Tepper L. Gill, Departments of Electrical & Computer Engineering and Mathematics, Howard University, Washington DC 20059 U.S.A.
email: `tgill@howard.edu`

BANACH SPACES FOR THE SCHWARTZ DISTRIBUTIONS

Abstract

This paper is a survey of a new family of Banach spaces $\mathcal{B}$ that provide the same structure for the Henstock-Kurzweil (HK) integrable functions as the L^p spaces provide for the Lebesgue integrable functions. These spaces also contain the wide sense Denjoy integrable functions. They were first use to provide the foundations for the Feynman formulation of quantum mechanics. It has recently been observed that these spaces contain the test functions $\mathcal{D}$ as a continuous dense embedding. Thus, by the Hahn-Banach theorem, $\mathcal{D}' \subset \mathcal{B}'$.

A new family that extend the space of functions of bounded mean oscillation $BMO[\mathbb{R}^n]$, to include the HK-integrable functions are also introduced.

1   Introduction

Since the work of Henstock [11] and Kurzweil [19], the most important finitely additive measure is the one generated by the Henstock-Kurzweil integral (HK-integral). It generalizes the Lebesgue, Bochner and Pettis integrals. The HK-integral is equivalent to the Denjoy and Perron integrals. However, it is much easier to understand (and learn) compared to the these and the Lebesgue integral. It provides useful variants of the same theorems that have made the Lebesgue integral so important. We assume that the reader is acquainted with this integral, but more detail can be found in Gill and Zachary [5]. (For

different perspectives, see Gordon [6], Henstock [10], Kurzweil [19], or Pfeffer [28].)

The most important factor preventing the widespread use of the HK-integral in mathematics, engineering and physics is the lack of a Banach space structure comparable to the L^p spaces for the Lebesgue integral. The purpose of this paper is to provide a survey of some new classes of Banach spaces, which have this property and some with other interesting properties, but all contain the HK-integrable functions.

The first two classes are the KS^p and the SD^p spaces, $1 \leq p \leq \infty$. These are all separable spaces that contain the corresponding L^p spaces as dense, continuous, compact embeddings. We have recently discovered that these two classes also contain the test functions $\mathcal{D}$ as a continuous dense embedding. This implies the each dual space contains the Schwartz distributions. The family of SD^p spaces also have the remarkable property that $\|D^\alpha f\|_{SD} = \|f\|_{SD}$.

The other main classes of spaces $\mathcal{Z}^p$ and the $\mathcal{Z}^{-p}$ spaces, $1 \leq p \leq \infty$, are related to the space of functions of bounded mean oscillation, BMO. We also introduce an extended version of this space, which we call the space of functions of weak bounded mean oscillation, BMO^w.

We provide a few applications of the first two families of spaces, which either provide simpler solutions to old problems or solve open problems.

The main tool for the work in this paper had its beginnings in 1965, when Gross [9] proved that every real separable Banach space contains a separable Hilbert space as a continuous dense embedding, which is the support of a Gaussian measure. This was a generalization of Wiener's theory, which used the (densely embedded Hilbert) Sobolev space $\mathbb{H}^1_0[0,1] \subset \mathbb{C}_0[0,1]$. In 1972, Kuelbs [18] generalized Gross' theorem to include the Hilbert space rigging $\mathbb{H}^1_0[0,1] \subset \mathbb{C}_0[0,1] \subset L^2[0,1]$. For our purposes, a general version of this theorem can be stated as:

Theorem 1.1. (Gross-Kuelbs)*Let $\mathcal{B}$ be a separable Banach space. Then there exist separable Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and a positive trace class operator T_{12} defined on $\mathcal{H}_2$ such that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ all as continuous dense embeddings, with $\left(T_{12}^{1/2}u,\ T_{12}^{1/2}v\right)_1 = (u,\ v)_2$ and $\left(T_{12}^{-1/2}u,\ T_{12}^{-1/2}v\right)_2 = (u,\ v)_1$.*

A proof can be found in [5]. The space $\mathcal{H}_1$ is part of the abstract Wiener space method for extending Wiener measure to separable Banach spaces. The space $\mathcal{H}_2$ is a major tool in the construction Banach spaces for HK-integable functions. (We call it the natural Hilbert space for $\mathcal{B}$.)

1.1   Summary

In Section 2, we establish a number of background results to make the paper self contained. The major result is Corollary 2.8 (from Theorem 2.7). It allows us to construct the path integral in the manner originally suggested by Feynman. In Section 3, after a few examples, we construct the KS-spaces and derive some of their important properties. In Section 4, we construct the SD-spaces and discuss their properties. In Section 5 we discuss the family of spaces related to the functions of bounded mean oscillation.

In Section 6, we give a few applications. The first application uses KS^2 to provide a simple solution to the generator (with unbounded coefficients) problem for Markov processes. The second application uses KS^2 and Theorem 2.7 to construct the Feynman path integral. The third application uses SD^2 to provide the best possible a priori bound for the nonlinear term of the Navier-Stokes equation.

2   Background

In this section, we provide some background results, which are required in the paper.

2.1   Banach Space Basics

Let $\mathcal{B}$ be a Banach space, with dual $\mathcal{B}^*$.

Definition 2.1. *The sequence $(u_n)_{n=1}^{\infty} \subset \mathcal{B}$ is called a Schauder basis (S-basis) for $\mathcal{B}$ if $\|u_n\|_{\mathcal{B}} = 1$ and, for each $u \in \mathcal{B}$, there is a unique sequence (a_n) of scalars such that*

$$u = \lim_{n \to \infty} \sum_{k=1}^{n} a_k u_k = \sum_{k=1}^{\infty} a_k u_k.$$

For example, if $\mathcal{B} = L^p[0,1]$, $1 < p < \infty$, the family of vectors

$$\{1, \cos(2\pi t), \sin(2\pi t) \cos(4\pi t), \sin(4\pi t), \dots \}$$

is a S-basis for $\mathcal{B}$.

Definition 2.2. *A duality map $\mathcal{J} : \mathcal{B} \mapsto \mathcal{B}^*$, is a set*

$$\mathcal{J}(u) = \left\{ u^* \in \mathcal{B}^* \,\Big|\, \langle u, u^* \rangle = \|u\|_{\mathcal{B}}^2 = \|u^*\|_{\mathcal{B}'}^2 \right\}, \ \forall u \in \mathcal{B}.$$

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. If $u \in L^p[\Omega] = \mathcal{B}$, $1 < p < \infty$, then the standard example is

$$u^* = \mathcal{J}(u)(x) = \|u\|_p^{2-p} |u(x)|^{p-2} u(x) \in L^q[\Omega], \; \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

Furthermore,

$$\langle u, u^* \rangle = \|u\|_p^{2-p} \int_\Omega |u(x)|^p \, d\lambda_n(x) = \|u\|_p^2 = \|u^*\|_q^2 \tag{2.1}$$

It is easy to see that every Banach space with an S-basis is separable.

2.2   Bounded Operator Extension

We now consider the problem of operator extensions from $\mathcal{B}$ to $\mathcal{H}(= \mathcal{H}_2)$. It is not hard to see that, since $\mathcal{B}$ is a continuous dense embedding in $\mathcal{H}$, every closed densely defined linear operator on $\mathcal{B}$ has a closed densely defined extension to $\mathcal{H}$, $\mathcal{C}[\mathcal{B}] \xrightarrow{\text{ext}} \mathcal{C}[\mathcal{H}]$ (see Theorem 2.7 for a proof). In this section, we show that this also holds for bounded linear operators, $L[\mathcal{B}] \xrightarrow{\text{ext}} L[\mathcal{H}]$. This important result depends on the following theorem by Lax [20]. It is not well known, so we include a proof.

Theorem 2.3. (Lax's Theorem) *Let $\mathcal{B}$ be a separable Banach space continuously and densely embedded in a Hilbert space $\mathcal{H}$ and let T be a bounded linear operator on $\mathcal{B}$ which is symmetric with respect to the inner product of $\mathcal{H}$ (i.e., $(Tu, v)_\mathcal{H} = (u, Tv)_\mathcal{H}$ for all $u, v \in \mathcal{B}$). Then:*

1. *The operator T is bounded with respect to the $\mathcal{H}$ norm and*

   $$\|T^*T\|_\mathcal{H} = \|T\|_\mathcal{H}^2 \leqslant k \|T\|_\mathcal{B}^2,$$

   where k is a positive constant.

2. *The spectrum of T relative to $\mathcal{H}$ is a subset of the spectrum of T relative to $\mathcal{B}$.*

3. *The point spectrum of T relative to $\mathcal{H}$ is a equal to the point spectrum of T relative to $\mathcal{B}$.*

PROOF. To prove (1), let $u \in \mathcal{B}$ and, without loss, we can assume that $k = 1$ and $\|u\|_\mathcal{H} = 1$. Since T is selfadjoint,

$$\|Tu\|_\mathcal{H}^2 = (Tu, Tu) = \left(u, T^2 u\right) \leqslant \|u\|_\mathcal{H} \left\|T^2 u\right\|_\mathcal{H} = \left\|T^2 u\right\|_\mathcal{H}.$$

Thus, we have $\|Tu\|_{\mathcal{H}}^4 \leqslant \|T^4 u\|_{\mathcal{H}}$, so it is easy to see that $\|Tu\|_{\mathcal{H}}^{2n} \leqslant \|T^{2n} u\|_{\mathcal{H}}$ for all n. It follows that:

$$\|Tu\|_{\mathcal{H}} \leqslant (\|T^{2n} u\|_{\mathcal{H}})^{1/2n} \leqslant (\|T^{2n} u\|_{\mathcal{B}})^{1/2n}$$
$$\leqslant (\|T^{2n}\|_{\mathcal{B}})^{1/2n} (\|u\|_{\mathcal{B}})^{1/2n} \leqslant \|T\|_{\mathcal{B}} (\|u\|_{\mathcal{B}})^{1/2n}.$$

Letting $n \to \infty$, we get that $\|Tu\|_{\mathcal{H}} \leqslant \|T\|_{\mathcal{B}}$ for u in a dense set of the unit ball of $\mathcal{H}$. It follows that

$$\|T\|_{\mathcal{H}} = \sup_{\|u\|_{\mathcal{H}} \leqslant 1} \|Tu\|_{\mathcal{H}} \leqslant \|T\|_{\mathcal{B}}.$$

To prove (2), suppose $\lambda_0 \notin \sigma_T$, the spectrum of T over $\mathcal{B}$ so that $T - \lambda_0 I$ has a bounded inverse S on $\mathcal{B}$. Since T is symmetric on $\mathcal{H}$, so is S. It follows from (1) that T and S extend to bounded linear operators $\bar{T}$ and $\bar{S}$ on $\mathcal{H}$. We also see that $\lambda_0 \notin \sigma_{\bar{T}}$. It follows from this that $\bar{T}$ has inverse $\bar{S}$ and the spectrum of $\bar{T}$ on $\mathcal{H}$ is a subset of the spectrum of T on $\mathcal{B}$ (i.e., $\sigma_{\bar{T}} \subset \sigma_T$).

To prove (3), suppose that $\lambda \in \sigma_p$, the point spectrum of T, so that $T - \lambda I$ has a finite dimensional null space N and $dimN = dim\{\mathcal{B}/J\}$, where $J = (T - \lambda I)(\mathcal{B})$.

Since T is symmetric, every vector in J is orthogonal to N. Conversely, from $dimN = dim\{\mathcal{B}/J\}$ we see that J contains all vectors that are orthogonal to N. It follows that, $(T - \lambda I)$ is a one-to-one, onto mapping of $J \to J$, so that $T - \lambda I = S$ has an inverse on J, which is bounded (on J) by the Closed Graph Theorem. It follows that the extension $\bar{S}$ of S to the closure of $J, \bar{J}$ in $\mathcal{H}$ is bounded on $\bar{J}$. This means that $(\bar{T} - \lambda I)$ is the orthogonal compliment of N over $\mathcal{H}$, so that λ belongs to the point spectrum of $\bar{T}$ on $\mathcal{H}$ and the null space of $(\bar{T} - \lambda I)$ over $\mathcal{H}$ is N. It follows that the point spectrum of T is unchanged on extension to $\mathcal{H}$. $\qquad\square$

Let $\mathcal{H}$ be the natural Hilbert space for $\mathcal{B}$, let $\mathbf{J}$ be the standard linear mapping from $\mathcal{H} \to \mathcal{H}^*$ and let $\mathbf{J}_{\mathcal{B}}$ be its restriction to $\mathcal{B}$. Since $\mathcal{B}$ is a continuous dense embedding in $\mathcal{H}$, $\mathbf{J}_{\mathcal{B}}$ is a (conjugate) isometric isomorphism of $\mathcal{B}$ onto $\mathbf{J}_{\mathcal{B}}(\mathcal{B}) \subset \mathcal{B}^*$.

Definition 2.4. *If $u \in \mathcal{B}$, set $u_h = \mathbf{J}_{\mathcal{B}}(u)$ and define*

$$\mathcal{B}_h^* = \{u_h \in \mathcal{B}^* : u \in \mathcal{B}\},$$

so that $\langle u, u_h \rangle = (u, u)_{\mathcal{H}} = \|u\|_{\mathcal{H}}^2$. It is clear from our construction of $\mathcal{B}_h^*$ that the mapping taking $\mathcal{B} \to \mathcal{B}_h^*$ is a (conjugate) isometric isomorphism. We call $\mathcal{B}_h^*$ the h-representation for $\mathcal{B}$ in $\mathcal{B}^*$.

Its easy to prove that following result.

Theorem 2.5. *If $\mathcal{B}$ is a reflexive Banach space. Then $\mathcal{B}_h^*$ is bijectively related to $\mathcal{B}^*$.*

Remark 2.6. *In general, the embedding of $\mathcal{B}_h^*$ is a proper subspace of $\mathcal{B}^*$.*

We can now state and prove the following fundamental theorem.

Theorem 2.7. *Let $\mathcal{B}$ be a separable Banach space. If $A \in \mathcal{C}[\mathcal{B}]$, then there is a unique operator $A^* \in \mathcal{C}[\mathcal{B}]$ satisfying:*

1. $(aA)^* = \bar{a}A^*$,

2. $A^{**} = A$,

3. $(A^* + B^*) = A^* + B^*$,

4. $(AB)^* = B^* A^*$ *on* $D(A^*) \bigcap D(B^*)$,

5. $(A^* A)^* = A^* A$ *on* $D(A^* A)$ *(self adjoint)*,

6. *if* $A \in L[\mathcal{B}]$, *then* $\|A^* A\|_{\mathcal{H}} \leq k \|A^* A\|_{\mathcal{B}}$ *and*

7. *if* $A \in L[\mathcal{B}]$, *then* $\|A^* A\|_{\mathcal{B}} \leq c \|A\|_{\mathcal{B}}^2$, *for some constant c.*

PROOF. Recall that $\mathbf{J}$ is the natural linear mapping from $\mathcal{H}_2 = \mathcal{H} \to \mathcal{H}^*$ and $\mathbf{J}_{\mathcal{B}}$ is the restriction of $\mathbf{J}$ to $\mathcal{B}$, so that $\mathbf{J}_{\mathcal{B}}(\mathcal{B}) = \mathcal{B}_{\mathbf{h}}^*$. If $A \in \mathcal{C}[\mathcal{B}]$, then $A' : \mathcal{B}^* \to \mathcal{B}^*$ and $A' \mathbf{J}_{\mathcal{B}} : \mathcal{B} \to \mathcal{H}^*$. Since $\mathcal{B}$ is dense in $\mathcal{H}$, $\mathcal{B}_h^*$ is dense in $\mathcal{H}^*$. It follows that $A' \mathbf{J}_{\mathcal{B}}$ is a closed densely define operator on $\mathcal{H}^*$, $\mathbf{J}_{\mathcal{B}}^{-1} A' \mathbf{J}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is a closed and densely defined linear operator on $\mathcal{B}$. We define $A^* = [\mathbf{J}_{\mathcal{B}}^{-1} A' \mathbf{J}_{\mathcal{B}}] \in \mathcal{C}[\mathcal{B}]$. If $A \in L[\mathcal{B}]$, $A^* = \mathbf{J}_{\mathcal{B}}^{-1} A' \mathbf{J}_{\mathcal{B}}$ is defined on all of $\mathcal{B}$ so that, by the Closed Graph Theorem, $A^* \in L[\mathcal{B}]$. The proofs of (1)-(3) are straight forward. To prove (4), let $u \in D(A^*) \bigcap D(B^*)$, then

$$
\begin{aligned}
(BA)^* u &= [\mathbf{J}_{\mathcal{B}}^{-1} (BA)' \mathbf{J}_{\mathcal{B}}] u = [\mathbf{J}_{\mathcal{B}}^{-1} A' B' \mathbf{J}_{\mathcal{B}}] u \\
&= \left[\mathbf{J}_{\mathcal{B}}^{-1} A' \mathbf{J}_{\mathcal{B}} \right] \left[\mathbf{J}_{\mathcal{B}}^{-1} B' \mathbf{J}_{\mathcal{B}} \right] u = A^* B^* u.
\end{aligned}
\tag{2.2}
$$

If we replace B by A^* in equation (2.2), noting that $A^{**} = A$, we also see that $(A^* A)^* = A^* A$, proving (5). To prove (6), we first see that:

$$
\langle A^* Av, \mathbf{J}_{\mathcal{B}}(u) \rangle = \langle A^* Av, u_h \rangle = (A^* Av, u)_{\mathcal{H}} = (v, A^* Au)_{\mathcal{H}},
$$

so that $A^* A$ is symmetric. Thus, by Theorem 2.3 (Lax), $A^* A$ has a bounded extension to $\mathcal{H}$ and $\|A^* A\|_{\mathcal{H}} = \|A\|_{\mathcal{H}}^2 \leqslant k \|A^* A\|_{\mathcal{B}}$, where k is a positive constant. We also have that

$$
\|A^* A\|_{\mathcal{B}} \leqslant \|A^*\|_{\mathcal{B}} \|A\|_{\mathcal{B}} \leqslant \|\mathbf{J}_{\mathcal{B}}\|_{\mathcal{B}^*} \|\mathbf{J}_{\mathcal{B}}^{-1}\|_{\mathcal{B}} \|A'\|_{\mathcal{B}^*} \|A\|_{\mathcal{B}} = c \|A\|_{\mathcal{B}}^2, \tag{2.3}
$$

proving (7). It also follows that

$$\|A\|_{\mathcal{H}} \leq \sqrt{ck}\, \|A\|_{\mathcal{B}}.\tag{2.4}$$

$\square$

If $c = 1$ and equality holds in (2.3) for all $A \in L[\mathcal{B}]$, then $L[\mathcal{B}]$ is a C^*-algebra. In this case, $\mathcal{B}$ is a Hilbert space. Thus, in general the inequality in (2.3) is strict. From (2.4), we see the following.

Corollary 2.8. *Let $\mathcal{B}$ be a separable Banach space. If $A \in L[\mathcal{B}]$, then there is a unique operator $\bar{A} \in L[\mathcal{H}]$ (i.e., $L[\mathcal{B}] \xrightarrow{ext} L[\mathcal{H}]$).*

Theorem 2.9. (Polar Representation) *Let $\mathcal{B}$ be a separable Banach space. If $A \in \mathbb{C}[\mathcal{B}]$, then there exists a partial isometry U and a self-adjoint operator T, $T = T^*$, with $D(T) = D(A)$ and $A = UT$.*

PROOF. Let $\bar{A}$ be the closed densely defined extension of A to $\mathcal{H}$. On $\mathcal{H}$, $\bar{T}^2 = \bar{A}^* \bar{A}$ is self-adjoint and there exist a unique partial isometry $\bar{U}$, with $\bar{A} = \bar{U}\bar{T}$. Thus, the restriction to $\mathcal{B}$ gives us $A = UT$ and U is a partial isometry on $\mathcal{B}$. (It is easy to check that $A^*A = T^2$.) $\square$

3   The Kuelbs-Steadman KS^p Spaces

3.1   Special Constructions

Our first construction is based on an extension of a norm due to Alexiewicz [2]. we first recall that the HK-integral is equivalent to the strict Denjoy integral (see Henstock [10] or Pfeffer [28]). In the one-dimensional case, Alexiewicz [2] has shown that the class $D(\mathbb{R})$, of Denjoy integrable functions, can be normed in the following manner: for $f \in D(\mathbb{R})$, define $\|f\|_D$ by

$$\|f\|_{A_1} = \sup_s \left| \int_{-\infty}^s f(r)d\lambda(r) \right|.$$

It is clear that this is a norm, and it is known that $D(\mathbb{R})$ is not complete (see Alexiewicz [2]). If we replace $\mathbb{R}$ by $\mathbb{R}^n$, let $[a_i, b_i] \subset \bar{\mathbb{R}} = [-\infty, \infty]$, and define $[\mathbf{a}, \mathbf{b}] \in \bar{\mathbb{R}}^n$ by $[\mathbf{a}, \mathbf{b}] = \prod_{k=1}^n [a_i, b_i]$. If $f \in D(\mathbb{R}^n)$ we define the norm of f by (see [25]):

$$\|f\|_{A_n} = \sup_{x_1 \cdots x_n} \left| \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(\mathbf{y})d\lambda_n(\mathbf{y}) \right|.\tag{3.1}$$

Lemma 3.1. *If $HK[\mathbb{R}^n]$ is the completion of $D(\mathbb{R}^n)$ in the above norm, then $L^1_{Loc}[\mathbb{R}^n] \subset HK[\mathbb{R}^n]$, as a continuous embedding.*

The completion of $\mathbb{C}_0[\mathbb{R}]$ in the L^1 norm leads to all absolutely integrable functions (i.e., the norm limit of L^1 functions is an L^1 function). The completion of $\mathbb{C}_0[\mathbb{R}]$ in the Alexiewicz norm leads to $HK[\mathbb{R}^n]$, but the class of HK-integrable functions is a proper subset of $HK[\mathbb{R}^n]$. This is the case for all suggested norms and Hönig has conjectured that there is no "natural norm" for the HK-integrable functions (see [12]). He suggests that the following is the best we can hope for:

Lemma 3.2. *(Hönig) Let $HK[a, b]$ be the space of HK-integrable functions on $[a, b] \subset \mathbb{R}$ and let $\mathbb{C}_a[a, b]$ be the continuous functions f on $[a, b]$, with $f(a) = 0$. If $\|\cdot\|_{A_1}$ is the Alexiewicz norm:*

$$\|f\|_{A_1} = \sup_s \left| \int_a^s f(r) d\lambda(r) \right|,$$

then the completion of $HK[a, b]$ is the set of all distributions that are weak derivatives of functions in $\mathbb{C}_a[a, b]$.

Talvila [31] independently obtained the same result for $\mathbb{R}$ and used it to motivate his definition of a distributional integral.

Definition 3.3. *If there is a continuous function $F(x)$ with real limits at infinity such that $F'(x) = f(x)$ (weak derivative), then the distributional integral of $f(x)$ is defined to be $D \int_{-\infty}^{\infty} f(x) dx = F(\infty) - F(-\infty)$.*

Talvila shows that the Alexiewicz norm leads to a Banach space of integrable distributions that is isometrically isomorphic to the space of continuous functions on the extended real line with uniform norm. He also shows that the dual space can be identified with the space of functions of bounded variation.

The following two spaces are also closely related to $HK[\mathbb{R}^n]$.

Theorem 3.4. *Let $\{u_i\}_{i=1}^{\infty} \subset \mathbb{C}_0^1[\mathbb{R}^n]$ be S-basis for $\mathcal{B} = L^1[\mathbb{R}^n]$ or an orthonormal basis for $\mathcal{H} = L^2[\mathbb{R}^n]$.*

1. *Then, for $L^1[\mathbb{R}^n]$,*

$$\|f\|_{A_n^1} = \sup_i \left| \int_{\mathbb{R}^n} f(\mathbf{x}) u_i(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|,$$

defines a weaker norm on $L^1[\mathbb{R}^n]$ and $f \in L^1[\mathbb{R}^n] \Rightarrow \|f\|_{A_n^1} \leq \|f\|_1$, and

2. If $\mathcal{B}_{A_n^1}$ is the completion of $\mathcal{B}$ in this norm, then $L_{Loc}^1[\mathbb{R}^n] \subset \mathcal{B}_{A_n^1}$, as a continuous embedding.

3. For $L^2[\mathbb{R}^n]$

$$\|f\|_{A_n^2} = \sup_i \left| \int_{\mathbb{R}^n} f(\mathbf{x}) u_i(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|,$$

defines a weaker norm on $L^2[\mathbb{R}^n]$ and $f \in L^2[\mathbb{R}^n] \Rightarrow \|f\|_{A_n^2} \leq \|f\|_2$.

4. If $\mathcal{H}_{A_n^2}$ is the completion of $\mathcal{H}$ in this norm, then $L_{Loc}^2[\mathbb{R}^n] \subset \mathcal{H}_{A_n^2}$, as a continuous embedding.

Remark 3.5. *The last two spaces are close but not the same. A proof is actually required to show that $\mathcal{B}_{A_n^1}$ and $\mathcal{H}_{A_n^2}$ contains the HK-integrable functions. The condition $\{u_i\}_{i=1}^\infty \subset \mathbb{C}_0^1[\mathbb{R}^n]$ is sufficient and the proof is the same as for Theorem 4.9 (see Section 4.3).*

3.2 General Construction

For our first general construction, fix n, and let $\mathbb{Q}^n$ be the set $\{\mathbf{x} = (x_1, x_2 \cdots, x_n) \in \mathbb{R}^n\}$ such that x_i is rational for each i. Since this is a countable dense set in $\mathbb{R}^n$, we can arrange it as $\mathbb{Q}^n = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots\}$. For each l and i, let $\mathbf{B}_l(\mathbf{x}_i)$ be the closed cube centered at $\mathbf{x}_i$, with sides parallel to the coordinate axes and edge $e_l = \frac{1}{2^l \sqrt{n}}, l \in \mathbb{N}$. Now choose the natural order which maps $\mathbb{N} \times \mathbb{N}$ bijectively to $\mathbb{N}$, and let $\{\mathbf{B}_k, \ k \in \mathbb{N}\}$ be the resulting set of (all) closed cubes $\{\mathbf{B}_l(\mathbf{x}_i) \ |(l, i) \in \mathbb{N} \times \mathbb{N}\}$ centered at a point in $\mathbb{Q}^n$. Let $\mathcal{E}_k(\mathbf{x})$ be the characteristic function of $\mathbf{B}_k$, so that $\mathcal{E}_k(\mathbf{x})$ is in $L^p[\mathbb{R}^n] \cap L^\infty[\mathbb{R}^n]$ for $1 \leq p < \infty$. Define $F_k(\,\cdot\,)$ on $L^1[\mathbb{R}^n]$ by

$$F_k(f) = \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_n(\mathbf{x}). \tag{3.2}$$

It is clear that $F_k(\,\cdot\,)$ is a bounded linear functional on $L^p[\mathbb{R}^n]$ for each k, $\|F_k\|_\infty \leq 1$ and, if $F_k(f) = 0$ for all k, $f = 0$ so that $\{F_k\}$ is fundamental on $L^p[\mathbb{R}^n]$ for $1 \leq p \leq \infty$. Fix $t_k > 0$ such that $\sum_{k=1}^\infty t_k = 1$ and define a measure $d\mathbf{P}(\mathbf{x}, \mathbf{y})$ on $\mathbb{R}^n \times \mathbb{R}^n$ by:

$$d\mathbf{P}(\mathbf{x}, \mathbf{y}) = \left[\sum_{k=1}^\infty t_k \mathcal{E}_k(\mathbf{x}) \mathcal{E}_k(\mathbf{y}) \right] d\lambda_n(\mathbf{x}) d\lambda_n(\mathbf{y}).$$

We first construct our Hilbert space. Define an inner product $(\,\cdot\,)$ on $L^1[\mathbb{R}^n]$ by

$$
\begin{aligned}
(f,g) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(\mathbf{x})\overline{g(\mathbf{y})}d\mathbf{P}(\mathbf{x},\mathbf{y}) \\
&= \sum_{k=1}^{\infty} t_k \left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\lambda_n(\mathbf{x}) \right] \overline{\left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{y})g(\mathbf{y})d\lambda_n(\mathbf{y}) \right]}.
\end{aligned}
\tag{3.3}
$$

We use a particular choice of t_k in Gill and Zachary [5], which is suggested by physical analysis in another context. We call the completion of $L^1[\mathbb{R}^n]$, with the above inner product, the Kuelbs-Steadman space, $KS^2[\mathbb{R}^n]$. Following suggestions of Gill and Zachary, Steadman [30] constructed this space by adapting an approach developed by Kuelbs [18] for other purposes. Her interest was in showing that $L^1[\mathbb{R}^n]$ can be densely and continuously embedded in a Hilbert space which contains the HK-integrable functions. To see that this is the case, let $f \in D[\mathbb{R}^n]$, then:

$$
\|f\|_{KS^2}^2 = \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\lambda_n(\mathbf{x}) \right|^2 \leqslant \sup_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\lambda_n(\mathbf{x}) \right|^2 \leqslant \|f\|_D^2 \,,
$$

so $f \in KS^2[\mathbb{R}^n]$.

Theorem 3.6. *For each p, $1 \leqslant p \leqslant \infty$, $KS^2[\mathbb{R}^n] \supset L^p[\mathbb{R}^n]$ as a dense subspace.*

PROOF. By construction, $KS^2[\mathbb{R}^n]$ contains $L^1[\mathbb{R}^n]$ densely, so we need only show that $KS^2[\mathbb{R}^n] \supset L^q[\mathbb{R}^n]$ for $q \neq 1$. If $f \in L^q[\mathbb{R}^n]$ and $q < \infty$, we have

$$
\begin{aligned}
\|f\|_{KS^2} &= \left[\sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\lambda_n(\mathbf{x}) \right|^{\frac{2q}{q}} \right]^{1/2} \\
&\leqslant \left[\sum_{k=1}^{\infty} t_k \left(\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) \left| f(\mathbf{x}) \right|^q d\lambda_n(\mathbf{x}) \right)^{\frac{2}{q}} \right]^{1/2} \\
&\leqslant \sup_k \left(\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) \left| f(\mathbf{x}) \right|^q d\lambda_n(\mathbf{x}) \right)^{\frac{1}{q}} \leqslant \|f\|_q \,.
\end{aligned}
$$

Hence, $f \in KS^2[\mathbb{R}^n]$. For $q = \infty$, first note that $vol(\mathbf{B}_k)^2 \leq \left[\frac{1}{2\sqrt{n}} \right]^{2n}$, so we

have

$$\|f\|_{KS^2} = \left[\sum_{k=1}^{\infty} t_k \left| \int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|^2 \right]^{1/2}$$

$$\leqslant \left[\left[\sum_{k=1}^{\infty} t_k [vol(\mathbf{B}_k)]^2 \right] [ess\sup |f|]^2 \right]^{1/2} \leqslant \left[\frac{1}{2\sqrt{n}}\right]^n \|f\|_{\infty}.$$

Thus $f \in KS^2[\mathbb{R}^n]$, and $L^{\infty}[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$. $\qquad\square$

Before proceeding to additional study, we construct the $KS^p[\mathbb{R}^n]$ spaces, for $1 \leq p \leq \infty$.

To construct $KS^p[\mathbb{R}^n]$ for all p and for $f \in L^p$, define:

$$\|f\|_{KS^p} = \begin{cases} \left\{ \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|^p \right\}^{1/p}, & 1 \leqslant p < \infty, \\ \sup_{k \geqslant 1} \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|, & p = \infty. \end{cases}$$

It is easy to see that $\|\cdot\|_{KS^p}$ defines a norm on L^p. If KS^p is the completion of L^p with respect to this norm, we have:

Theorem 3.7. *For each q, $1 \leqslant q \leqslant \infty$, $KS^p[\mathbb{R}^n] \supset L^q[\mathbb{R}^n]$ as a dense continuous embedding.*

PROOF. As in the previous theorem, by construction $KS^p[\mathbb{R}^n]$ contains $L^p[\mathbb{R}^n]$ densely, so we need only show that $KS^p[\mathbb{R}^n] \supset L^q[\mathbb{R}^n]$ for $q \neq p$. First, suppose that $p < \infty$. If $f \in L^q[\mathbb{R}^n]$ and $q < \infty$, we have

$$\|f\|_{KS^p} = \left[\sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|^{\frac{qp}{q}} \right]^{1/p}$$

$$\leqslant \left[\sum_{k=1}^{\infty} t_k \left(\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) |f(\mathbf{x})|^q d\lambda_n(\mathbf{x}) \right)^{\frac{p}{q}} \right]^{1/p}$$

$$\leqslant \sup_k \left(\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) |f(\mathbf{x})|^q d\lambda_n(\mathbf{x}) \right)^{\frac{1}{q}} \leqslant \|f\|_q.$$

Hence, $f \in KS^p[\mathbb{R}^n]$. For $q = \infty$, we have

$$\|f\|_{KS^p} = \left[\sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|^p \right]^{1/p}$$

$$\leqslant \left[\left[\sum_{k=1}^{\infty} t_k [vol(\mathbf{B}_k)]^p \right] [ess\sup |f|]^p \right]^{1/p} \leqslant M \|f\|_{\infty}.$$

Thus $f \in KS^p[\mathbb{R}^n]$, and $L^{\infty}[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n]$. The case $p = \infty$ is obvious. $\quad\square$

Theorem 3.8. *For KS^p, $1 \leq p \leq \infty$, we have:*

1. *If $f, g \in KS^p$, then $\|f + g\|_{KS^p} \leq \|f\|_{KS^p} + \|g\|_{KS^p}$ (Minkowski inequality).*

2. *If K is a weakly compact subset of L^p, it is a compact subset of KS^p.*

3. *If $1 < p < \infty$, then KS^p is uniformly convex.*

4. *If $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, then the dual space of KS^p is KS^q.*

5. *$KS^\infty \subset KS^p$, for $1 \leq p < \infty$.*

PROOF. The proof of (1) follows from the classical case for sums. The proof of (2) follows from the fact that, if $\{f_m\}$ is any weakly convergent sequence in K with limit f, then

$$\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) \left[f_m(\mathbf{x}) - f(\mathbf{x})\right] d\lambda_n(\mathbf{x}) \to 0$$

for each k. It follows that $\{f_m\}$ converges strongly to f in KS^p.

The proof of (3) follows from a modification of the proof of the Clarkson inequalities for l^p norms.

In order to prove (4), observe that, for $p \neq 2$, $1 < p < \infty$, the linear functional

$$L_g(f) = \|g\|_{KS^p}^{2-p} \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) g(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|^{p-2} \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{y}) f(\mathbf{y})^* d\lambda_n(\mathbf{y})$$

is a unique duality map on KS^q for each $g \in KS^p$ and that KS^p is reflexive from (3). To prove (5), note that $f \in KS^\infty$ implies that $\left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|$ is uniformly bounded for all k. It follows that $\left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|^p$ is uniformly bounded for each p, $1 \leq p < \infty$. It is now clear from the definition of KS^∞ that:

$$\left[\sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|^p \right]^{1/p} \leq \|f\|_{KS^\infty} < \infty$$

$\square$

Note that, since $L^1[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n]$ and $KS^p[\mathbb{R}^n]$ is reflexive for $1 < p < \infty$, we see that the second dual $\left\{L^1[\mathbb{R}^n]\right\}^{**} = \mathfrak{M}[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n]$. Recall that $\mathfrak{M}[\mathbb{R}^n]$ is the space of bounded finitely additive set functions defined on the Borel sets $\mathfrak{B}[\mathbb{R}^n]$.

In many applications, it is convenient to formulate problems on one of the standard Sobolev spaces $W^{m,p}(\mathbb{R}^n)$. We first recall that a function f such that $\int_K |f(\mathbf{x})|^p \, d\lambda_n(\mathbf{x}) < \infty$ for every compact set K in $\mathbb{R}^n$ is said to be in $L_{\text{loc}}^p(\mathbb{R}^n)$. We can easily see that $L_{\text{loc}}^p(\mathbb{R}^n) \subset KS^q(\mathbb{R}^n)$, $1 \leq q \leq \infty$, for all p, $1 \leq p \leq \infty$. This means that $KS^q(\mathbb{R}^n)$ contains a large class of distributions (see Adams [1]).

Theorem 3.9. *For each p, $1 \leq p \leq \infty$, the test functions $\mathcal{D} \subset KS^p(\mathbb{R}^n)$ as a continuous embedding.*

PROOF. Since $KS^\infty(\mathbb{R}^n)$ is continuously embedded in $KS^p(\mathbb{R}^n)$, $1 \leq q < \infty$, it suffices to prove the result for $KS^\infty(\mathbb{R}^n)$. Suppose that $\phi_j \to \phi$ in $\mathcal{D}[\mathbb{R}^n]$, so that there exist a compact set $K \subset \mathbb{R}^n$, containing the support of $\phi_j - \phi$ and $D^\alpha \phi_j$ converges to $D^\alpha \phi$ uniformly on K for every multi-index α. Let $L = \{l \in \mathbb{N} : \text{the support of } \mathcal{E}_l, \; stp\{\mathcal{E}_l\} \subset K\}$, then

$$\lim_{j \to \infty} \|D^\alpha \phi - D^\alpha \phi_j\|_{KS} = \lim_{j \to \infty} \sup_{l \in L} \left| \int_{\mathbb{R}^n} [D^\alpha \phi(x) - D^\alpha \phi_j(x)] \mathcal{E}_l(x) \, d\lambda_n(x) \right|$$

$$\leqslant \left[\frac{1}{2\sqrt{n}} \right]^n \lim_{j \to \infty} \sup_{x \in K} |D^\alpha \phi(x) - D^\alpha \phi_j(x)| = 0.$$

It follows that $\mathcal{D}[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n]$ as a continuous embedding, for $1 \leq p \leq \infty$. Thus, by the Hahn-Banach theorem, we see that the Schwartz distributions, $\mathcal{D}'[\mathbb{R}^n] \subset [KS^p(\mathbb{R}^n)]'$, for $1 \leq p \leq \infty$. □

We close this section with the following result that will be important later.

Lemma 3.10. *Let the Fourier transform, $\mathfrak{F}$ and the convolution operator, $\mathfrak{C}$ be defined on $L^1[\mathbb{R}^n]$. Then each has a bounded extension to the linear operators on $KS^2[\mathbf{R}^n]$.*

PROOF. From Corollary 2.8, every bounded linear operator on $L^1[\mathbb{R}^n]$ extends to a bounded linear operator on $KS^2[\mathbb{R}^n]$. The theorem applies to $\mathfrak{F}$ and $\mathfrak{C}$. □

Remark 3.11. *It should be noted that this theorem also implies (via the Closed Graph Theorem) that both operators have bounded extensions to all $L^p[\mathbf{R}^n]$ spaces for $1 \leq p \leq \infty$, as restrictions from $KS^2[\mathbb{R}^n]$. This is the first easy proof of the extension based on operator theory. The traditional proof for $\mathfrak{F}$ is obtained via rather deep methods of (advanced) real analysis. The identification of the range of $\mathfrak{F}$, for each p, still relies on traditional methods.*

4   The Jones SD^p Spaces

For our second class of spaces, we begin with the construction of a special class of functions in $\mathbb{C}_c^\infty[\mathbb{R}^n]$ (see Jones, [19] page 249).

4.1   The remarkable Jones functions

Definition 4.1. *For $x \in \mathbb{R}$, $0 \le y < \infty$ and $1 < a < \infty$, define the Jones functions $g(x,y)$, $h(x)$ by:*

$$g(x,y) = \exp\left\{-y^a e^{iax}\right\},$$

$$h(x) = \begin{cases} \displaystyle\int_0^\infty g(x,y)dy, & x \in \left(-\frac{\pi}{2a}, \frac{\pi}{2a}\right) \\[2mm] 0, & otherwise. \end{cases}$$

The following properties of g are easy to check:

1.
$$\frac{\partial g(x,y)}{\partial x} = -iay^a e^{iax} g(x,y),$$

2.
$$\frac{\partial g(x,y)}{\partial y} = -ay^{a-1} e^{iax} g(x,y),$$

so that

3.
$$iy\frac{\partial g(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial x}.$$

It is also easy to see that $h(x) \in L^1[-\frac{\pi}{2a}, \frac{\pi}{2a}]$ and,

$$\frac{dh(x)}{dx} = \int_0^\infty \frac{\partial g(x,y)}{\partial x}dy = \int_0^\infty iy\frac{\partial g(x,y)}{\partial y}dy. \tag{4.1}$$

Integration by parts in the last expression in (1.1) shows that $h'(x) = -ih(x)$, so that $h(x) = h(0)e^{-ix}$ for $x \in (-\frac{\pi}{2a}, \frac{\pi}{2a})$. Since $h(0) = \int_0^\infty \exp\{-y^a\}dy$, an additional integration by parts shows that $h(0) = \Gamma(\frac{1}{a} + 1)$. For each $k \in \mathbb{N}$ let $a = a_k = \pi 2^{k-1}$, $h(x) = h_k(x)$, $x \in (-\frac{1}{2^k}, \frac{1}{2^k})$ and set $\varepsilon_k = \frac{1}{2^{k+1}}$.

Let $\mathbb{Q}$ be the set of rational numbers in $\mathbb{R}$ and for each $x^i \in \mathbb{Q}$, define

$$f_k^i(x) = f_k(x - x^i) = \begin{cases} c_k \exp\left\{\frac{\varepsilon_k^2}{|x-x^i|^2 - \varepsilon_k^2}\right\}, & \left|x - x^i\right| < \varepsilon_k, \\ 0, & \left|x - x^i\right| \ge \varepsilon_k, \end{cases}$$

where c_k is the standard normalizing constant. It is clear that the support of f_k^i is

$$\mathrm{spt}(f_k^i) \subset [-\varepsilon_k, \varepsilon_k] = [-\tfrac{1}{2^{k+1}}, \tfrac{1}{2^{k+1}}] = I_k^i.$$

If we set $\chi_k^i(x) = (f_k^i * h_k)(x)$, its support is $\mathrm{spt}(\chi_k^i) \subset [-\tfrac{1}{2^{k+1}}, \tfrac{1}{2^{k+1}}]$. For $x \in \mathrm{spt}(\chi_k^i)$, we can also write $\chi_k^i(x) = \chi_k(x - x^i)$ as:

$$\chi_k^i(x)$$
$$= \int_{I_k^i} f_k \left[(x - x^i) - z \right] h_k(z) dz$$
$$= \int_{I_k^i} h_k \left[(x - x^i) - z \right] f_k(z) dz$$
$$= e^{-i(x - x^i)} \int_{I_k^i} e^{iz} f_k(z) dz.$$

Thus, if $\alpha_{k,i} = \int_{I_k^i} e^{iz} f_k^i(z) dz$, we can now define:

$$\xi_k^i(x) = \alpha_{k,i}^{-1} \overline{\chi}_k^i(x) = \left\{ \begin{array}{ll} \frac{1}{n} e^{i(x - x^i)}, & x \in I_k^i \\ 0, & x \notin I_k^i, \end{array} \right.$$

so that $\left| \xi_k^i(x) \right| < \frac{1}{n}$.

4.2   The Construction

To construct our space on $\mathbb{R}^n$, let $\mathbb{Q}^n$ be the set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$, such that for each j, $1 \le j \le n$, the component x_j is rational. Since this is a countable dense set in $\mathbb{R}^n$, we can arrange it as $\mathbb{Q}^n = \left\{ \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \cdots \right\}$. For each k and i, let $\mathbf{B}_k(\mathbf{x}^i)$ be the closed cube centered at $\mathbf{x}^i$ with edge $e_k = \frac{1}{2^k \sqrt{n}}$.

We choose the natural order which maps $\mathbb{N} \times \mathbb{N}$ bijectively to $\mathbb{N}$:

$$\{(1,1),\ (2,1),\ (1,2),\ (1,3),\ (2,2),\ (3,1),\ (3,2),\ (2,3),\ \ldots\}$$

and let $\{\mathbf{B}_m,\ m \in \mathbb{N}\}$ be the set of closed cubes $\mathbf{B}_k(\mathbf{x}^i)$ with $(k, i) \in \mathbb{N} \times \mathbb{N}$ and $\mathbf{x}^i \in \mathbb{Q}^n$. For each $\mathbf{x} \in \mathbf{B}_m$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, we define $\mathcal{E}_m(\mathbf{x})$ by :

$$\mathcal{E}_m(\mathbf{x}) = \left(\xi_k^i(x_1), \xi_k^i(x_2) \ldots \xi_k^i(x_n) \right).$$

It is easy to show that, for $m = (k, i)$,

$$|\mathcal{E}_m(\mathbf{x})| < 1, \quad \mathbf{x} \in \prod_{j=1}^{n} I_k^i,$$
$$\mathcal{E}_m(\mathbf{x}) = 0, \quad \mathbf{x} \notin \prod_{j=1}^{n} I_k^i.$$

It is also easy to see that $\mathcal{E}_m(\mathbf{x})$ is in $L^p[\mathbb{R}^n]^n = \mathbf{L}^p[\mathbb{R}^n]$ for $1 \le p \le \infty$. Define $F_m(\,\cdot\,)$ on $\mathbf{L}^p[\mathbb{R}^n]$ by

$$F_m(f) = \int_{\mathbb{R}^n} \mathcal{E}_m(\mathbf{x}) \cdot f(\mathbf{x}) d\lambda_n(\mathbf{x}).$$

It is clear that $F_m(\,\cdot\,)$ is a bounded linear functional on $\mathbf{L}^p[\mathbb{R}^n]$ for each m with $\|F_m\| \le 1$. Furthermore, if $F_m(f) = 0$ for all m, $f = 0$ so that $\{F_m\}$ is a fundamental sequence of functionals on $\mathbf{L}^p[\mathbb{R}^n]$ for $1 \le p \le \infty$.

Set $t_m = \frac{1}{2^m}$ so that $\sum_{m=1}^{\infty} t_m = 1$ and define a inner product $(\,\cdot\,)$ on $\mathbf{L}^1[\mathbb{R}^n]$ by

$$(f, g) = \sum_{m=1}^{\infty} t_m \left[\int_{\mathbb{R}^n} \mathcal{E}_m(\mathbf{x}) \cdot f(\mathbf{x}) d\lambda_n(\mathbf{x}) \right] \overline{\left[\int_{\mathbb{R}^n} \mathcal{E}_m(\mathbf{y}) \cdot g(\mathbf{y}) d\lambda_n(\mathbf{y}) \right]}.$$

The completion of $\mathbf{L}^1[\mathbb{R}^n]$ with the above inner product is a Hilbert space, which we denote as $SD^2[\mathbb{R}^n]$. For our next theorem, we recall that $\mathfrak{M}[\mathbb{R}^n]$ is the space of all (finite) complex measures on $\mathfrak{B}[\mathbb{R}^n]$ that are absolutely continuous with respect to Lebesgue measure λ_n and that, a sequence of measures $(\mu_j) \subset \mathfrak{M}[\mathbb{R}^n]$, converges weakly to a measure $\mu \in \mathfrak{M}[\mathbb{R}^n]$ if and only if, for every bounded continuous function h on $\mathbb{R}^n$, $h \in \mathbb{C}(\mathbb{R}^n)$, $\lim_{j \to \infty} \left| \int_{\mathbb{R}^n} h(\mathbf{x}) d\mu_j(\mathbf{x}) - \int_{\mathbb{R}^n} h(\mathbf{x}) d\mu(\mathbf{x}) \right| = 0$. The proofs of the following theorem are the same as for the $KS^p[\mathbb{R}^n]$ spaces, so we omit them.

Theorem 4.2. *For each p, $1 \le p \le \infty$, we have:*

1. *The space $SD^2[\mathbb{R}^n] \supset \mathbf{L}^p[\mathbb{R}^n]$ as a continuous, dense and* <u>*compact*</u> *embedding.*

2. *The space $SD^2[\mathbb{R}^n] \supset \mathfrak{M}[\mathbb{R}^n]$, the space of finitely additive measures on $\mathbb{R}^n$, as a continuous dense and* <u>*compact*</u> *embedding.*

Definition 4.3. *We call $SD^2[\mathbb{R}^n]$ the Jones strong distribution Hilbert space on $\mathbb{R}^n$.*

In order to justify our definition, let α be a multi-index of nonnegative integers, $\alpha = (\alpha_1,\ \alpha_2,\ \cdots \alpha_k)$, with $|\alpha| = \sum_{j=1}^{k} \alpha_j$. If D denotes the standard partial differential operator, let $D^\alpha = D^{\alpha_1} D^{\alpha_2} \cdots D^{\alpha_k}$.

Theorem 4.4. *Let $\mathcal{D}[\mathbb{R}^n]$ be $\mathbb{C}_c^\infty[\mathbb{R}^n]$ equipped with the standard locally convex topology (test functions).*

1. *If $\phi_j \to \phi$ in $\mathcal{D}[\mathbb{R}^n]$, then $\phi_j \to \phi$ in the norm topology of $SD^2[\mathbb{R}^n]$, so that $\mathcal{D}[\mathbb{R}^n] \subset SD^2[\mathbb{R}^n]$ as a continuous dense embedding.*

2. *If $T \in \mathcal{D}'[\mathbb{R}^n]$, then $T \in SD^2[\mathbb{R}^n]'$, so that $\mathcal{D}'[\mathbb{R}^n] \subset SD^2[\mathbb{R}^n]'$ as a continuous dense embedding.*

3. *For any f, $g \in SD^2[\mathbb{R}^n]$ and any multi-index α, $(D^\alpha f, g)_{SD} = (-i)^\alpha (f, g)_{SD}$.*

PROOF. The proofs of (1) and (2) are easy. To prove (3), we use the fact that each $\mathcal{E}_m \in \mathbb{C}_c^\infty[\mathbb{R}^n]$. Thus, for any $f \in SD^2[\mathbb{R}^n]$ we have:

$$\int_{\mathbb{R}^n} \mathcal{E}_m(\mathbf{x}) \cdot D^\alpha f(\mathbf{x}) d\lambda_n(\mathbf{x}) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} D^\alpha \mathcal{E}_m(\mathbf{x}) \cdot f(\mathbf{x}) d\lambda_n(\mathbf{x}).$$

An easy calculation shows that:

$$(-1)^{|\alpha|} \int_{\mathbb{R}^n} D^\alpha \mathcal{E}_m(\mathbf{x}) \cdot f(\mathbf{x}) d\lambda_n(\mathbf{x}) = (-i)^{|\alpha|} \int_{\mathbb{R}^n} \mathcal{E}_m(\mathbf{x}) \cdot f(\mathbf{x}) d\lambda_n(\mathbf{x}).$$

It now follows that, for any $\mathbf{g} \in SD^2[\mathbb{R}^n]$, $(D^\alpha f, \mathbf{g})_{SD^2} = (-i)^{|\alpha|}(f, \mathbf{g})_{SD^2}$.
□

4.3 Functions of Bounded Variation

The objective of this section is to show that every HK-integrable function is in $SD^2[\mathbb{R}^n]$. To do this, we need to discuss a certain class of functions of bounded variation. For functions defined on $\mathbb{R}$, the definition of bounded variation is unique. However, for functions on $\mathbb{R}^n$, $n \geq 2$, there are a number of distinct definitions.

The functions of bounded variation in the sense of Cesari are well known to analysts working in partial differential equations and geometric measure theory (see Leoni [21]).

Definition 4.5. *A function $f \in L^1[\mathbb{R}^n]$ is said to be of bounded variation in the sense of Cesari or $f \in BV_c[\mathbb{R}^n]$, if $f \in L^1[\mathbb{R}^n]$ and each i, $1 \leq i \leq n$, there exists a signed Radon measure μ_i, such that*

$$\int_{\mathbb{R}^n} f(\mathbf{x}) \frac{\partial \phi(\mathbf{x})}{\partial x_i} d\lambda_n(\mathbf{x}) = -\int_{\mathbb{R}^n} \phi(\mathbf{x}) d\mu_i(\mathbf{x}),$$

for all $\phi \in \mathbb{C}_0^\infty[\mathbb{R}^n]$.

The functions of bounded variation in the sense of Vitali [34], are well known to applied mathematicians and engineers with interest in error estimates associated with research in control theory, financial derivatives, high speed networks, robotics and in the calculation of certain integrals. (See, for

example [17], [24], [27] or [26] and references therein.) For the general definition, see Yeong ([34], p. 175). We present a definition that is sufficient for continuously differentiable functions.

Definition 4.6. *A function f with continuous partials is said to be of bounded variation in the sense of Vitali or $f \in BV_v[\mathbb{R}^n]$ if for all intervals (a_i, b_i), $1 \leq i \leq n$,*

$$V(f) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left| \frac{\partial^n f(\mathbf{x})}{\partial x_1 \partial x_2 \cdots \partial x_n} \right| d\lambda_n(\mathbf{x}) < \infty.$$

Definition 4.7. *We define $BV_{v,0}[\mathbb{R}^n]$ by:*

$$BV_{v,0}[\mathbb{R}^n] = \{f(\mathbf{x}) \in BV_v[\mathbb{R}^n] : f(\mathbf{x}) \to 0, \text{ as } x_i \to -\infty\},$$

where x_i is any component of $\mathbf{x}$.

The following two theorems may be found in [34]. (See p. 184 and 187, where the first is used to prove the second.) Recall that, if $[a_i, b_i] \subset \bar{\mathbb{R}} = [-\infty, \infty]$, we define $[\mathbf{a}, \mathbf{b}] \in \bar{\mathbb{R}}^n$ by $[\mathbf{a}, \mathbf{b}] = \prod_{k=1}^{n} [a_i, b_i]$. (The notation (RS) means Riemann-Stieltjes.)

Theorem 4.8. *Let f be HK-integrable on $[\mathbf{a}, \mathbf{b}]$ and let $g \in BV_{v,0}[\mathbb{R}^n]$, then fg is HK-integrable and*

$$(HK) \int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})g(\mathbf{x})d\lambda_n(\mathbf{x}) = (RS) \int_{[\mathbf{a},\mathbf{b}]} \left\{ (HK) \int_{[\mathbf{a},\mathbf{x}]} f(\mathbf{y})d\lambda_n(\mathbf{y}) \right\} dg(\mathbf{x})$$

Theorem 4.9. *Let f be HK-integrable on $[\mathbf{a}, \mathbf{b}]$ and let $g \in BV_{v,0}[\mathbb{R}^n]$, then fg is HK-integrable and*

$$\left| (HK) \int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})g(\mathbf{x})d\lambda_n(\mathbf{x}) \right| \leq \|f\|_{A_n} V_{[\mathbf{a},\mathbf{b}]}(g)$$

Lemma 4.10. *The space $HK[\mathbb{R}^n]$, of all HK-integrable functions is contained in $SD^2[\mathbb{R}^n]$.*

PROOF. Since each $\mathcal{E}_m(\mathbf{x})$ is continuous and differentiable, $\mathcal{E}_m(\mathbf{x}) \in BV_{v,0}[\mathbb{R}^n]$, so that for $f \in HK[\mathbb{R}^n]$,

$$\|f\|_{\mathbf{SD}^2}^2 = \sum_{m=1}^{\infty} t_m \left| \int_{\mathbb{R}^n} \mathcal{E}_m(\mathbf{x}) \cdot f(\mathbf{x})d\mathbf{x} \right|^2 \leqslant \sup_m \left| \int_{\mathbb{R}^n} \mathcal{E}_m(\mathbf{x}) \cdot f(\mathbf{x})d\mathbf{x} \right|^2$$

$$\leqslant \|f\|_{A_n}^2 [\sup_m V(\mathcal{E}_m)]^2 < \infty.$$

It follows that $f \in SD^2[\mathbb{R}^n]$. $\square$

4.4 The General Case, SD^p, $1 \leq p \leq \infty$

To construct $SD^p[\mathbb{R}^n]$ for all p and for $\mathbf{f} \in \mathbf{L}^p$, define:

$$\|\mathbf{f}\|_{SD^p} = \begin{cases} \left\{ \sum\limits_{m=1}^{\infty} t_m \left| \int_{\mathbb{R}^n} \mathcal{E}_m(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|^p \right\}^{1/p}, & 1 \leqslant p < \infty, \\ \sup\limits_{m \geqslant 1} \left| \int_{\mathbb{R}^n} \mathcal{E}_m(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|, & p = \infty. \end{cases}$$

It is easy to see that $\|\cdot\|_{SD^p}$ defines a norm on $\mathbf{L}^p$. If SD^p is the completion of $\mathbf{L}^p$ with respect to this norm, we have:

Theorem 4.11. *For each q, $1 \leqslant q \leqslant \infty$, $SD^p[\mathbb{R}^n] \supset \mathbf{L}^q[\mathbb{R}^n]$ as a continuous dense embedding.*

Theorem 4.12. *For SD^p, $1 \leq p \leq \infty$, we have:*

1. *If $p^{-1} + q^{-1} = 1$, then the dual space of $SD^p[\mathbb{R}^n]$ is $SD^q[\mathbb{R}^n]$.*

2. *For all $f \in SD^p[\mathbb{R}^n], g \in SD^q[\mathbb{R}^n]$ and all multi-index α, $\langle D^\alpha f, g \rangle = (-i)^{|\alpha|} \langle f, g \rangle$.*

3. *The test function space $\mathcal{D}[\mathbb{R}^n]$ is contain in $SD^p[\mathbb{R}^n]$ as a continuous dense embedding.*

4. *If K is a weakly compact subset of $\mathbf{L}^p[\mathbb{R}^n]$, it is a strongly compact subset of $SD^p[\mathbb{R}^n]$.*

5. *The space $SD^\infty[\mathbb{R}^n] \subset SD^p[\mathbb{R}^n]$.*

Remark 4.13. *The fact that the families $KS^p[\mathbb{R}^n]$ and $SD^p[\mathbb{R}^n]$ are separable and contain L^∞ as a compact embedding, makes it clear that the relationship between analysis and topology is not as straight forward as one would expect from past history. Thus, from an analysis point of view they are big, but from a topological point of view they are relatively small (separable).*

5   Zachary Spaces

In this section, we discuss one new space and two other families of spaces that naturally flow from the existence of a Banach space structure for functions with a bounded integral.

5.1   Functions of Bounded and Weak Bounded Mean Oscillation

We first define the class of functions of bounded mean oscillation (BMO) using the sharp maximal function ($M^{\#}$). In the following section, we define a weak maximal function (M^w) and use it construct the class of functions BMO^w, which extends BMO to include the functions with a bounded integral.

5.1.1   Sharp maximal function and BMO

Definition 5.1. *Let $f \in L^1_{\mathrm{loc}}[\mathbb{R}^n]$ and let Q be a cube in $\mathbb{R}^n$.*

1. *We define the average of f over Q by*

$$\underset{Q}{Avg}\, f = \frac{1}{\lambda_n\,[Q]} \int_Q f(\mathbf{y})d\lambda_n(\mathbf{y}).$$

2. *We defined the sharp maximal function $M^{\#}(f)(\mathbf{x})$, by*

$$M^{\#}(f)(\mathbf{x}) = \sup_Q \frac{1}{\lambda_n\,[Q]} \int_Q \left| f(\mathbf{y}) - \underset{Q}{Avg}\, f \right| d\lambda_n(\mathbf{y}),$$

   where the supremum is over all cubes containing $\mathbf{x}$.

3. *If $M^{\#}(f)(\mathbf{x}) \in L^{\infty}[\mathbb{R}^n]$, we say that f is of bounded mean oscillation. More precisely, the space of functions of bounded mean oscillation are defined by:*

$$BMO[\mathbb{R}^n] = \left\{ f \in L^1_{loc}[\mathbb{R}^n] : \ M^{\#}(f) \in L^{\infty}[\mathbb{R}^n] \right\}$$

   and

$$\|f\|_{BMO} = \left\| M^{\#}(f) \right\|_{L^{\infty}}.$$

We may also obtain an equivalent definition of $BMO[\mathbb{R}^n]$ using balls, but for our purposes, cubes are natural (see Grafakos [7] p. 546). We note that $BMO[\mathbb{R}^n]$ is not a Banach space and is not separable. In the next section, we construct the space $BMO^w[\mathbb{R}^n]$, that contains $BMO[\mathbb{R}^n]$ as a continuous dense embedding.

5.1.2   Weak maximal function and BMO^w

Definition 5.2. *Let $f \in L^1_{\mathrm{loc}}[\mathbb{R}^n]$ and let Q be a cube in $\mathbb{R}^n$.*

1. *We define f_{aQ} over Q by*

$$f_{aQ} = \left| \frac{1}{\lambda_n\,(Q)} \int_Q f(\mathbf{y})d\lambda_n\,(\mathbf{y}) \right| = \left| \underset{Q}{Avg}\, f \right|.$$

2. *We defined the weak maximal function $M^w(f)(\mathbf{x})$, by*

$$M^w(f)(\mathbf{x}) = \sup_Q \left| \frac{1}{\lambda_n(Q)} \int_Q [f(\mathbf{y}) - f_{aQ}] d\lambda_n(\mathbf{y}) \right|,$$

where the supremum is over all cubes containing $\mathbf{x}$.

3. *If $M^w(f)(\mathbf{x}) \in L^\infty[\mathbb{R}^n]$, we say that f is of weak bounded mean oscillation. We define BM by*

$$BM = \left\{ f(\mathbf{x}) \in L^1_{loc}[\mathbb{R}^n] : M^w(f)(\mathbf{x}) \in L^\infty[\mathbb{R}^n] \right\}$$

and define a seminorm on BM by $\|f\|_{BMO^w} = \|M^w(f)\|_\infty$.

Definition 5.3. *We define $BMO^w[\mathbb{R}^n]$ to be the completion of BM in the seminorm $\|\cdot\|_{BMO^w}$.*

Remark 5.4. *If $\|f\|_{BMO^w} = 0$, then for every cube Q containing $\mathbf{x}$,*

$$\frac{1}{\lambda_n(Q)} \int_Q f(\mathbf{y}) d\lambda_n(\mathbf{y}) - f_{aQ} = 0,$$

so that

$$\frac{1}{\lambda_n(Q)} \int_Q f(\mathbf{y}) d\lambda_n(\mathbf{y}) = \left| \frac{1}{\lambda_n(Q)} \int_Q f(\mathbf{y}) d\lambda_n(\mathbf{y}) \right|.$$

Since $a = |a|$ if and only if $a \geq 0$, we see that $f(\mathbf{x})$ is a nonnegative constant (a.e). It follows that $BMO^w[\mathbb{R}^n]$ is not a Banach space, but becomes one if we identify terms that differ by a constant.

Theorem 5.5. *The space $BMO[\mathbb{R}^n] \subset BMO^w[\mathbb{R}^n]$ as a continuous dense embedding.*

PROOF. It is easy to see that $BMO[\mathbb{R}^n]$ is a dense subset. To prove that its a continuous embedding, we note that

$$M^w(f)(\mathbf{x}) =$$

$$\sup_Q \frac{1}{\lambda_n(Q)} \left| \int_Q \left[f(\mathbf{y}) - \left| \underset{Q}{Avg} f \right| \right] d\lambda_n(\mathbf{y}) \right|$$

$$\leqslant \sup_Q \frac{1}{\lambda_n(Q)} \int_Q \left| f(\mathbf{y}) - \left| \underset{Q}{Avg} f \right| \right| d\lambda_n(\mathbf{y})$$

$$\leqslant \sup_Q \frac{1}{\lambda_n(Q)} \int_Q \left| f(\mathbf{y}) - \underset{Q}{Avg} f \right| d\lambda_n(\mathbf{y}) = M^\#(f)(\mathbf{x}).$$

$\square$

Corollary 5.6. *The space* $BMO^w[\mathbb{R}^n] \subset KS^\infty[\mathbb{R}^n]$ *as a continuous dense embedding.*

PROOF. The proof is easy since $L^1_{\mathrm{loc}}[\mathbb{R}^n] \cup L^\infty[\mathbb{R}^n] \subset KS^\infty[\mathbb{R}^n]$, with $L^\infty[\mathbb{R}^n]$ dense and

$$\|M^w(f)\|_{KS^\infty}$$
$$= \sup_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) M^w(f)(\mathbf{x}) d\lambda_n(\mathbf{x}) \right| \leqslant \left[\frac{1}{2\sqrt{n}} \right]^n \|M^w(f)\|_{L^\infty} = \left[\frac{1}{2\sqrt{n}} \right]^n \|f\|_{M^w}.$$

$\square$

5.2   Zachary Functions of Bounded Mean Oscillation

We now construct another class of functions. Let the family of cubes $\{Q_k\}$ centered at each rational point in $\mathbb{R}^n$ be the ones generated by the indicator functions $\{\mathcal{E}_k(\mathbf{x})\}$, for $KS^2[\mathbb{R}^n]$. Let $f \in L^1_{\mathrm{loc}}[\mathbb{R}^n]$ and as before, we define f_{ak} by

$$f_{ak} = \left| \frac{1}{\lambda_n[Q_k]} \int_{Q_k} f(\mathbf{y}) d\lambda_n(\mathbf{y}) \right| = \left| \frac{1}{\lambda_n[Q_k]} \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{y}) f(\mathbf{y}) d\lambda_n(\mathbf{y}) \right|.$$

Definition 5.7. *If* p, $1 \leq p < \infty$ *and* $t_k = 2^{-k}$, *we define* $\|f\|_{\mathcal{Z}^p}$ *by*

$$\|f\|_{\mathcal{Z}^p} = \left\{ \sum_{k=1}^\infty t_k \left| \frac{1}{\lambda_n[Q_k]} \int_{Q_k} [f(\mathbf{y}) - f_{ak}] \, d\lambda_n(\mathbf{y}) \right|^p \right\}^{1/p}.$$

The set of functions for which $\|f\|_{\mathcal{Z}^p} < \infty$ *is called the Zachary functions of bounded mean oscillation and order* p, $1 \leq p < \infty$. *If* $p = \infty$, *we say that* $f \in \mathcal{Z}^\infty[\mathbb{R}^n]$ *if*

$$\|f\|_{\mathcal{Z}^\infty} = \sup_k \left| \frac{1}{\lambda_n[Q_k]} \int_{Q_k} [f(\mathbf{y}) - f_{ak}] \, d\lambda_n(\mathbf{y}) \right| < \infty.$$

The following theorem shows how the Zachary spaces are related to the space of functions of Bounded mean oscillation $BMO[\mathbb{R}^n]$. (We omit proofs.)

Theorem 5.8. *If* $\mathcal{Z}^p[\mathbb{R}^n]$ *is the class of Zachary functions of bounded mean oscillation and order* p, $1 \leq p \leq \infty$, *then* $\mathcal{Z}^p[\mathbb{R}^n]$ *is a linear space and*

1. $\|\lambda f\|_{\mathcal{Z}^p} = |\lambda| \, \|f\|_{\mathcal{Z}^p}$.

2. $\|f + g\|_{\mathcal{Z}^p} \leqslant \|f\|_{\mathcal{Z}^p} + \|g\|_{\mathcal{Z}^p}$.

3. $\|f\|_{\mathcal{Z}^p} = 0, \Rightarrow f \geq 0$ *is a nonnegative constant (a.e.).*

4. *The space* $\mathcal{Z}^\infty[\mathbb{R}^n] \subset \mathcal{Z}^p[\mathbb{R}^n]$, $1 \leq p < \infty$, *as a dense continuous embedding.*

5. $BMO^w[\mathbb{R}^n] \subset \mathcal{Z}^\infty[\mathbb{R}^n]$, *as a dense continuous embedding.*

PROOF. The first four are clear. To prove (5), suppose that $f \in BMO^w[\mathbb{R}^n]$, then by definition of $\|\cdot\|_{BMO^w}$, the supremum is over the set of all cubes in $\mathbb{R}^n$. Since this set is much larger then the countable number used to define $\|\cdot\|_{\mathcal{Z}^\infty}$. It follows that $BMO^w[\mathbb{R}^n] \subset \mathcal{Z}^\infty[\mathbb{R}^n]$ as a dense continuous embedding. $\square$

We now consider the Carleson measure characterization of $BMO[\mathbb{R}^n]$ which will prove useful in construction another class of Zachary spaces that are Banach spaces (see Koch and Tataru [15]). If $u(\mathbf{x}, t)$ is a solution of the heat equation:

$$u_t - \Delta u = 0, \ u(\mathbf{x}, 0) = f(\mathbf{x}),$$

where $f \in L^1_{loc}[\mathbb{R}^n]$, it can be shown that

$$\|f\|_{BMO} = \sup_{\mathbf{x}, r} \left\{ \frac{1}{\lambda_n[Q(\mathbf{x}, r)]} \int_{Q(\mathbf{x}, r)} \int_0^{r^2} |\nabla u(\mathbf{y}, t)|^2 \, dt d\lambda_n(\mathbf{y}) \right\}^{1/2},$$

where the gradient is in the weak sense. Since

$$\sup_{k, r} \left\{ \frac{1}{\lambda_n[Q_k]} \int_{Q_k} \left| \int_0^{r^2} \nabla u(\mathbf{y}, t) dt \right|^2 d\lambda_n(\mathbf{y}) \right\}^{1/2}$$

$$\leq \sup_{k, r} \left\{ \frac{1}{\lambda_n[Q_k]} \int_{Q_k} \int_o^{r^2} |\nabla u(\mathbf{y}, t)|^2 dt d\lambda_n(\mathbf{y}) \right\}^{1/2},$$

we see that we can also define the seminorms on $BMO^w[\mathbb{R}^n]$ and $\mathcal{Z}^p[\mathbb{R}^n]$ by:

$$\|f\|_{BMO^w} = \sup_{\mathbf{x}, r} \left\{ \frac{1}{\lambda_n[Q(\mathbf{x}, r)]} \int_{Q(\mathbf{x}, r)} \left| \int_0^{r^2} \nabla u(\mathbf{y}, t) dt \right|^2 d\lambda_n(\mathbf{y}) \right\}^{1/2}$$

$$\|f\|_{\mathcal{Z}^p} = \sup_r \left\{ \sum_{k=1}^\infty t_k \frac{1}{\lambda_n[Q_k]} \int_{Q_k} \left| \int_0^{r^2} \nabla u(\mathbf{y}, t) dt \right|^p d\lambda_n(\mathbf{y}) \right\}^{1/p},$$

5.3   The Space of Functions $BMO^{-1}[\mathbb{R}^n]$

The class of functions $BMO^{-1}[\mathbb{R}^n]$, are defined as those for which:

$$\|f\|_{BMO^{-1}} = \sup_{\mathbf{x},r} \left\{ \frac{1}{\lambda_n\left[Q(\mathbf{x},r)\right]} \int_{Q(\mathbf{x},r)} \int_0^{r^2} |u(\mathbf{y},t)|^2 \, dt d\lambda_n(\mathbf{y}) \right\}^{1/2} < \infty.$$

It is known that $BMO^{-1}[\mathbb{R}^n]$ is a Banach space in the above norm.

Definition 5.9. *We say $f \in \mathcal{Z}^{-p}[\mathbb{R}^n]$, $1 \le p < \infty$ if*

$$\|f\|_{\mathcal{Z}^{-p}} = \sup_r \left\{ \sum_{k=1}^\infty t_k \frac{1}{\lambda_n\left[Q_k\right]} \int_{Q_k} \left| \int_0^{r^2} u(\mathbf{y},t)dt \right|^p d\lambda_n(\mathbf{y}) \right\}^{1/p} < \infty.$$

If $p = \infty$, we say that $f \in \mathcal{Z}^{-\infty}[\mathbb{R}^n]$ if

$$\|f\|_{\mathcal{Z}^{-\infty}} = \sup_{k,r} \left\{ \frac{1}{\lambda_n\left[Q_k\right]} \int_{Q_k} \left| \int_0^{r^2} u(\mathbf{y},t)dt \right| d\lambda_n(\mathbf{y}) \right\} < \infty.$$

Theorem 5.10. *For the class of spaces $\mathcal{Z}^{-p}[\mathbb{R}^n]$, we have:*

1. *For each p, $1 \le p \le \infty$, $\mathcal{Z}^{-p}[\mathbb{R}^n]$ is a Banach space.*

2. *The space $\mathcal{Z}^{-\infty}[\mathbb{R}^n] \subset \mathcal{Z}^{-p}[\mathbb{R}^n]$, $1 \le p < \infty$, as a dense continuous embedding.*

3. *The space $BMO^{-1}[\mathbb{R}^n] \subset \mathcal{Z}^{-\infty}[\mathbb{R}^n]$ as a dense continuous embedding.*

PROOF. The first two are obvious. To prove (3), if $f \in BMO^{-1}[\mathbb{R}^n]$, then

$$\|f\|_{\mathcal{Z}^{-\infty}} = \sup_{k,r} \left\{ \frac{1}{\lambda_n\left[Q_k\right]} \int_{Q_k} \left| \int_0^{r^2} u(\mathbf{y},t)dt \right| d\lambda_n(\mathbf{y}) \right\}^{2/2}$$

$$\le \sup_{k,r} \left\{ \frac{1}{\lambda_n\left[Q_k\right]} \int_{Q_k} \int_0^{r^2} |u(\mathbf{y},t)|^2 dt d\lambda_n(\mathbf{y}) \right\}^{1/2}$$

$$\le \sup_{\mathbf{x},r} \left\{ \frac{1}{\lambda_n\left[Q\left(\mathbf{x},r\right)\right]} \int_{Q(\mathbf{x},r)} \int_0^{r^2} |u(\mathbf{y},t)|^2 dt d\lambda_n(\mathbf{y}) \right\}^{1/2} = \|f\|_{BMO^{-1}}.$$

$$\square$$

Remark 5.11. *We could also define $BMO^{-w}[\mathbb{R}^n]$ by:*

$$\|f\|_{BMO^{-w}} = \sup_{\mathbf{x},r} \left\{ \frac{1}{\lambda_n \left[Q\left(\mathbf{x},r\right) \right]} \int_{Q(\mathbf{x},r)} \left| \int_0^{r^2} u(\mathbf{y},t)dt \right|^2 d\lambda_n\left(\mathbf{y}\right) \right\}^{1/2}.$$

It is easy to see that $BMO^{-w}[\mathbb{R}^n]$ is a Banach space and that $BMO^{-1}[\mathbb{R}^n] \subset BMO^{-w}[\mathbb{R}^n] \subset \mathcal{Z}^{-\infty}[\mathbb{R}^n]$ as a continuous embeddings. We <u>conjecture</u> that $BMO^{-w}[\mathbb{R}^n]$ and $\mathcal{Z}^{-\infty}[\mathbb{R}^n]$ allow us to replace solutions of the <u>heat equation</u> by those of the Schrödinger equation: $iu_t - \Delta u = 0$, $u(\mathbf{x},0) = f(\mathbf{x})$.

6   Applications

In this section, we consider a few applications associated with the families $KS^p[\mathbb{R}^n]$ and $SD^p[\mathbb{R}^n]$, $1 \le p \le \infty$. In each case, we either solve an open problem or provide a substantial improvement in methods used in a given area.

6.1   Markov Processes

In the study of Markov processes, it is well-known that semigroups associated with processes whose generators have unbounded coefficients, are not necessarily strongly continuous when defined on $\mathbb{C}_b[\mathbb{R}^n]$, the space of bounded continuous functions, or $\mathbb{UBC}[\mathbb{R}^n]$, the bounded uniformly continuous functions. These are the natural spaces on which to formulate the theory. The problem is that the generator of such a semigroup does not exist in the standard sense (is not C_0). As a consequence, a number of equivalent weaker definitions have been developed in the literature. A good discussion of this and related problems see Lorenzi and Bertoldi [22]. The following is one version of convergence used to a define semigroups using the generator in these cases.

Definition 6.1. *A sequence of functions $\{f_k\}$ in $\mathbb{C}_b[\mathbb{R}^n]$ is said to converge to f in the mixed topology, written τ^M-$\lim f_k = f$, if and only if $\sup_{k \in \mathbb{N}} \|f_k\|_\infty \leqslant M$ and $\|f_k - f\|_\infty \to 0$ uniformly on every compact subset of $\mathbb{R}^n$.*

It is clear that the family of bounded continuous functions, $\mathbb{C}_b[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n]$ as a continuous dense embedding.

Theorem 6.2. *If $\{f_k\}$ converges to f in the mixed topology on $\mathbb{C}_b[\mathbb{R}^n]$, then $\{f_n\}$ converges to f in the norm topology of $KS^p[\mathbb{R}^n]$ for each $1 \le p \le \infty$.*

PROOF. It suffices to prove the result for KS^∞. Since $\sup_{k \in \mathbb{N}} \|f_k\|_{KS^\infty} \le \sup_{k \in \mathbb{N}} \|f_k\|_{\mathbb{C}_b}$, we must prove that τ^M-$\lim f_k = f \Rightarrow \lim_{k \to \infty} \|f_k - f\|_{KS^\infty} = 0$. This follows from the fact that each cube used in the definition of the $KS^\infty[\mathbb{R}^n]$ norm, is a compact subset of $\mathbb{R}^n$. □

Theorem 6.3. *Suppose that $\hat{T}(t)$ is a transition semigroup defined on $\mathbb{C}_b[\mathbb{R}^n]$, with weak generator $\hat{A}$. Let $T(t)$ be the extension of $\hat{T}(t)$ to $KS^p[\mathbb{R}^n]$. Then $T(t)$ is strongly continuous, and the extension A of $\hat{A}$ to $KS^p[\mathbb{R}^n]$ is the strong generator of $T(t)$.*

Proof. Since $\mathbb{C}_b[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n]$ as a continuous embedding, for $1 \le p \le \infty$, we can apply Corollary 2.8 to show that $\hat{T}(t)$ has a bounded extension to $KS^2[\mathbb{R}^n]$. It is easy to see that the extended operator $T(t)$ is a semigroup. Since the τ^M topology on $\mathbb{C}_b[\mathbb{R}^n]$ is stronger then the norm topology on $KS^p[\mathbb{R}^n]$, we see that the generator A, of $T(t)$ is strong.   □

6.2   Feynman Path Integral

Feynman's introduction of his path integral into quantum theory created a major mathematical problem, that centered around the nonexistence of a measure for his integral. A number of analysts, beginning with Henstock [10] have advocated the HK-integral as a perfect substitute for this problem (see also Muldowney [23]). Since then, a number of researchers have addressed the problem. A fairly complete list of papers and books can be found in Gill and Zachary [5]. The book by Johnson and Lapidus [13] also contains additional sources.

The space $L^2[\mathbb{R}^n]$ is perfect for the Heisenberg and Schrödinger formulations of quantum mechanics, but fails for the Feynman formulation. In addition, neither the physically intuitive nor computationally efficient methods of Feynman are revealed on $L^2[\mathbb{R}^n]$. In this section we briefly show that $KS^2[\mathbb{R}^n]$ is the natural Hilbert space for the Feynman formulation of quantum mechanics. This space makes it possible to preserve all the physically intuitive and computational advantages discovered by Feynman and to represent the Heisenberg and Schrödinger formulations.

We assume that the reader is familiar with the HK-integral, but give a brief discussion in one dimension to establish notation. (A full discussion with proofs and some interesting examples can be found in Gill and Zachary [5].)

Definition 6.4. *Let $[a, b] \subset \mathbb{R}$, let $\delta(t)$ map $[a, b] \to (0, \infty)$, and let $\mathbf{P} = \{t_0, \tau_1, t_1, \tau_2, \cdots, \tau_n, t_n\}$, where $a = t_0 \leqslant \tau_1 \leqslant t_1 \leqslant \cdots \leqslant \tau_n \leqslant t_n = b$. We call $\mathbf{P}$ an HK-partition for δ, if for $1 \leqslant i \leqslant n$, $t_{i-1}, t_i \in (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$.*

Definition 6.5. *The function $f(t)$, $t \in [a, b]$, is said to have a HK-integral if there is an number $F[a, b]$ such that, for each $\varepsilon > 0$, there exists a function δ from $[a, b] \to (0, \infty)$ such that, whenever $\mathbf{P}$ is a HK-partition for δ, then (with $\Delta t_i = t_i - t_{i-1}$)*

$$\left| \sum\nolimits_{i=1}^{n} \Delta t_i f(\tau_i) - F[a, b] \right| < \varepsilon.$$

To understand Feynman's path integral in a natural setting, we consider the free particle in non-relativistic quantum theory in $\mathbb{R}^3$:

$$i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t} - \frac{\hbar^2}{2m}\Delta\psi(\mathbf{x},t) = 0, \ \psi(\mathbf{x},s) = \delta(\mathbf{x}-\mathbf{y}). \qquad (6.1)$$

The solution can be computed directly:

$$\psi(\mathbf{x},t) = K\left[\mathbf{x},\ t;\ \mathbf{y},\ s\right] = \left[\frac{2\pi i\hbar(t-s)}{m}\right]^{-3/2}\exp\left[\frac{im}{2\hbar}\frac{|\mathbf{x}-\mathbf{y}|^2}{(t-s)}\right].$$

Feynman wrote the above solution to equation (6.1) as

$$K\left[\mathbf{x},\ t;\ \mathbf{y},\ s\right] = \int_{\mathbf{x}(s)=y}^{\mathbf{x}(t)=x}\mathcal{D}\mathbf{x}(\tau)\exp\left\{\frac{im}{2\hbar}\int_s^t \left|\frac{d\mathbf{x}}{dt}\right|^2 d\tau\right\}, \qquad (6.2)$$

where

$$\int_{\mathbf{x}(s)=y}^{\mathbf{x}(t)=x}\mathcal{D}\mathbf{x}(\tau)\exp\left\{\frac{im}{2\hbar}\int_s^t \left|\frac{d\mathbf{x}}{dt}\right|^2 d\tau\right\} =:$$

$$\lim_{N\to\infty}\left[\frac{m}{2\pi i\hbar\varepsilon(N)}\right]^{3N/2}\int_{\mathbb{R}^3}\prod_{j=1}^N d\mathbf{x}_j \exp\left\{\frac{i}{\hbar}\sum_{j=1}^N\left[\frac{m}{2\varepsilon(N)}\left(\mathbf{x}_j-\mathbf{x}_{j-1}\right)^2\right]\right\}, \qquad (6.3)$$

with $\varepsilon(N) = (t-s)/N$. Equation (6.3) is an attempt to define an integral over the space of all continuous paths of the exponential of an integral of the classical Lagrangian on configuration space. This approach has led to a new approach for quantizing physical systems, called the path integral method.

Since $L^2[\mathbb{R}^3]$ is the standard state space for quantum physics, from a strictly mathematical point of view equation (6.3) has two major problems:

1. The kernel $K\left[\mathbf{x},\ t;\ \mathbf{y},\ s\right]$ and $\delta(\mathbf{x})$ are not in $L^2[\mathbb{R}^3]$.

2. The kernel $K\left[\mathbf{x},\ t;\ \mathbf{y},\ s\right]$ cannot be used to define a measure.

Since $KS^2[\mathbb{R}^3]$ contains the space of measures $\mathfrak{M}[\mathbb{R}^3]$, it follows that all the approximating sequences for the Dirac measure converge strongly to it in the $KS^2[\mathbb{R}^n]$ topology. (For example, $[\sin(\lambda\cdot\mathbf{x})/(\lambda\cdot\mathbf{x})] \in KS^2[\mathbb{R}^n]$ and converges strongly to $\delta(\mathbf{x})$.) Thus, the finitely additive set function defined on the Borel sets (Feynman kernel [3]): (with $m = 1$ and $\hbar = 1$)

$$\mathbb{K}_{\mathbf{f}}[t,\mathbf{x};s,B] = \int_B (2\pi i(t-s))^{-n/2}\exp\{i|\mathbf{x}-\mathbf{y}|^2\big/2(t-s)\}d\lambda_3(\mathbf{y})$$

is in $KS^2[\mathbb{R}^n]$ and $\|\mathbb{K}_{\mathbf{f}}[t,\mathbf{x};s,B]\|_{KS} \leqslant 1$, while $\|\mathbb{K}_{\mathbf{f}}[t,\mathbf{x};s,B]\|_{\mathfrak{M}} = \infty$ (the total variation norm) and

$$\mathbb{K}_{\mathbf{f}}[t,\mathbf{x};s,B] = \int_{\mathbb{R}^3}\mathbb{K}_{\mathbf{f}}[t,\mathbf{x};\tau,d\lambda_3(\mathbf{z})]\mathbb{K}_{\mathbf{f}}[\tau,\mathbf{z};s,B], \ \text{(HK-integral)}.$$

Definition 6.6. *Let* $\mathbf{P}_n = \{t_0, \tau_1, t_1, \tau_2, \cdots, \tau_n, t_n\}$ *be a HK-partition for a function* $\delta_n(s)$, $s \in [0, t]$ *for each* n, *with* $\lim_{n \to \infty} \Delta\mu_n = 0$ *(mesh). Set* $\Delta t_j = t_j - t_{j-1}, \tau_0 = 0$ *and, for* $\psi \in KS^2[\mathbb{R}^n]$, *define*

$$\int_{\mathbf{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau) \,;\, \mathbf{x}(0)] = e^{-\lambda t} \sum_{k=0}^{[\![\lambda t]\!]} \frac{(\lambda t)^k}{k!} \left\{ \prod_{j=1}^{k} \int_{\mathbf{R}^n} \mathbb{K}_{\mathbf{f}}[t_j, \mathbf{x}(\tau_j) \,;\, t_{j-1}, d\mathbf{x}(\tau_{j-1})] \right\},$$

and

$$\int_{\mathbf{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau) \,;\, \mathbf{x}(0)]\psi[\mathbf{x}(0)] = \lim_{\lambda \to \infty} \int_{\mathbf{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau) \,;\, \mathbf{x}(0)]\psi[\mathbf{x}(0)] \quad (6.4)$$

whenever the limit exists.

Remark 6.7. *In the above definition we have used the Poisson process. This is not accidental but appears naturally from a physical analysis of the information that is knowable in the micro-world (see [5]). It has been suggested by Kolokoltsov [16] that such jump processes provide another way to give meaning to Feynman diagrams.*

By Corollary 2.8 (with $\mathcal{B} = L^2[\mathbb{R}^n]$), the convolution is a well defined bounded operation on $KS^2[\mathbb{R}^n]$, so that the following is elementary.

Theorem 6.8. *The function* $\psi(\mathbf{x}) = 1 \in KS^2[\mathbb{R}^n]$ *and*

$$\int_{\mathbb{R}^{n[s,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau) \,;\, \mathbf{x}(s)] = \mathbb{K}_{\mathbf{f}}[t, \mathbf{x} \,;\, s, \mathbf{y}] = \frac{1}{\sqrt{[2\pi i(t-s)]^n}} \exp\{i|\mathbf{x} - \mathbf{y}|^2 \big/ 2(t-s)\}.$$

Remark 6.9. *The above result is what Feynman was trying to obtain without the appropriate space. When a potential is present, one uses the standard perturbation methods (i.e., Trotter-Kato theorems). A more general (sum over paths) result, that covers all application areas can be found in [5]. canonically*

It is clear that the position operator $\mathbf{x}$, *and the momentum operator* $\mathbf{p}$ *have closed densely defined extensions to* KS^2. *Treating the Fourier transform as an unitary operator, it has a bounded (unitary) extension to* KS^2 *so that* $\mathbf{x}$ *and* $\mathbf{p}$ *are still canonically conjugate pairs. Thus, both the Heisenberg and Schrödinger theories also have natural formulations on* KS^2. *It is in this sense that we say that* KS^2 *is the most natural Hilbert space for quantum mechanics.*

If we replace the Feynman kernel by the heat kernel, we have:

$$\mathbb{K}_{\mathbf{h}}[t, \mathbf{x} \,;\, s, B] = \int_B (2\pi(t-s))^{-n/2} \exp\{-|\mathbf{x} - \mathbf{y}|^2 \big/ 2(t-s)\} d\lambda_3(\mathbf{y})$$

is in $KS^2[\mathbb{R}^n]$ and

$$\mathbb{K}_{\mathbf{h}}[t,\mathbf{x}\,;\,s,B] = \int_{\mathbb{R}^3}\mathbb{K}_{\mathbf{h}}[t,\mathbf{x}\,;\,\tau,d\lambda_3(\mathbf{z})]\mathbb{K}_{\mathbf{h}}[\tau,\mathbf{z}\,;\,s,B], \quad (\text{HK-integral}).$$

Theorem 6.10. *For the function $\psi(\mathbf{x}) \equiv 1 \in KS^2[\mathbb{R}^n]$, we have*

$$\int_{\mathbb{R}^{n[s,t]}}\mathbb{K}_{\mathbf{h}}[\mathcal{D}\mathbf{x}(\tau)\,;\,\mathbf{x}(s)] = \mathbb{K}_{\mathbf{h}}[t,\mathbf{x}\,;\,s,\mathbf{y}] = \frac{1}{\sqrt{[2\pi(t-s)]^n}}\exp\{-|\mathbf{x}-\mathbf{y}|^2\big/2(t-s)\}.$$

This result implies that all known results for the Weiner path integral also have extensions to $KS^2[\mathbb{R}^n]$, with initial data in $HK[\mathbb{R}^n]$. Furthermore, the strong continuity of the semigroup generating the heat equation means that the integral can still be concentrated on the space of continuous paths.

6.3 Examples

If we treat $K[\mathbf{x},\ t;\ \mathbf{y},\ s]$ as the kernel for an operator acting on good initial data, then a partial solution has been obtained by a number of workers. (See [5] for references to all the important contributions in this direction.) The standard method is to compute the Wiener path integral for the problem under consideration and then use analytic continuation in the mass to provide a rigorous meaning for the Feynman path integral. The standard reference is Johnson and Lapidus [13]. The following example provides a path integral representation for a problem that cannot be solved using analytic continuation via a Gaussian kernel. (For the general non-Gaussian case, see [5].) It is shown that, if the vector $\mathbf{A}$ is constant, $\mu = mc/\hbar$, and $\boldsymbol{\beta}$ is the standard beta matrix of relativistic quantum theory, then the solution to the square-root equation for a spin 1/2 particle:

$$i\hbar\partial\psi(\mathbf{x},t)/\partial t = \left\{\boldsymbol{\beta}\sqrt{c^2\left(\mathbf{p}-\tfrac{e}{c}\mathbf{A}\right)^2 + m^2c^4}\right\}\psi(\mathbf{x},t),\ \psi(\mathbf{x},0) = \psi_0(\mathbf{x}),$$

is given by:

$$\psi(\mathbf{x},t) = \mathbf{U}[t,0]\psi_0(\mathbf{x}) = \int_{\mathbb{R}^3}\exp\left\{\frac{ie}{2\hbar c}\,(\mathbf{x}-\mathbf{y})\cdot\mathbf{A}\right\}\mathbf{K}\,[\mathbf{x},t\,;\,\mathbf{y},0]\,\psi_0(\mathbf{y})d\mathbf{y},$$

where

$$\mathbf{K}\,[\mathbf{x},t\,;\,\mathbf{y},0] = \frac{ict\mu^2\beta}{4\pi}\begin{cases} \dfrac{-H_2^{(1)}\left[\mu\left(c^2t^2-||\mathbf{x}-\mathbf{y}||^2\right)^{1/2}\right]}{[c^2t^2-||\mathbf{x}-\mathbf{y}||^2]}, & ct < -||\mathbf{x}-\mathbf{y}||, \\[2ex] \dfrac{-2iK_2\left[\mu\left(||\mathbf{x}-\mathbf{y}||^2-c^2t^2\right)^{1/2}\right]}{\pi[||\mathbf{x}-\mathbf{y}||^2-c^2t^2]}, & c\,|t| < ||\mathbf{x}-\mathbf{y}||, \\[2ex] \dfrac{H_2^{(2)}\left[\mu\left(c^2t^2-||\mathbf{x}-\mathbf{y}||^2\right)^{1/2}\right]}{[c^2t^2-||\mathbf{x}-\mathbf{y}||^2]}, & ct > ||\mathbf{x}-\mathbf{y}||. \end{cases}$$

The function $K_2(\,\cdot\,)$ is a modified Bessel function of the third kind of second order, while $H_2^{(1)}$, $H_2^{(2)}$ are Hankel functions (see Gradshteyn and Ryzhik [8]). Thus, we have a kernel that is far from the standard form. This example can be found in [5], where we only considered the kernel for the Bessel function term. In that case, it was shown that, under appropriate conditions, this term will reduce to the free-particle Feynman kernel and, if we set $\mu = 0$, we get the kernel for a (spin 1/2) massless particle.

6.4   The Navier-Stokes Problem

In this section, we use $SD^2[\mathbb{R}^3]$ to provide the strongest possible a priori estimate for the nonlinear term of the classical Navier-Stokes equation.

6.4.1   Introduction

Let $[L^2(\mathbb{R}^3)]^3$ be the Hilbert space of square integrable functions on $\mathbb{R}^3$, let $\mathbb{H}[\mathbb{R}^3]$ be the completion of the set of functions in $\left\{ \mathbf{u} \in \mathbb{C}_0^\infty[\mathbb{R}^3]^3 \mid \nabla \cdot \mathbf{u} = 0 \right\}$ which vanish at infinity with respect to the inner product of $[L^2(\mathbb{R}^3)]^3$. The classical Navier-Stokes initial-value problem (on $\mathbb{R}^3$ and all $T > 0$) is to find a function $\mathbf{u} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ and $p : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ such that

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(t) \text{ in } (0,T) \times \mathbb{R}^3,$$
$$\nabla \cdot \mathbf{u} = 0 \text{ in } (0,T) \times \mathbb{R}^3 \text{ (in the weak sense)}, \qquad (6.5)$$
$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \mathbb{R}^3.$$

The equations describe the time evolution of the fluid velocity $\mathbf{u}(\mathbf{x},t)$ and the pressure p of an incompressible viscous homogeneous Newtonian fluid with constant viscosity coefficient ν in terms of a given initial velocity $\mathbf{u}_0(\mathbf{x})$ and given external body forces $\mathbf{f}(\mathbf{x},t)$.

Let $\mathbb{P}$ be the (Leray) orthogonal projection of $(L^2[\mathbb{R}^3])^3$ onto $\mathbb{H}[\mathbb{R}^3]$ and define the Stokes operator by: $\mathbf{A}\mathbf{u} =: -\mathbb{P}\Delta \mathbf{u}$, for $\mathbf{u} \in D(\mathbf{A}) \subset \mathbb{H}^2[\mathbb{R}^3]$, the domain of $\mathbf{A}$. If we apply $\mathbb{P}$ to equation (6.5), with $B(\mathbf{u},\mathbf{u}) = \mathbb{P}(\mathbf{u} \cdot \nabla)\mathbf{u}$, we can recast it into the standard form:

$$\partial_t \mathbf{u} = -\nu \mathbf{A}\mathbf{u} - B(\mathbf{u},\mathbf{u}) + \mathbb{P}\mathbf{f}(t) \text{ in } (0,T) \times \mathbb{R}^3,$$
$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \mathbb{R}^3, \qquad (6.6)$$

where the orthogonal complement of $\mathbb{H}$ relative to $\{L^2(\mathbb{R}^3)\}^3$, $\{\mathbf{v} : \mathbf{v} = \nabla q,\ q \in \mathbb{H}^1[\mathbb{R}^3]\}$, is used to eliminate the pressure term (see [4] or [[29], [32],[33]).

Definition 6.11. *We say that a velocity vector field in $\mathbb{R}^3$ is* <u>reasonable</u> *if for $0 \leq t < \infty$, there is a continuous function $m(t) > 0$, depending only on t and a constant M_0, which may depend on $\mathbf{u}_0$ and f, such that*

$$0 < m(t) \leqslant \|\mathbf{u}(t)\|_{\mathbb{H}} \leq M_0.$$

The above definition formalizes the requirement that the fluid has nonzero but bounded positive definite energy. However, this condition still allows the velocity to approach zero at infinity in a weaker norm.

6.4.2 The Nonlinear Term: A Priori Estimates

The difficulty in proving the existence and uniqueness of global-in-time strong solutions for equation (6.6) is directly linked to the problem of getting good a priori estimates for the nonlinear term $B(\mathbf{u}, \mathbf{u})$. For example, using standard methods on $\mathbb{H}$, the following estimates are known. If $\mathbf{u}, \mathbf{v} \in D(\mathbf{A})$, a typical bound in the $\mathbb{H}$ norm for the nonlinear term in equation (6.6) can be found in Sell and You [29] (see page 366):

$$\max\left\{\|B(\mathbf{u}, \mathbf{v})\|_{\mathbb{H}}, \ \|B(\mathbf{v}, \mathbf{u})\|_{\mathbb{H}}\right\} \leqslant C_0 \left\|\mathbf{A}^{5/8}\mathbf{u}\right\|_{\mathbb{H}} \left\|\mathbf{A}^{5/8}\mathbf{v}\right\|_{\mathbb{H}}.$$

In this section, we show how $SD^2[\mathbb{R}^3]$ allows us to obtain the best possible a priori estimates. Let $\mathbb{H}_{sd}$ be the closure of $\mathbb{H} \cap SD^2[\mathbb{R}^3]$ in the SD^2 norm.

Theorem 6.12. *Suppose that the reasonable vector field $\mathbf{u}(\mathbf{x}, t) \in SD^2[\mathbb{R}^3] \cap D(\mathbf{A})$ satisfies (6.6) and $\mathbf{A}$ is the Stokes operator. Then*

   1.

$$\langle \nu\mathbf{A}\mathbf{u}, \mathbf{u}\rangle_{\mathbb{H}_{sd}} = 3\nu \|\mathbf{u}\|_{\mathbb{H}_{sd}}^2. \tag{6.7}$$

   2. There exists a constants $M_1, M_2; M_i = M_i(\mathbf{u}_0, \mathbf{f})) > 0$, such that

$$\left|\langle B(\mathbf{u}, \mathbf{u}), \mathbf{u}\rangle_{\mathbb{H}_{sd}}\right| \leq M_1 \|\mathbf{u}\|_{\mathbb{H}_{sd}}^3 \tag{6.8}$$

   3. and

$$max\{\|B(\mathbf{u}, \mathbf{v})\|_{\mathbb{H}_{sd}}, \ \|B(\mathbf{v}, \mathbf{u})\|_{\mathbb{H}_{sd}}\} \leqslant M_2 \|\mathbf{u}\|_{\mathbb{H}_{sd}} \|\mathbf{v}\|_{\mathbb{H}_{sd}}. \tag{6.9}$$

PROOF. From the definition of the inner product, for $\mathbb{H}_{sd}[\mathbb{R}^3]$ we have

$$\langle \nu\mathbf{A}\mathbf{u}, \mathbf{u}\rangle_{\mathbb{H}_{sd}} = \nu \sum_{m=1}^{\infty} t_m \left[\int_{\mathbb{R}^3} \mathcal{E}_m(\mathbf{x}) \cdot \mathbf{A}\mathbf{u}(\mathbf{x})d\lambda_3(\mathbf{x})\right] \overline{\left[\int_{\mathbb{R}^3} \mathcal{E}_m(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y})d\lambda_3(\mathbf{y})\right]}.$$

Using the fact that $\mathbf{u} \in D(\mathbf{A})$, it follows that

$$\int_{\mathbb{R}^3} \mathcal{E}_m(\mathbf{y}) \cdot \partial_{y_j}^2 \mathbf{u}(\mathbf{y}) d\lambda_3(\mathbf{y}) = \int_{\mathbb{R}^3} \partial_{y_j}^2 \mathcal{E}_m(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) d\lambda_3(\mathbf{y}) = (i)^2 \int_{\mathbb{R}^3} \mathcal{E}_m(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) d\lambda_3(\mathbf{y}).$$

Using this in the above equation and summing on j, we have $(\mathbf{A} = -\mathbb{P}\Delta)$

$$\int_{\mathbb{R}^3} \mathcal{E}_m(\mathbf{y}) \cdot \mathbf{A}\mathbf{u}(\mathbf{y}) d\lambda_3(\mathbf{y}) = 3 \int_{\mathbb{R}^3} \mathcal{E}_m(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) d\lambda_3(\mathbf{y}).$$

It follows that

$$\langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle_{\mathbb{H}_{sd}}$$
$$= 3 \sum_{m=1}^{\infty} t_m \left[\int_{\mathbb{R}^3} \mathcal{E}_m(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) d\lambda_3(\mathbf{x}) \right] \overline{\left[\int_{\mathbb{R}^3} \mathcal{E}_m(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) d\lambda_3(\mathbf{y}) \right]}$$
$$= 3 \left\| \mathbf{u} \right\|_{\mathbb{H}_{sd}}^2.$$

This proves (6.7). To prove (6.8), let

$$b(\mathbf{u}, \mathbf{v}, \mathcal{E}_m) = \int_{\mathbb{R}^3} (\mathbf{u}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{x})) \cdot \mathcal{E}_m(\mathbf{x}) d\lambda_3(\mathbf{x})$$

and define the vector $\mathbf{I}$ by $\mathbf{I} = [1, 1, 1]^t$. We start with integration by parts and $\nabla \cdot \mathbf{u} = 0$, to get

$$b(\mathbf{u}, \mathbf{v}, \mathcal{E}_m) = -b(\mathbf{u}, \mathcal{E}_m, \mathbf{v}) = -i \int_{\mathbb{R}^3} (\mathbf{u}(\mathbf{x}) \cdot \mathbf{I}) (\mathcal{E}_m(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})) d\lambda_3(\mathbf{x}).$$

From the above equation, we have $(m \leftrightarrow (k, i))$

$$|b(\mathbf{u}, \mathbf{v}, \mathcal{E}_m)| \leqslant \sqrt{3} \int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x})| |\mathbf{v}(\mathbf{x})| d\lambda_3(\mathbf{x}) \sup_k \|\mathcal{E}_m\|_{\infty}$$
$$\leqslant C_1 \|\mathbf{u}\|_{\mathbb{H}} \|\mathbf{v}\|_{\mathbb{H}}.$$

We also have:
$$\left| \int_{\mathbb{R}^3} \mathbf{w}(\mathbf{x}) \cdot \mathcal{E}_m(\mathbf{x}) d\lambda_3(\mathbf{x}) \right| \leqslant C_2 \|\mathbf{w}\|_{\mathbb{H}}.$$

If we combine the last two results, we get that:

$$\left| \langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{\mathbb{H}_{sd}} \right|$$
$$\leqslant \sum_{m=1}^{\infty} t_m |b(\mathbf{u}, \mathbf{v}, \mathcal{E}_m)| \overline{\left| \int_{\mathbb{R}^3} \mathbf{w}(\mathbf{y}) \cdot \mathcal{E}_m(\mathbf{y}) d\lambda_3(\mathbf{y}) \right|} \qquad (6.10)$$
$$\leqslant C \|\mathbf{u}\|_{\mathbb{H}} \|\mathbf{v}\|_{\mathbb{H}} \|\mathbf{w}\|_{\mathbb{H}}.$$

Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are reasonable velocity vector fields, there is a constant M_1 depending on $\mathbf{u}_0, \mathbf{v}_0, \mathbf{w}_0$ and f, such that

$$C \left\| \mathbf{u} \right\|_{\mathbb{H}} \left\| \mathbf{v} \right\|_{\mathbb{H}} \left\| \mathbf{w} \right\|_{\mathbb{H}} \leq M_1 \left\| \mathbf{u} \right\|_{\mathbb{H}_{sd}} \left\| \mathbf{v} \right\|_{\mathbb{H}_{sd}} \left\| \mathbf{w} \right\|_{\mathbb{H}_{sd}}. \tag{6.11}$$

If $\mathbf{w} = \mathbf{v} = \mathbf{u}$, we have that:

$$\left| \langle B(\mathbf{u}, \mathbf{u}), \mathbf{u} \rangle_{\mathbb{H}_{sd}} \right| \leqslant M_1 \left\| \mathbf{u} \right\|_{\mathbb{H}_{sd}}^3.$$

This proves (6.8). The proof of (6.9) is a straight forward application of (6.10) and (6.11). □

Conclusion

In this survey we have shown have constructed a number of classes Banach spaces KS^p, $1 \leq p \leq \infty$, SD^p, $1 \leq p \leq \infty$, $\mathcal{Z}^p$, $1 \leq p \leq \infty$, $\mathcal{Z}^{-p}$, $1 \leq p \leq \infty$ and BMO^w. These spaces are of particular interest because they contain the Henstock-Kurzweil integrable functions. (They also contain all functions that are integrable via any of the classical integrals.) The KS^p and SD^p class contain the standard L^p spaces as dense compact embeddings. They also contain the test functions $\mathcal{D}[\mathbb{R}^n]$, that $\mathcal{D}'[\mathbb{R}^n]$ is a continuous embedding into the dual space. The SD^p class has the remarkable property that $\left\| D^\alpha f \right\|_{SD} = \left\| f \right\|_{SD}$, for every index α.

In the analytical theory of Markov processes, it is well-known that, in general, the semigroup $T(t)$ associated with the process is not strongly continuous on $\mathbb{C}_b[\mathbb{R}^n]$, the space of bounded continuous functions or $\mathbb{UBC}[\mathbb{R}^n]$, the bounded uniformly continuous functions. We have shown that the weak generator defined by the mixed locally convex topology on $\mathbb{C}_b[\mathbb{R}^n]$ is a strong generator on $KS^p[\mathbb{R}^n]$ (e.g., $T(t)$ is strongly continuous on $KS^p[\mathbb{R}^n]$ for $1 \leq p \leq \infty$).

We also have used KS^2 to construct the free-particle path integral in the manner originally intended by Feynman. It is shown in [5] that KS^2 has a claim as the natural representation space for the Feynman formulation of quantum theory in that it allows representations for both the Heisenberg and Schrödinger representations for quantum mechanics, a property not shared by L^2.

We also have used SD^2 to to provided the strongest a priori bounds for the nonlinear term of the classical Navier-Stokes equation.

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, (1975).

[2] A. Alexiewicz, *Linear functionals on Denjoy-integrable functions*, Colloq. Math., **1** (1948), 289-293.

[3] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, (1965).

[4] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations,* 2nd Edition, Vol. II, Springer Tracts in Natural Philosophy, Vol. **3**9 Springer, New York, 1997.

[5] T. L. Gill and W. W. Zachary, *Functional Analysis and the Feynman operator Calculus*, Springer New York, (2016).

[6] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron and Henstock*, Graduate Studies in Mathematics, Vol. 4, Amer. Math. Soc., (1994).

[7] L. Grafakos, *Classical and Modern Fourier Analysis,* Person Prentice-Hall, New Jersey, 2004.

[8] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, (1965).

[9] L. Gross, Abstract Wiener spaces, Proc. Fifth Berkeley Symposium on Mathematics Statistics and Probability, 1965, pp. 31?42. MR 35:3027

[10] R. Henstock, *The General Theory of Integration*, Clarendon Press, Oxford, (1991).

[11] R. Henstock, *A Riemann-type integral of Lebesque power*, Canadian Journal of Mathematics **20** (1968), 79-87.

[12] C. S. Hönig, *There is no natural Banach space norm on the space of Kurzweil-Henstock-Denjoy-Perron integrable functions*, 30° Seminário Brasileiro de Análise, (1989), 387-397.

[13] G. W. Johnson and M. L. Lapidus, *The Feynman Integral and Feynman's Operational Calculus*, Oxford U. Press, New York, (2000).

[14] F. Jones, *Lebesgue Integration on Euclidean Space*, Revised Edition, Jones and Bartlett Publishers, Boston, (2001).

[15] H. Koch and D. Tataru. *Well-posedness for the Navier-Stokes equations,* Adv. Math. **157**(1) (2001), 22-35.

[16] V. Kolokoltsov, *A new path integral representation for the solutions of the Schrödinger equation*, Math. Proc. Cam. Phil. Soc. **32** (2002), 353-375.

[17] V. Koltchinskii, C.T. Abdallah, M. Ariola, P. Dorato and D. Panchenko, *Improved Sample Complexity Estimates for Statistical Learning Control of Uncertain Systems,* IEEE Trans. Automatic Control **45** (2000), 2383-2388.

[18] J. Kuelbs, *Gaussian measures on a Banach space,* Journal of Functional Analysis **5** (1970), 354–367.

[19] J. Kurzweil, *Nichtabsolut konvergente Integrale*, Teubner-Texte zür Mathematik, Band **26**, Teubner Verlagsgesellschaft, Leipzig, (1980).

[20] P. D. Lax, *Symmetrizable linear tranformations,* Comm. Pure Appl. Math. **7** (1954), 633-647.

[21] G. Leoni, *A First Course in Sobolev Spaces,* AMS Graduate Studies in Math. **105**, Providence, R.I, 2009.

[22] L. Lorenzi and M. Bertoldi, *Analytical Methods for Markov Semigroups*, Monographs and Textbooks in Pure and Applied Mathematics, Chapman & Hall/CRC, New York, (2007).

[23] P. Muldowney, *A Modern Theory of Random Variation*, Wiley, (2012).

[24] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods,* SIAM, (1992).

[25] K. M. Ostaszewski, *The space of Henstock integrable functions of two variables*, Internat. J. Math. and Math. Sci. **11** (1988), 15?22.

[26] A. Papageorgiou and J.G. Traub, *Faster Evaluation of Multidimensional Integrals,* Computers in Physics, Nov., (1997), 574-578.

[27] S. Paskov and J.G. Traub, *Faster Valuation of Financial Derivatives,* Journal of Portfolio Management, **22** (1995), 113-120.

[28] W. F. Pfeffer, *The Riemann Approach to Integration: Local Geometric Theory*, Cambridge Tracts in Mathematics **109**, Cambridge University Press, (1993).

[29] G. R. Sell and Y. You, *Dynamics of evolutionary equations,* Applied Mathematical Sciences, Vol. **1**43, Springer, New York, 2002.

[30] V. Steadman, *Theory of operators on Banach spaces*, Ph.D thesis, Howard University, 1988.

[31] E. Talvila, *The distributional Denjoy integral*, Real Analysis Exchange, **33** (2008), 51-82.

[32] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis,* AMS Chelsea Pub., Providence, RI, 2001.

[33] R. Temam, *Infinite dimensional dynamical systems in mechanics and physics,* Applied Mathematical Sciences, Vol. **6**8, Springer, New York, 1988.

[34] L. Tuo-Yeong, *Henstock-Kurweil Integration on Euclidean Spaces,* Series in Real Analysis-Vol 12 World Scientific, New Jersey, (2011).