott College, Decatur, GA 30030 U.S.A. email: rbayless@agnesscott.edu

ERGODIC PROPERTIES OF RATIONAL FUNCTIONS THAT PRESERVE LEBESGUE MEASURE ON $\mathbb{R}$

Abstract

We prove that all negative generalized Boole transformations are conservative, exact, pointwise dual ergodic, and quasi-finite with respect to Lebesgue measure on the real line. We then provide a formula for computing the Krengel, Parry, and Poisson entropy of all conservative rational functions that preserve Lebesgue measure on the real line.

1 Introduction

Ergodic properties of rational maps that preserve an infinite measure are the subject of much recent study as in [4], [6], [8], [9], [11] [21], [24], and [25]. They also have classical roots and were studied earlier in [2], [5], [12], [13], [14], and [22]. In this paper we study rational maps that preserve Lebesgue measure, λ, on $\mathbb{R}$. This class of maps is often referred to as generalized Boole transformations.

The classical Boole transformation was originally studied in [7], and, over a century later, was shown to be conservative and ergodic in [5]. Since then a generalized version of the Boole transformation has been studied in [2], [13] and [14]. Generalized Boole transformations have the form

$$G(x) = x + \beta + \sum_{k=1}^{N} \frac{p_k}{t_k - x}, \tag{1.1}$$

Mathematical Reviews subject classification: Primary: 37A40, 37A35
Key words: entropy, infinite measure, rational functions, generalized Boole transformations
Received by the editors January 30, 2017
Communicated by: Zbigniew Nitecki

where $\beta, t_k, p_k \in \mathbb{R}$ and $p_k > 0$ for all $k = 1, ..., N$. It is known that $\pm G$ gives a complete characterization of all rational functions that preserve λ ([13], [18]). From now on we refer to $+G$ as a positive generalized Boole transformation, and we continue using the notation G for such transformations. Similarly, we refer to $-G$ as a negative generalized Boole transformation, and we will denote these transformations by S.

The ergodic properties of positive generalized Boole transformations have been well studied. In [14] Li and Schweiger showed that if $\beta = 0$, then G is conservative and ergodic. In [2] Aaronson studied positive generalized Boole transformations under the umbrella of inner functions on the upper half-plane and proved that if $\beta = 0$, then G is exact and pointwise dual ergodic with return sequence $a_n(T) \sim \frac{1}{\pi}\sqrt{\frac{2n}{\sum_{k=1}^{N} p_k}}$. If $\beta \neq 0$, then G is totally dissipative and non-ergodic. More recently, Aaronson and Park showed in [4] that if $\beta = 0$, then G is quasi-finite.

A key assumption in the above results for positive generalized Boole transformations is that the constant term β must be 0. In Section 2, we show that the situation is different for negative generalized Boole transformations, and we prove:

Theorem A. If S is a negative generalized Boole transformation, then S is: (1) exact, (2) conservative, and (3) pointwise dual ergodic with respect to λ.

Let $R : \mathbb{R} \to \mathbb{R}$ be a conservative rational function that preserves λ. Since any λ-preserving rational map is a generalized Boole transformation, either R is a positive generalized Boole transformation with $\beta = 0$ or R is a negative generalized Boole transformation with any $\beta \in \mathbb{R}$.

In Section 3, we compute the entropy of conservative rational functions that preserve λ on $\mathbb{R}$. Three independent definitions have been suggested for entropy in the infinite setting. They are Krengel entropy ([12]), Parry entropy ([16]), and Poisson entropy ([11], [21]). It is known to be difficult to compute these entropies, and, in particular, to show they are finite. In Theorem 3.4, we prove that $h_{\mathrm{Kr}}(R) = h_{\mathrm{Pa}}(R) = h_{\mathrm{Po}}(R)$. Then, we prove the following theorem which provides a formula for computing the entropy.

Theorem B. If R is a conservative rational function that preserves λ, then

$$h_{\mathrm{Kr}}(R) = \int_{\mathbb{R}} \log |R'(x)| d\lambda(x). \tag{1.2}$$

2 Ergodic properties of λ-preserving rational functions

2.1 Preliminary definitions and notation

Let $(X, \mathcal{B}, m, T)$ denote a σ-finite measure space, $(X, \mathcal{B}, m)$, together with a transformation $T : X \to X$ such that $T^{-1}\mathcal{B} \subseteq \mathcal{B}$. We assume throughout that $(X, \mathcal{B}, m, T)$ is *measure-preserving* (i.e. $m(T^{-1}A) = m(A)$ for all $A \in \mathcal{B}$). We say $(X, \mathcal{B}, m, T)$ is *n-to-1* if for almost every $x \in X$, the set $T^{-1}(x)$ contains precisely n distinct points. Given a nonsingular n-to-1 system, we call a partition $\mathcal{P} = \{P_i\}_{i=1}^n$ of X a *Rohlin partition* for T if $T : P_i \to X$ is one-to-one and onto for each $i = 1, ..., n$. From now on when we write T we mean that $(X, \mathcal{B}, m, T)$ is an infinite measure-preserving system and T is n-to-1.

We say T is *ergodic* if for any set $A \in \mathcal{B}$ such that $T^{-1}A = A$, we have $m(A) = 0$ or $m(A^c) = 0$. We say T is *exact* if for any set $A \in \mathcal{B}$ such that $T^{-n}(T^n(A)) = A$ for all $n > 0$, we have $m(A) = 0$ or $m(A^c) = 0$. It is clear that if T is exact, then T is ergodic, but in general the converse does not hold.

A set $A \in \mathcal{B}$ is called *wandering* for T if the sets $\{T^{-i}A\}_{i=0}^\infty$ are pairwise disjoint. We say T is *conservative* if there does not exist a wandering set of positive measure. Since $m(X) = \infty$, then conservativity is not automatic, because the preimages of a set $A \in \mathcal{B}$ have plenty of room to "wander" throughout an infinite measure space. A set $A \in \mathcal{B}$ is called a *sweep-out set* for T if $\bigcup_{n=0}^\infty T^{-n}A = X \mod m$. The following theorem relates the existence of sweep-out sets to the conservativity of measure-preserving transformations.

Theorem 2.1 (Maharam's Recurrence Theorem, [15]). *Suppose $(X, \mathcal{B}, m, T)$ is a measure-preserving system. If there exists a sweep-out set $A \in \mathcal{B}$ with $m(A) < \infty$, then T is conservative.*

Given $(X, \mathcal{B}, m, T)$ with Rohlin partition $\mathcal{P} = \{P_i\}_{i=1}^n$, we denote each branch $T|_{P_i}$ by T_i. We define the *Jacobian* of T by

$$J_T(x) = \sum_{i=1}^n \mathbb{1}_{P_i}(x) \frac{dmT_i}{dm}(x). \tag{2.1}$$

The *Perron-Frobenius operator* (or *dual operator*) $\widehat{T} : L^1(m) \to L^1(m)$ is defined by the finite sum

$$\widehat{T}f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{J_T(y)}. \tag{2.2}$$

We let ψ_i denote the inverse of T restricted to P_i (i.e. $\psi_i = T^{-1}|_{P_i}$), and we write (2.2) as

$$\widehat{T}f(x) = \sum_{i=1}^{n} f(\psi_i(x)) \cdot J_{\psi_i}(x). \tag{2.3}$$

Note that if $X = \mathbb{R}$, $m = \lambda$, and T is piecewise C^1, then $J_T(x) = |T'(x)|$ and $J_{\psi_i}(x) = |\psi_i'(x)|$.

We say T is *pointwise dual ergodic* if T is conservative, ergodic and there exist constants $a_n(T)$ such that for all $f \in L^1(m)$ we have

$$\frac{1}{a_n(T)} \sum_{k=0}^{n-1} \widehat{T}^k f \to \int_X f dm \quad \text{a.e. as } n \to \infty. \tag{2.4}$$

As stated in the Introduction, positive and negative generalized Boole transformations give a complete characterization of rational functions that preserve Lebesgue measure on the real line ([13], [18]). Let

$$G(x) = x + \beta + \sum_{k=1}^{N} \frac{p_k}{t_k - x} \quad \text{and} \quad S(x) = -x - \beta - \sum_{k=1}^{N} \frac{p_k}{t_k - x}, \tag{2.5}$$

where $\beta, t_k, p_k \in \mathbb{R}$, and $p_k > 0$. We assume throughout this paper that the poles $\{t_i\}_{i=1}^N$ are in ascending order. That is, $t_i < t_{i+1}$ for all $i = 1, ..., N-1$.

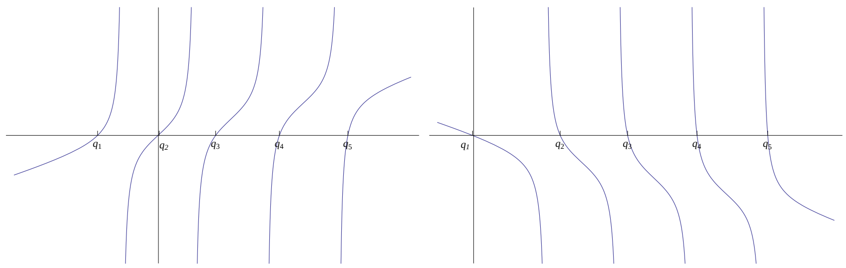
Let R denote a rational function that preserves Lebesgue measure (i.e. $R = G$ or $R = S$). Let $q_1, ..., q_{N+1}$ denote the roots of R in ascending order, so $R(q_i) = 0$. We define a partition $\mathfrak{Q} = \{Q_1, ..., Q_{N+1}\}$ of $\mathbb{R}$ such that $Q_i = [q_i, q_{i+1})$ for $i = 1, ..., N$ and $Q_{N+1} = (-\infty, q_1) \cup [q_{N+1}, \infty)$. Note that there is exactly one pole between each q_i and q_{i+1}. Namely, $t_i \in Q_i$ for $i = 1, ..., N$. We further note that R is an $(N+1)$-to-1 mapping with respect to λ on $\mathbb{R}$ with Rohlin partition $\mathfrak{Q}$. The general shape of R and the partition $\mathfrak{Q}$ are depicted in Figure 1.

We develop a bit more notation related to $\mathfrak{Q}$ that will be used throughout this paper. We let $\psi_i = R^{-1}|_{Q_i}$, so $\psi_i : \mathbb{R} \to Q_i$ is 1-to-1 and onto for $i = 1, ..., N+1$. We denote the refinement $Q_{i_1} \cap R^{-1}Q_{i_2} \cap ... \cap R^{-(n-1)}Q_{i_n}$ by $Q_{i_1 i_2 ... i_n}$, and let

$$\psi_{i_1 ... i_n} = R^{-1}|_{Q_{i_1 ... i_n}}, \quad \text{so} \quad \psi_{i_1 ... i_n} = \psi_{i_1 ... i_{n-1}} \circ \psi_{i_n}. \tag{2.6}$$

For convenience, we define one more piece of notation and let

$$\psi_{i^{[k]}} = \underbrace{\psi_i \circ \psi_i \circ ... \circ \psi_i}_{k-\text{times}}. \tag{2.7}$$



	Left: $R = G$	Right: $R = S$
Number of Poles	4	4
Number of Roots	5	5
Number of Fixed Points	3	5

Figure 1: Examples of conjugated R when $N = 4$.

Let $w_1, ..., w_{N-1}$ (or w_{N+1}) denote the fixed points of R (note that G has $N-1$ fixed points, while S has $N+1$ fixed points in $\mathbb{R}$). For the rest of this paper it is convenient to conjugate R so that $w_1 = 0$. That is, $R(0) = 0$ and all other fixed points are positive. Let $\phi(x) = x - w_1$ and $\phi^{-1}(x) = x + w_1$. From now on we replace R with $\tilde{R} = \phi \circ R \circ \phi^{-1}$. Note that Figure 1 depicts the shape of a conjugated R.

2.2 Proofs of Ergodic properties

Lemma 2.2. *If S is a negative generalized Boole transformation, then the second iterate of S is a positive generalized Boole transformation and has the form*

$$S^2(x) = x + \sum_{k=1}^{N^2+2N} \frac{\rho_k}{\tau_k - x}, \tag{2.8}$$

where $\tau_k, \rho_k \in \mathbb{R}$ and $\rho_k > 0$.

Proof. Let G and S have the form in (2.5). Since S is a rational function that preserves λ, any iterate of S is also a λ-preserving rational function, so by results recalled above $S^2 = \pm G$. Now, we want to show that $S^2 = +G$. Note that

$$\lim_{x \to -\infty} (G(x) - x) = \beta \text{ and } \lim_{x \to -\infty} (S(x) - x) = \infty. \tag{2.9}$$

Thus, given a λ-preserving rational function, we can check this limit to determine if it has the form of G or S, and if it has the form of G, then we also obtain the constant β. We check this limit for S^2, noting that

$$S^2(x) = \underbrace{x + \sum_{k=1}^{N} \frac{p_k}{t_k - x}}_{I} \underbrace{- \sum_{k=1}^{N} \frac{p_k}{t_k - S(x)}}_{II} . \tag{2.10}$$

We have $\lim_{x\to-\infty}(S^2(x) - x) = 0$. Therefore, S^2 is a positive generalized Boole transformation with constant $\beta = 0$. □

Proof of Theorem A part (1). Given Lemma 2.2, we appeal to the aforementioned results in [2] and [14] to conclude that S^2 is exact. It is a simple exercise to show that if the second iterate of a measure-preserving transformation is exact, then so is the original transformation. Furthermore, since S is exact, it is also ergodic. □

Before proving Theorem A part (2), we provide some motivation. It is known that any *invertible*, ergodic map that preserves a non-atomic σ-finite measure is necessarily conservative, and the only invertible, dissipative, ergodic transformation of a σ-finite measure space is isomorphic to translation on the integers with counting measure. However, the generalized Boole transformations are non-invertible, and it is shown in [3] (remark on pg. 22 and Proposition 6.4.8) that there exist non-invertible, totally dissipative, ergodic measure-preserving transformations. Thus, conservativity is not immediate in this setting.

Proof of Theorem A part (2). We will show that the set $A = \bigcup_{i=1}^{N} Q_i = [q_1, q_{N+1})$ is a sweep-out set for S and apply Maharam's Recurrence Theorem.

Note that $\psi_{(N+1)}(q_1) = q_{N+1}$, so in order to study the inverse images of the endpoints of A we need only consider the sequence $\{\psi_{(N+1)^{[k]}}(q_{N+1})\}_{k\geq 0}$. It is convenient to define two separate sequences corresponding to the even (positive) and the odd (negative) terms. We will denote the even terms by $\{V_k\}_{k\geq 0}$ such that $V_0 = q_{N+1}$ and $V_k = \psi_{(N+1)^{[2k]}}(q_{N+1})$. The odd terms will be denoted by $\{W_k\}_{k\geq 0}$ such that $W_0 = q_1$ and $W_k = \psi_{(N+1)^{[2k-1]}}(q_{N+1})$. We have

$$(W_{\lceil k/2\rceil}, V_{\lfloor k/2\rfloor}] = \bigcup_{j=0}^{k} S^{-j}(A), \quad \text{or} \quad [W_{k/2}, V_{k/2}) = \bigcup_{j=0}^{k} S^{-j}(A), \tag{2.11}$$

depending on whether k is odd or even. In order to show A is a sweep-out set, we need to show

$$\lim_{k\to\infty} V_k = \infty \quad \text{and} \quad \lim_{k\to\infty} W_k = -\infty. \tag{2.12}$$

We provide the details for the first limit in (2.12), and the second limit follows by a similar argument. Since $V_k = \psi_{(N+1)^{[2k]}}(q_{N+1})$, we have that $V_k = S^2(V_{k+1})$. By the proof of Lemma 2.2 we have

$$V_k = V_{k+1} + \sum_{i=1}^{\mathfrak{N}} \frac{\rho_i}{\tau_i - V_{k+1}}, \tag{2.13}$$

where $\mathfrak{N} = N^2 + 2N$, $\tau_i, \rho_i \in \mathbb{R}$ and $\rho_i > 0$.

We now show that there exists a $c > 0$ such that $V_k \geq c\sqrt{k}$, which implies $V_k \to \infty$ as $k \to \infty$ (this argument has been adapted from [14]). First, for $k \geq 0$ we have $V_k > 0$ and $V_k \in Q_{N+1}$. Thus, $V_k > \tau_i$ for all $i = 1, ..., \mathfrak{N}$. By (2.13) we have that for $k \geq 0$

$$V_k \leq V_{k+1} - \frac{\rho_{\mathfrak{N}}}{V_{k+1}}. \tag{2.14}$$

Multiplying both sides by V_{k+1} and using the quadratic formula yields

$$4V_{k+1}^2 \geq 2V_k^2 + 2V_k\sqrt{V_k^2 + 4\rho_{\mathfrak{N}}} + 4\rho_{\mathfrak{N}}. \tag{2.15}$$

We note that $\sqrt{V_k^2 + 4\rho_{\mathfrak{N}}} \geq V_k$, so (2.15) implies

$$V_{k+1}^2 \geq V_k^2 + \rho_{\mathfrak{N}}. \tag{2.16}$$

By induction on (2.16) we have $V_k \geq \sqrt{\rho_{\mathfrak{N}}} \cdot \sqrt{k}$. Thus, $V_k \to \infty$. A similar argument shows $W_k \to -\infty$. □

PROOF OF THEOREM A PART (3). Let $A = [q_1, q_{N+1})$ as in the proof of part (2). By Lemma 2.2 and the aforementioned results in [2] we have that S^2 is pointwise dual ergodic with return sequence $a_n(S^2) \sim \frac{1}{\pi}\sqrt{\frac{2n}{\sum_{k=1}^{N^2+N} \rho_k}}$, so we have

$$\frac{1}{a_n(S^2)} \sum_{k=0}^{n-1} \widehat{(S^2)}^k \mathbb{1}_A(x) \to \lambda(A) \quad \text{a.e. as } n \to \infty. \tag{2.17}$$

We want to show there exists a sequence $a_n(S)$ such that

$$\frac{1}{a_n(S)} \sum_{k=0}^{n-1} \widehat{S}^k \mathbb{1}_A(x) \to \lambda(A) \quad \text{a.e. as } n \to \infty, \tag{2.18}$$

and pointwise dual ergodicity will follow from Hurewicz's Ergodic Theorem [10]. For notational convenience, we will write $a_n = a_n(S^2)$ and $b_n = a_n(S)$.

Let
$$b_n = \begin{cases} 2a_{\frac{n}{2}} & \text{if } n \text{ is even} \\ 2a_{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

If n is even:

$$\frac{1}{b_n}\sum_{k=0}^{n-1}\widehat{S}^k \mathbb{1}_A(x) = \frac{1}{2a_{\frac{n}{2}}}\left(\sum_{k=0}^{\frac{n}{2}-1}\widehat{(S^2)}^k \mathbb{1}_A(x) + \sum_{k=0}^{\frac{n}{2}-1}\widehat{(S^2)}^k (\widehat{S}\mathbb{1}_A)(x)\right). \tag{2.19}$$

If n is odd:

$$\frac{1}{b_n}\sum_{k=0}^{n-1}\widehat{S}^k \mathbb{1}_A(x) = \frac{1}{2a_{\frac{n-1}{2}}}\left(\mathbb{1}_A(x)+\sum_{k=1}^{\frac{n-1}{2}}\widehat{(S^2)}^k \mathbb{1}_A(x) + \sum_{k=0}^{\frac{n-1}{2}-1}\widehat{(S^2)}^k (\widehat{S}\mathbb{1}_A)(x)\right). \tag{2.20}$$

Since S^2 is pointwise dual ergodic, then along the two sequences of even and odd terms we have,

$$\frac{1}{b_n}\sum_{k=0}^{n-1}\widehat{S}^k \mathbb{1}_A(x) \to \frac{1}{2}\left(\lambda(A) + \int_{\mathbb{R}}\widehat{S}\mathbb{1}_A(x)d\lambda(x)\right) = \lambda(A), \tag{2.21}$$

λ-a.e. as $n \to \infty$. Thus, S is pointwise dual ergodic.

□

Corollary 2.3. *If S is a negative generalized Boole transformation, then the return sequence $b_n \sim \sqrt{2}a_n$, where a_n is the return sequence for S^2.*

Proof. By the definitions of a_n and b_n given in the proof of Theorem A part (3) we have

$$\lim_{n\to\infty}\frac{b_{2n}}{a_{2n}} = \lim_{n\to\infty}\frac{b_{2n-1}}{a_{2n-1}} = \sqrt{2}. \tag{2.22}$$

□

3 Entropy

3.1 The induced transformation

One technique commonly used to study conservative infinite measure-preserving transformations is inducing on a finite-measure sweep-out set. Let $A \in \mathcal{B}$ be a sweep-out set for $(X, \mathcal{B}, m, T)$. For $x \in X$ define $\phi_A(x) = \min\{n :$

$T^n(x) \in A\}$. That is, $\phi_A(x)$ is the *first-hitting-time* of x to A. If $x \in A$, then $\phi_A(x)$ is often referred to as the *first-return-time* of x to A. The *induced transformation*, $T_A : A \to A$, is defined by

$$T_A(x) = T^{\phi_A(x)}(x) \text{ for } x \in A.$$

If $(X, \mathcal{B}, m, T)$ is a measure-preserving system and A is a sweep-out set for T, then T_A is a measure-preserving transformation of $(A, \mathcal{B}|_A, m|_A)$, where $\mathcal{B}|_A = \{B \cap A : B \in \mathcal{B}\}$ and $m|_A(B) = m(A \cap B)$.

We develop some notation to describe precise hitting-times to A. Let $\mathfrak{A}$ denote the *first-return partition* of A. That is, $\mathfrak{A} = \{A_k\}$, where

$$A_k = \{x \in A : \phi_A(x) = k\} = A \cap T^{-k}A \setminus \bigcup_{j=1}^{k-1} T^{-j}A. \tag{3.1}$$

Let $\mathfrak{B} = \{B_k\}$ be a similar partition of A^c. That is,

$$B_k = \{x \in A^c : \phi_A(x) = k\} = A^c \cap T^{-k}A \setminus \bigcup_{j=1}^{k-1} T^{-j}A = T^{-k}A \setminus \bigcup_{j=0}^{k-1} T^{-j}A. \tag{3.2}$$

We now turn our attention back to conservative rational functions that preserve Lebesgue measure with sweep-out set $A = [q_1, q_{N+1})$. We partition each atom A_k of $\mathfrak{A}$ into N sets $A_{k,i}$ for $i = 1, .., N$ such that $R^k : A_{k,i} \to A$ is one-to-one and onto. That is, we let $A_{k,i} = \psi_i \circ \psi_{(N+1)^{[k-1]}}(A)$. Figure 2 shows how $A_{k,i}$ and B_k move under the ψ maps. Each solid arrow depicts a 1-to-1 and onto mapping, and the dashed arrows indicate $(N-2)$ individual 1-to-1 and onto mappings.

3.2 Preliminaries on entropy

We quickly recall the definition of entropy for transformations preserving a finite measure and point the reader to [17] or [23] for a more in-depth discussion. Let $(\Omega, \mathcal{C}, \mu)$ be a finite-measure space, and let $\alpha = \{a_i\}$ be a countable partition of Ω. The *entropy of* α is defined by

$$H(\alpha) = -\sum_{i=0}^{\infty} m(a_i) \log(m(a_i)). \tag{3.3}$$

If T is a measure-preserving transformation of $(\Omega, \mathcal{C}, \mu)$, then $T^{-n}\alpha$ denotes the partition $\{T^{-n}a_i\}$. The *entropy of T with respect to* α is defined by

$$h(T, \alpha) = \lim_{n\to\infty} \frac{1}{n} H(\alpha \vee T^{-1}\alpha \vee ... \vee T^{-(n-1)}\alpha). \tag{3.4}$$

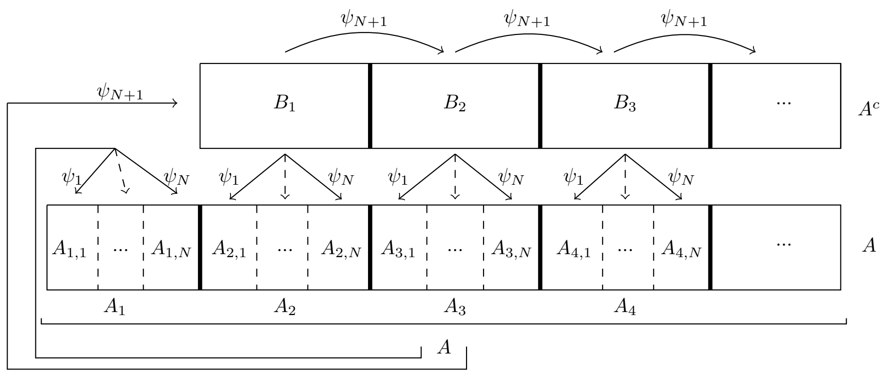


Figure 2: How hitting-time sets move under ψ_j, $j = 1, ..., N+1$.

The *entropy of the transformation* T is defined by

$$h(T) = \sup h(T, \alpha), \tag{3.5}$$

where the supremum is taken over all finite partitions α.

Krengel was the first to extend the notion of entropy to infinite measure-preserving transformations (see [12]). He provided the following definition.

Definition 3.1 ([12])**.** Let $(X, \mathcal{B}, m, T)$ be a conservative σ-finite measure-preserving system. Let $A \in \mathcal{B}$ such that $0 < m(A) < \infty$. Define

$$h_{\mathrm{Kr}}(T) = \sup_{A} h(T_A, m|_A). \tag{3.6}$$

Krengel also proved the following theorem which provides a useful simplification of Definition 3.1 in the case where A is a sweep-out set.

Theorem 3.1 ([12])**.** *Let $(X, \mathcal{B}, m, T)$ be a conservative σ-finite measure-preserving system. If $A \in \mathcal{B}$ such that $0 < m(A) < \infty$, and A is a sweep-out set for T, then*

$$h_{\mathrm{Kr}}(T) = h(T_A, m|_A). \tag{3.7}$$

We note that Krengel's definition of entropy is equivalent to Abramov's formula for entropy in the finite measure-preserving case. Also, we have written $m|_A$, to emphasize that we are considering the measure, m, restricted to A (not normalized).

3.3 Other entropy definitions and quasi-finiteness

In addition to the Krengel entropy, two other definitions of entropy have been suggested for infinite measure-preserving transformations. In this section, we quickly recall these definitions and show that all three entropies coincide for conservative rational functions that preserve Lebesgue measure on $\mathbb{R}$.

In 1969 Parry provided a new extension of entropy to transformations preserving an infinite measure ([16]). Before stating Parry's definition, we need a few definitions. Let $(X, \mathcal{B}, m, T)$ be a measure-preserving system. Let $\mathcal{C}$ be a sub-σ-algebra of $\mathcal{B}$. If $f \in L^1(m)$, then $d\mu = f dm$ defines a measure such that $\mu(A) = \int_A f dm$. By the Radon-Nikodym Theorem there exists a function $E(f|\mathcal{C})$ such that

$$\int_C E(f|\mathcal{C}) dm = \int_C f dm \quad \text{for all } C \in \mathcal{C}. \tag{3.8}$$

For $A \in \mathcal{B}$ we define $m(A|\mathcal{C}) = E(\mathbb{1}_A|\mathcal{C})$. If $\alpha = \{a_i\}$ is a measurable partition of X, then we define the conditional information of α given $\mathcal{C}$ to be

$$I(\alpha|\mathcal{C}) = -\sum_{a_i \in \alpha} \log(m(a_i|\mathcal{C})) \cdot \mathbb{1}_{a_i}. \tag{3.9}$$

Finally, the conditional entropy of α given $\mathcal{C}$ is defined by

$$H(\alpha|\mathcal{C}) = \int_X I(\alpha|\mathcal{C}) dm. \tag{3.10}$$

Given a partition α we write $\hat{\alpha}$ to denote the σ-algebra generated by α. That is, elements of $\hat{\alpha}$ are unions of the atoms in α. For more information on the information function and conditional entropy see [16] or [17]. We now state Parry's definition of entropy for infinite measure-preserving transformations.

Definition 3.2 (Parry Entropy, [16])**.** Let $(X, \mathcal{B}, m, T)$ be a σ-finite measure-preserving system. The *Parry entropy* of T is defined by

$$h_{\mathrm{Pa}}(T) = \sup\{H(\alpha|\widehat{T^{-1}\alpha})\}, \tag{3.11}$$

where the supremum is taken over all measurable partitions α such that $\hat{\alpha}$ is σ-finite and $T^{-1}\alpha \leq \alpha$.

More recently, another definition of entropy for infinite measure-preserving transformations has been suggested. The Poisson suspension, $(X^*, \mathcal{B}^*, m^*, T_*)$, of a system preserving a σ-finite measure, $(X, \mathcal{B}, m, T)$, is a method of associating a probability-preserving transformation to a possibly infinite measure-preserving system. We have that $(X^*, \mathcal{B}^*, m^*, T_*)$ is a point process in which identical particles propagate according to T, do not interact with one another, and the expected number of particles in each set $E \in \mathcal{B}$ is determined (in a Poisson manner) by $m(E)$. A formal description of the Poisson suspension is given in [21] and [11].

Definition 3.3 (Poisson Entropy, [21])**.** The *Poisson entropy* of an infinite measure-preserving transformation is defined as the Kolmogorov entropy of the Poisson suspension. That is, $h_{\mathrm{Po}}(T) = h(T_*)$.

The three definitions of entropy for infinite measure-preserving transformations coincide in the case when T is a quasi-finite transformation ([16], [11]).

Definition 3.4 (Quasi-Finite, [12])**.** Suppose $(X, \mathcal{B}, m, T)$ is a conservative measure-preserving system. The map T is called *quasi-finite* if there exists a sweep-out set $A \in \mathcal{B}$ with $m(A) < \infty$ such that the first return time partition, $\mathfrak{A} = \{A_k\}$, has finite entropy.

A related property is called log lower bounded, and the following definition can be found in [4].

Definition 3.5. Given a conservative, ergodic, infinite measure-preserving system, $(X, \mathcal{B}, m, T)$, we set

$$\mathcal{F}_{\log} = \left\{ A \in \mathcal{B} : 0 < m(A) < \infty \text{ and } \int_A \log(\phi_A) dm < \infty \right\}. \tag{3.12}$$

The transformation T is called *log-lower bounded* (LLB) if $\mathcal{F}_{\log} \neq \emptyset$.

The following Lemma is stated as a remark in [4], and the details of the proof are outlined in a slightly different context in [1].

Lemma 3.2. *If T is log lower bounded, then T is quasi-finite.*

Aaronson and Park proved the following theorem which provides an equivalence between LLB transformations and pointwise dual ergodic transformations with a specific condition on the return sequence.

Theorem 3.3 ([4])**.** *If T is a conservative, pointwise dual ergodic, infinite measure-preserving transformation of $(X, \mathcal{B}, m)$, then*

$$T \text{ is LLB} \iff \sum_{n=1}^{\infty} \frac{1}{n a_n(T)} < \infty. \tag{3.13}$$

Theorem 3.4. *All conservative rational functions that preserve Lebesgue measure are log-lower bounded (and quasi-finite). Thus,*

$$h_{\mathrm{Kr}}(R) = h_{\mathrm{Pa}}(R) = h_{\mathrm{Po}}(R). \tag{3.14}$$

Proof. The results in [2] combined with the proof of Theorem A part (3) show $a_n(R) \sim c\sqrt{n}$, where $c \in \mathbb{R}$ and $c > 0$. Thus, we apply Theorem 3.3 and obtain that R is LLB, and therefore R is quasi-finite by Lemma 3.2. Thus, the three entropy definitions coincide by the results of [16] and [11]. □

3.4 Entropy formula

Before proving Theorem B we give a little motivation and history for the integral formula. The following definition can be found in [22].

Definition 3.6. Let $I = [a, b]$ be a closed interval in $\mathbb{R}$. Let $\mathfrak{T}_{Ren}(I)$ denote the class of all transformations $T : I \to I$ such that there exists a partition into subintervals $\{I_j : j \in J\}$ satisfying the following properties:

1. (piecewise differentiable and surjective) $T|_{I_j}$ is C^2 and $\overline{T(I_j)} = I$ for all j. Each I_j contains exactly one fixed point of T.
2. (expanding) There exists a $\rho > 1$ such that $|T'(x)| \geq \rho$ for all $x \in I_j$.
3. (Adler's condition) $\left|\frac{T''(x)}{T'(x)^2}\right|$ is bounded on $\bigcup_{j \in J} I_j$.

If $T \in \mathfrak{T}_{Ren}(I)$, then T satisfies Renyi's condition, and T preserves an absolutely continuous finite measure, μ ([19], [22]). Furthermore, we can compute the entropy of T via the following formula.

Theorem 3.5 (Rohlin's Formula, [20])**.** *Let $I = [a, b]$ be a closed interval of $\mathbb{R}$. If $T \in \mathfrak{T}_{Ren}(I)$ and μ is invariant for T, then*

$$h(T) = \int_I \log |T'(x)| d\mu(x). \tag{3.15}$$

Lemma 3.6. *Suppose R is a conservative rational function that preserves λ on $\mathbb{R}$ and let $A = [q_1, q_{N+1})$. Then the induced transformation, $R_A \in \mathfrak{T}_{Ren}(A)$, and*

$$h(R_A) = \int_A \log |R_A'(x)| d\lambda|_A(x). \tag{3.16}$$

Proof. We want to show that R_A satisfies (1)-(3) of Definition 3.6. Consider the partition $\{A_{k,i}\}$ defined above. To show (1) we note that if $x \in A_{k,i}$, then $R_A = R^k$, so $R_A : A_{k,i} \to A$ is one-to-one and onto. Furthermore, R is piecewise smooth on $\mathbb{R}$, so R_A is C^2 on each $A_{k,i}$. To show (2) we note,

$$|R'(x)| = 1 + \sum_{i=1}^{N} \frac{p_i}{(t_i - x)^2}. \tag{3.17}$$

We have $|R'(x)| > 1$ for all $x \in \mathbb{R}$, but $|R'(x)| \to 1$ as $x \to \infty$. The set $A = [q_1, q_{N+1})$, however, is bounded away from ∞. Therefore, there exists a constant $\rho > 1$ such that $\inf_{x \in A} |R'(x)| \geq \rho$, so by the chain rule $|R'_A(x)| \geq \rho > 1$ for all $x \in A$. Finally, (3) was shown in [14] for the case when $R = G$ and $\beta = 0$. We will modify the argument to show (3) in the case when $R = S$. Let $x \in A_{k,i}$. The chain rule yields

$$\left| \frac{(S)''_A(x)}{((S)'_A(x))^2} \right| \leq \sum_{j=1}^{k} \left| \frac{(S)''((S)^{k-j}(x))}{((S)'((S)^{k-j}(x)))^2} \right|. \tag{3.18}$$

A calculation shows $|(S)''(y)((S)'(y))^{-2}|$ is bounded and decreases for large $|y|$ satisfying $|(S)''(y)((S)'(y))^{-2}| \leq M|y|^{-3}$. Since $x \in A_k$, we know $(S)^{k-j}(x) \in B_j$ (as in (3.2)). From our study of A in (2.11), we have the following two cases:

1. If j is even, then $B_j = (V_{(j/2)-1}, V_{j/2}) \cup \{W_{j/2}\}$.
2. If j is odd, then $B_j = (W_{\lceil j/2 \rceil}, W_{\lfloor j/2 \rfloor}) \cup \{V_{\lfloor j/2 \rfloor}\}$.

Therefore, $(S)^{k-j}(x) \in [V_{\lfloor j/2 \rfloor - 1}, V_{\lceil j/2 \rceil}] \cup [W_{\lceil j/2 \rceil}, W_{\lfloor j/2 \rfloor}]$. Starting with the right-hand side of (3.18) we have

$$\begin{aligned} \sum_{j=1}^{k} \left| \frac{(S)''((S)^{k-j}(x))}{((S)'((S)^{k-j}(x)))^2} \right| &\leq M \sum_{j=1}^{k} |W_{\lfloor j/2 \rfloor}|^{-3} + |V_{\lfloor j/2 \rfloor - 1}|^{-3} \\ &\leq M \sum_{j=1}^{k} \frac{1}{c_2^3(\lfloor j/2 \rfloor)^{3/2}} + \frac{1}{c_1^3(\lfloor j/2 \rfloor - 1)^{3/2}}, \end{aligned} \tag{3.19}$$

where the second line comes from the proof of Theorem A part (2). We see that the limit as $k \to \infty$ of (3.19) is finite.

Therefore $R_A \in \mathfrak{T}_{Ren}(A)$, and the integral formula follows from Theorem 3.5. □

We are now ready to prove Theorem B.

PROOF OF THEOREM B. By the Theorem 3.1 and Lemma 3.6 we have

$$h_{\mathrm{Kr}}(R) = h(R_A) = \int_A \log |R'_A(x)| d\lambda|_A(x). \tag{3.20}$$

By Theorem 10.6 in [16] we have

$$\int_A \log |R'_A(x)| d\lambda|_A(x) \leq \int_{\mathbb{R}} \log |R'(x)| d\lambda(x). \tag{3.21}$$

Now, by the definition of Parry entropy and Lemma 10.5 in [16] it follows that

$$\int_{\mathbb{R}} \log |R'(x)| d\lambda(x) \leq h_{\mathrm{Pa}}(R). \tag{3.22}$$

We already proved that $h_{\mathrm{Kr}}(R) = h_{\mathrm{Pa}}(R)$ in Theorem 3.4. Therefore,

$$h_{\mathrm{Kr}}(R) = \int_{\mathbb{R}} \log |R'(x)| d\lambda(x). \tag{3.23}$$

□

Acknowledgment. These results form a part of the author's PhD thesis which was completed at The University of North Carolina, Chapel Hill. The author would like to thank her advisor, Professor Jane M. Hawkins, for her guidance on this project. The author would also like to thank the anonymous referees for their helpful suggestions to improve this paper.

References

[1] J. Aaronson, *Rational ergodicity and a metric invariant for Markov shifts*, Israel J. Math., **27** (1977), 93–123.

[2] J. Aaronson, *Ergodic theory for inner functions of the upper half plane*, Ann. Inst. H. Poincaré Sect. B (N.S.), **14(3)** (1978), 233–253.

[3] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, American Mathematical Society, 1997.

[4] J. Aaronson and K. K. Park, *Predictability, entropy and information of infinite transformations*, Fund. Math., **206** (2009), 1–21.

[5] R. L. Adler and B. Weiss, *The ergodic infinite measure preserving transformation of Boole*, Israel J. Math., **16** (1973), 263–278.

[6] D. Blackmore, J. Golenia, A. K. Prykarpatsky, and Y. A. Prykarpatsky, *Invariant measures for discrete dynamical systems and ergodic properties of generalized Boole-type transformations*, Ukrainskyi Matematychnyi Zhurnal, **65(1)** (2013), 44–57.

[7] G. Boole, *On the comparison of transcendence with certain applications to the theory of definite integrals*, Philos. Trans. Roy. Soc. London, **147(3)** (1857), 745–803.

[8] H. Bruin, M. Nicol, and D. Terhesiu, *On young towers associated with infinite measure preserving transformations.*, Stoch. Dyn., **9** (2009), 625–655.

[9] H. Hu and S. Vaienti, *Absolutely continuous invariant measure for non-uniformly expanding maps*, Ergodic Theory Dynam. Systems, **29** (2009), 1185–1215.

[10] W. Hurewicz, *Ergodic theorem without invariant measure*, Ann. of Math., **45(2)** (1944), 192–206.

[11] É. Janvresse, T. Meyerovitch, E. Roy, and T. de la Rue, *Poisson suspensions and entropy for infinite transformations*, Trans. Amer. Math. Soc., **362(6)** (2010), 3069–3094.

[12] U. Krengel, *Entropy of conservative transformations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, **7** (1967), 161–181.

[13] G. Letac, *Which functions preserve Cauchy laws?*, Proc. Amer. Math. Soc., **67(2)** (1977, 277–286.

[14] T. Y. Li and F. Schweiger, *The generalized Boole's transformation is ergodic*, Manuscripta Math., **25(2)** (1978), 161–167.

[15] D. Maharam, *Incompressible transformations*, Fund. Math, **56** (1964), 35–50.

[16] W. Parry, *E*ntropy and Generators in Ergodic Theory, W.A. Benjamin, Inc., New York, NY, 1969.

[17] K. E. Petersen, *E*rgodic Theory, Cambridge University Press, New York, NY, 1983.

[18] G. Pólya and G. Szegő, *P*roblems and Theorems in Analysis I. Reprint of the 1978 English translation. English Translation by D. Aeppli. Springer-Verlag, 1998.

[19] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar, **8** (1957), 477–493.

[20] V. A. Rohlin, *Exact endomorphisms of a lebesgue space*, Amer. Math. Soc. Trans., **36(1)** (1964), 1–36.

[21] E. Roy, *Poisson suspensions and infinite ergodic theory*, Ergodic Theory Dynam. Systems, **29(2)** (2009), 667–683.

[22] M. Thaler, *Transformations on* [0, 1] *with infinite invariant measures*, Israel J. Math., **46(1)** (1983), 67–96.

[23] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, Springer-Verlag, New York, NY, 1982.

[24] R. Zweimüller, *Ergodic properties of infinite measure-preserving interval maps with indifferent fixed points*, Ergodic Theory Dynam. Systems, **20(5)** (2000), 1519–1549.

[25] R. Zweimüller, *Measure preserving transformations similar to markov shifts*, Israel J. Math., **173** (2009), 421–443.