PLENARY LECTURE

Real Analysis Exchange
Vol. 43(1), 2017, pp. 43–50
DOI: 10.14321/realanalexch.41.1.0043

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SOME RESULTS ABOUT BIG AND LITTLE LIP

Abstract

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. We examine the relationship between the so-called "big Lip" and "little lip" functions: $\mathrm{Lip}\ f$ and $\mathrm{lip}\ f$.

1 Introduction

Throughout this note we will assume that f is a continuous, real-valued function defined on $\mathbb{R}$. Recall that f is Lipschitz (on $\mathbb{R}$) if there exists $M > 0$ such that $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$. If there is such a constant M, we will say that f is M-Lipschitz. Lipschitz functions are rather well behaved:

Theorem 1.1 (Rademacher, 1919)**.** *If f is Lipschitz, then f is differentiable a.e. on $\mathbb{R}$.*

One can weaken the Lipschitz assumption in Rademacher's Theorem and still reach the same conclusion by requiring that f satisfy a local Lipschitz condition. For this we need the so-called "big Lip" function defined as follows:

$$\mathrm{Lip}\ f(x) = \limsup_{r \to 0^+} \frac{M_f(x,r)}{r},$$

where

$$M_f(x,r) = \sup\{|f(x) - f(y)| \colon |x - y| \le r\}.$$

Then it is not hard to show the following:

Mathematical Reviews subject classification: Primary: 26A27
Key words: Lipschitz conditions, nondifferentiability
Received by the editors November 10, 2017
Communicated by: Paul D. Humke

Basic Result: f is M-Lipschitz if and only if Lip $f(x) \leq M$ for all $x \in \mathbb{R}$.

More interesting is the following generalization of Rademacher's Theorem:

Theorem 1.2 (Rademacher-Stepanov, 1923)**.** *Suppose that f is continuous on $\mathbb{R}$. Then f is differentiable a.e. on $L_f = \{x \in \mathbb{R}\colon Lip\ f(x) < \infty\}$.*

In particular, if Lip $f(x) < \infty$ for all $x \in \mathbb{R}$, then f is differentiable a.e. on $\mathbb{R}$.

If we replace the lim sup in the definition of Lip f with a lim inf, we get the so-called "little lip" function:

$$\text{lip } f(x) = \liminf_{r \to 0^+} \frac{M_f(x,r)}{r}.$$

The Basic Result above remains true with Lip f replaced with lip f. On the other hand, the Rademacher-Stepanov theorem fails spectacularly if we replace L_f with $l_f = \{x \in \mathbb{R}\colon \text{lip } f(x) < \infty\}$ as the following result shows:

Theorem 1.3 (Balogh and Csörnyei, 2006, [1])**.**

1. *There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ with lip $f(x) = 0$ a.e. on $\mathbb{R}$, but such that f is nowhere differentiable on $\mathbb{R}$.*
2. *There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a set $A \subset \mathbb{R}$ of positive measure such that lip $f(x) < \infty$ for all $x \in \mathbb{R}$, but f is nowhere differentiable on A.*

It is possible to make the exceptional set $E = \{\text{lip } f(x) \neq 0\}$ in part (1) quite small:

Theorem 1.4 (Hanson, 2012,[4])**.** *There exists a set $S \subset \mathbb{R}$ of Hausdorff dimension 0 and a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that lip $f(x) = 0$ for all $x \in \mathbb{R} \backslash S$ and f is nowhere differentiable on $\mathbb{R}$.*

In both Theorem 1.3(1) and Theorem 1.4 the function f is constructed so that Lip $f(x) = \infty$ for all $x \in \mathbb{R}$. This highlights the fact that Lip f and lip f can behave very differently. Off of a small exceptional set S we have Lip $f = \infty$ and lip $f = 0$.

However, it is not possible to construct a function f such that Lip $f(x) = \infty$ and lip $f(x) < \infty$ for <u>all</u> $x \in \mathbb{R}$. This follows from the following result, which allows us to recover a version of the Rademacher-Stepanov Theorem involving the little lip function.

Theorem 1.5 (Balogh and Csörnyei, 2006,[1]).

1. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\mathbb{R} \backslash l_f$ is countable, then f is differentiable on a set of positive measure.*

2. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $\mathbb{R} \backslash l_f$ is countable and lip f is locally integrable, then f is differentiable a.e. on $\mathbb{R}$.*

The integrability condition in part (2) of this result is quite sharp. For example, in part (2) of Theorem 1.3 the construction can be carried out so that f is locally in L^p for every $p < 1$.

2 Characterizing N_f

In this section we consider the problem of characterizing sets of non-differentiability for functions f with Lip f (or lip f) finite everywhere on $\mathbb{R}$. To streamline the exposition we introduce the following notation: We define

$$N_f = \{x\colon f \text{ is not differentiable at } x\},$$

and let Lip $\mathbb{R} = \{f\colon L_f = \mathbb{R}\}$ and lip $\mathbb{R} = \{f\colon l_f = \mathbb{R}\}$.

We would like to characterize N_f for functions in Lip $\mathbb{R}$ and lip $\mathbb{R}$. In the case of Lip $\mathbb{R}$ the work has essentially been accomplished by Zahorski, who proved the following beautiful result:

Theorem 2.1 (Zahorski, 1942, [10]).

1. *$E = N_f$ for some continuous $f : \mathbb{R} \to \mathbb{R}$ if and only if $E = E_1 \cup E_2$, where E_1 is G_δ, E_2 is $G_{\delta\sigma}$, and $|E_2| = 0$.*

2. *$E = N_f$ for some Lipschitz $f : \mathbb{R} \to \mathbb{R}$ if and only if $|E| = 0$ and E is $G_{\delta\sigma}$.*

Note: The set E is G_δ if E can be written as the intersection of countably many open sets. A $G_{\delta\sigma}$ set is a countable union of G_δ sets. Furthermore, we use $|E|$ to denote the Lebesgue measure of E.

Using the Rademacher-Stepanov Theorem we can reframe part (2) of Zahorski's Theorem as follows:

Theorem 2.2. *$E = N_f$ for some $f \in$ Lip $\mathbb{R}$ if and only if $|E| = 0$ and E is $G_{\delta\sigma}$.*

Characterizing N_f for functions in lip $\mathbb{R}$ appears to be more difficult. Assume for the moment that $f \in \text{lip}\,\mathbb{R}$. Then f is continuous so by part (1) of Zahorski's Theorem it follows that $N_f = E_1 \cup E_2$, where E_1 is G_δ, E_2 is $G_{\delta\sigma}$ and $|E_2| = 0$. Additionally, it follows from the proof of part (1) of Theorem 1.4 that $|N_f \cap (a,b)| < b - a$ for all $(a,b) \subset \mathbb{R}$. We introduce the following:

Definition 2.3. *A subset E of $\mathbb{R}$ is trim if $|E \cap (a,b)| < b - a$ for all open intervals (a,b).*

Based on our work so far we can make the following conjecture:

Conjecture 1: $E = N_f$ for some $f \in \text{lip}\,\mathbb{R}$ if and only if $E = E_1 \cup E_2$ where E_1 is trim G_δ, E_2 is $G_{\delta\sigma}$ and $|E_2| = 0$.

Of course, the forward direction of the conjecture has been established. In the other direction the following is known:

Theorem 2.4 (Hanson, 2016,[5])**.**

1. *If E is closed and nowhere dense, then there exists $f \in$ lip $\mathbb{R}$ such that $N_f = E$.*
2. *If E is trim and G_δ, then there exists $f \in$ lip $\mathbb{R}$ such that $|E \triangle N_f| = 0$.*

3 Characterizing L_f and l_f

Another interesting problem to consider is that of characterizing L_f and l_f for continuous functions f. As with the case of characterizing N_f, this problem is more straightforward for L_f. For this case we have the following:

Theorem 3.1. *$E = L_f$ for some continuous function $f : \mathbb{R} \to \mathbb{R}$ if and only if E is F_σ.*

Proving the forward direction of this result is a straightford exercise. The reverse direction follows easily from a result of Piranian ([9]).

Moving on to the problem of characterizing l_f, it is easy to show that if f is continuous, then l_f is a $G_{\delta\sigma}$ set, which leads to the following natural conjecture:

Conjecture 2: $E = l_f$ for some continuous f if and only if E is a $G_{\delta\sigma}$ set.

A partial result in this direction is the following:

Theorem 3.2 (Buczolich, Hanson, Rmoutil, Zürcher [2])**.** *If E is either F_σ or G_δ, then there exists continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $l_f = E$.*

The proof of the G_δ case in this theorem is already quite involved. It appears to be quite challenging to extend the proof to the $G_{\delta\sigma}$ case.

4 Connections between lip f and quasiconformal mappings

There is a nice connection between big and little lip and the theory of quasiconformal functions. In this section we assume that Ω and Ω' are open, connected subsets of $\mathbb{R}^n$ with $n \geq 2$ and $f : \Omega \to \Omega'$ is an orientation preserving homeomorphism. Let $||Df||$ denote the operator norm of the matrix of partial derivatives of f and J_f the determinant of this matrix. Then the analytic definition of a quasiconformal mapping is the following:

Definition 4.1. *f is quasiconformal on Ω if $f \in W^{1,n}_{loc}(\Omega)$ and there is a constant $K \geq 1$ such that $||Df(x)||^n \leq KJ_f(x)$ a.e. on Ω.*

Quasiconformal functions can also be characterized using a more geometric approach. Define

$$H_f(x) = \limsup_{r\to 0^+} \frac{M_f(x,r)}{m_f(x,r)},$$

where $M_f(x,r) = \sup_{|x-y|=r} |f(x) - f(y)|$ and $m_f(x,r) = \inf_{|x-y|=r} |f(x) - f(y)|$.

The function H_f is known as the linear dilatation of f. The following classical result relates H_f to the analytic definition of quasiconfomality:

Theorem 4.2 (Gehring, 1960)**.** *f is quasiconformal on Ω if and only if there exists $K \geq 1$ such that $H_f(x) \leq K$ for all $x \in \Omega$.*

A few years later Gehring showed that the hypotheses on H_f can be weakened a bit and still give the same conclusion:

Theorem 4.3 (Gehring, 1962,[3])**.** *Suppose that $S \subset \Omega$ with σ-finite $n-1$ dimensional measure, $H_f(x) < \infty$ for all $x \in \Omega \backslash S$ and there is a $K < \infty$ such that $H_f(x) \leq K$ a.e. on Ω. Then f is quasiconformal on Ω.*

Taking the same approach with H_f as we did with Lip f we define a "lim inf" version of the linear dilatation as follows:

$$h_f(x) = \liminf_{r\to 0^+} \frac{M_f(x,r)}{m_f(x,r)}.$$

Another way to weaken the hypotheses in Theorem 4.6 is by replacing H_f with h_f. In [6] Heinonen and Koskela showed, surprisingly enough, that doing so does not affect the conclusion:

Theorem 4.4 (Heinonen, Koskela 1995)**.** *f is quasiconformal on Ω if and only if $h_f(x) \leq K < \infty$ for all $x \in \Omega$.*

More recently Kallunki and Koskela, [7] showed that Theorem 4.3 is also true with H_f replaced by h_f:

Theorem 4.5 (Kallunki, Koskela, 2000)**.** *Suppose that S is a subset of Ω with σ-finite $n-1$ dimensional measure, $h_f(x) < \infty$ for all $x \in \Omega \backslash S$ and there is a $K < \infty$ such that $h_f(x) \leq K$ a.e. on Ω. Then f is quasiconformal on Ω.*

The last two theorems give the impression that h_f and H_f are interchangeable. However, this changes when we consider the following:

Theorem 4.6. *If $H_f(x) < \infty$ a.e. on Ω, then f is differentiable a.e. on Ω.*

This result is an easy consequence of the Rademacher Stepanov Theorem and the Lebesgue Differentiation Theorem. A proof of it can be found in ([8]). Because of its dependence on the Rademacher-Stepanov Theorem, it may not be surprising to learn that Theorem 4.6 is not true if we replace H_f with h_f:

Theorem 4.7 (Hanson,2012,[4])**.** *Let $n \geq 2$. There exists a homeomorphism $g : (0,1)^n \to \mathbb{R}^n$ and a set $S \subset (0,1)^n$ such that*

1. *$dim_{\mathcal{H}}(S) \leq n-1$*
2. *$h_g(x) = 1$ for all $x \in (0,1)^n \backslash S$*
3. *$H_g(x) = \infty$ for all $x \in (0,1)^n$*
4. *g is nowhere differentiable.*

Theorem 4.7 follows directly from Theorem 1.4 by defining $g(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{n-1}, x_n + f(x_1))$, where f is constructed as in Theorem 1.4. In the opposite direction the following result is an analogue of Theorem 1.5 (part (2)).

Theorem 4.8 (Kallunki, Koskela, 2000)**.** *Suppose that $n = 2$, $h_f(x) < \infty$ for all $x \in \Omega \backslash S$, where S has σ-finite length and $h_f \in L^2_{loc}(\Omega)$. Then f is differentiable a.e. on Ω.*

In examining the proof of Theorem 4.8 it seems that there is good evidence that the integrability condition on h_f can be weakened, leading to the following conjecture:

Conjecture 3: Theorem 4.8 remains true if we replace the assumption $h_f \in L^2_{loc}(\Omega)$ with $h_f \in L^1_{loc}(\Omega)$.

5 Additional Questions

In addition to Conjectures 1-3, there are many interesting questions concerning the big and little lip functions and their relationship to each other. A few of them are listed below. We use the notation $l_f^\infty = \mathbb{R}\backslash l_f$ and $L_f^\infty = \mathbb{R}\backslash L_f$.

Q1: Is it possible to characterize l_f and L_f for monotone functions?

Q2: For which pairs of sets $\{E, G\}$ does there exist f such that $l_f = E$ and $L_f = G$?

Q3: Assume E is a G_δ set. We know that there is a continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $E = L_f^\infty = l_f^\infty$. Does there exist $f\colon \mathbb{R} \to \mathbb{R}$ such that $E = L_f^\infty = l_f^\infty = N_f$?

Q4: Given a G_δ set E of measure zero, there is a continuous, monotone function $f\colon \mathbb{R} \to \mathbb{R}$ such that $E = l_f^\infty$.

1. Can we require that $E = N_f$ as well?
2. If we do not require monotonicity, is $E = N_f = l_f^\infty$ (in case the above should fail) or even $E = N_f = l_f^\infty = L_f^\infty$ possible?

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