PLENARY LECTURE

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LINEAR SUBSPACES OF HYPERCYCLIC VECTORS

Abstract

In my talk I presented results from [5, 6, 7] on the existence of hypercyclic algebras for convolution operators acting on the space of entire functions.

The search for large algebraic structures (e.g., dense linear spaces, closed subspaces of infinite dimension, or infinitely generated algebras) in non-linear settings has drawn increasing interest throughout various fields [1, 4]. This has been the case, in particular, for Linear Dynamics, ever since the conception of the notion of a *hypercyclic vector.* Recall that the *Invariant Subset Problem* asks whether every operator on a (separable, infinite-dimensional) Hilbert space has a non-trivial closed invariant subset; that is, a set that is neither $\{0\}$ nor the whole space and is mapped into itself by the operator. C. Read constructed an operator on ℓ_1 without non-trivial closed sets, showing that the Invariant Subset Problem has a negative answer in the general setting of Banach spaces ([9], [10]). The notion of a hypercyclic vector arose naturally in this study of invariant subsets. An operator T on a topological vector space X is said to be *hypercyclic* provided for some vector $f \in X$ its orbit

$$\mathrm{Orb}\{f, T\} = \{f, Tf, T^2 f, \dots\}$$

is dense in X. Such f is called a *hypercyclic vector* for T. In this way, an operator T lacks non-trivial closed invariant subsets if and only if *all* non-zero vectors in X are hypercyclic for T. Hence the search for large linear structures

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within the set $HC(T)$ of hypercyclic vectors for a given operator T has been an important task in Linear Dynamics.

The first examples of hypercyclic operators have been of convolution operators on the space $X = H(\mathbb{C})$ of entire functions on the complex plane, endowed with the compact-open topology. By a convolution operator here we mean an operator that commutes with each translation τ_a, $a \in \mathbb{C}$ given by

$$\tau_a(f)(z) = f(z+a) \quad (f \in H(\mathbb{C}), z \in \mathbb{C}).$$

G. D. Birkhoff showed in 1929 that each τ_a $(a \neq 0)$ is hypercyclic on $H(\mathbb{C})$ and two decades later MacLane showed the hypercyclicity of the operator D of complex differentiation. Both results were beautifully unified in 1991 by Godefroy and Shapiro, who noted that each entire function $\Phi(z) = \sum a_n z^n \in H(\mathbb{C})$ of exponential type (i.e., satisfying $|\Phi(z)| \leq Ae^{B|z|}$ for all $z \in \mathbb{C}$, for some scalars A and B) induces an operator

$$\Phi(D) : H(\mathbb{C}) \to H(\mathbb{C}),\ \Phi(D)f = \sum a_n D^n f.$$

So for instance, $\tau_a = \Phi(D)$ for $\Phi(z) = e^{az}$, while $D = \Phi(D)$ for $\Phi(z) = z$. They showed that each convolution operator is of this form, and that it is hypercyclic unless it is a scalar multiple of the identity.

We should mention that in general the set $HC(T)$ is either empty or contains a dense, T-invariant linear subspace. And in the latter case the set $HC(T)$ sometimes contains (but zero) linear subspaces that are both closed and infinite-dimensional, such as when T is a convolution operator, but not always, see [3, 8].

The search for hypercyclic algebras for convolution operators on $H(\mathbb{C})$ started with the work of Aron et al [2], who showed in 2007 that no translation operator τ_a on $H(\mathbb{C})$ can support a hypercyclic algebra, in a very strong way: for any integer $p > 1$ and any $f \in H(\mathbb{C})$, the power f^p cannot be hypercyclic for τ_a. Indeed, by Hurwicz' theorem and since τ_a is multiplicative it follows that the multiplicity of each root of each non-constant function in the closure of the orbit $\mathrm{Orb}(f^p, \tau_a)$ must be an integer multiple of p. MacLane's operator D, they showed, behaves very differently: the collection of entire functions f for which every power f^n $(n = 1, 2, \dots)$ is hypercyclic for the operator D of complex differentiation is residual in $H(\mathbb{C})$!

A few years later Shkarin [11] showed that $HC(D)$ contains both a hypercyclic subspace and a hypercyclic algebra, and with a different approach Bayart and Matheron [3] also showed that the set of entire functions that generate a hypercyclic algebra for D is residual in $H(\mathbb{C})$. From Ansari's theorem and the León-Müller theorem it follows that $T = \lambda D^n$ supports a hypercyclic algebra for each positive integer n and each unimodular scalar λ, so it was natural to ask whether any other convolution operators support hypercycic algebras. Following the approach by Bayart and Matheron, we showed in [5] that $P(D)$ supports a hypercyclic algebra whenever P is a non-constant polynomial vanishing at zero. The proof no longer works if $P(0) \neq 0$ or if P is not a polynomial.

The results above prompted several questions seeking to understand the essential reasons why a translation $\tau_a = \Phi(D)$ with $\Phi(z) = e^{az}$ failed to support a hypercyclic algebra. Was it because in this case $\Phi(0) \neq 0$? Was it because Φ is not a polynomial, or that it has exponential growth order one? Was it because of τ_a being multiplicative, or being a composition operator? In [6] we answered the above questions in the negative with an approach that exploits the rich supply of eigenvalues that the hypercyclic convolution operators on $H(\mathbb{C})$ enjoy, showing many new ones supporting hypercyclic algebras. Such is the case for $T = aI + D$ whenever the scalar a is in the closed unit disc, for $T = \sin(D)$ and for $T = e^D - aI$ with $0 < a \leq 1$ (so in particular τ_1 is the uniform limit on bounded subsets of $H(\mathbb{C})$ of operators supporting hypercyclic algebras). More generally, we showed the existence of hypercyclic algebras for convolution operators $\Phi(D)$ whose symbol Φ satisfies the following geometric condition: *the level set* $\{z : \ |\Phi(z)| = 1\}$ *contains a non-trivial strictly convex arc* Γ *for which the convex hull of the set* $\Gamma \cup \{0\}$*, except* $\Gamma \cup \{0\}$*, is mapped by* Φ *into the open unit disc.* In response to the last question, we showed that any translation τ_a $(a \neq 0)$ supports a hypercyclic algebra on the space $C^\infty(\mathbb{R}, \mathbb{C})$ of complex-valued smooth functions on the real line and endowed with the topology given by the seminorms

$$p_n(f) = \sup_{0 \leq k \leq n} \sup_{|x| \leq n} |f^{(k)}(x)| \ (f \in C^\infty(\mathbb{R}, \mathbb{C}), n \in \mathbb{N}),$$

noting first the following zero-one law for any hypercyclic multiplicativate operator T on a separable F-algebra X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$: either T supports no hypercyclic algebra or else each of its hypercyclic vectors generates a hypercyclic algebra, with the latter occuring precisely when for each non-constant polynomial P with coefficients in $\mathbb{K}$ and vanishing at zero the map $\widehat{P} : X \to X$, $f \mapsto P(f)$, has dense range. In particular, each translation τ_a with $0 \neq a \in \mathbb{R}$ is hypercyclic on $C^\infty(\mathbb{R}, \mathbb{R})$ but fails to support a hypercyclic algebra. All hypercyclic algebras obtained, as in the constructions by Shkarin and by Bayart and Matheron, were singly generated, and thus topologically small as they are contained in the union of two hyperplanes of $H(\mathbb{C})$. This further motivated the following question, first raised by Aron [3, p. 217], also in [4, 1]:

Does D support a hypercyclic algebra that is not finitely generated?

In [7] we provide a positive answer:

Theorem (J.B.-D.Papathanasiou)**.** *The MacLane operator D supports a dense hypercyclic algebra that is not finitely generated. Moreover,*

(a) *For each $N \in \mathbb{N}$ the set of $f = (f_j)_{j=1}^N$ in $H(\mathbb{C})^N$ that generate a hypercyclic algebra for D that is not contained in a hypercyclic algebra induced by k generators with $k < N$ is residual in $H(\mathbb{C})^N$, and*

(b) *The set of $f = (f_j)_{j=1}^\infty \in H(\mathbb{C})^{\mathbb{N}}$ that generate a dense hypercyclic algebra for D that is not contained in a finitely generated hypercyclic algebra for D is residual in $H(\mathbb{C})^{\mathbb{N}}$.*

We also established that the same is true for those convolution operators $\Phi(D)$ whose symbol Φ satisfies the geometric condition on the level set $\{z : |\Phi(z)| = 1\}$ mentioned above. And when a polynomial P satisfies this geometric condition the operator $P(\frac{d}{dx})$ supports a dense hypercyclic algebra on $C^\infty(\mathbb{R},\mathbb{C})$ that is not finitely generated, where $\frac{d}{dx}$ denotes real differentiation. Finally, we showed that many F-algebras X such as $C^\infty(\mathbb{R},\mathbb{C})$ support the following $0-1$ law: either no weakly mixing multiplicative operator supports a hypercyclic algebra, or else every weakly mixing multiplicative operator supports a dense, non-finitely generated hypercyclic algebra. In particular, every non-trivial translation on $C^\infty(\mathbb{R},\mathbb{C})$ supports a hypercyclic algebra that is not finitely generated.

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