INROADS

Real Analysis Exchange
Vol. 42(2), 2017, pp. 429–432

Hajrudin Fejzić,* Department of Mathematics, California State University, San Bernardino, CA 92407, U.S.A. email: hfejzic@csusb.edu

A NOTE ON MONOTONICITY THEOREMS FOR APPROXIMATELY CONTINUOUS FUNCTIONS

We say that $f(x)$ is approximately continuous at a if there is a measurable set V such that a is a density point of V and the restriction $f|_V(x)$ is continuous at a. If in addition $f|_V(x)$ is differentiable at a then we say that $f(x)$ is approximately differentiable at a, and we denote the derivative by $f'_{ap}(a)$. It is well known, Theorem 2.5 [2], that $f'_{ap}(x) \geq 0$ at every point x of an interval I implies that f is nondecreasing on I. See [2], page 107 for the proof of this result. The first part of this note is to provide a simple proof of Theorem 2.5. Since the conditions used in our proof are much weaker than those of Theorem 2.5, our Theorem 2 can be also regarded as its generalization.

Definition 1. *Let A be a measurable set and $\lambda(A)$ its Lebesque measure. The upper and lower right densities of a point x with respect to A are $d^+(A,x) = \limsup_{h\to 0^+} \lambda(A\cap(x,x+h))/h$ and $d_+(A,x) = \liminf_{h\to 0^+} \lambda(A\cap(x,x+h))/h$ respectively. The upper and lower left densities are defined as $d^-(A,x) = \limsup_{h\to 0^+} \lambda(A\cap(x-h,x))/h$ and $d_-(A,x) = \liminf_{h\to 0^+} \lambda(A\cap(x-h,x))/h$ respectively. When the two lower densities are 1, we say that x is a density point of A.*

Let $E_y = \{x : f(x) < y\}$, and $E^y = \{x : f(x) > y\}$. The condition that f is approximately continuous at a implies that if $a \in E_y$ then a is a density point of E_y and if $a \in E^y$, then a is a density point of E^y. Moreover if f is approximately continuous on an interval I, then it's measurable (See Theorem 5.2 [2]). In proving monotonicity results, the assumption $f'_{ap}(a) \geq 0$ on an interval can be replaced with the weaker assumption that $f'_{ap}(a) > 0$, for

Mathematical Reviews subject classification: Primary: 26A24
Key words: approximately continuous functions, monotonicity theorems
Received by the editors January 25, 2017
Communicated by: Brian S. Thomson
*The research for this paper was supported by CSUSB 2016 Mini-grant.

the weaker assumption is true for $f_n(x) = f(x) + \frac{1}{n}x$. Now the monotonicity of $f_n(x)$ for every n implies the monotonicity of f. The assumption $f'_{ap}(a) > 0$, implies that $d_+(\{x > a : f(x) \geq f(a)\}, a) = 1$.

Theorem 2. *Let f be a measurable function defined on an open interval I, such that for $x \in E_y$ we have $d^-(E_y, x) > 1/2$ while for $x \in E^y$ we have $d^+(E^y, x) > 1/2$. In addition suppose that for all but countably many $x \in I$, $d^+(\{z > x : f(z) \geq f(x)\}, x) > 1/2$. Then f is nondecreasing on I.*

PROOF. Suppose to the contrary that there is $a < b$ such that $f(a) > f(b)$. Let $f(a) > y > f(b)$. We will show that there is $a < r < b$ such that $f(r) = y$ and $d^+(\{z > r : f(z) \geq f(r)\}, r) \leq 1/2$. Since the last inequality can hold for only countably many r, and we have uncountably many choices for y, we have a contradiction.

Let $A = \{x : f(x) \geq y\}$. Consider a continuous function g on $[a, b]$ defined by

$$g(x) = \lambda(A \cap (a, x)) - \frac{1}{2}(x - a).$$

Then $g(a) = 0$ and since $a \in E^y \subset A$, by assumption we have that $d^+(A, a) \geq d^+(E^y, a) > 1/2$ which implies that $\limsup_{x \to a^+} \frac{g(x)-g(a)}{x-a} > 0$. Thus if r denotes the point where $g(x)$ attains its maximum, then $a < r \leq b$. Hence for every $a \leq z < r$, $g(z) \leq g(r)$. This implies that $\frac{\lambda(A \cap (z,r))}{r-z} \geq 1/2$. Taking $\liminf_{z \to r^-}$ we see that

$$d_-(A, r) \geq 1/2.$$

If $r \in E_y$, then by assumption we would have $d^-(E_y, r) > 1/2$, but E_y is disjoint from A so this is impossible. Hence $r \notin E_y$ and thus $r < b$.

So for all $r < z \leq b$, $g(r) \geq g(z)$ which implies that $\frac{\lambda(A \cap (r,z))}{z-r} \leq 1/2$. Taking $\limsup_{z \to r^+}$ we see that

$$d^+(A, r) \leq 1/2.$$

Since $E^y \subset A$ we also have $d^+(E^y, r) \leq 1/2$. If $r \in E^y$, then by assumption $d^+(E^y, r) > 1/2$, which is a contradiction. Hence $r \notin E_y \cup E^y$, that is $f(r) = y$ and $A \supset \{z > r : f(z) \geq f(r)\}$.

□

Another monotonicity theorem for approximately continuous functions was obtained by Ornstein in 1971, [6].

Ornestein Theorem. *Let f be approximately continuous on an interval I. Suppose that for all (but countably many) $x \in I$, $d^+(\{w > x : f(w) \geq f(x)\}, x) > 0$. Then f is nondecreasing.*

The Ornestein theorem provides a positive answer to a question from 1937 by A. J. Ward.

Scottish book problem # 157. *Let f be approximately continuous on an interval I. Suppose that for all $x \in I$, the approximate upper right derivate, $D^+_{ap} f(x)$ is positive. Is $f(x)$ monotone increasing?*

See [1] on problem 157 and [3] for a different proof of Ornestein theorem. In the second part of this note, using similar ideas as in our proof of Theorem 2 we present a shorter proof of Ornestein theorem. Moreover we prove "the parenthetical version" of Ornestein's theorem which is stronger than the original version.

PROOF. Suppose to the contrary that there is $a < b$ such that $f(a) > f(b)$. Let $f(b) < y < f(a)$. We will show that there is $a < r < b$ such that $f(r) = y$ and $d^+(\{w > r : f(w) \geq f(r)\}, r) = 0$. Since the last equality can hold for only countable many r, and we have uncountably many choices for y, we have a contradiction.

Let $A = \{x : f(x) \geq y\}$, and for any positive integer n let $C_n = \{t : f(t) > y - \frac{y-f(b)}{n}\}$ so that C_n's are nested and $A = \cap_{n=1}^{\infty} C_n$. Consider a continuous function

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \lambda(C_n \cap (a, x)) + \lambda(A \cap (a, x)) - (x - a).$$

Since $g(a) = 0$, and a is a right density point of A we have that $\lim_{x \to a^+} \frac{g(x)-g(a)}{x-a} \geq \lim_{x \to a^+} \frac{2\lambda(A \cap (a,x))}{x-a} - 1 = 1$. Therefore if r is the point where g attains its maximum then $a < r \leq b$. Thus for every $a \leq z < r$, we have $g(z) \leq g(r)$ which implies that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda(C_n \cap (z, r))}{r - z} + \frac{\lambda(A \cap (z, r))}{r - z} \geq 1. \tag{1}$$

If there is an integer n such that $r \notin C_n$, then approximate continuity implies that $d^-(C_{n+1}, r) = 0$. From (1) we get $1 - \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} \frac{\lambda(C_{n+1} \cap (z,r))}{r-z} + \frac{\lambda(C_{n+1} \cap (z,r))}{r-z} \geq 1$. Taking $\limsup_{z \to r^-}$ we obtain a contradiction $1 - \frac{1}{2^{n+1}} \geq 1$.

Thus $r \in C_n$ for all n and thus also in $A = \cap_{n=1}^{\infty} C_n$. Hence $r < b$ and $g(w) \leq g(r)$ for all $r < w \leq b$. This implies that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda(C_n \cap (r, w))}{w - r} + \frac{\lambda(A \cap (r, w))}{w - r} \leq 1. \tag{2}$$

Since $r \in C_n$ for all n, approximate continuity of f implies that $\lim_{w\to r} \frac{\lambda(C_n \cap (r,w))}{w-r} = 1$. Taking $\limsup_{w\to r^+}$ in (2) we obtain $1+d^+(A,r) \le 1$, that is $d^+(A,r) = 0$.

If $f(r) > y$ then by the approximate continuity of f we would have $d_+(A,r) = 1$. Therefore $f(r) = y$, $A \supset \{w > r : f(w) \ge f(r)\}$.

□

References

[1] K. Beanland, P. Humke and T. Richards, *On Scottish Book 157,* Real Anal. Exchange, **41(2)** (2016), 331–346.

[2] A. Bruckner, *Differentiation of Real Functions,* CRM Monograph Series, American Mathematical Society (1994).

[3] C. Freiling, P. Humke and R. O'Malley, *An Alternate Solution to Scottish Book 157,* Real Anal. Exchange, (2016), 403–408.

[4] C. Freiling, P. Humke and R. O'Malley, *Approximetely continuous functions have approximate extrema, a new proof,* Real Anal. Exchange, (2016), 409-414.

[5] R. O'Malley, *Approximate maxima,* Fund. Math., **1** (1977), 75–81.

[6] D. Ornstein, *A characterization of monotone functions,* Illinois J. Math., **15** (1971), 73-76.